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Riemannian Submersions and Lattices in 2-step Nilpotent Lie Groups

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We consider simply connected, 2-step nilpotent Lie groups N, all of which are diffeomorphic to Euclidean spaces via the Lie group exponential map exp: $\mathcal{N} \to N$. We show that every such N with a suitable left invariant metric is the base space of a Riemannian submersion and homomorphism $\rho : N^* \to N$, where the fibers of ρ are flat, totally geodesic Euclidean spaces. The left invariant metric and Lie algebra of N^* are obtained from N by constructing a Lie algebra $\mathfrak{G}$ whose Killing form B is negative semidefinite. If B is negative definite, then we show that N^* admits a (cocompact) lattice subgroup Γ^*. Moreover $\Gamma = \rho(\Gamma^*)$ is a lattice in N if $\Gamma^* \cap \mathrm{Ker}(\rho)$ is a lattice in $\mathrm{Ker}(\rho)$. Conversely, if N admits a lattice Γ, then N^* admits a lattice Γ^* such that $\Gamma = \rho(\Gamma^*)$. In this case the Riemannian submersion and homomorphism $\rho : N^* \to N$ induces a Riemannian submersion $\rho' : \Gamma^* \backslash N^* \to \Gamma \backslash N$ whose fibers are flat, totally geodesic tori. The idea underlying the proof is that every 2-step nilpotent Lie algebra is isomorphic to a standard metric 2-step nilpotent Lie algebra, which we define and discuss.
We also use a criterion of Mal'cev to derive conditions that guarantee the existence of lattices in N. We apply these conditions to prove the existence of lattices in simply connected, 2-step nilpotent Lie groups N that arise from Lie triple systems with compact center in $\mathfrak{so}(n, \mathbb{R})$, the Lie algebra of skew symmetric linear transformations of $\mathbb{R}^n$ with the standard inner product. Lie triple systems with compact center include subspaces of $\mathfrak{so}(n, \mathbb{R})$ that arise from finite dimensional real representations of Clifford algebras or compact Lie groups. The center of the Lip triple system is trivial for representations of Clifford algebras and compact semisimple Lie groups.

Introduction.

For finite dimensional real vector spaces U, V a linear map $j : U \to \mathrm{End}(V)$ is called *skew symmetrizable* if there exists an inner product $\langle\, ,\, \rangle_V$ on V such that the elements of $j(U)$ are skew symmetric relative to $\langle\, ,\, \rangle_V$. If j is injective in addition, then we may define a (positive definite) inner product $\langle\, ,\, \rangle_U$

on U by $\langle u, u^* \rangle = -\operatorname{trace} j(u)j(u^*)$. We obtain a unique 2-step nilpotent Lie algebra structure $[\,,\,]$ on $\mathcal{N} = V \oplus U$ (orthogonal direct sum) such that U is contained in the center of $\mathcal{N}$ and $\langle [X,Y], Z\rangle_U = \langle j(Z)(X), Y\rangle_V$ for all X, Y in V and Z in U. In section 2 we show that the isomorphism type of $\{\mathcal{N}, [,]\}$ depends only on the image $j(U) = W$ in $\operatorname{End}(V)$ and not on the particular injective, skew symmetrizable linear map $j : U \to \operatorname{End}(V)$ or the choice of inner product $\langle\,,\,\rangle_V$.

In view of the statements above we are led to give special attention to metric 2-step nilpotent Lie algebras $\mathcal{N} = \mathbb{R}^n \oplus W$, (orthogonal direct sum), where $\mathbb{R}^n$ has a fixed inner inner product $\langle\,,\,\rangle$, and W is a p-dimensional subspace of $\mathfrak{so}(n,\mathbb{R})$ with the inner product $\langle X, Y\rangle^* = -\operatorname{trace}(XY)$. The subspace W is by definition in the center of $\mathcal{N}$, and the 2-step nilpotent bracket operation on $\mathbb{R}^n$ is defined uniquely by the condition $\langle [v,w], Z\rangle^* = \langle Z(v), w\rangle$ for all v, w in $\mathbb{R}^n$ and all Z in W. Such a Lie algebra $\mathcal{N} = \mathbb{R}^n \oplus W$ is called a *standard metric 2-step nilpotent* Lie algebra.

In section 2.6 we show that every 2-step nilpotent Lie algebra $\mathcal{N}$ is isomorphic as a Lie algebra to one of the standard metric 2-step nilpotent Lie algebras $\mathcal{N} = \mathbb{R}^n \oplus W$. The isomorphism is not uniquely determined but depends on the choice of a special basis of $\mathcal{N}$.

Let W be a subspace of $\mathfrak{so}(n,\mathbb{R})$ and let $\mathfrak{G}$ be the subalgebra of $\mathfrak{so}(n,\mathbb{R})$ generated by W. Let $\mathcal{N}^* = \mathbb{R}^n \oplus \mathfrak{G}$ and $\mathcal{N} = \mathbb{R}^n \oplus W$ be the associated standard metric 2-step nilpotent Lie algebras. Let N^* and N denote the corresponding simply connected metric 2-step nilpotent Lie groups. The orthogonal projection $\pi : \mathcal{N}^* \to \mathcal{N}$ is a surjective Lie algebra homomorphism that lifts to a surjective Lie group homomorphism $\rho : N^* \to N$ such that $d\rho = \pi$.

In section 3 we show that $\rho : N^* \to N$ is a Riemannian submersion with flat, simply connected, totally geodesic fibers. Moreover, if N admits a lattice Γ, necessarily cocompact, then N^* admits a lattice Γ^* such that $\rho(\Gamma^*) = \Gamma$, and there is an induced Riemannian submersion $\rho' : \Gamma^*\backslash N^* \to \Gamma\backslash N$ whose fibers are flat, totally geodesic tori that are all isometric. Moreover, $\operatorname{Ker}(\rho) \cap \Gamma^*$ is a lattice in $\operatorname{Ker}(\rho)$. Conversely, if $\operatorname{Ker}(\rho) \cap \Gamma^*$ is a lattice in $\operatorname{Ker}(\rho)$ for some lattice Γ^* in N^*, then $\Gamma = \rho(\Gamma^*)$ is a lattice in N.

Lattices in a simply connected nilpotent Lie group N are not guaranteed to exist. In fact, lattices never exist for a "generic" simply connected 2-step nilpotent Lie group N of dimension n whose center has dimension $p \geq 3$, provided that n is sufficiently large relative to p. See the beginning of section 4 and [E3] for more details. Mal'cev [Ma] has shown that a simply connected nilpotent Lie group N admits a lattice $\Leftrightarrow$ the Lie algebra $\mathcal{N}$ has a basis with rational structure constants. It is of interest to differential geometers to have criteria that guarantee the existence of lattices in the corresponding simply connected groups N.

For finite dimensional real vector spaces U, V a linear map $j : U \to \mathrm{End}(V)$ will be called *rational* if there exist bases $\mathcal{B}_U$ for U and $\mathcal{B}_V$ for V such that $j(Z)(\mathbb{Q} - \mathrm{span}(\mathcal{B}_V)) \subseteq \mathbb{Q} - \mathrm{span}(\mathcal{B}_V)$ for all Z in $\mathcal{B}_U$. Let $j : U \to \mathrm{End}(V)$ be an injective, rational and skew symmetrizable linear map, and let $\{\mathcal{N} = V \oplus U, [\ ,\ ]\}$ be the 2-step nilpotent Lie algebra defined as above. If N is the simply connected 2-step nilpotent Lie group with Lie algebra $\mathcal{N}$, then the Mal'cev criterion implies that N admits a lattice (Proposition 2.7). In the Propositions of (4.2) and (4.3) we use this result to show that lattices in simply connected 2-step nilpotent Lie groups N always exist in certain special situations:

1) Fix an inner product on $\mathbb{R}^n$, and let $\mathfrak{G}$ be a subalgebra of $\mathfrak{so}(n, \mathbb{R})$ that is the Lie algebra of a compact, connected subgroup of $SO(n, \mathbb{R})$. We show in Appendix 1 that there exists a basis $\mathcal{C}'$ of $\mathfrak{G}$ with the following properties: a) The structure constants of $\mathcal{C}'$ lie in $\mathbb{Z}$, b) Any finite dimensional real $\mathfrak{G}$-module U admits a basis $\mathcal{B}_U$ such that the elements of $\mathcal{C}'$ leave invariant $\mathbb{Z} - \mathrm{span}(\mathcal{B}_U)$. This is a slight generalization of a result in [R1], where the proof is given in the case that $\mathfrak{G}$ is semisimple.

Now let W be a subspace of $\mathfrak{G}$ that admits a basis in $\mathfrak{G}_{\mathbb{Q}} = \mathbb{Q} - \mathrm{span}(\mathcal{C}')$ and let $\mathcal{N} = \mathbb{R}^n \oplus W$ be the corresponding standard metric 2-step nilpotent Lie algebra. Then the simply connected Lie group N with Lie algebra $\mathcal{N}$ admits a lattice Γ (Proposition 4.2).

Remark. Any semisimple subalgebra $\mathfrak{G}$ of $\mathfrak{so}(n, \mathbb{R})$ is the Lie algebra of a compact, connected subgroup G of $\mathrm{SO}(n, \mathbb{R})$. See for example [Mo, p. 614].

If $\mathfrak{H}$ is any subalgebra of $\mathfrak{so}(n, \mathbb{R})$, then $\mathfrak{G} = [\mathfrak{H}, \mathfrak{H}]$ is semisimple (cf. Appendix 1). If $\rho : G \to GL(\mathbb{R}^n)$ is a representation of a compact semisimple Lie group , then the Lie algebra of $\rho(G)$ is a semisimple subalgebra of $\mathfrak{so}(n, \mathbb{R})$ for any $\rho(G)$-invariant inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$.

2) Let $C\ell(m)$ denote the real negative definite Clifford algebra determined by $\mathbb{R}^m$ with the standard inner product. Let $j : C\ell(m) \to \mathrm{End}(\mathbb{R}^n)$ denote a representation of $C\ell(m)$; that is, $j(Z)^2 = -|Z|^2\,\mathrm{Id}$ for all Z in $\mathbb{R}^m$. If $j : \mathbb{R}^m \to \mathrm{End}(\mathbb{R}^n)$ also denotes the restriction of the Clifford representation, then j is injective, skew symmetrizable and rational. Let $W = j(\mathbb{R}^m)$, which lies in $\mathfrak{so}(n, \mathbb{R})$ for a suitable inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$. If $\mathcal{N} = \mathbb{R}^n \oplus W$, defined as above, then N admits a lattice. See the corollary of (4.3c).

These spaces N of Heisenberg type arising from representations of Clifford algebras were first studied seriously by A. Kaplan in [K1,2]. The existence of lattices in some of these groups N is known, but the treatment in [K1] is brief. See [CD] for a different proof of the existence of lattices in a space N of Heisenberg type.

3) A subspace W of $\mathfrak{so}(n,\mathbb{R})$ is a *Lie triple system* if $[X,[Y,Z]] \in W$ whenever X, Y and $Z \in W$. The Lie triple systems W in $\mathfrak{so}(n,\mathbb{R})$ provide a rich class of examples whose associated 2-step nilpotent Lie groups N admit lattices (see 4.3).

If W is a Lie triple system in $\mathfrak{so}(n,\mathbb{R})$, then $\mathfrak{H} = W + [W,W]$ is a subalgebra of $\mathfrak{so}(n)$. Moreover, $\mathfrak{H}$ is semisimple $\Leftrightarrow \{0\} = Z(W) = \{X \in W : [X,Y] = 0$ for all $Y \in W\}$. Conversely, if $\mathfrak{H}$ is any subalgebra of $\mathfrak{so}(n,\mathbb{R})$, then $\mathfrak{H} = W \oplus [W,W]$, (direct sum), for some Lie triple system W. See Appendix 2 for more details.

If W is a Lie triple system in $\mathfrak{so}(n,\mathbb{R})$, then gWg^{-1} is also a Lie triple system in $\mathfrak{so}(n,\mathbb{R})$ for all elements g in $\mathrm{SO}(n,\mathbb{R})$. If we regard W and gWg^{-1} as equivalent for all g in $\mathrm{SO}(n,\mathbb{R})$, then one may show that there are only finitely many equivalence classes of Lie triple systems in $\mathfrak{so}(n,\mathbb{R})$ with $Z(W) = \{0\}$.

The Lie triple system examples of 3) actually contain the examples of 2) and 1). Clearly, any Lie subalgebra W of $\mathfrak{so}(n,\mathbb{R})$ is a Lie triple system. In section 2.5 we show that if $j : C\ell(m) \to \mathrm{End}(\mathbb{R}^n)$ is a representation of the Clifford algebra $C\ell(m)$, then $W = j(\mathbb{R}^m)$ is a Lie triple system in $\mathfrak{so}(n,\mathbb{R})$ for any W-invariant inner product $\langle\, , \rangle$ on $\mathbb{R}^n$. Moreover, we show in Appendix 3 that $\mathfrak{H} = W \oplus [W,W]$ is isomorphic to $\mathfrak{so}(m+1,\mathbb{R})$ if $m \neq 3$. If $m = 3$, then either $\mathfrak{H} = W \oplus [W,W]$ is isomorphic to $\mathfrak{so}(4,\mathbb{R})$ or $W = [W,W] = \mathfrak{H}$ is isomorphic to $\mathfrak{so}(3,\mathbb{R})$.

The statements above from Appendices 2 and 3 are intended to motivate the study of 2-step nilpotent Lie groups N with left invariant metrics whose Lie algebras $\mathcal{N}$ are standard metric 2-step nilpotent Lie algebras. These appendices may be found on the author's website at (www.math.unc.edu).

The paper is organized as follows. Section 1 contains basic material about 2-step nilpotent Lie algebras and lattices in simply connected 2-step nilpotent Lie groups. In section 2 we prove the facts stated in the first paragraph of the introduction, and we introduce standard metric 2-step nilpotent Lie algebras. In section 2.6 we show that every 2-step nilpotent Lie algebra is isomorphic to one of these. The Riemannian submersion result, stated earlier, is proved in section 3. The existence of lattices in the two situations mentioned above is proved in section 4.

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1. Lattices in nilpotent Lie groups.

1.1. Basic information.

General references for the material in this section are [CG] and [R2] as well as the original paper of Mal'cev ([Ma]).

(1.1a) Definition and exponential map. A Lie algebra $\mathcal{N}$ is *nilpotent* if $\mathcal{N}^k = \{0\}$ for some positive integer k, where $\mathcal{N}^0 = \mathcal{N}$ and $\mathcal{N}^k = [\mathcal{N}, \mathcal{N}^{k-1}]$ for all $k \geq 1$. The nilpotent Lie algebra $\mathcal{N}$ is said to be *k-step* if $\mathcal{N}^k = \{0\}$ but $\mathcal{N}^{k-1} \neq \{0\}$.

We consider only nilpotent Lie algebras over $\mathbb{R}$. If N is a simply connected nilpotent Lie group with Lie algebra $\mathcal{N}$, then the Lie group exponential map $\exp : \mathcal{N} \to N$ is a diffeomorphism and we let $\log : N \to \mathcal{N}$ denote its inverse.

(1.1b) Multiplication formula. For elements X and Y in $\mathcal{N}$ the Campbell-Baker-Hausdorff formula says that $\exp(X) \cdot \exp(Y) = \exp(X + Y + P(X,Y)(X) + Q(X,Y)(Y))$, where $P(X,Y)$ and $Q(X,Y)$ are finite polynomials in $\operatorname{ad} X$ and $\operatorname{ad} Y$ ([V]). If $\mathcal{N}$ is 2-step nilpotent, the case that concerns us in this paper, the formula becomes

$$\exp(X)\exp(Y) = \exp(X + Y + \tfrac{1}{2}[X,Y]) \qquad \text{for all } X, Y \text{ in } \mathcal{N}$$
$$\log(nn^*) = \log(n) + \log(n^*) + \tfrac{1}{2}[\log(n), \log(n^*)] \quad \text{for all } n, n^* \in N.$$

For elements n, n^* in N let $[n, n^*] = nn^*n^{-1}n^{*-1}$. We obtain

$$[\exp(X), \exp(Y)] = \exp([X,Y]) \qquad \text{for all } X, Y \text{ in } \mathcal{N}.$$

1.2. The Mal'cev criterion for lattices.

(1.2a) Definition of lattice. A lattice Γ in a connected Lie group H is a discrete subgroup such that $\Gamma\backslash H$ possesses a finite measure invariant under the action of H.

If N is a simply connected nilpotent Lie group, then every lattice Γ of N is cocompact; that is, $\Gamma\backslash N$ is compact ([R2, Theorem 2.1]). Not every simply connected nilpotent Lie group N admits a lattice Γ; see for example [R2, Remark 2.14] or section 4.1 of this paper for the 2-step case.

(1.2b) Existence Theorem ([M], [R2], [CG]). A simply connected Lie group N admits a lattice Γ if and only if there exists a basis $\mathcal{B} = \{\xi_1, \xi_2, ...\xi_n\}$ of the Lie algebra $\mathcal{N}$ such that $[\xi_i, \xi_j] = \sum_{k=1}^{n} C_{ij}^k \xi_k$, where the constants $\{C_{ij}^k\}$ are rational numbers. In addition,

a) If $\mathcal{B} = \{\xi_1, \xi_2, ...\xi_n\}$ is a basis of $\mathcal{N}$ with rational structure constants as above, and if L is a vector lattice of $\mathcal{N}$ contained in $\mathcal{N}_{\mathbb{Q}} = \mathbb{Q} - \mathrm{span}(\mathcal{B})$, then the subgroup Γ of N generated by $\exp(L)$ is a lattice in N. Moreover, $\mathbb{Q} - \mathrm{span}(\mathcal{B}) = \mathbb{Q} - \mathrm{span}(\log \Gamma)$.

b) If Γ is a lattice of N, then $\mathcal{N}_{\mathbb{Q}} = \mathbb{Q} - \mathrm{span}(\log \Gamma)$ is a Lie algebra over $\mathbb{Q}$, and there exists a $\mathbb{Q}$-basis $\mathcal{B} = \{\xi_1, \xi_2, ...\xi_n\}$ of $\mathcal{N}_{\mathbb{Q}}$ that is also an $\mathbb{R}$-basis of $\mathcal{N}$.

c) If Γ_1 and Γ_2 are lattices in N, then $\mathbb{Q}-\mathrm{span}(\log \Gamma_1) = \mathbb{Q}-\mathrm{span}(\log \Gamma_2)$ if and only if Γ_1 and Γ_2 are *commensurable*; that is, $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2.

(1.2c) Commensurability Example [CG, Theorem 5.4.2]. Let Γ be a lattice in a simply connected, nilpotent Lie group N. Then there exist lattices Γ_1 and Γ_2 in N such that

a) Γ_1 is a finite index subgroup of Γ and Γ is a finite index subgroup of Γ_2.

b) $\Lambda_1 = \log(\Gamma_1)$ and $\Lambda_2 = \log(\Gamma_2)$ are vector lattices in $\mathcal{N}$.

1.3. Rationality.

We first define rational structures, subalgebras and subgroups in a general setting and then specialize to the case of nilpotent Lie algebras.

(1.3a) Rational Lie algebra structures. Let $\mathcal{B} = \{\xi_1, \xi_2, ...\xi_n\}$ be a basis of an arbitrary real Lie algebra $\mathfrak{H}$ such that $[\xi_i, \xi_j] = \sum_{k=1}^{n} C_{ij}^k \xi_k$, where the constants $\{C_{ij}^k\}$ are rational numbers. Then $\mathfrak{H}_{\mathbb{Q}} = \mathbb{Q} - \mathrm{span}(\mathcal{B})$ is a Lie algebra over $\mathbb{Q}$, and $\mathfrak{H}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to $\mathfrak{H}$. One calls $\mathfrak{H}_{\mathbb{Q}}$ a *rational* structure on $\mathfrak{H}$.

Examples of rational structures.

Example 1. Let $\mathcal{N}$ be a nilpotent Lie algebra . By the discussion in (1.2b) and (1.2c) the rational structures on $\mathcal{N}$ correspond bijectively to the commensurability classes of lattices in N, the simply connected Lie group with Lie algebra $\mathcal{N}$.

Example 2. Let V be a finite dimensional real vector space, and let $\mathcal{B}$ be a basis of V. Let $\mathrm{End}(V)$ be the real Lie algebra of endomorphisms of V with Lie bracket $[\,,\,]$ defined by $[T,S] = TS - ST$. Let $\mathrm{End}_{\mathcal{B}}(V)_{\mathbb{Q}} = \{T \in \mathrm{End}(V) : T$ has a matrix with entries in $\mathbb{Q}$ relative to the basis $\mathcal{B}\}$. Then $\mathrm{End}_{\mathcal{B}}(V)_{\mathbb{Q}}$ is a rational structure for $\mathrm{End}(V)$.

Example 3 (Real Chevalley bases and rational structures). Let $\mathfrak{G}^{\mathbb{C}}$ denote a complex semisimple Lie algebra. If $\mathfrak{A}$ is a Cartan subalgebra of $\mathfrak{G}^{\mathbb{C}}$ with roots $\Phi \subseteq \mathfrak{A}^*$ and simple roots $\Delta \subseteq \Phi$, then $\mathfrak{A}$ defines a *Chevalley basis* $\mathcal{C} = \{H^*_\alpha, y_\beta : \alpha \in \Delta, \beta \in \Phi\}$ of $\mathfrak{G}^{\mathbb{C}}$ whose structure constants lie in $\mathbb{Z}$ and which has the properties that i) $\mathrm{ad}\, A(y_\beta) = \beta(A) y_\beta$ for $A \in \mathfrak{A}$, $\beta \in \Phi$ ii) $\{H^*_\alpha : \alpha \in \Delta\}$ is a basis of $\mathfrak{A}$ such that $-H^*_\alpha = H^*_{-\alpha}$ for all $\alpha \in \Delta$. See [Hu, pp.143-146] for a definition and a more detailed discussion. The Chevalley basis $\mathcal{C}$ is not unique, but there are "natural " Chevalley bases $\mathcal{C}$ for the simple, complex Lie algebras $\mathfrak{G}^{\mathbb{C}}$ in the classification $\mathfrak{A}_n, \mathfrak{B}_n, \mathfrak{C}_n$, and $\mathfrak{D}_n$. See for example [He, pp. 186-191].

The complex semisimple Lie algebra $\mathfrak{G}^{\mathbb{C}}$ has a real subalgebra $\mathfrak{G}$ whose Killing form is negative definite and whose complexification is $\mathfrak{G}^{\mathbb{C}}$. The subalgebra $\mathfrak{G}$ is called a *compact real form* of $\mathfrak{G}^{\mathbb{C}}$ and is unique up to isomorphism. Conversely, if $\mathfrak{G}$ is a real subalgebra whose Killing form is negative definite, then $\mathfrak{G}^{\mathcal{C}}$ is semisimple and $\mathfrak{G}$ is a compact real form of $\mathfrak{G}^{\mathbb{C}}$.

A compact real form $\mathfrak{G}$ of a complex semisimple Lie algebra $\mathfrak{G}^{\mathbb{C}}$ may be constructed as follows. From a Chevalley basis $\mathcal{C} = \{H^*_\alpha, y_\beta : \alpha \in \Delta, \beta \in \Phi\}$ for $\mathfrak{G}^{\mathbb{C}}$ one defines a "real" Chevalley basis $\mathcal{C}_{\mathbb{R}} = \{iH^*_\alpha, u_\beta, v_\beta : \alpha \in \Delta, \beta \in \Phi\}$, where $u_\beta = y_\beta - y_{-\beta}$ and $v_\beta = iy_\beta + iy_{-\beta}$ for all $\beta \in \Phi$. If $\mathfrak{G} = \mathbb{R} - \mathrm{span}(\mathcal{C}_{\mathbb{R}})$, then $\mathfrak{G}$ is a compact real form for $\mathfrak{G}^{\mathcal{C}}$, and $\mathcal{C}_{\mathbb{R}}$ is a basis of $\mathfrak{G}$ with structure constants in $\mathbb{Z}$. Any compact real form of $\mathfrak{G}^{\mathbb{C}}$ arises in this way for a suitable choice of Chevalley basis $\mathcal{C}$. See [B] and [C]. See also [He, pp. 181-182] for further discussion, where u_β and v_β are replaced by iu_β and $-iv_\beta$ respectively.

The rational structure $\mathfrak{G}_{\mathbb{Q}}$ for $\mathfrak{G}$ given by $\mathfrak{G}_{\mathbb{Q}} = \mathbb{Q} - \mathrm{span}(\mathcal{C}_{\mathbb{R}})$ will be useful later in Proposition 4.2 for constructing examples of simply connected, 2-step nilpotent Lie groups N that admit lattices.

Chevalley bases adapted to involutions [B]. Let $\mathfrak{G}$ be a Lie subalgebra of $\mathfrak{so}(n,\mathbb{R})$ whose Killing form is negative definite. Let $\theta : \mathfrak{G} \to \mathfrak{G}$ be a Lie algebra automorphism such that $\theta^2 = \mathrm{Id}$. Then we can choose a Chevalley basis $\mathcal{C} = \{H_\alpha^*, y_\beta : \alpha \in \Delta, \beta \in \Phi\}$ for the complexification $\mathfrak{G}^{\mathbb{C}}$ such that

a) The real Chevalley basis $\mathcal{C}_{\mathbb{R}} = \{iH_\alpha^*, u_\beta, v_\beta : \alpha \in \Delta, \beta \in \Phi\}$ is a basis for $\mathfrak{G}$.

b) $\mathfrak{G}$ admits a basis $\mathfrak{B} \subseteq \mathbb{Z} - \mathrm{span}(\mathcal{C}_{\mathbb{R}})$ such that $\theta(\xi) = \pm\xi$ for all $\xi \in \mathfrak{B}$. In particular, the $+1$ and -1 eigenspaces of θ are rational relative to the rational structure $\mathfrak{G}_{\mathbb{Q}} = \mathbb{Q} - \mathrm{span}(\mathcal{C}_{\mathbb{R}})$ on $\mathfrak{G}$.

Assertion b) will be proved below in lemma 2 of (4.3c). The proof of b) follows from Proposition 3.7 and the preceding discussion in section 3 of [B]. See also Proposition 14.3 of [R2, pp. 215-220].

(1.3b) Rational subalgebras and subgroups.

Definition of rational structure.

Let $\mathfrak{H}$ be a real Lie algebra, and let $\mathfrak{H}_{\mathbb{Q}}$ be a rational structure on $\mathfrak{H}$. Let H be a Lie group with Lie algebra $\mathfrak{H}$.

A subspace or subalgebra $\mathfrak{H}^*$ of $\mathfrak{H}$ is *rational* with respect to $\mathfrak{H}_{\mathbb{Q}}$ if $\mathfrak{H}_{\mathbb{Q}}$ contains a basis of $\mathfrak{H}^*$. A connected subgroup H^* of H is *rational* with respect to $\mathfrak{H}_{\mathbb{Q}}$ if its Lie algebra $\mathfrak{H}^*$ is rational with respect to $\mathfrak{H}_{\mathbb{Q}}$.

We now specialize to nilpotent Lie algebras.

Proposition. [CG, §5.4] *Let $\mathcal{N}$ be a nilpotent Lie algebra, and let $\mathcal{N}_{\mathbb{Q}}$ be a rational structure on $\mathcal{N}$. Let N be the simply connected nilpotent Lie group with Lie algebra $\mathcal{N}$. Let Γ be a lattice of N such that $\mathcal{N}_{\mathbb{Q}} = \mathbb{Q} - \mathrm{span}(\log\Gamma)$.*

A subalgebra $\mathcal{N}^*$ of $\mathcal{N}$ is rational with respect to $\mathcal{N}_{\mathbb{Q}}$ if and only if $\Gamma^* = \Gamma \cap N^*$ is a lattice in $N^* = \exp(\mathcal{N}^*)$.

(1.3c) Examples of rational subalgebras ([CG, §5.2]). Let $\mathcal{N}$ be a nilpotent Lie algebra, and let N be the simply connected nilpotent Lie group with Lie algebra $\mathcal{N}$. The following subalgebras are rational with respect to the rational structure $\mathcal{N}_{\mathbb{Q}} = \mathbb{Q} - \mathrm{span}(\log\Gamma)$ for *any* lattice Γ of N.

1) The center $\mathcal{Z}$ of $\mathcal{N}$.

2) The subalgebras $\mathcal{N}^k = [\mathcal{N}, \mathcal{N}^{k-1}]$, where $\mathcal{N}^0 = \mathcal{N}$ (descending central series).

3) The subalgebras $\mathcal{N}_k = \{X \in \mathcal{N} : [X, \mathcal{N}] \subseteq \mathcal{N}_{k-1}\}$, where $\mathcal{N}^0 = \{0\}$ (ascending central series).

(1.3d) Remark. If $\mathcal{Z}$ denotes the center of a nilpotent Lie algebra $\mathcal{N}$, and Z denotes the center of the corresponding simply connected Lie group N, then $\exp : \mathcal{Z} \to Z$ and $\log : Z \to \mathcal{Z}$ are group isomorphisms as well as diffeomorphisms since $\mathcal{Z}$ and Z are abelian. Here, addition is the group operation on $\mathcal{Z}$. In particular, if Γ^* is a lattice in Z, then $\log \Gamma^*$ is a vector lattice in $\mathcal{Z}$. Combining this fact with the proposition in (1.3b) and 1) of (1.3c) it follows that if Γ is a lattice in N, then $(\log \Gamma) \cap \mathcal{Z} = \log(\Gamma \cap Z)$ is a vector lattice in $\mathcal{Z}$. Similarly, if $N^* = \exp([\mathcal{N}, \mathcal{N}])$, then $(\log \Gamma) \cap [\mathcal{N}, \mathcal{N}] = \log(\Gamma \cap N^*)$ is a vector lattice in $[\mathcal{N}, \mathcal{N}]$.

(1.3e) A useful rational basis of $\mathcal{N}$.

Proposition. *Let N be a 2-step simply connected nilpotent Lie group with Lie algebra $\mathcal{N}$. Let Γ be a lattice in N and let $\mathcal{N}_{\mathbb{Q}} = \mathbb{Q} - \mathrm{span}(\log \Gamma)$. Then there exists a basis $\mathcal{B} = \{X_1, X_2, ...X_n, Z_1, Z_2, ...Z_p\}$ of $\mathcal{N}$ such that*

1) $\mathcal{B} \subseteq \log \Gamma$.
2) $\{Z_1, Z_2, ...Z_p\}$ *is a basis of $\mathcal{Z}$ such that* $\mathbb{Q} - \mathrm{span}\{(\log \Gamma) \cap \mathcal{Z}\} = \mathbb{Q} - \mathrm{span}\{Z_1, Z_2, ...Z_p\}$.
3) *If* $q = \dim([\mathcal{N}, \mathcal{N}])$, *then* $\{Z_1, Z_2, ...Z_q\}$ *is a basis of* $[\mathcal{N}, \mathcal{N}]$ *such that* $\mathbb{Q} - \mathrm{span}\{(\log \Gamma) \cap [\mathcal{N}, \mathcal{N}]\} = \mathbb{Q} - \mathrm{span}\{Z_1, Z_2, ...Z_q\}$.
4) $\mathbb{Q} - \mathrm{span}(\mathcal{B}) = \mathbb{Q} - \mathrm{span}(\log \Gamma) = \mathcal{N}_{\mathbb{Q}}$.
5) *The structure constants of $\mathcal{B}$ lie in $\mathbb{Z}$.*

We first prove a stronger result for the special case that $\Lambda = \log \Gamma$ is a vector lattice in $\mathcal{N}$. This lemma will also be useful later in the proof of the main result of section 3.

Lemma. *Let N and Γ be as above and suppose that $\Lambda = \log \Gamma$ is a vector lattice in $\mathcal{N}$. Then there exists a $\mathbb{Z}$-basis $\mathcal{B} = \{X_1, X_2, ...X_n, Z_1, Z_2, ...Z_p\}$ of Λ such that*

1) *$\mathcal{B}$ is also an $\mathbb{R}$-basis of $\mathcal{N}$.*
2) $\{Z_1, Z_2, ...Z_p\}$ *is a basis of $\mathcal{Z}$ such that* $(\log \Gamma) \cap \mathcal{Z} = \mathbb{Z} - \mathrm{span}\{Z_1, Z_2, ...Z_p\}$.
3) *If* $q = \dim([\mathcal{N}, \mathcal{N}])$, *then* $\{Z_1, Z_2, ...Z_q\}$ *is a basis of* $[\mathcal{N}, \mathcal{N}]$ *such that* $(\log \Gamma) \cap [\mathcal{N}, \mathcal{N}] \subseteq \mathbb{Z} - \mathrm{span}\{Z_1, Z_2, ...Z_q\}$ *and* $\mathbb{Q} - \mathrm{span}\{(\log \Gamma) \cap [\mathcal{N}, \mathcal{N}]\} = \mathbb{Q} - \mathrm{span}\{Z_1, Z_2, ...Z_q\}$.

4) $\mathbb{Q} - \operatorname{span}(\mathcal{B}) = \mathbb{Q} - \operatorname{span}(\log \Gamma)$.

5) The structure constants of $\mathcal{B}$ lie in $\mathbb{Z}$.

Proof of the Lemma. Since $\mathcal{N}$ is 2-step nilpotent it follows that $\Lambda' = \operatorname{ad}(\Lambda)$ is a finitely generated free abelian group in $\operatorname{ad}(\mathcal{N}) \subseteq \operatorname{End}(\mathcal{N})$. Let $\{X_1', X_2', ..., X_n'\}$ be a $\mathbb{Z}$-basis for Λ', and let $\{X_1, X_2, ..., X_n\}$ be elements of Λ such that $\operatorname{ad} X_i = X_i'$ for $1 \le i \le n$. By (1.3d) $(\log \Gamma) \cap \mathcal{Z}$ is a vector lattice in $\mathcal{Z}$, and $(\log \Gamma) \cap [\mathcal{N}, \mathcal{N}]$ is a vector lattice in $[\mathcal{N}, \mathcal{N}]$. Since $P = (\log \Gamma) \cap \mathcal{Z}$ and $Q = (\log \Gamma) \cap [\mathcal{N}, \mathcal{N}]$ are finitely generated free $\mathbb{Z}$-modules with $Q \subseteq P$, the invariant factor theorem says that there exist $\mathbb{Z}$-bases $\{Z_1, Z_2, ...Z_p\}$ for P and $\{Z_1^*, Z_2^*, ...Z_q^*\}$ for Q such that $Z_i^* = m_i Z_i$ for $1 \le i \le q$, where $\{m_1, m_2, ...m_q\}$ are positive integers such that m_i divides m_{i+1} for all i. If $P = (\log \Gamma) \cap \mathcal{Z}$ and $A = \mathbb{Z}\text{-span}\{X_1, X_2, ..., X_n\}$, then it is easy to prove that $\{X_1, X_2, ..., X_n\}$ is a $\mathbb{Z}$-basis for A, $A \cap P = \{0\}$ and $\Lambda = A \oplus P$.

The discussion above shows that $\mathcal{B} = \{X_1, X_2, ...X_n, Z_1, Z_2, ...Z_p\}$ is a $\mathbb{Z}$-basis of Λ satisfying assertions 2) and 3) of the Lemma. We prove 1) and 4). Clearly $\mathbb{Q} - \operatorname{span}(\mathcal{B}) = \mathbb{Q} - \operatorname{span}(\Lambda) = \mathbb{Q} - \operatorname{span}(\log \Gamma)$. The elements of $\mathcal{B}$ are linearly independent over $\mathbb{Q}$, and hence $\mathcal{B}$ is a $\mathbb{Q}$-basis of $\mathcal{N}_{\mathbb{Q}} = \mathbb{Q} - \operatorname{span}(\log \Gamma)$. By the first remarks in (1.3) $\mathcal{N}_{\mathbb{Q}}$ is a Lie algebra over $\mathbb{Q}$ and $\mathcal{N}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to $\mathcal{N}$. It follows that the structure constants of $\mathcal{B}$ lie in $\mathbb{Q}$, and $\mathcal{B}$ is an $\mathbb{R}$-basis of $\mathcal{N}$. This proves 1) and 4).

It remains only to show that the structure constants of $\mathcal{B}$ lie in $\mathbb{Z}$. In this case (1.1b) implies that if $X \in \log \Gamma$ and $Y \in \log \Gamma$, then $[X, Y] \in \log \Gamma$. Hence $\mathbb{Z} - \operatorname{span}(\mathcal{B}) = \Lambda = \log \Gamma$ is closed under Lie brackets and, in particular, $[\mathcal{B}, \mathcal{B}] \subseteq (\log \Gamma) \cap \mathcal{Z} = \mathbb{Z} - \operatorname{span}\{Z_1, Z_2, ...Z_p\}$ $\square$

Proof of the Proposition. Let Γ be a lattice of N. By (1.2c) there exists a finite index subgroup Γ^* of Γ such that $\Lambda^* = \log \Gamma^*$ is a vector lattice in $\mathcal{N}$. Let $\mathcal{B} = \{X_1, X_2, ...X_n, Z_1, Z_2, ...Z_p\}$ be a $\mathbb{Z}$-basis for Λ^* satisfying the five conditions of the lemma. By (1.2b) $\mathbb{Q} - \operatorname{span}(\mathcal{B}) = \mathbb{Q} - \operatorname{span}(\log \Gamma^*) = \mathbb{Q} - \operatorname{span}(\log \Gamma)$. Assertions 4) and 5) of the Proposition now follow from assertions 4) and 5) of the lemma while assertion 1) of the Proposition holds since $\mathcal{B} \subseteq \log \Gamma^*$ and $\Gamma^* \subseteq \Gamma$. Since $(\log \Gamma^*) \cap \mathcal{Z}$ and $(\log \Gamma) \cap \mathcal{Z}$ are vector lattices in $\mathcal{Z}$, with the first contained in the second, it follows that $\mathbb{Q} - \operatorname{span}\{(\log \Gamma^*) \cap \mathcal{Z}\} = \mathbb{Q} - \operatorname{span}\{(\log \Gamma) \cap \mathcal{Z}\}$. Similarly $\mathbb{Q} - \operatorname{span}\{(\log \Gamma^*) \cap [\mathcal{N}, \mathcal{N}]\} = \mathbb{Q} - \operatorname{span}\{(\log \Gamma) \cap [\mathcal{N}, \mathcal{N}]\}$. Assertions 2) and 3) of the Proposition now follow from assertions 2) and 3) of the lemma. $\square$

2. Standard and involutive metric 2-step nilpotent Lie algebras.

2.1. Abelian and nonabelian factors of a 2-step nilpotent Lie algebra.

It is useful to observe that one may always split off an abelian Lie algebra $\mathcal{E}$ from a 2-step nilpotent Lie algebra $\mathcal{N}$ and reduce consideration to the case that $[\mathcal{N}, \mathcal{N}] = \mathcal{Z}$, the center of $\mathcal{N}$. We call the ideals $\mathcal{E}$ and $\mathcal{N}^*$ in the proposition below the *abelian* and *nonabelian* factors of $\mathcal{N}$.

Proposition. *Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra with center $\mathcal{Z}$. Then there exist ideals $\mathcal{N}^*$ and $\mathcal{E}$ of $\mathcal{N}$ with $\mathcal{E} \subseteq \mathcal{Z}$ such that*

1) $\mathcal{N} = \mathcal{N}^* \oplus \mathcal{E}$ and $\mathcal{Z} = [\mathcal{N}, \mathcal{N}] \oplus \mathcal{E}$.

2) $\mathcal{N}^*$ is a 2-step nilpotent Lie algebra such that $[\mathcal{N}, \mathcal{N}] = [\mathcal{N}^*, \mathcal{N}^*] = \mathcal{Z}^*$, the center of $\mathcal{N}^*$.

3) The ideals $\mathcal{N}^*$ and $\mathcal{E}$ are uniquely determined up to isomorphism by 1).

4) If $\mathcal{N}$ has a basis $\mathcal{B}$ with rational structure constants, then $\mathcal{N}^*$ has a basis $\mathcal{B}^*$ with integer structure constants.

From 1) of the Proposition above we obtain

Corollary. *Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra with center $\mathcal{Z}$. Then $\mathcal{N}$ has a trivial abelian factor $\Leftrightarrow [\mathcal{N}, \mathcal{N}] = \mathcal{Z}$.*

Proof of the Proposition. We begin by proving 3). Suppose that we can write $\mathcal{N} = \mathcal{N}_1^* \oplus \mathcal{E}_1 = \mathcal{N}_2^* \oplus \mathcal{E}_2$,where $\{\mathcal{N}_1^*, \mathcal{E}_1\}$ and $\{\mathcal{N}_2^*, \mathcal{E}_2\}$ satisfy the hypotheses 1) and 2) of the Proposition. If $\mathcal{V}$ is a subspace of $\mathcal{N}$ such that $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$, then $\mathcal{N} = \mathcal{V} \oplus [\mathcal{N}, \mathcal{N}] \oplus \mathcal{E}_i$ for $i = 1, 2$. Let $T : \mathcal{N} \to \mathcal{N}$ be a linear isomorphism such that $T = \mathrm{Id}$ on $\mathcal{V} \oplus [\mathcal{N}, \mathcal{N}]$ and $T(\mathcal{E}_1) = \mathcal{E}_2$. It is easy to check that T is a Lie algebra isomorphism, and hence T induces a Lie algebra isomorphism $\bar{T} : \mathcal{N} / \mathcal{E}_1 \to \mathcal{N} / \mathcal{E}_2$. However, $\mathcal{N} / \mathcal{E}_1 \cong \mathcal{N}_1^*$ and $\mathcal{N}/\mathcal{E}_2 \cong \mathcal{N}_2^*$ by 1). This proves that $\mathcal{N}_1^* \cong \mathcal{N}_2^*$, and $\mathcal{E}_1^* \cong \mathcal{E}_2^*$ since $\mathcal{E}_1^*$ and $\mathcal{E}_2^*$ are abelian Lie algebras of the same dimension by 1).

To prove the existence of $\mathcal{N}^*$ and $\mathcal{E}$ we choose $\mathcal{E}$ to be any subspace of $\mathcal{Z}$ such that $\mathcal{Z} = [\mathcal{N}, \mathcal{N}] \oplus \mathcal{E}$. Let $\mathcal{V}$ be a subspace of $\mathcal{N}$ such that $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$. If $\mathcal{N}^* = \mathcal{V} \oplus [\mathcal{N}, \mathcal{N}]$, then $\mathcal{N}^*$ and $\mathcal{E}$ satisfy 1) and it follows that $[\mathcal{N}, \mathcal{N}]$

$= [\mathcal{N}^*, \mathcal{N}^*] \subseteq \mathcal{Z}^* \subseteq \mathcal{Z}$. Let Z^* be any element of $\mathcal{Z}^*$ and write $Z^* = X + Z$, where $X \in \mathcal{V}$ and $Z \in [\mathcal{N}, \mathcal{N}] \subseteq \mathcal{Z}^*$. It follows that $X \in \mathcal{V} \cap \mathcal{Z}^* \subseteq \mathcal{V} \cap \mathcal{Z}$ $= \{0\}$. We conclude that $\mathcal{Z}^* = [\mathcal{N}, \mathcal{N}]$, which proves 2).

We prove 4). Let $\mathcal{B}$ be a basis of $\mathcal{N}$ with rational structure constants. If N is the simply connected Lie group with Lie algebra $\mathcal{N}$, then by (1.2b) N admits a lattice Γ such that $\mathbb{Q} - \mathrm{span}(\mathcal{B}) = \mathbb{Q} - \mathrm{span}(\log \Gamma)$. Let $\mathcal{B}' = \{X_1, X_2, ...X_n, Z_1, Z_2, ...Z_p\}$ be a basis of $\mathcal{N}$ such that $\mathbb{Q} - \mathrm{span}(\mathcal{B}') = \mathbb{Q} - \mathrm{span}(\log \Gamma) = \mathbb{Q} - \mathrm{span}(\mathcal{B})$, and $\mathcal{B}'$ satisfies the conditions of the proposition in (1.3e). Let $\mathcal{V} = \mathbb{R} - \mathrm{span}(X_1, X_2, ...X_n)$, $\mathcal{E} = \mathbb{R} - \mathrm{span}(Z_{q+1}, ..., Z_p)$ and $\mathcal{N}^* = \mathbb{R} - \mathrm{span}(X_1, X_2, ...X_n, Z_1, Z_2, ...Z_q)$, where $q = \dim[\mathcal{N}, \mathcal{N}]$ and $p = \dim \mathcal{Z}$. Then $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$, $\mathcal{N}^* = \mathcal{V} \oplus [\mathcal{N}, \mathcal{N}]$ and $\mathcal{N} = \mathcal{N}^* \oplus \mathcal{E}$ by the properties of $\mathcal{B}'$ from (1.3e). The ideals $\mathcal{N}^*$ and $\mathcal{E}$ satisfy 1) and 2) of the Proposition, as we observed in the previous paragraph. If $\mathcal{B}^* = \{X_1, X_2, ...X_n, Z_1, Z_2, ...Z_q\}$, then $\mathcal{B}^*$ is a basis of $\mathcal{N}^*$ with integer structure constants by (1.3e). We have proved 4) for a particular choice of $\mathcal{N}^*$, but $\mathcal{N}^*$ is uniquely determined up to isomorphism by 1) □

2.2. Basic structure of metric 2-step nilpotent Lie algebras.

(2.2a) The bracket operation determines a linear map. Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra with center $\mathcal{Z}$. Given an inner product $\langle\, , \rangle$ on $\mathcal{N}$ we write $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$, where $\mathcal{V} = \mathcal{Z}^\perp$, the orthogonal complement of $\mathcal{Z}$. For each element Z of $\mathcal{Z}$ we obtain a skew symmetric transformation $j(Z) : \mathcal{V} \to \mathcal{V}$ defined by the equation

$$\langle j(Z)X, Y\rangle = \langle [X, Y], Z\rangle \qquad \text{for all } X, Y \text{ in } \mathcal{V}. \qquad (*)$$

If $\{N, \langle\, , \rangle\}$ denotes the simply connected Lie group with Lie algebra $\mathcal{N}$ and corresponding left invariant metric $\langle\, , \rangle$, then the geometry of $\{N, \langle\, , \rangle\}$ can be expressed in terms of the maps $\{j(Z) : Z \in \mathcal{Z}\}$. See [E1] for further details. This approach to studying the geometry of $\{N, \langle\, , \rangle\}$ was introduced by A. Kaplan in [K1] and [K2] in the case that $\{N, \langle\, , \rangle\}$ is of Heisenberg type (see example 2) of (2.4) below).

It is evident that $j : \mathcal{Z} \to \mathfrak{so}(\mathcal{V})$ is a linear map, where $\mathfrak{so}(\mathcal{V})$ denotes the vector space of skew symmetric linear transformations of $\mathcal{V}$.

(2.2b) Constructing bracket operations from linear maps. Let V and U be finite dimensional real inner product spaces and let $\mathcal{N} = V \oplus U$ be the orthogonal direct sum. Let $j : U \to \mathfrak{so}(V)$ be a linear map. Now define a bracket structure $[\, ,]$ on $\mathcal{N}$ by $(*)$ to make $\{\mathcal{N}, \langle\, , \rangle\}$ a metric 2-step nilpotent Lie algebra in which U is contained in the center of $\mathcal{N}$.

Let V, U and $j : U \to \mathsf{so}(V)$ be as above. Let c be a positive number and let V_c, U_c denote the same vector spaces with the original inner products multiplied by c^2. Clearly $j(U) \subseteq \mathsf{so}(V_c)$ for every positive number c. If $\mathcal{N}_c$ denotes the orthogonal direct sum $V_c \oplus U_c$ relative to the new inner products, then it is easy to check that the corresponding bracket operation $[\,,\,]_c$ defined by $(*)$ above is unchanged; that is $[\,,\,]_c = [\,,\,]$ for every c.

(2.2c) Abelian factors and Euclidean de Rham factors. By the proposition in (2.1) the next result shows that the dimension of the abelian factor of $\mathcal{N}$ equals the dimension of the Euclidean de Rham factor of $\{N, \langle\,,\,\rangle\}$ for any left invariant metric $\langle\,,\,\rangle$ on N.

Proposition. *Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra, and let N be the simply connected nilpotent Lie group with Lie algebra $\mathcal{N}$. Let $\mathcal{Z}$ denote the center of $\mathcal{N}$. Then the following are equivalent :*

1) $[\mathcal{N}, \mathcal{N}]$ has codimension $p \geq 0$ in $\mathcal{Z}$.

2) Let $\langle\,,\,\rangle$ denote an inner product on $\mathcal{N}$. Let $\mathcal{V} = \mathcal{Z}^\perp$ and let $j : \mathcal{Z} \to \mathsf{so}(\mathcal{V})$ be the linear map defined above by $(*)$. Then the kernel of j has dimension p.

3) Let $\langle\,,\,\rangle$ denote an inner product on $\mathcal{N}$ and also the corresponding left invariant metric on N. Then the Euclidean de Rham factor of $\{N, \langle\,,\,\rangle\}$ has dimension p, the dimension of the abelian factor of $\mathcal{N}$.

Proof. The equivalence of 1) and 2) follows immediately from the fact that $j(Z) = 0 \Leftrightarrow Z$ is orthogonal to $[\mathcal{N}, \mathcal{N}]$ for an element Z of $\mathcal{Z}$. This fact is an immediate consequence of $(*)$ in (2.2a). The equivalence of 2) and 3) follows directly from Proposition 2.7 of [E1] and the proposition in (2.1) above. $\square$

Corollary. *Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra, and let N be the simply connected nilpotent Lie group with Lie algebra $\mathcal{N}$. Then the following are equivalent.*

1) $\mathcal{N}$ has a trivial abelian factor.

2) There exists an inner product $\langle\,,\,\rangle$ on $\mathcal{N}$ such that the linear map $j : \mathcal{Z} \to \mathsf{so}(\mathcal{V})$ from $(*)$ is injective, where $\mathcal{V} = \mathcal{Z}^\perp$.

3) For every inner product $\langle\,,\,\rangle$ on $\mathcal{N}$ the linear map $j : \mathcal{Z} \to \mathsf{so}(\mathcal{V})$ from $(*)$ is injective, where $\mathcal{V} = \mathcal{Z}^\perp$.

Proof. The implications 3) $\Rightarrow$ 2) $\Rightarrow$ 1) $\Rightarrow$ 3) follow immediately from the proposition above. $\square$

2.3. External direct sum constructions and partial uniqueness.

We can extend the construction of (2.2b) in a way that will be useful for examples, two of which we describe below. Let V and U be finite dimensional real vector spaces, and let $j : U \to \mathrm{End}(V)$ be a linear map. The map j will be called *skew symmetrizable* if there exists an inner product $\langle\, ,\rangle$ on V such that the elements of $j(U)$ are skew symmetric relative to $\langle\, ,\rangle$. If j is skew symmetrizable, then an inner product $\langle\, ,\rangle$ on $\mathcal{N} = V \oplus U$ will be said to be *j-admissible* if a) V and U are orthogonal relative to $\langle\, ,\rangle$ and b) The elements of $j(U)$ are skew symmetric relative to $\langle\, ,\rangle$. If $\langle\, ,\rangle$ is any j-admissible inner product on $\mathcal{N}$, then we may define a 2-step nilpotent Lie algebra structure on $\mathcal{N}$ as in (2.2b).

(2.3a) A partial uniqueness result.

Proposition. *Let V and U be finite dimensional real vector spaces, and let $j : U \to \mathrm{End}(V)$ be a skew symmetrizable linear map. Let $\langle\, ,\rangle_1$ and $\langle\, ,\rangle_2$ be j-admissible inner products on $\mathcal{N} = V \oplus U$, and let $[\, ,]_1$ and $[\, ,]_2$ denote the corresponding 2-step nilpotent Lie algebra structures on $\mathcal{N}$ given by $(*)$ in (2.2a) with U contained in the center of $\mathcal{N}$. If $\langle\, ,\rangle_1 = c^2\langle\, ,\rangle_2$ on U for some positive constant c, then $\{\mathcal{N}, [\, ,]_1\}$ is Lie algebra isomorphic to $\{\mathcal{N}, [\, ,]_2\}$.*

As an immediate consequence we obtain the following

Corollary. *Let V and U be finite dimensional real vector spaces, and let $j : U \to \mathrm{End}(V)$ be a skew symmetrizable linear map. Then every inner product $\langle\, ,\rangle_U$ on U determines a unique 2-step nilpotent Lie algebra structure given by $(*)$ in (2.2a) on the vector space $\mathcal{N} = V \oplus U$.*

Proof of the proposition. By the discussion at the end of (2.2b) it suffices to consider the case $c = 1$; that is $\langle\, ,\rangle_1 = \langle\, ,\rangle_2$ on U. Let $S : V \to V$ be the linear transformation such that $\langle v, w\rangle_2 = \langle Sv, w\rangle_1$ for all v, w in V. Then S is positive definite and symmetric relative to both $\langle\, ,\rangle_1$ and $\langle\, ,\rangle_2$ on V. For every Z in U the linear transformation $j(Z)$ commutes with S since $j(Z)$ is skew symmetric on V with respect to both $\langle\, ,\rangle_1$ and $\langle\, ,\rangle_2$. Write $V = V_1 \oplus V_2 \oplus ...V_N$, where $S = \lambda_i^2\,\mathrm{Id}$ on each V_i, and $\lambda_i \neq \lambda_j$ for $i \neq j$. The subspaces $\{V_i\}$ are orthogonal relative to both $\langle\, ,\rangle_1$ and $\langle\, ,\rangle_2$ since S is symmetric relative to both $\langle\, ,\rangle_1$ and $\langle\, ,\rangle_2$.

Define a linear isomorphism $\varphi : \mathcal{N} \to \mathcal{N}$ by $\varphi = \sqrt{S} = \lambda_i\,\mathrm{Id}$ on each V_i, and $\varphi = \mathrm{Id}$ on U. We assert that $\varphi : \{\mathcal{N}, [\, ,]_2\} \to \{\mathcal{N}, [\, ,]_1\}$ is a Lie algebra isomorphism.

Lemma.

$$[V_i, V_j]_1 = [V_i, V_j]_2 = \{0\} \text{ if } i \neq j.$$

Proof. The skew symmetric transformations in $j(U)$ leave each eigenspace V_i of S invariant since the elements of $j(U)$ commute with S. Let $\langle , \rangle^*$ denote the restriction of $\langle\, , \,\rangle_1$ and $\langle\, , \,\rangle_2$ to U. For Z in U and $i \neq j$ we have $\langle [V_i, V_j]_1, Z\rangle^* = \langle j(Z)(V_i), V_j\rangle_1 \subseteq \langle V_i, V_j\rangle_1 = \{0\}$. Hence $[V_i, V_j]_1 = \{0\}$, and a similar argument shows that $[V_i, V_j]_2 = \{0\}$. $\square$

Now let v, w in V and Z in U be given. Write $v = \sum_{i=1}^N v_i$ and $w = \sum_{i=1}^N w_i$, where $v_i, w_i \in V_i$ for each i. Then $[v, w]_1 = \sum_{i=1}^N [v_i, w_i]_1$ and $[v, w]_2 = \sum_{i=1}^N [v_i, w_i]_2$ by the lemma above. Hence $\langle \varphi[v, w]_2, Z\rangle^* = \langle [v, w]_2, Z\rangle^* = \sum_{i=1}^N \langle [v_i, w_i]_2, Z\rangle^* = \sum_{i=1}^N \langle j(Z)v_i, w_i\rangle_2 = \sum_{i=1}^N \lambda_i^2 \langle j(Z)v_i, w_i\rangle_1$ since $\langle\, , \,\rangle_2 = \lambda_i^2 \langle\, , \,\rangle_1$ on each V_i. On the other hand, since $\varphi = \lambda_i \operatorname{Id}$ on each V_i, we obtain from the lemma $\langle [\varphi v, \varphi w]_1, Z\rangle^* = \sum_{i=1}^N \lambda_i^2 \langle [v_i, w_i]_1, Z\rangle^* = \sum_{i=1}^N \lambda_i^2 \langle j(Z)v_i, w_i\rangle_1$. We conclude that $\varphi[v, w]_2 = [\varphi v, \varphi w]_1$ for all $v, w \in V$, which completes the proof since U lies in the center of $\{\mathcal{N}, [\, ,],]_1\}$ and $\{\mathcal{N}, [\, , \,]_2\}$. $\square$

(2.3b) Standard external direct sum examples. By (2.2c) the linear map j in the corollary of (2.3a) is injective $\Leftrightarrow$ $\mathcal{N}$ has trivial abelian factor. If the map j is injective, then there is a family of preferred inner products $\langle\, , \,\rangle_U$ on U that are unique up to scaling by positive constants. We obtain a family of external direct sum examples $\mathcal{N} = V \oplus U$ with trivial abelian factor that we call *standard*.

Proposition 1. *Let V and U be finite dimensional real vector spaces, and let $j : U \to \operatorname{End}(V)$ be an injective skew symmetrizable linear map. Let $\langle\, , \,\rangle_V$ be an inner product on V such that the elements of $j(U)$ are skew symmetric relative to $\langle\, , \,\rangle_V$. For any positive constant c let $\langle\, , \,\rangle_c$ be the inner product on U defined by $\langle u, u^*\rangle_c = -c^2 \operatorname{trace} j(u)j(u^*)$ for any elements u, u^* in U. Let $[\, , \,]_c$ be the 2-step nilpotent Lie algebra structure on $\mathcal{N} = V \oplus U$ defined in (2.2b). Then $\mathcal{N}$ has no abelian factor, and $[\, , \,]_c$ is independent, up to isomorphism, of the choice of inner product $\langle\, , \,\rangle_V$ on V and the positive constant c.*

Proof. For any positive number c the symmetric bilinear form $\langle\, , \,\rangle_c$ on U is positive definite since $j : U \to \operatorname{End}(V)$ is injective and skew symmetrizable. The Lie algebra structure $[\, , \,]$ on $\mathcal{N}$ is unique up to isomorphism by the proposition in (2.3a). Finally, $\{\mathcal{N}, [\,,\,]\}$ has no abelian factor by the corollary in (2.2c). $\square$

Next, we sharpen the result above to show that the isomorphism type of the 2-step nilpotent Lie algebra $\mathcal{N} = V \oplus U$ in Proposition 1 depends only on the image $W = j(U) \subseteq \operatorname{End}(V)$. More precisely, we have

Proposition 2. *Let V, U_1 and U_2 be finite dimensional real vector spaces, and let $j_1 : U_1 \to \operatorname{End}(V)$ and $j_2 : U_2 \to \operatorname{End}(V)$ be injective skew symmetrizable linear maps such that $j_1(U_1) = j_2(U_2) = W \subseteq \operatorname{End}(V)$. Let $[\,,\,]_1$ and $[\,,\,]_2$ be any 2-step nilpotent Lie algebra structures on $\mathcal{N}_1 = V \oplus U_1$ and $\mathcal{N}_2 = V \oplus U_2$ constructed as in Proposition 1. Then $\{\mathcal{N}_1, [\,,\,]_1\}$ and $\{\mathcal{N}_2, [\,,\,]_2\}$ are isomorphic as Lie algebras.*

Proof. It suffices to prove this in the case that $j_1 : U_1 \to \operatorname{End}(V)$ is any injective skew symmetrizable linear map, $U_2 = j_1(U_1) = W$ and j_2 is the inclusion map $i : W \to \operatorname{End}(V)$. Fix an inner product $\langle\,,\,\rangle_V$ on V such that $W = j_1(U_1) \subseteq \mathfrak{so}(V, \langle\,,\,\rangle_V)$. Define inner products $\langle\,,\,\rangle_1$ on U_1 and $\langle\,,\,\rangle_2$ on $W = U_2$ by $\langle u, u^*\rangle_1 = -\operatorname{trace} j_1(u) j_1(u^*)$ for u, u^* in U_1 and $\langle Z, Z^*\rangle_2 = -\operatorname{trace} j_2(Z) j_2(Z^*) = -\operatorname{trace}(ZZ^*)$ for Z, Z^* in $U_2 = W$. Let $[\,,\,]_1$ and $[\,,\,]_2$ be the corresponding 2-step nilpotent Lie algebra structures on $\mathcal{N}_1 = V \oplus U_1$ and $\mathcal{N}_2 = V \oplus U_2 = V \oplus W$ defined as in (2.2b).

Let $\varphi : \mathcal{N}_1 \to \mathcal{N}_2$ be the linear isomorphism defined by $\varphi = \operatorname{Id}$ on V and $\varphi = j_1$ on U_1. We assert that $\varphi : \{\mathcal{N}_1, [\,,\,]_1\} \to \{\mathcal{N}_2, [\,,\,]_2\}$ is a Lie algebra isomorphism. Let elements X, Y in V and Z in W be given. Let u_1 in U_1 be the unique element such that $j_1(u_1) = Z$. We compute $\langle \varphi[X,Y]_1, Z\rangle_2 = \langle j_1[X,Y]_1, j_1(u_1)\rangle_2 = -\operatorname{trace}(j_1[X,Y]_1\, j_1(u_1)) = \langle [X,Y]_1, u_1\rangle_1 = \langle j_1(u_1)(X), Y\rangle_V = \langle Z(X), Y\rangle_V = \langle j_2(Z)(X), Y\rangle_V = \langle [X,Y]_2, Z\rangle_2 = \langle [\varphi X, \varphi Y]_2, Z\rangle_2$. Hence $\varphi[X,Y]_1 = [\varphi X, \varphi Y]_2$ since $Z \in W$ was arbitrary. It follows that $\varphi : \{\mathcal{N}_1, [\,,\,]_1\} \to \{\mathcal{N}_2, [\,,\,]_2\}$ is a Lie algebra isomorphism since U_1 and W are contained in the centers of $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively. $\square$

2.4. Examples of skew symmetrizable linear maps.

Example 1. Subspaces of $\mathfrak{so}(\mathbf{n}, \mathbb{R})$. Let W be a nonzero subspace of $\mathfrak{so}(n, \mathbb{R})$, and let $j : W \to \mathfrak{so}(n, \mathbb{R})$ be the inclusion map. Then j is a skew symmetrizable linear map.

Example 2. Representations of compact Lie groups. Let G be a compact connected Lie group, and let $\rho : G \to GL(V)$ be a representation of G on a finite dimensional real vector space V. Let $U = \mathfrak{G}$, the Lie algebra of G, and let $j = d\rho : \mathfrak{G} \to \mathrm{End}(V)$ be the induced representation. Let $\langle\, , \rangle_{\mathcal{V}}$ be any inner product on V that is invariant under $\rho(G)$. Then the elements of $j(\mathfrak{G})$ are skew symmetric relative to $\langle\, , \rangle_{\mathcal{V}}$. Any choice of inner product on $\mathfrak{G}$ determines a j-admissible inner product on $\mathcal{N} = V \oplus \mathfrak{G}$.

In geometric applications it is often desirable to choose a $\mathfrak{G}$-invariant inner product $\langle\, , \rangle_{\mathfrak{G}}$ on $\mathfrak{G}$; that is, $\mathrm{ad}\, X$ is skew symmetric relative to $\langle\, , \rangle_{\mathfrak{G}}$ for all X in $\mathfrak{G}$. For example, let $\langle X, Y\rangle_{\mathfrak{G}} = -\operatorname{trace} j(X)\, j(Y)$ for X, Y in $\mathfrak{G}$ if j is injective, which yields a standard external direct sum example. If $\mathfrak{G}$ is semisimple, then we may set $\langle\, , \rangle_{\mathfrak{G}} = -B$, where B is the Killing form on $\mathfrak{G}$. More generally, if $\mathfrak{G}$ is semisimple, and $\mathfrak{G}_1 \oplus \mathfrak{G}_2 \oplus \ldots \oplus \mathfrak{G}_N$ is the decomposition of $\mathfrak{G}$ into simple ideals, then

a) The ideals $\{\mathfrak{G}_i\}$ are orthogonal relative to any $\mathfrak{G}$-invariant inner product $\langle\, , \rangle_{\mathfrak{G}}$ on $\mathfrak{G}$.

b) On simple ideals $\mathfrak{G}_i$ of $\mathfrak{G}$ the $\mathfrak{G}$-invariant inner products $\langle\, , \rangle_{\mathfrak{G}}$ are all positive multiples of $-B$.

Remark. If G is a compact, connected, semisimple Lie group and $j = d\rho : \mathfrak{G} \to \mathrm{End}(V)$ is injective, then the 2-step nilpotent structures on $\mathcal{N} = V \oplus \mathfrak{G}$ defined as in (2.3) by $\mathfrak{G}$-invariant inner products on $\mathfrak{G}$ are all isomorphic.

The proof of this statement in full generality requires a generalization of the proposition in (2.3a), which we omit. We consider only the case that G and $\mathfrak{G}$ are simple.

If G is a simple compact Lie group and the representation $\rho : G \to GL(V)$ is nontrivial, then $j = d\rho : \mathfrak{G} \to \mathrm{End}(V)$ is injective, and from b) we conclude that there exists a positive constant d such that $-d^2 B(X, Y) = -\operatorname{trace} j(X)\, j(Y)$ for all X, Y in $\mathfrak{G}$. Hence, b) also implies that for every $\mathfrak{G}$-invariant inner product $\langle\, , \rangle$ on $\mathfrak{G}$ there exists a positive number c such that $\langle X, Y\rangle = -c^2 \operatorname{trace} j(X)\, j(Y)$ for all X, Y in $\mathfrak{G}$. By Proposition 1 in (2.3b) the 2-step nilpotent Lie algebras on $\mathcal{N} = V \oplus \mathfrak{G}$ determined by $\mathfrak{G}$-invariant inner products on $\mathfrak{G}$ are all isomorphic to the standard Lie algebra where $c = 1$.

Example 3. Representations of negative definite real Clifford algebras. Let $\{\mathcal{N}, \langle\, , \rangle\}$ be a metric, 2-step nilpotent Lie algebra , and write $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$ where $\mathcal{Z}$ is the center of $\mathcal{N}$ and $\mathcal{V} = \mathcal{Z}^{\perp}$. Suppose that

$j(Z)^2 = -|Z|^2\,\mathrm{Id}$ for every Z in $\mathcal{Z}$, where $j(Z) : \mathcal{V} \to \mathcal{V}$ is the skew symmetric linear map defined by $(*)$ in (2.2a). The corresponding simply connected Lie group N with left invariant metric $\langle\, ,\, \rangle$ is called a space of *Heisenberg type*. These spaces were first introduced and studied seriously by A. Kaplan in [K1, K2]. Clearly $j : \mathcal{Z} \to \mathrm{End}(V)$ is injective for spaces of Heisenberg type.

By polarization, the identity $j(Z)^2 = -|Z|^2\,\mathrm{Id}$ for every Z in $\mathcal{Z}$ is equivalent to the identity $j(Z)\,j(Z^*) + j(Z^*)\,j(Z) = -2\langle Z, Z^*\rangle\,\mathrm{Id}$ on $\mathcal{V}$ for all Z, Z^* in $\mathcal{Z}$. Hence

$$\langle Z, Z^*\rangle = -\Big(\frac{1}{\dim \mathcal{V}}\Big)\,\mathrm{trace}\, j(Z)\,j(Z^*) \qquad \text{for all } Z, Z^* \text{ in } \mathcal{Z}. \qquad (\#)$$

For spaces of Heisenberg type the linear map $j : \mathcal{Z} \to \mathrm{End}(\mathcal{V})$ extends to a representation on $\mathcal{V}$ of $C\ell(\mathcal{Z})$, the negative definite real Clifford algebra determined by $\{\mathcal{Z}, \langle\, ,\, \rangle\}$.

Conversely, let $\{U, \langle , \rangle^*\}$ be an n-dimensional real inner product space, and let $C\ell(n)$ denote the negative definite real Clifford algebra determined by $\{U, \langle , \rangle^*\}$. Let $j : C\ell(n) \to \mathrm{End}(V)$ be a representation of $C\ell(n)$ on a finite dimensional real vector space V. By definition $j(Z)^2 = -|Z|^2\,\mathrm{Id}$ for every Z in U. The subgroup $G = \mathrm{Pin}(n)$ of $C\ell(n)$ generated by the unit vectors in U is a compact subgroup of the group of units of $C\ell(n)$. See for example [FH, pp. 307-312]. By a standard averaging procedure we may choose an inner product $\langle\, ,\, \rangle$ on V such that $j(G)$ is a compact subgroup of the orthogonal group $O(V, \langle\, ,\, \rangle)$. If Z is a unit vector in $\{U, \langle\, ,\, \rangle\}$, then $j(Z)$ is an orthogonal transformation of $\{V, \langle\, ,\, \rangle\}$ such that $j(Z)^2 = -\,\mathrm{Id}$. It follows that $j(Z)$ is also skew symmetric on $\{V, \langle\, ,\, \rangle\}$ since $j(Z)^t = j(Z)^{-1} = -j(Z)$. Hence $j(Z)$ is skew symmetric for all Z in U.

The inner products $\langle\, ,\, \rangle$ on V and $\langle , \rangle^*$ on U determine a j-admissible inner product on $\mathcal{N} = V \oplus U$. The corresponding 2-step nilpotent Lie algebra structure $[\, ,\,]$ on $\mathcal{N}$ is standard in the sense of (2.3b) by the proposition in (2.3a) and the identity $(\#)$ above.

Remarks. (See Appendix 3 for details) a) The isomorphism type of the Clifford algebra $C\ell(n)$ depends only on n and not on the real inner product space$\{U, \langle , \rangle^*\}$. Moreover, $C\ell(n)$ becomes a Lie algebra where the bracket is given by $[a, b] = (1/2)\{ab - ba\}$. If $U^* = U \oplus [U, U]$, then U^* is a Lie subalgebra of $C\ell(n)$ that is isomorphic to $\mathfrak{so}(n+1, \mathbb{R})$. If $j : C\ell(n) \to \mathrm{End}(V)$ is a representation relative to Clifford multiplication in $C\ell(n)$ and composition in $\mathrm{End}(V)$, then $\varphi = (1/2)\,j : C\ell(n) \to \mathrm{End}(V)$ is a Lie algebra homomorphism.

b) The irreducible representations $j : C\ell(n) \to \mathrm{End}(V)$ are uniquely determined up to equivalence if $n \not\equiv 3 \pmod 4$. If $n \equiv 3 \pmod 4$, then $C\ell(n)$ has two inequivalent irreducible representations.

2.5. Standard metric 2-step nilpotent Lie algebras.

Motivated by the statement and proof of Proposition 2 in (2.3b) we now describe a simple family of examples of metric 2-step nilpotent Lie algebras $\{\mathcal{N}, \langle\, ,\, \rangle\}$. In (2.6) we shall see that any 2-step nilpotent Lie algebra $\mathcal{N}$ is isomorphic as a Lie algebra to one of these metric examples.

Fix a positive integer n, and let $\{V, \langle\, ,\, \rangle_{\mathcal{V}}\}$ be an n-dimensional real inner product space. Let $\mathfrak{so}(V)$ denote the Lie algebra of skew symmetric linear transformations of $\{V, \langle\, ,\, \rangle_{\mathcal{V}}\}$, and equip $\mathfrak{so}(V)$ with the positive definite inner product $\langle , \rangle^*$ given by $\langle X, Y\rangle^* = -\operatorname{trace}(XY)$ for elements X, Y of $\mathfrak{so}(V)$. Note that $\langle , \rangle^*$ is a constant multiple of the Killing form of $\mathfrak{so}(V)$. If $V = \mathbb{R}^n$ with the standard inner product, then we let $\mathfrak{so}(n, \mathbb{R})$ denote $\mathfrak{so}(V)$.

Let W be a p-dimensional subspace of $\mathfrak{so}(V)$, and let $\{\mathcal{N}, \langle\, ,\, \rangle\}$ denote the orthogonal direct sum of V and W. Let $[\, ,\,]$ be the 2-step nilpotent structure on $\mathcal{N} = V \oplus W$ such that W is contained in the center of $\mathcal{N}$, and for elements X, Y of V, $[X, Y]$ is the unique element of W such that $\langle [X, Y], Z\rangle^* = \langle Z(X), Y\rangle_{\mathcal{V}}$ for every element Z of W.

Remark. The bracket operation $[\, ,\,]$ depends upon the choice of the subspace W. It is easy to see from the definition that $[\mathcal{N}, \mathcal{N}] = W$ for any one of these metric 2-step nilpotent Lie algebras $\{\mathcal{N}, \langle\, ,\, \rangle\}$, $\mathcal{N} = V \oplus W$. Moreover, $W = \mathcal{Z}$, the center of $\mathcal{N}$, if and only if for every nonzero vector X of V there exists a vector Z of W such that $Z(X)$ is nonzero. We omit the details.

Terminology. A metric 2-step nilpotent Lie algebra $\{\mathcal{N}, \langle\, ,\, \rangle\}$ will be called *standard* if it arises in the manner above. A metric 2-step nilpotent Lie algebra $\{\mathcal{N}, \langle\, ,\, \rangle\}$ will be called *involutive* if it is standard and the subspace W of $\mathfrak{so}(V)$ is actually a subalgebra of $\mathfrak{so}(V)$. Let $\{N, \langle\, ,\, \rangle\}$ be the simply connected 2-step nilpotent Lie group with Lie algebra $\mathcal{N}$ and corresponding left invariant metric. We say that $\{N, \langle\, ,\, \rangle\}$ is a *standard* metric 2-step nilpotent Lie group (respectively an *involutive* metric 2-step nilpotent Lie group) if $\{\mathcal{N}, \langle\, ,\, \rangle\}$ has the corresponding property.

Examples from Lie triple systems.

Let n be any positive integer, and let W be a subspace of $\mathfrak{so}(n,\mathbb{R})$ such that $[X,[Y,Z]] \in W$ for all elements X, Y, Z of W. The subspace W is called a *Lie triple system* in $\mathfrak{so}(n,\mathbb{R})$, and it is well known from the theory of Riemannian symmetric spaces that $X = \exp(W)$ is a totally geodesic submanifold of the special orthogonal group $\mathrm{SO}(n,\mathbb{R})$ equipped with a biinvariant Riemannian metric, where $\exp : \mathfrak{so}(n,\mathbb{R}) \to \mathrm{SO}(n,\mathbb{R})$ is the matrix exponential map. Conversely, if X is a totally geodesic submanifold of $\mathrm{SO}(n,\mathbb{R})$ that contains the identity I, then $X = \exp(W)$ for some Lie triple system W. Any totally geodesic submanifold Y of $\mathrm{SO}(n,\mathbb{R})$ is isometric by a left translation to a totally geodesic submanifold X that contains the identity.

Let W be a Lie triple system in $\mathfrak{so}(n,\mathbb{R})$ and define $Z(W) = \{X \in W : [X,Y] = 0 \text{ for all } Y \in W\}$. We call $Z(W)$ the *center* of W, and we say that W has *compact center* if $\exp(Z(W))$ is a compact subset of $SO(n,\mathbb{R})$. Note that $\exp(Z(W))$ is a connected abelian subgroup of $SO(n,\mathbb{R})$ for any Lie triple system W.

The Lie triple systems in $\mathfrak{so}(n,\mathbb{R})$ form an important class of examples whose structure is described in more detail in Appendix 2. We now list two important examples.

Example 1. Representations of compact Lie groups. Let G be a compact connected Lie group, and let $\rho : G \to GL(V)$ be a representation of G on a finite dimensional real vector space V. Let $\langle\, ,\, \rangle_{\mathcal{V}}$ be any inner product on V that is invariant under $\rho(G)$. If $j = d\rho : \mathfrak{G} \to \mathrm{End}(V)$ is the induced representation of the Lie algebra $\mathfrak{G}$ of G, then $W = j(\mathfrak{G})$ is a subalgebra of $\mathfrak{so}(V)$, the skew symmetric linear transformations of $\{V, \langle\, ,\, \rangle_{\mathcal{V}}\}$.

A subalgebra W of $\mathfrak{so}(V)$ is clearly a Lie triple system in $\mathfrak{so}(V)$. In this case W has compact center, and if G is semisimple, then W has trivial center; that is, $Z(W) = \{0\}$.

More generally, any Lie subgroup G of the orthogonal group $\mathrm{SO}(V)$ is a totally geodesic submanifold of $\mathrm{SO}(V)$ (possibly immersed) with respect to any biinvariant metric on $\mathrm{SO}(V)$. A subgroup G is an imbedded totally geodesic submanifold if G is closed in $\mathrm{SO}(V)$. A biinvariant metric on $\mathrm{SO}(V)$ is unique up to positive multiples.

Example 2. Representations of real Clifford algebras. Let $\{U, \langle\, ,\, \rangle\}$ be an n-dimensional real inner product space, and let $C\ell(n)$ denote the negative definite real Clifford algebra determined by $\{U, \langle\, ,\, \rangle\}$. Let $j : C\ell(n) \to \mathrm{End}(V)$ be a representation of $C\ell(n)$ on a finite dimensional real vector space V, where multiplication in $\mathrm{End}(V)$ is composition. As in (2.3) choose an inner product $\langle\, ,\, \rangle$ on V such that $W = j(U) \subseteq \mathfrak{so}(V)$. We assert that W is a Lie triple system with trivial center.

To see that W is a Lie triple system let $\{u_1, u_2, ...u_n\}$ be an orthonormal basis of U. By the discussion in (2.4) we obtain

$$j(u_i)j(u_k) = -j(u_k)j(u_i) \text{ if } i \neq k \text{ and } j(u_i)^2 = -\operatorname{Id} \text{ for all } i. \qquad (*)$$

Hence $[W, W] = \mathrm{span}\{[j(u_i), j(u_k)] : 1 \leq i, k \leq n\} = \mathrm{span}\{j(u_i)j(u_k) : 1 \leq i < k \leq n\}$. If i, k, ℓ are all distinct, then $j(u_\ell)$ commutes with $j(u_i)j(u_k)$ by $(*)$. If $\ell = i$ or k, then $j(u_i)\, j(u_k)\, j(u_\ell) = \pm j(u_i)$ or $\pm j(u_k)$. It follows that $[W, [W, W]] = \mathrm{span}\{[j(u_i)\, j(u_k), j(u_\ell)] : 1 \leq i < k \leq n,\ 1 \leq \ell \leq n\} = \mathrm{span}\{j(u_i) : 1 \leq i \leq n\} = W$. Hence W is a Lie triple system.

We show that the Lie triple sytem $W = j(U)$ has trivial center. Let $u \in U$ be an element such that $j(u)$ commutes with $j(u^*)$ for all $u^* \in U$. If $\langle u, u^* \rangle = 0$, then $j(u)j(u^*) = -j(u^*)j(u)$ by the discussion of example 3 of (2.4). Hence if $\langle u, u^* \rangle = 0$, then $j(u)j(u^*) = 0$, which implies that $0 = j(u)j(u^*)j(u)j(u^*) = j(u)^2 j(u^*)^2 = |u|^2 |u^*|^2 \operatorname{Id}$. We conclude that $u = 0$ and W has trivial center.

If $n = \dim U$, then for $n \neq 3$ the totally geodesic subspace $X = \exp(W)$ of $\mathrm{SO}(V)$ with a biinvariant metric is a sphere of dimension n. This follows from the well known facts that $\mathfrak{G} = W \oplus [W, W]$ is isomorphic as a Lie algebra to $\mathfrak{so}(n+1, \mathbb{R})$ and $\mathcal{K} = [W, W]$ is isomorphic to $\mathfrak{so}(n, \mathbb{R})$. If $n = 3$, then X could be either a 3-sphere or a 2-sphere. For details, see for example Proposition 3 in Appendix 3 and Lemma 2 of Proposition 3.

2.6. Metrizing 2-step nilpotent Lie algebras into standard form.

We show next that every 2-step nilpotent Lie algebra $\mathcal{N}$ with an appropriate inner product $\langle\, ,\, \rangle$ is a standard metric 2-step nilpotent Lie algebra.

Proposition. *Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra of dimension $n+q$ such that $[\mathcal{N}, \mathcal{N}]$ has dimension $q \geq 1$. Then*

1) There exists a q-dimensional subspace W of $\mathfrak{so}(n, \mathbb{R})$ such that $\mathcal{N}$ is isomorphic as a Lie algebra to the standard metric 2-step nilpotent Lie algebra $\mathcal{N}^* = \mathbb{R}^n \oplus W$.

2) If $\mathcal{N}$ admits a basis with rational structure constants, then we may choose W to have a basis whose matrices have entries in $\mathbb{Z}$ relative to the standard orthonormal basis $\{e_1, e_2, ..., e_n\}$ of $\mathbb{R}^n$.

Proof. Let $\{Z_1, Z_2, ..., Z_q\}$ be a basis of $[\mathcal{N}, \mathcal{N}]$ and extend it to a basis $\mathcal{B} = \{X_1, X_2, ..., X_n, Z_1, Z_2, ..., Z_q\}$ of $\mathcal{N}$. Let $[X_i, X_j] = \sum_{k=1}^{q} C_{ij}^k Z_k$ for $1 \le i, j \le n$; $1 \le k \le q$ and suitable matrices $\{C^1, C^2, ...C^q\}$ in $\mathfrak{so}(n, \mathbb{R})$. Let $W = \text{span}\{C^1, C^2, ...C^q\} \subseteq \mathfrak{so}(n, \mathbb{R})$, and let $\mathcal{N}^* = \mathbb{R}^n \oplus W$ denote the standard metric 2-step nilpotent Lie algebra determined by W and the usual inner product on $\mathbb{R}^n$. We will show that $\mathcal{N}$ is isomorphic as a Lie algebra to $\mathcal{N}^* = \mathbb{R}^n \oplus W$. $\square$

Lemma. *The matrices $\{C^1, C^2, ...C^q\}$ are linearly independent in $\mathfrak{so}(n, \mathbb{R})$.*

Proof of the Lemma. Let $\{\alpha_1, \alpha_2, ...\alpha_q\}$ be real numbers such that $0 = \sum_{k=1}^{q} \alpha_k C^k$. Let $Z^* = \sum_{k=1}^{q} \alpha_k Z_k$. Define $\langle\, , \rangle$ to be the inner product on $[\mathcal{N}, \mathcal{N}]$ that makes $\{Z_1, Z_2, ..Z_q\}$ an orthonormal basis. Then for $1 \le r, s \le n$ we have $\langle [X_r, X_s], Z^* \rangle = \sum_{k=1}^{q} \alpha_k C_{rs}^k = \left(\sum_{k=1}^{q} \alpha_k C^k\right)_{rs} = 0$. Hence Z^* is orthogonal to $[\mathcal{N}, \mathcal{N}] = \text{span}\{Z_1, Z_2, ..Z_q\}$. It follows that $Z^* = 0$, which implies that $\alpha_k = 0$ for all k. $\square$

Proof of the Proposition. 1) Let $\{e_1, e_2, ..., e_n\}$ be the standard orthonormal basis of $\mathbb{R}^n$. Let $\{\rho_1, \rho_2, ..., \rho_q\}$ be the basis of W such that $\langle \rho_\alpha, C^\beta \rangle = \delta_{\alpha\beta}$ for $1 \le \alpha, \beta \le q$, where $\langle\, , \rangle$ denotes the standard inner product defined on $\mathfrak{so}(n, \mathbb{R})$ in (2.5). Let $T : \mathcal{N} \to \mathcal{N}^*$ be the unique linear isomorphism such that $T(X_i) = e_i$ for $1 \le i \le n$ and $T(Z_\alpha) = -\rho_\alpha$ for all $1 \le \alpha \le q$. We show that T is a Lie algebra isomorphism. It suffices to show that $T([X_i, X_j]) = [T(X_i), T(X_j)]^* = [e_i, e_j]^*$ for all $1 \le i, j \le n$, where $[\, ,]$ and $[\, ,]^*$ denote the Lie brackets in $\mathcal{N}$ and $\mathcal{N}^*$ respectively. Note that $\langle [e_i, e_j]^*, C^k \rangle = \langle C^k(e_i), e_j \rangle = C_{ji}^k$. Furthermore, since each C^r is skew symmetric we have $\langle T([X_i, X_j]), C^k \rangle = \langle \sum_{r=1}^{q} C_{ij}^r T(Z_r), C^k \rangle = \langle \sum_{r=1}^{q} C_{ji}^r \rho_r, C^k \rangle = C_{ji}^k = \langle [e_i, e_j]^*, C^k \rangle$ for $1 \le i, j \le n$. Hence $T([X_i, X_j]) = [e_i, e_j]^*$ for all $1 \le i, j \le n$ since $\{C^1, C^2, ...C^q\}$ is a basis for W. This completes the proof of 1).

2) If $\mathcal{N}$ admits a basis with rational structure constants, then the simply connected Lie group N with Lie algebra $\mathcal{N}$ admits a lattice Γ by the Mal'cev criterion. By the proposition in (1.3e) we may choose a basis $\mathcal{B}' = \{X_1, X_2, ..., X_m, Z_1, Z_2, ..., Z_p\}$ for $\mathcal{N}$ such that $\{Z_1, Z_2, ..., Z_p\}$ is a basis for $\mathcal{Z}$, $\{Z_1, Z_2, ..., Z_q\}$ is a basis for $[\mathcal{N}, \mathcal{N}]$ and $[X_i, X_j] = \sum_{k=1}^{q} C_{ij}^k Z_k$ for $1 \le i, j \le m$; $1 \le k \le q$, where the constants $\{C_{ij}^k\}$ lie in $\mathbb{Z}$. Hence the skew symmetric matrices $\{C^1, C^2, ..., C^q\}$ have entries in $\mathbb{Z}$.

If $\Lambda = \log \Gamma$ is a vector lattice in $\mathcal{N}$, then we may also choose $\mathcal{B}'$ to be a

$\mathbb{Z}$-basis of Λ by the lemma in (1.3e).

If $p = q$, then $m = n$ and we let $\mathcal{B} = \mathcal{B}'$. Otherwise we construct a basis $\mathcal{B} = \{X_1, X_2, ..., X_n, Z_1, Z_2, ..., Z_q\}$ from $\mathcal{B}'$ by setting $X_{m+i} = Z_{q+i}$ for $1 \leq i \leq p - q$. Now apply the construction in the proof of 1). $\square$

Remark. The proof shows that the isomorphism $T : \mathcal{N}^* \to \mathcal{N}$ in the proof of 1) is not unique but depends on a choice of basis $\mathcal{B} = \{X_1, X_2, ..., X_n, Z_1, Z_2, ..., Z_p\}$ for $\mathcal{N}$ as above.

2.7. Lattices and rational linear maps.

Let V be a finite dimensional real vector space, and let W be a subspace of $\mathrm{End}(V)$. Call W a *rational* subspace of $\mathrm{End}(V)$ if there exist bases $\mathcal{B}_V$ for V and $\mathcal{B}_W$ for W such that $Z\big(\mathbb{Q} - \mathrm{span}(\mathcal{B}_V)\big) \subseteq \mathbb{Q} - \mathrm{span}(\mathcal{B}_V)$ for all Z in $\mathcal{B}_W$. Equivalently, the matrix of Z relative to $\mathcal{B}_V$ has rational entries for all Z in $\mathcal{B}_W$. (Compare Example 2 in section 1.3a).

For finite dimensional real vector spaces U, V a linear map $j : U \to \mathrm{End}(V)$ will be called *rational* if there exist bases $\mathcal{B}_U$ for U and $\mathcal{B}_V$ for V such that $j(Z)\big(\mathbb{Q} - \mathrm{span}(\mathcal{B}_V)\big) \subseteq \mathbb{Q} - \mathrm{span}(\mathcal{B}_V)$ for all Z in $\mathcal{B}_U$.

Examples of skew symmetrizable, rational linear maps.

Example 1. Fix an inner product $\langle\, , \rangle$ on $\mathbb{R}^n$. Let $\mathfrak{so}(n, \mathbb{R})$ denote the Lie algebra of skew symmetric linear transformations of $\mathbb{R}^n$. Let W be a rational subspace of $\mathfrak{so}(n, \mathbb{R}) \subseteq \mathrm{End}(\mathbb{R}^n)$. Then the inclusion map $j : W \to \mathrm{End}(\mathbb{R}^n)$ is an injective, skew symmetrizable, rational linear map.

In particular, let $\langle\, , \rangle$ and $\mathcal{B}_V = \{e_1, e_2, ..., e_n\}$ denote the standard inner product and orthonormal basis of $V = \mathbb{R}^n$. Let $\{A_1, A_2, ..., A_p\}$ be arbitrary skew symmetric $n \times n$ matrices with rational coefficients. If W is the subspace of $\mathfrak{so}(n, \mathbb{R})$ spanned by the transformations whose matrices are $\{A_1, A_2, ..., A_p\}$ relative to $\mathcal{B}_V$, then W is a rational subspace of $\mathfrak{so}(n, \mathbb{R})$.

Example 2. Let W be a rational subspace of $\mathfrak{so}(n, \mathbb{R})$. Then for any g in $GL(\mathbb{R}^n)$ the inclusion map $j : gWg^{-1} \to \mathrm{End}(\mathbb{R}^n)$ is an injective, skew symmetrizable, rational linear map.

To see this, let W be a rational subspace of $\mathfrak{so}(n, \mathbb{R}^n)$ relative to bases $\mathcal{B}_V$ of $V = \mathbb{R}^n$ and $\mathcal{B}_W$ of W. The matrices of gWg^{-1} relative to the basis $g(\mathcal{B}_V)$ of $\mathbb{R}^n$ are skew symmetric since they are the same as the matrices of W relative to $\mathcal{B}_V$. It follows that $g\mathcal{B}_W g^{-1}$ is a basis of gWg^{-1} whose matrices relative to $g(\mathcal{B}_V)$ have rational entries. If $\langle\ ,\ \rangle_g$ denotes the inner product on $\mathbb{R}^n$ that makes $g(\mathcal{B}_V)$ an orthonormal basis, then $gWg^{-1} \subseteq \mathsf{so}(\mathbb{R}^n, \langle\ ,\ \rangle_g)$.

Example 3. Let W be a subalgebra of $\mathrm{End}(\mathbb{R}^n)$ whose Killing form B is negative definite. Then the inclusion map $j : W \to \mathrm{End}(\mathbb{R}^n)$ is an injective, skew symmetrizable, rational linear map.

To see this, note that $G = \exp(\mathfrak{G})$ is a compact subgroup of $GL(V)$ by remarks 3) and 5) following Proposition 4.2. If $\langle\ ,\ \rangle$ is any G-invariant inner product on V, then $\mathfrak{G} \subseteq \mathsf{so}(V, \langle\ ,\ \rangle)$. If $\mathcal{C}_{\mathbb{R}}$ is a real Chevalley basis of $\mathfrak{G}$, then by the proposition in Appendix 1 there exists a basis $\mathcal{B}_V$ of V such that the elements of $\mathcal{C}_{\mathbb{R}}$ leave invariant $\mathbb{Z} - \mathrm{span}(\mathcal{B}_V)$.

We introduce some terminology before stating the next result. We say that V is *irreducible* relative to a subspace W of $\mathrm{End}(V)$ if no proper subspace of V is invariant under all elements of W. We say that an inner product $\langle\ ,\ \rangle$ on V is *W-invariant* if $W \subseteq \mathsf{so}(V, \langle\ ,\ \rangle$.

Proposition. *Let $j : U \to \mathrm{End}(V)$ be an injective, rational and skew symmetrizable linear map, and let $\{\mathcal{N} = V \oplus U, [\ ,\ ]\}$ be the 2-step nilpotent Lie algebra defined in Proposition 1 of (2.3b). Suppose that V is irreducible relative to $W = j(U)$. Then $\mathcal{N}$ admits a basis with rational structure constants. In particular if N is the simply connected 2-step nilpotent Lie group with Lie algebra $\mathcal{N}$, then N admits a lattice.*

The proof of the Proposition follows immediately from Lemmas 2 and 3 below and the Mal'cev criterion for lattices from section 1. As an immediate consequence of the result above and the discussion preceding it we obtain the following

Corollary. *Fix an inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$ and let W be a rational subspace of $\mathsf{so}(n, \mathbb{R}) \subseteq \mathrm{End}(\mathbb{R}^n)$ such that $\mathbb{R}^n$ is irreducible relative to W. Let $\mathcal{N} = \mathbb{R}^n \oplus W$ be the corresponding standard metric 2-step nilpotent Lie algebra. Then $\mathcal{N}$ admits a basis with rational structure constants. If N is the simply connected 2-step nilpotent Lie group with Lie algebra $\mathcal{N}$, then N admits a lattice.*

Remark. If W is a "natural" rational subspaces of $\mathsf{so}(n, \mathbb{R})$ as defined above in Example 1, then the corollary contains a converse to 2) of the proposition in (2.6).

We now begin the proof of the Proposition. We first state three lemmas and then prove them in order.

Lemma 1. *Let $\{V, \langle\ ,\ \rangle\}$ be a finite dimensional real inner product space, and let $\{v_1, v_2, ..., v_n\}$ be a basis of V such that $\langle v_i, v_j\rangle \in \mathbb{Q}$ for all $1 \leq i,\ j \leq n$.*

1) Let $v \in V$ be a vector such that $\langle v, v_i \rangle \in \mathbb{Q}$ for all $1 \leq i \leq n$. Then $v \in \mathbb{Q} - \operatorname{span}\{v_1, v_2, ..., v_n\}$.

2) There exists an orthogonal basis $\{v_1^*, v_2^*, ..., v_n^*\}$ of V such that $\langle v_i^*, v_j^* \rangle \in \mathbb{Q}$ for all $1 \leq i, j \leq n$ and $\mathbb{Q} - \operatorname{span}\{v_1, v_2, ..., v_r\} = \mathbb{Q} - \operatorname{span}\{v_1^*, v_2^*, ..., v_r^*\}$ for all $1 \leq r \leq n$.

Lemma 2. *Let V be a finite dimensional real vector space, and let W be a rational subspace of $\operatorname{End}(V)$ with respect to bases $\{v_1, v_2, ..., v_n\}$ of V and $\{\xi_1, \xi_2, ..., \xi_p\}$ of W. Let $\langle\, , \rangle$ be an inner product on V such that $W \subseteq \mathsf{so}(V, \langle\, , \rangle)$ and $\langle v_i, v_j \rangle$ is a rational number for all $1 \leq i, j \leq n$. Let $\mathcal{N} = V \oplus W$ be the standard metric 2-step nilpotent Lie algebra as defined in Proposition 1 of (2.3b). Then $\{v_1, v_2, ..., v_n, \xi_1, \xi_2, ..., \xi_p\}$ is a basis for $\mathcal{N}$ with rational structure constants.*

Lemma 3. *Let V be a finite dimensional real vector space, and let W be a rational subspace of $\operatorname{End}(V)$ with respect to bases $\{v_1, v_2, ..., v_n\}$ of V and $\{\xi_1, \xi_2, ..., \xi_p\}$ of W. Suppose furthermore that V is irreducible relative to W.*

Let $\langle, \rangle^*$ be a W-invariant inner product on V; that is $W \subseteq \mathsf{so}(V, \langle, \rangle^*)$. Then there exists a positive constant c such that if $\langle\, , \rangle = c\langle, \rangle^*$, then $\langle\, , \rangle$ is W-invariant and $\langle v_i, v_j \rangle$ is a rational number for all $1 \leq i, j \leq n$.

Proof of Lemma 1. We omit the proof of 1). To prove 2) it suffices to find an orthogonal basis $\{v_1^*, v_2^*, ..., v_n^*\}$ of V such that $\mathbb{Q} - \operatorname{span}\{v_1, v_2, ..., v_r\} = \mathbb{Q} - \operatorname{span}\{v_1^*, v_2^*, ..., v_r^*\}$ for all $1 \leq r \leq n$. Set $v_1^* = v_1$ and proceed by induction on r. Suppose for some integer $r \geq 1$ we have found orthogonal vectors $\{v_1^*, v_2^*, ..., v_r^*\}$ such that $\mathbb{Q} - \operatorname{span}\{v_1, v_2, ..., v_s\} = \mathbb{Q} - \operatorname{span}\{v_1^*, v_2^*, ..., v_s^*\}$ for all $1 \leq s \leq r$. Define $v_{r+1}^* = v_{r+1} - \sum_{i=1}^{r} c_i v_i^*$, where $c_i = \langle v_{r+1}, v_i^* \rangle / \langle v_i^*, v_i^* \rangle \in \mathbb{Q}$. □

Proof of Lemma 2. By hypothesis the matrices of $\{\xi_1, \xi_2, ..., \xi_p\}$ relative to the basis $\{v_1, v_2, ..., v_n\}$ of V have entries in $\mathbb{Q}$. It follows that $\langle \xi_i, \xi_j \rangle = -\operatorname{trace}(\xi_i \xi_j) \in \mathbb{Q}$ for $1 \leq i, j \leq p$. Moreover, $\langle [v_i, v_j], \xi_k \rangle = \langle \xi_k(v_i), v_j \rangle \in \mathbb{Q}$ for $1 \leq i, j \leq n$ and $1 \leq k \leq p$ since $\langle v_i, v_j \rangle$ is a rational number for all $1 \leq i, j \leq n$. Hence $[v_i, v_j] \in \mathbb{Q} - \operatorname{span}\{\xi_1, \xi_2, ..., \xi_p\}$ for $1 \leq i, j \leq n$ by Lemma 1. □

Proof of Lemma 3.

Sublemma 3a. *If V is irreducible relative to a subspace W of $\mathsf{so}(V, \langle, \rangle^*)$,* *then the W-invariant inner products $\langle\,,\,\rangle$ on V have the form cB_o, where $B_o = \langle,\rangle^*$ and $c > 0$.*

Proof of Sublemma 3a. By hypothesis V admits a W-invariant inner product $\langle,\rangle^*$. Now let $\langle\,,\,\rangle_1$ and $\langle\,,\,\rangle_2$ be two W-invariant inner products on V, and let $S : V \to V$ be the linear transformation such that $\langle v, w\rangle_2 = \langle Sv, w\rangle_1$ for all $v, w \in V$. The transformation S is symmetric with respect to both $\langle\,,\,\rangle_1$ and $\langle\,,\,\rangle_2$, and S commutes with all elements of W. In particular, the elements of W leave invariant each eigenspace of S, and it follows that S is a multiple of the identity. □

Let $\langle,\rangle^*$ be the given W-invariant inner product on V. Let $\mathfrak{B}$ denote the $\mathbb{R}$-vector space of symmetric, bilinear forms on V (not necessarily positive definite). Define an action of $\mathrm{End}(V)$ on $\mathfrak{B}$ by $(XB)(v,w) = B(Xv,w) + B(v,Xw)$ for all $B \in \mathfrak{B}$ and all $v, w \in V$. It is easy to check that $X(YB) - Y(XB) = -[X,Y]B$ for all $X, Y \in \mathrm{End}\, V$ and $B \in \mathcal{B}$. We say that $B \in \mathfrak{B}$ is *W-invariant* if $XB = 0$ for all $X \in W$.

Sublemma 3b. *Let V be irreducible relative to a subspace W of $\mathfrak{so}(V, \langle,\rangle^*)$. Let $\mathfrak{W}$ be the subspace of $\mathfrak{B}$ consisting of W-invariant bilinear forms on V. Then $\dim_{\mathbb{R}} \mathfrak{W} = 1$ and $\mathfrak{W}$ is generated by a positive definite symmetric bilinear form B_o.*

Proof of Sublemma 3b. By the hypothesis of Lemma 2 we know that $\mathfrak{B}$ contains a positive definite symmetric bilinear form B_o. If $\dim_{\mathbb{R}} \mathfrak{W} =\geq 2$, then $\mathfrak{B}$ contains a symmetric bilinear form B such that $B_o + tB$ is $\mathbb{R}$-linearly independent from B_o for all nonzero t. However, for small nonzero t the form $B_o + tB$ is positive definite, which contradicts sublemma 3a. □

We are now ready to complete the proof of Lemma 3. Let $\{v_1, v_2, ..., v_n\}$ and $\{\xi_1, \xi_2, ..., \xi_p\}$ be bases of V and W as in the statement of the lemma, where V is irreducible relative to W. Let $\{v_1^*, v_2^*, ..., v_n^*\}$ be the basis of V^* that is dual to $\{v_1, v_2, ..., v_n\}$. For $1 \leq i \leq j \leq n$ let $\{B_{ij} = (1/2)(v_i^* \otimes v_j^* + v_j^* \otimes v_i^*)\}$ be the basis of $\mathfrak{B}$ defined by $B_{ij}(v_k, v_\ell) = 1$ if $\{k, \ell\} = \{i, j\}$ and $B_{ij}(v_k, v_\ell) = 0$ otherwise.

The hypotheses of Lemma 3 imply

$$\text{If } \mathfrak{B}_{\mathbb{Q}} = \mathbb{Q} - \mathrm{span}\{B_{ij}\},\ \text{then } \xi_\alpha(\mathfrak{B}_{\mathbb{Q}}) \subseteq \mathfrak{B}_{\mathbb{Q}} \qquad (*)$$
$$\text{for all } 1 \leq \alpha \leq p = \dim W.$$

Define a linear map $\xi : \mathfrak{B} \to \mathfrak{B}^p$ by $\xi(B) = (\xi_1(B), \xi_2(B), ..., \xi_p(B))$.

Note that $\xi(\mathfrak{B}_{\mathbb{Q}}) \subseteq \mathfrak{B}_{\mathbb{Q}}{}^{p}$ by $(*)$. Moreover, $\operatorname{Ker}(\xi) = \mathfrak{W}$, the subspace of $\mathfrak{B}$ consisting of W-invariant forms. It suffices to find a nonzero element B in $\operatorname{Ker}(\xi) \cap \mathfrak{B}_{\mathbb{Q}}$. For such an element, if we write $B = \sum_{i \leq j} q_{ij} B_{ij}$, where $q_{ij} \in \mathbb{Q}$, then $B(v_k, v_\ell) = q_{k\ell} \in \mathbb{Q}$ for all $1 \leq k \leq \ell \leq n$. By sublemma 3b, $B = cB_o$ for some nonzero real number c. If $c > 0$, then B is positive definite and W-invariant while if $c < 0$, then $-B$ is positive definite and W-invariant.

Let $N = \dim_{\mathbb{R}} \mathfrak{B}$. The basis $\{B_{ij} : i \leq j\}$ for $\mathfrak{B}$ defines a basis for $\mathfrak{B}^p$ in a natural way, and relative to these bases the linear map $\xi : \mathfrak{B} \to \mathfrak{B}^p$ has a $pN \times N$ matrix A whose entries lie in $\mathbb{Q}$ by $(*)$. The determinant of any $k \times k$ submatrix of A lies in $\mathbb{Q}$, and hence $\operatorname{rank}_{\mathbb{Q}}(A) = \operatorname{rank}_{\mathbb{R}}(A)$ and $\operatorname{nullity}_{\mathbb{Q}}(A) = \operatorname{nullity}_{\mathbb{R}}(A)$. By sublemma 3b $\operatorname{nullity}_{\mathbb{R}}(A) = 1$, and hence there exists a nonzero element B in $\operatorname{Ker}(\xi) \cap \mathfrak{B}_{\mathbb{Q}}$. The proof of Lemma 3 is complete. □

3. Riemannian submersions.

In this section we prove the following result. See (2.5) for terminology.

Theorem. *Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra , and let N denote the corresponding simply connected nilpotent Lie group with Lie algebra $\mathcal{N}$. Then there exists a left invariant metric $\langle\, ,\rangle$ on N, an involutive metric 2-step nilpotent Lie group $\{N^*, \langle,\rangle^*\}$ and a surjective homomorphism $\rho : N^* \to N$ with the following properties:*

1) *ρ is a Riemannian submersion whose fibers are simply connected, flat totally geodesic submanifolds of N^*. The fibers of ρ are the orbits of* $\operatorname{Ker}(\rho)$, *which is a simply connected, totally geodesic subgroup of the center Z^* of N^*.*

2) *If N admits a lattice Γ, then for a suitable choice of N^* there exists a lattice Γ^* of N^* such that $\rho(\Gamma^*) = \Gamma$ and $\Gamma^* \cap \operatorname{Ker}(\rho)$ is a lattice in* $\operatorname{Ker}(\rho)$.

3) *If N^* admits a lattice Γ^* such that $\Gamma^* \cap \operatorname{Ker}(\rho)$ is a lattice in* $\operatorname{Ker}(\rho)$, *then $\Gamma = \rho(\Gamma^*)$ is a lattice in N.*

4) *If there are lattices Γ^* in N^* and Γ in N such that $\rho(\Gamma^*) = \Gamma$, then ρ induces a Riemannian submersion $\rho' : \Gamma^* \backslash N^* \to \Gamma \backslash N$ whose fibers are flat, totally geodesic tori that are isometric to each other.*

Remark. As we shall see, the definition of $\mathcal{N}^*$ and N^* depends upon a representation of $\mathcal{N}$ as a standard metric 2-step nilpotent Lie algebra. This representation is not unique (cf. (2.6)), and it is an interesting problem to determine a representation that is "optimal" with respect to some reasonable constraint. The lack of uniqueness of N^* accounts for the wording in 2) of the proposition.

Proof. By the proposition in (2.6) $\mathcal{N}$ is isomorphic as a Lie algebra to a standard metric 2-step nilpotent Lie algebra $\mathcal{N}' = \mathbb{R}^n \oplus W$, where $\mathcal{N}$ has dimension $n+p$, $[\mathcal{N},\mathcal{N}]$ has dimension p, $\mathbb{R}^n$ is given the standard inner product and W is a p-dimensional subspace of $\mathfrak{so}(n,\mathbb{R})$. Let $\mathcal{N}$ be given the inner product $\langle\, , \rangle$ that makes this isomorphism also a linear isometry. Henceforth we regard $\{\mathcal{N}, <, \rangle\}$ as the standard metric 2-step nilpotent Lie algebra $\mathcal{N}' = \mathbb{R}^n \oplus W$.

Let $\mathfrak{G}$ denote the subalgebra of $\mathfrak{so}(n,\mathbb{R})$ generated by W, and let $\mathcal{N}^*$ be the involutive metric 2-step nilpotent Lie algebra $\mathbb{R}^n \oplus \mathfrak{G}$. Although $\mathcal{N}$ may be regarded as a linear subspace of $\mathcal{N}^*$ note that the bracket $[\, ,]$ on $\mathcal{N}$ is not the restriction of the bracket $[\, ,]^*$ on $\mathcal{N}^*$. For elements X, Y of $\mathbb{R}^n$ the bracket $[X,Y]^*$ will in general have a nonzero component in $W^\perp$, the orthogonal complement of W in $\mathfrak{G}$.

Regarding $\mathcal{N}$ as a subspace of $\mathcal{N}^*$ we let $\pi : \mathcal{N}^* \to \mathcal{N}$ be the surjective linear map such that π is the identity on $\mathbb{R}^n$ and $\pi : \mathfrak{G} \to W$ is the orthogonal projection relative to the canonical inner product on $\mathfrak{so}(n,\mathbb{R})$ (cf. (2.5)). We show that $\pi : \mathcal{N}^* \to \mathcal{N}$ is a surjective Lie algebra homomorphism and that the lifted homomorphism $\rho : N^* \to N$ with $\pi = d\rho$ satisfies the statements of the proposition.

Proof of 1) of the theorem.

We begin the proof of 1). From the definitions a routine argument yields

Lemma 1. *The map $\pi : \mathcal{N}^* \to \mathcal{N}$ is a surjective Lie algebra homomorphism and $\pi(\xi) = \xi$ for any vector $\xi \in \mathcal{N} = \mathrm{Ker}(\pi)^\perp$.*

If N^* is the simply connected nilpotent Lie group with Lie algebra $\mathcal{N}^*$, then there exists a unique homomorphism $\rho : N^* \to N$ such that $d\rho = \pi : \mathcal{N}^* \to \mathcal{N}$. The homomorphism ρ is surjective since $d\rho$ is surjective.

Lemma 2. $\mathrm{Ker}(\rho) = \exp^*(\mathrm{Ker}(\pi))$*, where* $\exp^* : \mathcal{N}^* \to N^*$ *is the Lie group exponential map of N^*. In particular,* $\mathrm{Ker}(\rho)$ *is a simply connected subgroup of Z^*, the center of N^*.*

Proof. Note that $\mathrm{Ker}(\pi) \subseteq \mathfrak{G} \subseteq \mathcal{Z}^*$, the center of $\mathcal{N}^*$, and the exponential maps $\exp : \mathcal{N} \to N$ and $\exp^* : \mathcal{N}^* \to N^*$ are diffeomorphisms satisfying $\rho \circ \exp^* = \exp \circ d\rho$. The proof is now straightforward. $\square$

Lemma 3. *The fibers of $\rho : N^* \to N$ are the orbits of* $\mathrm{Ker}(\rho)$ *and are flat, totally geodesic submanifolds of N^*. In particular,* $\mathrm{Ker}(\rho)$ *is a totally geodesic subgroup of N^*.*

Proof. Clearly $\rho^{-1}(\rho(n^*)) = n^*\dot{\mathrm{Ker}}(\rho)$ for all $n^* \in N^*$. Since $\mathrm{Ker}(\rho) \subseteq Z^*$ it follows from d) of (2.3) in [E1] that $\mathrm{Ker}(\rho)$ is a flat submanifold of N^*. From c) of (2.2) in [E1] and (2.10) of [E1] we see that $\mathrm{Ker}(\rho)$ is a totally geodesic submanifold of N^*. The orbits $n^*\dot{\mathrm{Ker}}(\rho)$ are also flat, totally geodesic submanifolds of N^* since left multiplication by an element n^* of N^* is an isometry.

To complete the proof of 1) of the theorem we must show that $\rho : N^* \to N$ is a Riemannian submersion. Given $n^* \in N^*$ let $X \subseteq T_{n^*}N^*$ be the kernel of $(d\rho)_{n^*} : T_{n^*}N^* \to T_nN$, where $n = \rho(n^*)$. Since $d\rho \circ dL_{n^*} = dL_{\rho(n^*)} \circ d\rho = dL_n \circ \pi$, it follows that $X = dL_{n^*}(\mathrm{Ker}(\pi))$ and hence $X^\perp = dL_{n^*}\big(\mathrm{Ker}(\pi)^\perp\big)$. Given $\xi \in X^\perp$ we write $\xi = dL_{n^*}(\xi')$ for some $\xi' \in \mathrm{Ker}(\pi)^\perp$ and note that $\pi(\xi') = \xi'$ by Lemma 1. Using Lemma 1 and the discussion above we obtain $|d\rho(\xi)| = |d\rho(dL_{n^*}(\xi'))| = |dL_n(\pi(\xi'))| = |dL_n(\xi')| = |\xi'| = |\xi|$ since L_{n^*} and L_n are isometries of N^* and N. This completes the proof of 1) $\square$

Proof of 2) of the theorem.

We begin the proof of 2). As in the proof of the proposition in (1.3e) we suppose first that $\Lambda = \log \Gamma$ is a vector lattice in $\mathcal{N}$.

Case 1 $\quad\Lambda = \log \Gamma$ is a vector lattice in $\mathcal{N}$

Lemma 4. *Let $\Lambda = \log \Gamma$ be a vector lattice in $\mathcal{N}$. Then we may assume that*

1) $\mathcal{N} = \mathbb{R}^n \oplus W$, a standard metric 2-step nilpotent Lie algebra such that $\mathbb{R}^n$ has the standard inner product and W is a subspace of $\mathfrak{so}(n,\mathbb{R})$. Moreover, W has a basis $\{Z_1, Z_2, ..., Z_p\}$ such that $[e_i, e_j] = \sum_{k=1}^p C_{ij}^k Z_k$, where $\{e_1, e_2, ..., e_n\}$ is the natural basis of $\mathbb{R}^n$, $C_{ij}^k \in \mathbb{Z}$ for every i, j, k , and $\langle Z_\alpha, C^\beta\rangle = -\delta_{\alpha\beta}$ for all $1 \le \alpha,\ \beta \le p$.

2) $\Lambda = \log \Gamma = \mathbb{Z} - \mathrm{span}\{e_1, e_2, ..., e_n, Z_1, Z_2, ..., Z_p\}$.

Proof. This follows from the proof of the proposition in (2.6). $\square$

Lemma 5. *Let $\mathcal{N} = \mathbb{R}^n \oplus W$ and $\{Z_1, Z_2, ..., Z_p\}$ be as in Lemma 4, and let $\mathfrak{G}$ be the subalgebra of $\mathfrak{so}(n,\mathbb{R})$ generated by W. Then there exists an* orthogonal *basis $\{Z_1^*, Z_2^*, ..., Z_{p+q}^*\}$ of $\mathfrak{G}$ such that*

1) $\mathbb{Q} - \mathrm{span}\{Z_1, Z_2, ..., Z_p\} = \mathbb{Q} - \mathrm{span}\{Z_1^*, Z_2^*, ..., Z_p^*\}$

2) $\{Z_1^*, Z_2^*, ..., Z_p^*\}$ is a basis of W
3) For $1 \le i \le p+q$ each matrix Z_i^* in $\mathfrak{so}(n,\mathbb{R})$ has entries in $\mathbb{Q}$.

Proof. We show first that each matrix Z_i, $1 \le i \le p$, has entries in $\mathbb{Q}$. Let $\{C^1, C^2, ..., C^p\}$ be the matrices in $\mathfrak{so}(n,\mathbb{R})$ defined in Lemma 4. These matrices are linearly independent since $\langle Z_i, C^j\rangle = -\delta_{ij}$, and hence $\{C^1, C^2, ..., C^p\}$ is a basis of W. The entries of C^i are integers for $1 \le i \le p$, and hence $\langle C^i, C^j\rangle = -\operatorname{trace}(C^i C^j) \in \mathbb{Z}$ for $1 \le i, j \le p$. By Lemma 4 $\langle Z_i, C^j\rangle = -\delta_{ij} \in \mathbb{Q}$ for $1 \le i, j \le p$. It follows from Lemma 1 of the proposition in (2.7) that $Z_i \in \mathbb{Q} - \operatorname{span}\{C^1, C^2, ..., C^p\}$ for $1 \le i \le p$, and therefore each Z_i has entries in $\mathbb{Q}$ since the matrices $\{C^1, C^2, ..., C^p\}$ have entries in $\mathbb{Z}$.

Let $\mathcal{B}^o$ denote the basis $\{Z_1, Z_2, ..., Z_p\}$ of $W = W^o$, and define inductively $W^{i+1} = W^i + [W^i, W^i]$. If $\mathcal{B}^i$ is a basis of W^i consisting of matrices with entries in $\mathbb{Q}$, then by adjoining brackets of basis elements we may extend $\mathcal{B}^i$ to a basis $\mathcal{B}^{i+1}$ of W^{i+1} consisting of matrices with entries in $\mathbb{Q}$. Since $\mathfrak{G} = W^i$ for some i we may extend $\{Z_1, Z_2, ..., Z_p\}$ to a basis $\{Z_1, Z_2, ..., Z_{p+q}$ of $\mathfrak{G}$ such that each matrix Z_j, $1 \le j \le p+q$, has entries in $\mathbb{Q}$.

By the discussion above $\langle Z_i, Z_j\rangle = -\operatorname{trace}(Z_i Z_j) \in \mathbb{Q}$ for $1 \le i, j \le p+q$. Now apply 2) of Lemma 1 in the proof of Proposition (2.7). We obtain an orthogonal basis $\{Z_1^*, Z_2^*, ..., Z_{p+q}^*\}$ of $\mathfrak{G}$ such that $\mathbb{Q} - \operatorname{span}\{Z_1, Z_2, ..., Z_r\} = \mathbb{Q} - \operatorname{span}\{Z_1^*, Z_2^*, ..., Z_r^*\}$ for every $1 \le r \le p+q$. Choosing $r = p$ proves 1). From 1) it follows that $\mathbb{R} - \operatorname{span}\{Z_1^*, Z_2^*, ..., Z_p^*\} = \mathbb{R} - \operatorname{span}\{Z_1, Z_2, ..., Z_p\} = W$, which proves 2). Finally, Z_i^* has entries in $\mathbb{Q}$ for $1 \le i \le p+q$ since $Z_i^* \in \mathbb{Q} - \operatorname{span}\{Z_1, Z_2, ..., Z_{p+q}\}$ and each Z_j has entries in $\mathbb{Q}$ for $1 \le j \le p+q$. This proves 3) of the lemma. □

Lemma 6. *Let $\{Z_1, Z_2, ..., Z_p\}$ be the elements of $\mathfrak{so}(n,\mathbb{R})$ defined in Lemma 4, and let $\{Z_{p+1}^*, Z_{p+2}^*, ..., Z_{p+q}^*\}$ be the elements of $\mathfrak{so}(n\,\mathbb{R})$ defined in Lemma 5. Let $\mathcal{B}^* = \{e_1, e_2, ..., e_n, Z_1, Z_2, ..., Z_p, Z_{p+1}^*, Z_{p+2}^*, ..., Z_{p+q}^*\}$. Then $\mathcal{B}^*$ is a basis with rational structure constants for the involutive metric 2-step nilpotent Lie algebra $\mathcal{N}^* = \mathbb{R}^n \oplus \mathfrak{G}$.*

Proof. Let $[\,,\,]^*$ denote the bracket operation in $\mathcal{N}^*$. Observe that $\langle Z_k^*, Z_j^*\rangle = -\operatorname{trace}(Z_k^* Z_j^*) \in \mathbb{Q}$ for all $1 \le j, k \le p+q$ by 3) of Lemma 5. Furthermore, $\langle [e_i, e_j]^*, Z_k^*\rangle = \langle Z_k^*(e_i), e_j\rangle \in \mathbb{Q}$ for $1 \le i, j \le n$ and $1 \le k \le p+q$ since Z_k^* has entries in $\mathbb{Q}$ and $\langle e_i, e_j\rangle = \delta_{ij}$. Now apply 1) of Lemma 1 in the proof of Proposition (2.7) and 1) of Lemma 5. We conclude that $[e_i, e_j]^* \in \mathbb{Q} - \operatorname{span}\{Z_1^*, Z_2^*, ..., Z_{p+q}^*\} = \mathbb{Q} - \operatorname{span}\{Z_1, Z_2, ..., Z_p, Z_{p+1}^*, Z_{p+2}^*, ..., Z_{p+q}^*\} \subseteq \mathbb{Q} - \operatorname{span}(\mathcal{B}^*)$. Hence $[\mathcal{B}^*, \mathcal{B}^*] \subseteq \mathbb{Q} - \operatorname{span}(\mathcal{B}^*)$. Finally, $\mathcal{B}^*$ is a basis of $\mathcal{N}^* = \mathbb{R}^n \oplus \mathfrak{G}$ since $\{e_1, e_2, ..., e_n\}$ is a basis of $\mathbb{R}^n$ and $\mathbb{R} -$ span$\{Z_1, Z_2, ..., Z_p, Z^*_{p+1}, Z^*_{p+2}, ..., Z^*_{p+q}\} = \mathbb{R} -$ span$\{Z^*_1, Z^*_2, ..., Z^*_{p+q}\} = \mathfrak{G}$ by 2) of Lemma 5. □

We are now ready to complete the proof of 2) of the theorem in the case that $\Lambda = \log \Gamma$ is a vector lattice in $\mathcal{N}$. By 2) of Lemma 4, $\Lambda = \log \Gamma = \mathbb{Z} -$ span$\{e_1, e_2, ..., e_n, Z_1, Z_2, ..., Z_p\}$. By Lemma 6 $\mathcal{N}^*_{\mathbb{Q}} = \mathbb{Q} -$ span$(\mathcal{B}^*)$ is a Lie algebra over $\mathbb{Q}$ such that $\mathcal{N}^*_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to $\mathcal{N}^*$. Let N^* denote the simply connected 2-step nilpotent Lie group with Lie algebra $\mathcal{N}^*$, and let $\exp^* : \mathcal{N}^* \to N^*$ denote the exponential map. If $L^* = \mathbb{Z} -$ span$(\mathcal{B}^*)$, then by a) of (1.2b) $\exp^*(L^*)$ generates a lattice Γ^* in N^*.

We assert that $\rho(\Gamma^*) = \Gamma$, where $\rho : N^* \to N$ is the surjective homomorphism and Riemannian submersion whose existence was established in assertion 1) of the theorem. This will complete the proof of assertion 2) of the theorem in the case that $\Lambda = \log \Gamma$. We recall from the proof of 1) that $\pi = d\rho : \mathcal{N}^* \to \mathcal{N}$ is the orthogonal projection, where we regard $\mathcal{N} = \mathbb{R}^n \oplus W$ as a vector subspace of $\mathcal{N}^* = \mathbb{R}^n \oplus \mathfrak{G}$. In particular, π fixes each of the elements in the set $\mathcal{B} = \{e_1, e_2, ..., e_n, Z_1, Z_2, ..., Z_p\}$, whose $\mathbb{Z} -$ span is Λ by Lemma 4, and π annihilates the remaining elements $\{Z^*_{p+1}, Z^*_{p+2}, ..., Z^*_{p+q}\}$ of $\mathcal{B}^*$, which are orthogonal to $\mathcal{B}$ by Lemma 5. It follows that $\pi(L^*) = \mathbb{Z} -$ span$(\mathcal{B}) = \Lambda$.

If $\exp : \mathcal{N} \to N$ and $\exp^* : \mathcal{N}^* \to N^*$ are the Lie group exponential maps, then $(\rho \circ \exp^*)(L^*) = (\exp \circ \pi)(L^*) = \exp(\Lambda) = \exp(\log \Gamma) = \Gamma$. Hence $\exp^*(L^*) \subseteq \rho^{-1}(\Gamma)$ and it follows that $\Gamma^* \subseteq \rho^{-1}\Gamma$ since $\exp^*(L^*)$ generates Γ^*. We have proved that $\rho(\Gamma^*) \subseteq \Gamma$. To prove that equality holds we note that for any element $\gamma \in \Gamma$, $\log \gamma \in \log(\Gamma) = \Lambda \subseteq L^*$. It follows that $\gamma^* = \exp^*(\log \gamma) \in \exp^*(L^*) \subseteq \Gamma^*$. Finally $\rho(\gamma^*) = (\rho \circ \exp^*)(\log \gamma)$ $= (\exp \circ \pi)(\log \gamma) = \exp(\log \gamma) = \gamma$ since π is the identity on $\log \Gamma \subseteq \mathcal{N}$. This proves that $\rho(\Gamma^*) = \Gamma$.

Case 2 $\quad$ Γ is an arbitrary lattice of N

By (1.2c) there exists a lattice Γ_o of N such that Γ is a finite index subgroup of Γ_o and $\log(\Gamma_o)$ is a vector lattice in $\mathcal{N}$. By case 1 there exists an involutive metric 2-step nilpotent Lie group $\{N^*, \langle\, ,\rangle^*\}$, a lattice Γ^*_o in N^* and a Riemannian submersion $\rho : N^* \to N$ such that $\rho(\Gamma^*_o) = \Gamma_o$ and ρ satisfies 1) of the theorem. Let $\Gamma^* = \rho^{-1}(\Gamma) \cap \Gamma^*_o$. Since $\rho(\Gamma^*_o) = \Gamma_o \supseteq \Gamma$ it follows that $\rho(\Gamma^*) = \Gamma$. Moreover, Γ^* has finite index in Γ^*_o since Γ has finite index in Γ_o. It follows that Γ^* is a lattice in N^*. □

Proof of 3) of the theorem.

Assertion 3) of the theorem and the remaining part of assertion 2) are consequences of the next result.

Lemma 7. *Let $\rho : H^* \to H$ be a surjective Lie homomorphism of noncompact connected Lie groups. Assume that* $\mathrm{Ker}(\rho)$ *is a connected Lie subgroup of H^*. Let Γ^* be a cocompact lattice in H^*. Then the following are equivalent:*

1) $\Gamma^* \cap \mathrm{Ker}(\rho)$ *is a cocompact lattice in* $\mathrm{Ker}(\rho)$.

2) $\rho(\Gamma^*)$ *is a cocompact lattice in H.*

If H^* is a simply connected nilpotent Lie group, then 1) and 2) are equivalent to

3) $\mathrm{Ker}(d\rho)$ *is a rational subalgebra of $\mathfrak{H}^*$ relative to the rational structure* $\mathfrak{H}^*_{\mathbb{Q}} = \mathbb{Q} - \mathrm{span}(\log \Gamma^*)$.

Proof. We first prove the equivalence of 1) and 3) in the case that H^* is a simply connected nilpotent Lie group. The Lie algebra of $\mathrm{Ker}(\rho)$ is $\mathrm{Ker}(d\rho)$ and hence $\exp\big(\mathrm{Ker}(d\rho)\big) \subseteq \mathrm{Ker}(\rho)$. Equality holds since $\exp\big(\mathrm{Ker}(d\rho)\big)$ is a simply connected Lie group by (1.1a) and (1.1b) and $\mathrm{Ker}(\rho)$ is connected by hypothesis. The equivalence of 1) and 3) now follows from (1.3b).

The proof of 1) $\Rightarrow$ 2) is contained in Lemma 5.1.4 of [CG]. We prove 2) $\Rightarrow$ 1). Since Γ^* is a cocompact lattice in H^* there exists a compact set D^* of H^* such that $\Gamma^* \cdot D^* = H^*$. Let $D = \rho(D^*)$ and $\Gamma = \rho(\Gamma^*)$. The set $\Gamma \cap D$ is finite since Γ is discrete and D is compact. Choose elements $\{\xi_1, \xi_2, ..., \xi_m\}$ in Γ^* such that $\Gamma \cap D = \{\rho(\xi_1), \rho(\xi_2), ..., \rho(\xi_m)\}$. Let C^* be the union of the sets $\{\xi_i^{-1}(D^*) : 1 \leq i \leq m\}$. Then C^* is a compact subset of H^*, and it suffices to show that $\{\Gamma^* \cap \mathrm{Ker}(\rho)\} \cdot \{C^* \cap \mathrm{Ker}(\rho)\} = \mathrm{Ker}(\rho)$.

It is enough to prove that $\mathrm{Ker}(\rho) \subseteq \{\Gamma^* \cap \mathrm{Ker}(\rho)\} \cdot \{C^* \cap \mathrm{Ker}(\rho)\}$ since the reverse inclusion is obvious. Given $\alpha \in \mathrm{Ker}(\rho)$ there exist elements $\gamma^* \in \Gamma^*$ and $d^* \in D^*$ such that $\alpha = \gamma^* d^*$. Then $e = \rho(\alpha) = \gamma d$, where $\gamma = \rho(\gamma^*) \in \Gamma$ and $d = \rho(d^*) \in D$. Hence $\gamma^{-1} = d \in \Gamma \cap D$, and there exists an element ξ_i, $1 \leq i \leq m$, such that $\gamma^{-1} = \rho(\xi_i)$. It follows that $\beta = \gamma^* \xi_i \in \Gamma^* \cap \mathrm{Ker}(\rho)$. Therefore $\alpha = \gamma^* d^* = \beta \xi_i^{-1} d^* = \beta c^*$, where $c^* = \xi_i^{-1} d^* \in C^*$. Since $c^* = \beta^{-1} \alpha \in \mathrm{Ker}(\rho) \cap C^*$, we conclude that $\mathrm{Ker}(\rho) \subseteq \{\Gamma^* \cap \mathrm{Ker}(\rho)\} \cdot \{C^* \cap \mathrm{Ker}(\rho)\}$. This completes the proof of 2) $\Rightarrow$ 1). $\square$

Proof of 4) of the theorem.

Let $\pi^* : N^* \to \Gamma^* \backslash N^*$ and $\pi : N \to \Gamma \backslash N$ be the projection maps and define $\rho' : \Gamma^* \backslash N^* \to \Gamma \backslash N$ by $\rho'(\pi^* n^*) = \pi(\rho n^*)$ or equivalently

a) $\rho' \circ \pi^* = \pi \circ \rho$

It is routine to verify that ρ' is well defined. From a) it follows that ρ' has maximal rank at every point since π and π^* are local isometries and ρ has maximal rank at every point. Hence

b) The fibers of ρ' are compact submanifolds of $\Gamma^* \backslash N^*$.

From a) we also obtain by straightforward arguments

c) $(\rho')^{-1}(\pi n) = \pi^*\big(\rho^{-1}(n)\big)$ for every $n \in N$.

From 1) of the proposition the fibers of $\rho : N^* \to N$ are flat, totally geodesic submanifolds of N^*. Since π^* is a local isometry we obtain from b) and c)

d) $(\rho')^{-1}(\pi n)$ is a compact, flat, totally geodesic submanifold of $\Gamma^* \backslash N^*$ for every $n \in N$.

We conclude the proof of 4) by showing that the fibers of ρ' are all isometric to the flat torus $\Gamma' \backslash Z'$, where $Z' = \mathrm{Ker}(\rho)$ and $\Gamma' = \Gamma^* \cap Z'$ is a lattice in Z' by Lemma 7. It follows from 1) of the proposition that $Z' = \mathrm{Ker}(\rho)$ is a simply connected, flat, totally geodesic submanifold of N^* that is contained in Z^*, the center of N^*. Hence Z' is isometric to a Euclidean space, $\Gamma' = \Gamma^* \cap Z'$ is a lattice of translations in Z' and $\Gamma' \backslash Z'$ is a flat torus.

Let $\pi' : Z' \to \Gamma' \backslash Z'$ denote the projection. For each $n \in N$ fix an element $n^* \in \rho^{-1}(n)$ and observe that $\rho^{-1}(n) = n^* Z'$. Define a map $T_n : (\rho')^{-1}(\pi n) = \pi^*\big(\rho^{-1}(n)\big) \to \Gamma' \backslash Z'$ by $T_n\big(\pi^*(n^* z')\big) = \pi'(z')$ for every $z' \in Z'$. It is straightforward to verify that T_n is well defined and a bijection. Since $T_n \circ \pi^* \circ L_{n^*} = \pi'$ it follows that T_n is a local isometry since π' and π^* are local isometries and L_{n^*} is an isometry. Therefore $T_n : (\rho')^{-1}(\pi n) \to \Gamma' \backslash Z'$ is an isometry for every $n \in N$, and the proof of 4) is complete.

4. Existence and nonexistence of lattices.

4.1. Nonexistence of lattices.

Typically lattices will not exist in a simply connected 2-step nilpotent Lie group N. A "generic" 2-step nilpotent Lie algebra with dimension n and center of dimension p will not admit a lattice if $p \geq 3$ and n is sufficiently large relative to p. See [E3] for details. For the convenience of the reader we outline a proof here.

It is shown in [E3] that the set of isomorphism classes of 2-step nilpotent Lie algebras with dimension n and center of dimension $p < n$ is an orbit space $X(p)/G$, where $G = GL(n, \mathbb{R})$ and $X(p)$ is a smooth manifold of dimension $pq + pD$, where $q = n - p$ and $D = (1/2)q(q-1)$. The set of elements in $X(p)$ with rational structure constants is a countable union of G-orbits. If

q is sufficiently large relative to p, then the dimension of $X(p)$ will be larger than the dimension of any of the G-orbits, and hence the union of countably many G-orbits will be a null set in $X(p)$.

4.2. Lattices constructed from compact subgroups of $GL(V)$.

Proposition. *Let V be a finite dimensional real vector space and let G be a compact, connected subgroup of $GL(V)$. Let $\langle\,,\,\rangle$ be a G-invariant inner product on V. Let $\mathcal{C}'$ be a basis of the Lie algebra such that*

a) the structure constants of $\mathcal{C}'$ lie in $\mathbb{Q}$

b) any finite dimensional real $\mathfrak{G}$-module U admits a basis $\mathcal{B}_U$ so that the elements of $\mathcal{C}'$ leave invariant $\mathbb{Q}$-span$(\mathcal{B}_U)$.

Let W be a rational subspace of $\mathfrak{G}$ relative to the rational structure $\mathfrak{G}_{\mathbb{Q}} = \mathbb{Q} - \mathrm{span}(\mathcal{C}')$ on $\mathfrak{G}$. Let $\mathcal{N} = V \oplus W$ be the corresponding standard metric 2-step nilpotent Lie algebra defined in (2.5). If N is the simply connected Lie group with Lie algebra $\mathcal{N}$, then N admits a lattice Γ.

Remarks. 1) A basis of $\mathcal{C}'$ of $\mathfrak{G}$ with the properties above always exists. In fact one may replace $\mathbb{Q}$ by $\mathbb{Z}$ in the statements of a) and b). See Appendix 1.

2) It is well known that G-invariant inner products on V exist since G is compact. The isomorphism type of $\mathcal{N} = V \oplus W$ is independent of the G-invariant inner product on V by the proposition in (2.3a).

3) If G is a compact, connected subgroup of $GL(V)$ and $\mathfrak{G}$ is the Lie algebra of G, then $G = \exp(\mathfrak{G})$, where $\exp : \mathrm{End}\, V \to GL(V)$ is the exponential map.

4) If G is a compact, connected Lie group, then G has a finite covering by $G' = T \times G^*$, where T is compact, connected and abelian and G^* is compact, connected and semisimple. For completeness we include a proof of this well known result in the first lemma of Appendix 1. Note that the group T is a finite index subgroup of the center of G since G^* has finite center.

5) The Killing form of a subalgebra $\mathfrak{H}$ of $\mathrm{End}(V)$ is negative definite $\Leftrightarrow$ $\mathfrak{H}$ is the Lie algebra of a compact semisimple subgroup H of $GL(V)$. Any Lie group H with Lie algebra $\mathfrak{H}$ is compact. See Proposition 6.6 and Corollary 6.7 of [He, pp. 132-133]. The connected Lie subgroup $H \subseteq GL(V)$ with Lie algebra $\mathfrak{H}$ is closed in the topology of $\mathrm{End}(V)$ by the semisimplicity of $\mathfrak{H}$. See Corollary 2 of [Mo, p.615]. Hence $H = \exp(\mathfrak{H})$ by 3).

In particular,

a) If G is any compact semisimple Lie group, and $\rho : G \to GL(V)$ is a nontrivial representation of G, then the Lie algebra $\mathfrak{G}$ of $\rho(G)$ has negative definite Killing form.

b) If $\langle\, ,\,\rangle$ is any inner product on V and $\mathfrak{H}$ is any nonabelian subalgebra of $so(V)$, then $\mathfrak{G} = [\mathfrak{H}, \mathfrak{H}]$ is semisimple and has negative definite Killing form. See Appendix 2 for further details.

6) The discussion of example 3 in (1.3a) explains how to compute a "standard" rational structure $\mathfrak{G}_{\mathbb{Q}}$ if $\mathfrak{G}$ is the compact real form of a complex simple Lie algebra $\mathfrak{G}^{\mathbb{C}}$ in the classification $\mathfrak{A}_n$, $\mathfrak{B}_n$, $\mathfrak{C}_n$ and $\mathfrak{D}_n$. From this case one may readily compute $\mathfrak{G}_{\mathbb{Q}}$ in the case that $\mathfrak{G}^{\mathbb{C}}$ is a direct sum of these classical complex simple Lie algebras.

Proof of Proposition 4.2. Let $\mathcal{C}'$ be a basis of $\mathfrak{G}$ with the properties stated in the proposition, and let $\mathfrak{G}_{\mathbb{Q}} = \mathbb{Q}-\text{span}(\mathcal{C}')$. Let W be a subspace of $\mathfrak{G}$ that is rational relative to the rational structure $\mathfrak{G}_{\mathbb{Q}}$ for $\mathfrak{G}$. Let $\mathcal{B} = \{\xi_1, \xi_2, ..., \xi_p\} \subseteq \mathfrak{G}_{\mathbb{Q}}$ be a basis of $\mathfrak{G}$ that contains a basis $\mathcal{B}_W = \{\xi_1, \xi_2, ..., \xi_q\}$ of W.

The elements of $\mathfrak{G}$, and in particular of W, are skew symmetric relative to the G-invariant inner product $\langle\, ,\,\rangle$ on V. Hence we can write V as a direct sum $V_1 \oplus V_2 ... \oplus V_N$, where each V_i is W-invariant and W-irreducible. The subspace W may be regarded as a subspace of $\text{End}(V_i)$ for each $1 \leq i \leq N$. By the hypothesis of the proposition, we may choose bases $\mathcal{B}_i$ in V_i such that any element of $\mathcal{B}$ has a matrix with rational entries relative to $\mathcal{B}_i$ for $1 \leq i \leq N$. Since $\mathcal{B}_W \subseteq \mathcal{B} \subseteq \mathfrak{G}_{\mathbb{Q}}$ it follows that W is a rational subspace of $\text{End}(V_i)$ relative to the bases $\mathcal{B}_W$ and $\mathcal{B}_i$ for each $1 \leq i \leq N$. By Lemma 3 in the proof of Proposition (2.7) we can find W-invariant inner products $\langle\, ,\,\rangle_i$ on V_i such that $\langle X_i, Y_i\rangle_i \in \mathbb{Q}$ for any two elements X_i, Y_i of $\mathcal{B}_i$. Let $\mathcal{B}_V = \{v_1, v_2, ..., v_n\}$ be the union of the bases $\{\mathcal{B}_i\}$, and let $\langle\, ,\,\rangle$ be the W-invariant inner product on $V = V_1 \oplus V_2 ... V_N$ such that $\langle\, ,\,\rangle = \langle\, ,\,\rangle_i$ on V_i and the subspaces $\{V_i\}$ are orthogonal.

The sets $\mathcal{B}_V$ and $\mathcal{B}_W$ are bases of V and W that satisfy the hypotheses of Lemma 2 in the proof of Proposition (2.7). By that result the basis $\mathcal{B}_V \cup \mathcal{B}_W = \{v_1, v_2, ..., v_n, \xi_1, \xi_2, ..., \xi_q\}$ for $\mathcal{N} = V \oplus W$ has rational structure constants. The proof of Proposition 4.2 is now complete by the Mal'cev criterion in (1.2b). $\square$

4.3. Lattices constructed from Lie triple systems.

(4.3a) Lie triple systems with compact center. Let $\mathfrak{G}$ be a finite dimensional Lie algebra over $\mathbb{R}$ whose Killing form $B_{\mathfrak{G}}$ is negative definite, and let G be a compact, connected Lie group with Lie algebra $\mathfrak{G}$. Let $Z(W) = \{X \in W : [X, Y] = 0 \text{ for all } Y \in W\}$. We call $Z(W)$ the *center* of W, and we say that W has *compact center* if $\exp(Z(W))$ is a compact subset of G, where $\exp : \mathfrak{G} \to G$ is the Lie group exponential map.

Remarks. 1) Note that $\exp(Z(W))$ is a connected abelian subgroup of G for any Lie triple system W since $Z(W)$ is an abelian subspace of $\mathfrak{G}$.

2) If W is a Lie triple system in $\mathfrak{G}$, then the compactness of $Z(W)$ does not depend on the choice of the Lie group G with Lie algebra $\mathfrak{G}$. Let G_1 and G_2 be two Lie groups with Lie algebra $\mathfrak{G}$, and let $\tilde{G}$ be a simply connected Lie group with Lie algebra $\mathfrak{G}$. Then $\tilde{G}$ is compact and is a finite cover of both G_1 and G_2. Hence $\exp_1(Z(W))$ is compact in $G_1 \Leftrightarrow \exp_2(Z(W))$ is compact in G_2.

3) We shall use the results of this section only in the case that $\mathfrak{G} = \mathfrak{so}(V)$, the skew symmetric linear transformations of a finite dimensional, real inner product space V.

(4.3b) Decomposition of Lie triple systems. We shall need the following result.

Proposition. *Let $\mathfrak{G}$ be a finite dimensional Lie algebra over $\mathbb{R}$ whose Killing form $B_{\mathfrak{G}}$ is negative definite, and let $\langle\, ,\, \rangle$ be any $\mathrm{ad}_{\mathfrak{G}}$-invariant inner product on $\mathfrak{G}$. Let G be a compact, connected Lie group with Lie algebra $\mathfrak{G}$. Let W be a Lie triple system in $\mathfrak{G}$. Then*

1) *The subspaces $[W, W]$ and $\mathfrak{H} = W + [W, W]$ are subalgebras of $\mathfrak{G}$.*

2) *Let $Z(W) = \{X \in W : [X, Y] = 0$ for all $Y \in W\}$, and let $\mathfrak{H} = W + [W, W]$. Then $Z(W)$ is the center of $\mathfrak{H}$ and $\mathfrak{H} = Z(W) \oplus \mathfrak{H}_0$, where $\mathfrak{H}_0 = [\mathfrak{H}, \mathfrak{H}]$ is a semisimple ideal of $\mathfrak{H}$. Moreover, the direct sum is orthogonal relative to $\langle\, ,\, \rangle$.*

3) *Let W_1 denote the orthogonal complement of $Z(W)$ in W relative to $\langle\, ,\, \rangle$. Then W_1 is a Lie triple system in $\mathfrak{G}$ and $\mathfrak{H}_0 = W_1 + [W_1, W_1]$. If W has compact center, then $H = \exp(\mathfrak{H})$ is a compact, connected subgroup of G, where $\exp : \mathfrak{G} \to G$ is the Lie group exponential map.*

Proof. The proofs of 1) and 2) may be found in Appendix 2. We prove only 3). Since W is the orthogonal direct sum $Z(W) \oplus W_1$ it follows that $[W, W] = [W_1, W_1]$ and $W_1,\, [W_1, W_1]] \subseteq [W,\, [W, W]] \subseteq W = Z(W) \oplus W_1$. To show that W_1 is a Lie triple system it suffices to show that $\langle A,\, [X, [Y, Z]]\rangle = 0$ for all $A \in Z(W)$ and all $X, Y, Z \in W_1$. However, $\langle A,\, [X, [Y, Z]]\rangle = -\langle \mathrm{ad}\, X(A), [Y, Z]\rangle = 0$ by the definition of $Z(W)$ and the $\mathrm{ad}_{\mathfrak{G}}$-invariance of $\langle\, ,\, \rangle$.

If $\mathfrak{H}_0^* = W_1 + [W_1, W_1]$, then $\mathfrak{H} = W + [W, W] = Z(W) + W_1 + [W_1, W_1] = Z(W) + \mathfrak{H}_0^*$. Moreover, $Z(W)$ is orthogonal to $[W_1, W_1]$ and hence to $\mathfrak{H}_0^*$ by the $\mathrm{ad}_{\mathfrak{G}}$-invariance of $\langle\, ,\, \rangle$. By the Jacobi identity $Z(W)$ commutes with $[W_1, W_1]$ since $Z(W)$ commutes with W_1. Hence $Z(W)$ commutes with $\mathfrak{H}_0^*$, which shows that $\mathfrak{H}_0 = [\mathfrak{H}, \mathfrak{H}] = [\mathfrak{H}_0^*, \mathfrak{H}_0^*] \subseteq \mathfrak{H}_0^*$ since W_1 is a Lie triple system and $\mathfrak{H}_0^*$ is a Lie algebra by 1). By 2) and the discussion above we have $Z(W) \oplus \mathfrak{H}_0^* = \mathfrak{H} = Z(W) \oplus \mathfrak{H}_0 \subseteq Z(W) \oplus \mathfrak{H}_0^*$, which shows that $\mathfrak{H}_0 = \mathfrak{H}_0^*$.

Suppose now that W has compact center; that is, $T_0 = \exp(Z(W))$ is a compact, connected abelian subgroup of G. Since $\mathfrak{H}_0$ is a semisimple subalgebra of $\mathfrak{G}$ it follows that $H_0 = \exp(\mathfrak{H}_0)$ is a compact, connected subgroup of G; see remark 5) following the statement of Proposition 4.2. Hence $H = T_0 H_0 = \exp(Z(W) \oplus \mathfrak{H}_0) = \exp(\mathfrak{H})$ is a compact, connected subgroup of G with Lie algebra $\mathfrak{H}$. $\square$

(4.3c) The Main Result.

Proposition. *Let $\{V, \langle\, ,\, \rangle\}$ be a finite dimensional, real inner product space, and let $\mathsf{so}(V)$ denote the Lie algebra of skew symmetric linear transformations of $\{V, \langle\, ,\, \rangle\}$. Let W be a Lie triple system in $\mathsf{so}(V)$ that has compact center. Let $\mathcal{N} = V \oplus W$ be the standard, 2-step nilpotent, metric Lie algebra defined in (2.5), and let N be the simply connected, 2-step nilpotent Lie group with Lie algebra $\mathcal{N}$. Then N admits a lattice Γ.*

Corollary. *Let $\{N, \langle\, ,\, \rangle\}$ be a simply connected 2-step nilpotent Lie group with a left invariant metric that is a space of Heisenberg type. Then N admits a lattice Γ.*

For another proof of the corollary, see [CD].

Proof of the Corollary. Let $\{N, \langle\, ,\, \rangle\}$ be a simply connected, 2-step nilpotent Lie group with a left invariant metric of Heisenberg type (cf. example 3 of (2.4)). Let $\{\mathcal{N}, [\, ,\,], \langle\, ,\, \rangle\}$ be the metric Lie algebra of $\{N, \langle\, ,\, \rangle\}$, and write $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$, where $\mathcal{Z}$ denotes the center of $\mathcal{N}$ and $\mathcal{V}$ is the orthogonal complement of $\mathcal{Z}$ in $\mathcal{N}$. Let $C\ell(\mathcal{Z})$ denote the negative definite real Clifford algebra determined by the real inner product space $\{\mathcal{Z}, \langle\, ,\, \rangle\}$. By the discussion of example 3 in (2.4) the linear map $j : \mathcal{Z} \to \mathsf{so}(\mathcal{V})$ extends to an algebra homomorphism $j : C\ell(\mathcal{Z}) \to \mathsf{so}(\mathcal{V}) \subseteq \mathrm{End}(\mathcal{V})$, where multiplication in $\mathrm{End}(\mathcal{V})$ is composition. By the discussion of example 2 in (2.5) the subspace $W = j(\mathcal{Z})$ of $\mathsf{so}(\mathcal{V})$ is a Lie triple system with trivial center in $\mathsf{so}(\mathcal{V})$.

Let $\mathcal{N}^* = \mathcal{V} \oplus W$ be the standard, metric, 2-step nilpotent Lie algebra constructed as in (2.5). By the Mal'cev criterion the Corollary will follow from the Proposition once we show that $\{\mathcal{N}^*, [\,,]^*\}$ and $\{\mathcal{N}, [\,,]\}$ are isomorphic as Lie algebras. By the definition in (2.5) of a standard, metric, 2-step nilpotent Lie algebra the inner product $\langle,\rangle^*$ on $W = j(\mathcal{Z})$ for $\mathcal{N}^*$ satisfies $\langle j(Z), j(Z^*)\rangle^* = -\operatorname{trace} j(Z)j(Z^*)$ for all Z, Z^* in $\mathcal{Z}$. Let $\langle\,,\rangle'$ be the inner product on $\mathcal{Z}$ such that $\langle Z, Z^*\rangle' = \langle j(Z), j(Z^*)\rangle^*$. Let $\mathcal{N}' = \mathcal{V} \oplus \mathcal{Z} = \mathcal{N}$ as a vector space, and let $\{\mathcal{N}', [\,,]'\}$ be the 2-step nilpotent Lie algebra constructed from the linear map $j : \{\mathcal{Z}, \langle\,,\rangle'\} \to \mathfrak{so}(\mathcal{V})$ as in (2.2b). Let $\varphi : \mathcal{N}' \to \mathcal{N}^*$ be the linear map defined by $\varphi(X+Z) = X + j(Z)$ for all $X \in \mathcal{V}$, $Z \in \mathcal{Z}$. It follows routinely from the definitions that $\varphi : \{\mathcal{N}', [\,,]'\} \to \{\mathcal{N}^*, [\,,]^*\}$ is a Lie algebra isomorphism. Next, note that $\langle\,,\rangle' = (\dim \mathcal{V})\langle\,,\rangle$ on $\mathcal{Z}$ by the discussion of example 3 in (2.4). Hence $\{\mathcal{N}', [\,,]'\}$ and $\{\mathcal{N}, [\,,]\}$ are isomorphic as Lie algebras by the proposition in (2.3a). We conclude that $\{\mathcal{N}^*, [\,,]^*\}$ and $\{\mathcal{N}, [\,,]\}$ are isomorphic as Lie algebras. $\square$

Proof of the Proposition. By the proposition in (4.3b) $\mathfrak{H} = W + [W, W]$ is a subalgebra of $\mathfrak{G} = \mathfrak{so}(V)$ and $H = \exp(\mathfrak{H})$ is a compact subgroup of $G = SO(V)$, the special orthogonal group of V. By the corollary to the proposition in Appendix 1 the Lie algebra $\mathfrak{H}$ has a basis $\mathcal{C}'$ that satisfies the hypotheses of Proposition 4.2. What remains is to show that if $\mathcal{C}'$ is chosen carefully, then the Lie triple system W is a rational subspace of $\mathfrak{H}$ with the rational structure $\mathfrak{H}_{\mathbb{Q}} = \mathbb{Q}\text{-span}(\mathcal{C}')$. One then applies Proposition 4.2 to complete the proof of the proposition in (4.3c).

We consider first the case that W has trivial center and then the general case that W has compact center.

Case 1 W has trivial center

To conform better to the standard notation of the literature we set $\mathfrak{P} = W$ and $\mathfrak{K} = [W, W] = [\mathfrak{P}, \mathfrak{P}]$ for the remainder of the proof.

We proceed in several steps.

Lemma 1. $[\mathfrak{K}, \mathfrak{P}] \subseteq \mathfrak{P}$, $[\mathfrak{P}, \mathfrak{P}] = \mathfrak{K}$ *and* $[\mathfrak{K}, \mathfrak{K}] \subseteq \mathfrak{K}$. *The Lie algebra* $\mathfrak{H} = \mathfrak{K} + \mathfrak{P}$ *is semisimple.*

Proof of Lemma 1. By the proposition in (4.3b) $\mathfrak{H} = \mathfrak{K} + \mathfrak{P} = W + [W, W]$ is a semisimple subalgebra of $\mathfrak{so}(V)$ The first two bracket relations follow from the definition of $\mathfrak{K}$ and the hypothesis that $\mathfrak{P}$ is a Lie triple system in $\mathfrak{so}(V)$. It remains only to prove that $[\mathfrak{K}, \mathfrak{K}] \subseteq \mathfrak{K}$.

Let P_1, P_2, P_1' and P_2' be arbitrary elements of $\mathfrak{P}$. To show that $[\mathfrak{K}, \mathfrak{K}] \subseteq \mathfrak{K}$

it suffices to show that $[[P_1, P_2], [P_1', P_2']] \in \mathfrak{K}$. If $X = [P_1', P_2']$, then $[[P_1, P_2], X] = \mathrm{ad}([P_1, P_2])(X) = [\mathrm{ad}\, P_1, \mathrm{ad}\, P_2](X) = \mathrm{ad}\, P_1(\mathrm{ad}\, P_2(X)) - \mathrm{ad}\, P_2(\mathrm{ad}\, P_1(X))$ by the Jacobi identity. Now $\mathrm{ad}\, P_2(X) \in [\mathfrak{P}, \mathfrak{K}] \subseteq \mathfrak{P}$ and hence $\mathrm{ad}\, P_1(\mathrm{ad}\, P_2(X)) \in [\mathfrak{P}, \mathfrak{P}] = \mathfrak{K}$. Similarly $\mathrm{ad}\, P_2(\mathrm{ad}\, P_1(X)) \in \mathfrak{K}$, which proves that $[\mathfrak{K}, \mathfrak{K}] \subseteq \mathfrak{K}$. □

For the notation used in the next result see the discussion of (1.3a).

Lemma 2. *Let $\mathfrak{G}$ be a Lie algebra whose Killing form is negative definite. Suppose that $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}$, direct sum, where $\mathfrak{K}$ and $\mathfrak{P}$ are subspaces of $\mathfrak{G}$ such that*

$$[\mathfrak{K}, \mathfrak{K}] \subseteq \mathfrak{K},\ [\mathfrak{K}, \mathfrak{P}] \subseteq \mathfrak{P} \ \textit{ and } \ [\mathfrak{P}, \mathfrak{P}] \subseteq \mathfrak{K}. \tag{$*$}$$

Then there exists a Chevalley basis $\mathcal{C} = \{H_\alpha^*, y_\beta : \alpha \in \Delta, \beta \in \Phi\}$ of the complexification $\mathfrak{G}^{\mathbb{C}}$ such that

a) The real Chevalley basis $\mathcal{C}_{\mathbb{R}} = \{iH_\alpha^*, u_\beta, v_\beta : \alpha \in \Delta, \beta \in \Phi\}$ is a basis for $\mathfrak{G}$, where $u_\beta = y_\beta - y_{-\beta}$ and $v_\beta = iy_\beta + iy_{-\beta}$.

b) There exist bases for $\mathfrak{K}$ and $\mathfrak{P}$ that are contained in $\mathbb{Z} - \mathrm{span}(\mathcal{C}_{\mathbb{R}})$. In particular $\mathfrak{K}$ and $\mathfrak{P}$ are rational subspaces of $\mathfrak{G}$ relative to the rational structure $\mathfrak{G}_{\mathbb{Q}} = \mathbb{Q} - \mathrm{span}(\mathcal{C}_{\mathbb{R}})$.

Remark. If $\theta : \mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P} \to \mathfrak{G}$ is the linear map given by $\theta(K+P) = K - P$ for all $K \in \mathfrak{K}$ and $P \in \mathfrak{P}$, then the bracket operations of $(*)$ imply that θ is a Lie algebra automorphism of $\mathfrak{G}$. Conversely, if θ is a Lie algebra automorphism of $\mathfrak{G}$ with $+1$ eigenspace $\mathfrak{K}$ and -1 eigenspace $\mathfrak{P}$, then the subspaces $\mathfrak{K}$ and $\mathfrak{P}$ satisfy the bracket relations of $(*)$.

Proof of Lemma 2. If $\mathfrak{G}^* = \mathfrak{K} \oplus \mathfrak{P}^*$, where $\mathfrak{P}^* = i\mathfrak{P}$, then $\mathfrak{G}^*$ is the noncompact, semisimple Lie algebra dual to $\mathfrak{G}$ (cf. [He, p.235]). The bracket relations $(*)$ for $\mathfrak{K}$ and $\mathfrak{P}$ imply the analogous bracket relations for $\mathfrak{K}$ and $\mathfrak{P}^*$. Hence the linear map $\theta^* : \mathfrak{G}^* = \mathfrak{K} \oplus \mathfrak{P}^* \to \mathfrak{G}^*$ given by $\theta(K + P^*) = K - P^*$ for all $K \in \mathfrak{K}$ and $P^* \in \mathfrak{P}^*$ is a Lie algebra automorphism of $\mathfrak{G}^*$. By Proposition 3.7 of [B] and its proof we may use the Cartan involution θ^* to construct a Chevalley basis $\mathcal{C} = \{H_\alpha^*, y_\beta : \alpha \in \Delta, \beta \in \Phi\}$ of $\mathfrak{G}^{*\mathbb{C}} = \mathfrak{G}^{\mathbb{C}}$ such that

a) The real Chevalley basis $\mathcal{C}_{\mathbb{R}} = \{iH_\alpha^*, u_\beta, v_\beta : \alpha \in \Delta, \beta \in \Phi\}$ is a basis for the compact, semisimple Lie algebra $\mathfrak{G}$.

b) $\mathfrak{K}$ has a basis in $\mathbb{Z} - \mathrm{span}(\mathcal{C}_{\mathbb{R}})$.

c) $\mathfrak{P}^* = i\mathfrak{P}$ has a basis in $i\{\mathbb{Z} - \mathrm{span}(\mathcal{C}_{\mathbb{R}})\}$.

The proof of Lemma 2 now follows immediately. □

Now, let $\mathfrak{H}_1 = \mathfrak{K} \cap \mathfrak{P}$. By Lemma 1 $\mathfrak{H}_1$ is an ideal of $\mathfrak{H} = \mathfrak{K} + \mathfrak{P}$ since $\mathfrak{H}_1$ is invariant under $\mathrm{ad}(\mathfrak{K})$ and $\mathrm{ad}(\mathfrak{P})$. We consider separately the cases a) $\mathfrak{H}_1 = \{0\}$ and b) $\mathfrak{H}_1 \neq \{0\}$. Suppose first that $\mathfrak{H}_1 = \{0\}$. The Killing form of $\mathfrak{H}$ is negative definite since $\mathfrak{H}$ is semisimple and by hypothesis $\mathfrak{H}$ is a direct sum $\mathfrak{K} \oplus \mathfrak{P}$. By lemmas 1 and 2, $W = \mathfrak{P}$ is a rational subspace of $\mathfrak{H}$ with the rational structure $\mathfrak{H}_{\mathbb{Q}} = \mathbb{Q}\text{-span}(\mathcal{C}_{\mathbb{R}})$. By the proposition in Appendix 1 and the remark that follows it, $\mathcal{C}' = \mathcal{C}_{\mathbb{R}}$ satisfies the hypotheses of Proposition 4.2. Now apply Proposition 4.2 to conclude the proof in this case.

We consider case b) where $\mathfrak{H}_1 = \mathfrak{K} \cap \mathfrak{P} \neq \{0\}$. By the argument of case a) it suffices to find a real Chevalley basis $\mathcal{C}_{\mathbb{R}}$ of $\mathfrak{H}$ such that W is a rational subspace of $\mathfrak{H}$ with the rational structure $\mathfrak{H}_{\mathbb{Q}} = \mathbb{Q}\text{-span}(\mathcal{C}_{\mathbb{R}})$.

In the remainder of the proof let B denote the Killing form of $\mathfrak{H}$. It follows that B is negative definite on $\mathfrak{H}$ since $\mathfrak{H}$ is semisimple. We fix the inner product $\langle\, , \rangle = -B$ on $\mathfrak{H}$.

Recall the following basic fact about Killing forms of Lie algebras

$$B([X, Y], Z]) = -B(Y, [X, Z]) \qquad \text{for all elements } X, Y, Z \text{ of } \mathfrak{H}. \tag{1}$$

Define $\mathfrak{H}_2$ to be the orthogonal complement of $\mathfrak{H}_1$ in $\mathfrak{H}$ relative to $-B$. Similarly, define $\mathfrak{K}_2$ and $\mathfrak{P}_2$ to be the orthogonal complements of $\mathfrak{H}_1$ in $\mathfrak{K}$ and $\mathfrak{P}$ respectively. Hence we have

$$\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2,\ \mathfrak{K} = \mathfrak{H}_1 \oplus \mathfrak{K}_2 \text{ and } \mathfrak{P} = \mathfrak{H}_1 \oplus \mathfrak{P}_2. \tag{2}$$

Remark. Since $\mathfrak{H}_1$ is an ideal of $\mathfrak{H}$ it follows that $\mathfrak{H}_2$ is also an ideal of $\mathfrak{H}$ since $B([X, \mathfrak{H}_2], \mathfrak{H}_1]) = -B(\mathfrak{H}_2, [X, \mathfrak{H}_1]) \subseteq -B(\mathfrak{H}_2, \mathfrak{H}_1) = \{0\}$ for all X in $\mathfrak{H}$ by (1). It follows that both $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are semisimple Lie algebras since the Killing form of an ideal $\mathfrak{A}$ of $\mathfrak{H}$ is the restriction of the Killing form B of $\mathfrak{H}$ to $\mathfrak{A}$. Hence $B_{\mathfrak{A}} = B$ is negative definite on $\mathfrak{A}$.

Lemma 3. *The Lie algebra $\mathfrak{H}_2$ is a direct sum $\mathfrak{H}_2 = \mathfrak{K}_2 \oplus \mathfrak{P}_2$. Moreover, $[\mathfrak{K}_2, \mathfrak{P}_2] \subseteq \mathfrak{P}_2$, $[\mathfrak{P}_2, \mathfrak{P}_2] \subseteq \mathfrak{K}_2$ and $[\mathfrak{K}_2, \mathfrak{K}_2] \subseteq \mathfrak{K}_2$.*

Lemma 4. *There exists a basis $\mathcal{C}$ of $\mathfrak{H}$ that satisfies the hypotheses of Proposition 4.2. If $\mathfrak{H}_{\mathbb{Q}} = \mathbb{Q}\text{-span}(\mathcal{C})$, then $\mathfrak{P} = W$ is a rational subspace of $\mathfrak{H}_{\mathbb{Q}}$.*

Proof of Proposition (4.3c). This is immediate when Proposition 4.2 and Lemma 4 are applied to $\mathcal{N} = \mathcal{V} \oplus W$. □

We now prove Lemmas 3 and 4.

Proof of Lemma 3. Since $\mathfrak{H}_1 = \mathfrak{P} \cap \mathfrak{K}$ it follows from (2) above that $\mathfrak{P}_2 \cap \mathfrak{K}_2 \subseteq (\mathfrak{P} \cap \mathfrak{K}) \cap (\mathfrak{P} \cap \mathfrak{K})^{\perp} = \{0\}$. Hence $\mathfrak{P}_2 \oplus \mathfrak{K}_2 \subseteq \mathfrak{H}_2 = \mathfrak{H}_1^{\perp}$. Note that $\mathfrak{H}_1 \oplus \mathfrak{H}_2 = \mathfrak{H} = \mathfrak{K} + \mathfrak{P} = (\mathfrak{H}_1 \oplus \mathfrak{K}_2) + (\mathfrak{H}_1 \oplus \mathfrak{P}_2) \subseteq \mathfrak{H}_1 \oplus \mathfrak{K}_2 \oplus \mathfrak{P}_2 \subseteq \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Hence all inclusions are equalities, and $\mathfrak{H}_2 = \mathfrak{K}_2 \oplus \mathfrak{P}_2$.

To show that $[\mathfrak{K}_2, \mathfrak{P}_2] \subseteq \mathfrak{P}_2$ it suffices to show that $[\mathfrak{K}, \mathfrak{P}_2] \subseteq \mathfrak{P}_2$ since $\mathfrak{K}_2 \subseteq \mathfrak{K}$. By (1) and (2) we know that $B([\mathfrak{K}, \mathfrak{P}_2], \mathfrak{H}_1) = -B(\mathfrak{P}_2, [\mathfrak{K}, \mathfrak{H}_1]) \subseteq -B(\mathfrak{P}_2, \mathfrak{H}_1) = \{0\}$ since $\mathfrak{H}_1$ is an ideal of $\mathfrak{H}$. By Lemma 2 and (2) we know that $[\mathfrak{K}, \mathfrak{P}_2] \subseteq [\mathfrak{K}, \mathfrak{P}] \subseteq \mathfrak{P} = \mathfrak{H}_1 \oplus \mathfrak{P}_2$. We conclude that $[\mathfrak{K}, \mathfrak{P}_2] \subseteq \mathfrak{P}_2$.

Next, $[\mathfrak{P}_2, \mathfrak{P}_2] \subseteq [\mathfrak{P}, \mathfrak{P}] = \mathfrak{K} = \mathfrak{H}_1 \oplus \mathfrak{K}_2$, and $B([\mathfrak{P}_2, \mathfrak{P}_2], \mathfrak{H}_1) = -B(\mathfrak{P}_2, [\mathfrak{P}_2, \mathfrak{H}_1]) \subseteq -B(\mathfrak{P}_2, \mathfrak{H}_1) = \{0\}$ by (1) and (2). Hence $[\mathfrak{P}_2, \mathfrak{P}_2] \subseteq \mathfrak{K}_2$.

Finally, $[\mathfrak{K}_2, \mathfrak{K}_2] \subseteq [\mathfrak{K}, \mathfrak{K}] \subseteq \mathfrak{K} = \mathfrak{H}_1 \oplus \mathfrak{K}_2$, and $B([\mathfrak{K}_2, \mathfrak{K}_2], \mathfrak{H}_1) = -B(\mathfrak{K}_2, [\mathfrak{K}_2, \mathfrak{H}_1]) \subseteq -B(\mathfrak{K}_2, \mathfrak{H}_1) = \{0\}$. This proves that $[\mathfrak{K}_2, \mathfrak{K}_2] \subseteq \mathfrak{K}_2$ and completes the proof of Lemma 3. □

Remark. The direct sum $\mathfrak{H}_2 = \mathfrak{K}_2 \oplus \mathfrak{P}_2$ is in fact an orthogonal direct sum relative to the inner product -B on $\mathfrak{H}$. If $K_2 \in \mathfrak{K}_2$ and $P_2 \in \mathfrak{P}_2$ are arbitrary elements, then by Lemma 3 $(\operatorname{ad} K_2 \circ \operatorname{ad} P_2)(\mathfrak{P}_2) \subseteq \mathfrak{K}_2$ and $(\operatorname{ad} K_2 \circ \operatorname{ad} P_2)(\mathfrak{K}_2) \subseteq \mathfrak{P}_2$. Moreover $\operatorname{ad} P_2(\mathfrak{H}_1) \subseteq [\mathfrak{P}_2, \mathfrak{H}_1] \subseteq [\mathfrak{H}_2, \mathfrak{H}_1] \subseteq \mathfrak{H}_2 \cap \mathfrak{H}_1 = \{0\}$ since $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are ideals of $\mathfrak{H}$. Since $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2 = \mathfrak{H}_1 \oplus \mathfrak{P}_2 \oplus \mathfrak{K}_2$ it follows that $B(K_2, P_2) = \operatorname{trace}(\operatorname{ad} K_2 \circ \operatorname{ad} P_2) = 0$. Hence $\mathfrak{K}_2$ and $\mathfrak{P}_2$ are orthogonal relative to $-B$.

Proof of Lemma 4. Applying Lemma 2 to $\mathfrak{H}_2 = \mathfrak{K}_2 \oplus \mathfrak{P}_2$ we can find a Chevalley basis $\mathcal{C}_2$ of $\mathfrak{H}_2^{\mathbb{C}}$ such that the subspaces $\mathfrak{K}_2$ and $\mathfrak{P}_2$ are rational relative to the rational structure $\mathfrak{H}_{2\mathbb{Q}} = \mathbb{Q} - \operatorname{span}(\mathcal{C}_{2\mathbb{R}})$ for $\mathfrak{H}_2$. Here $\mathcal{C}_{2\mathbb{R}}$ denotes the real Chevalley basis of $\mathfrak{H}_2$ determined by $\mathcal{C}_2$ as in example 3 of (1.3a). Choose any Chevalley basis $\mathcal{C}_1$ of $\mathfrak{H}_1^{\mathbb{C}}$ such that the corresponding real Chevalley basis $\mathcal{C}_{1\mathbb{R}}$ is a basis of $\mathfrak{H}_1$. Then $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ is a Chevalley basis of $\mathfrak{H}^{\mathbb{C}} = \mathfrak{H}_1^{\mathbb{C}} \oplus \mathfrak{H}_2^{\mathbb{C}}$, and $\mathcal{C}_{\mathbb{R}} = \mathcal{C}_{1\mathbb{R}} \cup \mathcal{C}_{2\mathbb{R}}$ is the corresponding real Chevalley basis of $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Define a rational structure $\mathfrak{H}_{\mathbb{Q}}$ for $\mathfrak{H}$ by $\mathfrak{H}_{\mathbb{Q}} = \mathbb{Q} - \operatorname{span}(\mathcal{C}_{\mathbb{R}}) = \mathbb{Q} - \operatorname{span}(\mathcal{C}_{1\mathbb{R}}) \oplus \mathbb{Q} - \operatorname{span}(\mathcal{C}_{2\mathbb{R}})$. Since $\mathfrak{P}_2$ is a rational subspace of $\mathfrak{H}_2$ relative to the rational structure $\mathbb{Q} - \operatorname{span}(\mathcal{C}_{2\mathbb{R}})$ for $\mathfrak{H}_2$ it follows that $\mathfrak{P} = \mathfrak{H}_1 \oplus \mathfrak{P}_2$ is a rational subspace of $\mathfrak{H}$ relative to $\mathfrak{H}_{\mathbb{Q}}$.

The Lie algebra $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ is semisimple by Lemma 1. By the remarks following the statement of Proposition 4.2 it follows that $H = \exp(\mathfrak{H})$ is a compact, connected subgroup of $SO(V)$, where $\exp : \mathfrak{so}(V) \to SO(V)$ is the exponential map. We conclude that the basis $\mathcal{C}_{\mathbb{R}}$ of $\mathfrak{H}$ satisfies the hypotheses of Proposition 4.2 by the proposition in Appendix 1 and the remark that follows it. □

We have completed the proof of Proposition 4.3 in the case that W has trivial center.

Case 2 W has compact center

Let $\mathfrak{H} = W + [W, W]$ and $\mathfrak{H}_0 = [\mathfrak{H}, \mathfrak{H}]$. Recall that $\mathfrak{H}$ and $\mathfrak{H}_0$ are Lie algebras and $\mathfrak{H}_0$ is semisimple by 1) and 2) of (4.3b). The proof of 3) in (4.3b) shows that $\mathfrak{H} = Z(W) \oplus \mathfrak{H}_0$ and $H = \exp(\mathfrak{H}) = T_0 \cdot H_0$ where $T_0 = \exp(Z(W))$ is a compact, connected abelian subgroup of $SO(V)$ and $H_0 = \exp(\mathfrak{H}_0)$ is a compact, connected semisimple subgroup of $SO(V)$. Moreover, T_0 commutes with H_0 since $Z(W)$ commutes with $\mathfrak{H}_0$. Hence H is a representation of $T_0 \times H_0$ on V.

Recall from 3) of (4.3b) that $W = Z(W) \oplus W_1$ (orthogonal direct sum), where W_1 is a Lie triple system in $\mathfrak{so}(V)$ and $\mathfrak{H}_0 = W_1 + [W_1, W_1]$. The center of the Lie triple system W_1 is trivial by 2) of (4.3b) since $\mathfrak{H}_0$ is semisimple. By Case 1 we may choose a real Chevalley basis $\mathcal{C}^0_{\mathbb{R}}$ of $\mathfrak{H}_0$ so that W_1 is a rational subspace of $\mathfrak{H}_{0\mathbb{Q}} = \mathbb{Q}\text{-span}(\mathcal{C}^0_{\mathbb{R}})$. Choose a basis $\{Z_1, ..., Z_k\}$ of $Z(W)$ so that $\exp(2\pi Z_i) = \mathrm{Id}$ in T_0 for all i. By the proposition in Appendix 1 the set $\mathcal{C}' = \{Z_1, ..., Z_k\} \cup \mathcal{C}^0_{\mathbb{R}}$ is a basis of $\mathfrak{H} = Z(W) \oplus \mathfrak{H}_0$ that satisfies the hypotheses of Proposition 4.2. Moreover, $W = Z(W) \oplus W_1$ is a rational subspace of $\mathfrak{H}$ with the rational structure $\mathfrak{H}_{\mathbb{Q}} = \mathbb{Q}\text{-span}(\mathcal{C}')$. Now apply Proposition 4.2. $\square$

Appendix 1.

Standard bases for representations of compact Lie groups. Let T^p be a p-torus, a compact, connected abelian Lie group of dimension $p \geq 1$, and let $\mathfrak{T}^p$ denote the Lie algebra of T^p. A basis $\mathcal{C}^p = \{Z_1, ..., Z_p\}$ will be called *standard* if $\exp(2\pi Z_i) = 1$ for $1 \leq i \leq p$, where $\exp : \mathfrak{T}^p \to T^p$ is the exponential map (and also a group homomorphism). Note that $\exp(\mathbb{R} Z_i)$ is a 1-torus for each i.

Proposition. *Let $G = T^p \times G^*$, where G^* is a compact, connected semisimple Lie group and $p \geq 0$. Let $\mathcal{C}' = \mathcal{C}^p \cup \mathcal{C}^*_{\mathbb{R}}$, where $\mathcal{C}^p$ is a standard basis of $\mathfrak{T}^p = LT^p$ and $\mathcal{C}^*_{\mathbb{R}}$ is a real Chevalley basis of $\mathfrak{G}^* = LG^*$. Then $\mathcal{C}' = \{X_1, X_2, ..., X_N\}$ has the following properties:*

1) $[X_i, X_j] = \displaystyle\sum_{i=1}^{N} C^k_{ij} X_k$, *where* $C^k_{ij} \in \mathbb{Z}$ *for all* i, j, k.

2) If $\rho : G \to GL(U)$ is a representation of G on a finite dimensional real vector space U, then there exists a basis $\mathcal{B}$ of U such that each element of $d\rho(\mathcal{C}')$ leaves invariant $\mathbb{Z}$-span$(\mathcal{B})$.

Corollary. *Let G be a compact, connected lie group with Lie algebra $\mathfrak{G}$. Then there exists a basis $\mathcal{C}' = \{X_1, X_2, ..., X_N\}$ of $\mathfrak{G}$ with the following properties :*

1) $[X_i, X_j] = \sum_{i=1}^{N} C_{ij}^k X_k$, *where* $C_{ij}^k \in \mathbb{Z}$ *for all* i, j, k.

2) *If $\rho : G \to GL(U)$ is a representation of G on a finite dimensional real vector space U, then there exists a basis $\mathcal{B}$ of U such that each element of $d\rho(\mathcal{C}')$ leaves invariant $\mathbb{Z} - \mathrm{span}(\mathcal{B})$.*

Remark. If $p = 0$ and G is semisimple in the proposition, then $\mathcal{C}' = \mathcal{C}_{\mathbb{R}}^*$ and the result follows from [R2], with the slightly weaker conclusion that the structure constants $\{C_{ij}^k\}$ of 1) lie in $\mathbb{Q}$. However, with some additional work one may conclude that $\{C_{ij}^k\} \subseteq \mathbb{Z}$. See [E2] for details and an alternate (but longer) proof of the proposition above in the case that G is semisimple.

Proof of the Corollary. We prove the corollary first. The first step is the following known result, whose proof we include for completeness.

Lemma. *Let G be a compact, connected Lie group. Then G has a finite covering $\pi : G' \to G$ such that $G' = T \times G^*$, where T is compact, connected and abelian and G^* is compact, connected and semisimple.*

Proof of the lemma. It is well known that $G = \mathfrak{Z} \oplus \mathfrak{G}^*$, where $\mathfrak{G}$ and $\mathfrak{G}^*$ are the Lie algebras of G and G^* and $\mathfrak{Z}$ is the center of $\mathfrak{G}$. See for example Lemma 2 of Appendix 2. The Killing form of $\mathfrak{G}^*$ is negative definite, and hence any Lie group with Lie algebra $\mathfrak{G}^*$ must be compact. (See the remarks following Proposition 4.2.) In particular, if G^* is the simply connected Lie group with Lie algebra $\mathfrak{G}^*$, then G^* is compact. If $\tilde{G} = \mathbb{R}^p \times G^*$, where $p = \dim \mathfrak{Z}$, then $\tilde{G}$ is a simply connected Lie group with Lie algebra $\mathfrak{G}$.

If $\pi : \tilde{G} \to G$ is the universal covering homomorphism, then $\mathrm{Ker}(\pi)$ is a discrete subgroup of $Z(\tilde{G}) = \mathbb{R}^p \times Z(G^*)$. If $\psi : Z(\tilde{G}) \to Z(G^*)$ is the projection homomorphism, then $\mathrm{Ker}(\psi)$ has finite index in $Z(\tilde{G})$ since $Z(G^*)$ is finite. Hence $H = \mathrm{Ker}(\pi) \cap \mathbb{R}^p = \mathrm{Ker}(\pi) \cap \mathrm{Ker}(\psi)$ has finite index in $\mathrm{Ker}(\pi) \cap Z(\tilde{G}) = \mathrm{Ker}(\pi)$. If $G' = \tilde{G}/H = (\mathbb{R}^p/H) \times G^*$, then G' is a finite cover of $G = \tilde{G}/\mathrm{Ker}(\pi)$, and hence G' is also compact. The group $T = \mathbb{R}^p/H$ is clearly abelian and connected, and T is compact since G' is compact. The group G' is connected since both T and G^* are connected. $\square$

Now let G be a compact, connected Lie group, and let $\pi' : G' \to G$ be a finite cover as in the lemma. If $\rho : G \to GL(U)$ is a representation of G on a finite dimensional real vector space U, then $\rho' = \pi' \circ \rho : G' \to GL(U)$ is a representation of G'. The corollary now follows from the proposition since G' and G have the same Lie algebra $\mathfrak{G}$ and $d\rho'(\mathfrak{G}) = d\rho(\mathfrak{G})$ in $\mathrm{End}(U)$. $\square$

Proof of the proposition. Let $G = T^p \times G^*$ and let $\mathcal{C}' = \mathcal{C}^p \cup \mathcal{C}^*_{\mathbb{R}}$, where $\mathcal{C}^p = \{Z_1, Z_2, ..., Z_p\}$ is a standard basis of $\mathfrak{T}^p = LT^p$ and $\mathcal{C}^*_{\mathbb{R}}$ is a real Chevalley basis of $\mathfrak{G}^* = LG^*$. Note that $\mathfrak{T}^p$ is the center of $\mathfrak{G} = \mathfrak{T}^p \oplus \mathfrak{G}^*$ since the semisimple Lie algebra $\mathfrak{G}^*$ has trivial center.

It is clear that $\mathcal{C}'$ is a basis of $\mathfrak{G}$ with structure constants in $\mathbb{Z}$ since the Chevalley basis $\mathcal{C}^*_{\mathbb{R}}$ has structure constants in $\mathbb{Z}$ (cf. [Hu, p. 145] or [B, section 3.2]). The basis $\mathcal{C}'$ therefore satisfies 1) of the proposition, and we need to show that $\mathcal{C}'$ also satisfies 2). We may assume that $p \geq 1$ by the remark after the statement of the proposition.

Lemma. *Let $\rho : G \to GL(U)$ be an irreducible representation of G on a finite dimensional real vector space U. Assume that $\rho(T^p)$ fixes no nonzero vectors of U. Let $\{Z_1, ..., Z_p\}$ be a standard basis of $\mathfrak{T}^p = LT^p$. Then*

1) There exists a nonzero linear map $\alpha : \mathfrak{T}^p \to \mathbb{R}$ such that $d\rho(Z)^2 = -\alpha(Z)^2 \,\mathrm{Id}$ on U for all Z in $\mathfrak{T}^p$.

2) There exists a linear map $J : U \to U$ such that $J^2 = -\mathrm{Id}$ and J commutes with $\rho(G)$. Moreover, for $1 \leq k \leq p$ there exists an integer n_k such that $d\rho(Z_k) = n_k J$ on U.

3) U has a complex structure such that $\rho(G) \subseteq \mathrm{End}_{\mathbb{C}}(U)$ and U is a complex $\mathfrak{G}^{\mathbb{C}}$-module.

4) There exists a basis $\mathcal{B}$ of U such that each element of $d\rho(\mathcal{C}')$ leaves invariant $\mathbb{Z} - \mathrm{span}(\mathcal{B})$.

For the moment we postpone the proof of the lemma and complete the proof of the Proposition. Let $G = T^p \times G^*$ as above, and let $\rho : G \to GL(U)$ be a representation of G on a finite dimensional real vector space U. Fix a $\rho(G)$-invariant inner product $\langle\, , \rangle$ on U. Let $U_0 = \{u \in U : \rho(t)u = u$ for all $t \in T^p\}$. Then U_0 is a G-module since $\rho(T^p)$ lies in the center of $\rho(G)$, and it follows that the orthogonal complement $U_0^\perp$ is also a G-module. By the definition of U_0, $\rho(T^p)$ fixes no nonzero vector of $U_0^\perp$. Write $U_0^\perp = U_1 \oplus ... \oplus U_N$, an orthogonal direct sum of irreducible G-modules. By 4) of the lemma we can find a basis $\mathcal{B}_k$ for U_k, $1 \leq k \leq N$, such that each element of $d\rho(\mathcal{C}')$ leaves invariant $\mathbb{Z} - \mathrm{span}(\mathcal{B}_k)$. If $\mathcal{B}_0^\perp = \bigcup_{k=1}^N \mathcal{B}_k$, then $\mathcal{B}_0^\perp$ is a basis of $U_0^\perp$ such that each element of $d\rho(\mathcal{C}')$ leaves invariant $\mathbb{Z} - \mathrm{span}(\mathcal{B}_0^\perp)$.

Now regard U_0 as a G^*-module, and consider the real Chevalley basis $\mathcal{C}^*_{\mathbb{R}}$ of $\mathfrak{G}^*$. Since G^* is semisimple we can apply [R1] to prove that there exists a basis $\mathcal{B}_0$ of U_0 such that every element of $\mathcal{C}^*_{\mathbb{R}}$ leaves invariant $\mathbb{Z}-\mathrm{span}(\mathcal{B}_0)$. See the remark following the statement of the Proposition. Since $d\rho(\mathfrak{T}^p) = \{0\}$ in $\mathrm{End}(U_0)$ by the definition of U_0 and since $\mathcal{C}' = \{Z_1, Z_2, ..., Z_p\} \cup \mathcal{C}^*_{\mathbb{R}}$, where $\{Z_1, Z_2, ..., Z_p\}$ is a standard basis of $\mathfrak{T}^p$, it follows that every element of $d\rho(\mathcal{C}')$ leaves invariant $\mathbb{Z} - \mathrm{span}(\mathcal{B}_0)$. Finally, if $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_0^{\perp}$, then $\mathcal{B}$ is a basis of $U = U_0 \oplus U_0^{\perp}$ such that every element of $d\rho(\mathcal{C}')$ leaves invariant $\mathbb{Z} - \mathrm{span}(\mathcal{B}_0)$. This will complete the proof of the Proposition. □

Proof of the lemma . For convenience we fix a $\rho(G)$-invariant inner product $\langle\, , \rangle$ on U. Then each element of $d\rho(\mathfrak{G}) \subseteq \mathrm{End}(U)$ is skew symmetric relative to $\langle\, , \rangle$ and has eigenvalues in $\mathbb{R}i = \{ai : a \in \mathbb{R}\}$.

1) Let $V = U^{\mathbb{C}}$, the complexification of U, and let $\rho : G \to GL(V)$ also denote the complex representation that extends the real representation $\rho : G \to GL(U)$. Since $\rho(T^p)$ is compact, connected and abelian there exists a nonzero eigenvector $v = u_1 + iu_2$ for $\rho(T^p)$, where $u_1, u_2 \in U$. Then v is an eigenvector for $d\rho(\mathfrak{T}^p) = L\rho(T^p)$, and there exists a linear map $\alpha : \mathfrak{T}^p \to \mathbb{R}$ such that $d\rho(Z) = \alpha(Z)iv$ for all $Z \in \mathfrak{T}^p$. If $U_\alpha = \{u \in U : d\rho(Z)^2 u = -\alpha(Z)^2 u$ for all $Z \in \mathfrak{T}^p\}$, then U_α is nonempty since it contains the real and imaginary parts u_1, u_2 of v. Note that $\rho(G)$ commutes with $d\rho(\mathfrak{T}^p)$ since $G = \exp(\mathfrak{G})$ and $\mathfrak{T}^p$ is the center of $\mathfrak{G}$. Hence $\rho(G)$ leaves U_α invariant, and $U = U_\alpha$ since U is an irreducible G-module. If $\alpha : \mathfrak{T}^p \to \mathbb{R}$ is zero, then $0 = d\rho(Z + Z')^2 \Rightarrow d\rho(Z)\, d\rho(Z') = 0$ for all Z, Z' in $\mathfrak{T}^p$. It follows that $d\rho(\mathfrak{T}^p)(U) \subseteq \mathrm{Ker}\, d\rho(\mathfrak{T}^p)$. Hence $\mathrm{Ker}\, d\rho(\mathfrak{T}^p) \neq \{0\}$ and $\rho(T^p)$ fixes all vectors in $\mathrm{Ker}\, d\rho(\mathfrak{T}^p)$, which contradicts the hypothesis on U.

2) By 1) $d\rho(\mathfrak{T}^p)$ is a 1-dimensional subspace of $\mathrm{End}(U)$ since α is nonzero. Hence if $\{Z_1, Z_2, ..., Z_p\}$ is the standard basis of $\mathfrak{T}^p$ from the statement of the lemma, then $d\rho(Z_k)$ is nonzero in $\mathrm{End}(U)$ for some $1 \le k \le p$.

We show that $n_k = \alpha(Z_k)$ is an integer for $1 \le k \le p$. If ci is an eigenvalue of $d\rho(Z_k)$ for some $1 \le k \le p$, then $e^{2\pi ci}$ is an eigenvalue of $\exp(d\rho(2\pi Z_k))$ $= \rho(\exp(2\pi Z_k)) = Id$ since $\exp(2\pi Z_k) = 1$ in T^p by the hypotheses on $\{Z_1, Z_2, ..., Z_p\}$. Here exp denotes both $\exp : \mathrm{End}(U) \to GL(U)$ and $\exp :$ $\mathfrak{T}^p = LT^p \to T^p$. Hence $1 = e^{2\pi ci}$, and we conclude that $c \in \mathbb{Z}$ and $d\rho(Z_k)^2$ has eigenvalue $-c^2 = -\alpha(Z_k)^2$.

Now fix k with $1 \le k \le p$ such that $d\rho(Z_k)$ is nonzero in $\mathrm{End}(U)$. Let $n_k = \alpha(Z_k) \in \mathbb{Z}$. Note that n_k is nonzero since $d\rho(Z_k)^2 = -\alpha(Z_k)^2\, \mathrm{Id}$ on U. If $J = (1/n_k)\, d\rho(Z_k)$, then clearly $J^2 = -\mathrm{Id}$ on U, and J commutes with $\rho(G)$ since $d\rho(\mathfrak{T}^p)$ commutes with $\rho(G)$. If $1 \le r \le p$, where r is arbitrary, then $d\rho(Z_r) = c_r J$ on U for some $c_r \in \mathbb{R}$ by the definition of J and the fact that $d\rho(\mathfrak{T}^k)$ is 1-dimensional in $\mathrm{End}(U)$. Hence by 1), $-\alpha(Z_r)^2\, \mathrm{Id} = d\rho(Z_r)^2$

$= -c_r^2 \,\mathrm{Id}$ on U, and we conclude that $c_r = \pm\alpha(Z_r) \in \mathbb{Z}$.

3) By 2) we may define a complex structure on U by setting $(a + bi)\, u = au + bJu$ for all $a, b \in \mathbb{R}$ and all $u \in U$. It follows that $\rho(G) \subseteq \mathrm{End}_{\mathbb{C}}(U)$ since $\rho(G)$ commutes with J. With the complex structure defined by J, U becomes a complex $\mathfrak{G}^{\mathbb{C}}$-module if we define $(X + iY)\, u = Xu + JYu$ for all $X, Y \in \mathfrak{G}$ and all $u \in U$.

4) We shall need the following

Sublemma. Let $\mathcal{L}$ be a finite dimensional complex semisimple Lie algebra, and let $\mathcal{C} = \{H_\alpha^*, y_\beta : \alpha \in \Delta, \beta \in \Phi\}$ be a Chevalley basis for $\mathcal{L}$ (cf. section 1.3a). Let V be a finite dimensional complex $\mathcal{L}$-module. Then there exists a $\mathbb{C}$-basis $\mathcal{B}_V$ of V such that every element of $\mathcal{C}$ leaves invariant $\mathbb{Z} - \mathrm{span}(\mathcal{B}_V)$.

Proof of the sublemma. If $\mathfrak{U}(\mathcal{L})$ denotes the universal enveloping algebra of $\mathcal{L}$, then V is also a $\mathfrak{U}(\mathcal{L})$-module. If $\mathfrak{U}(\mathcal{L})_{\mathbb{Z}}$ denotes the subring of $\mathfrak{U}(\mathcal{L})$ generated by $\{y_\beta^n/n! : \beta \in \Phi, n \in \mathbb{Z}+\}$, then there exists a basis $\mathcal{B}_V$ of V such that every element of $\mathfrak{U}(\mathcal{L})_{\mathbb{Z}}$ leaves invariant $\mathbb{Z} - \mathrm{span}(\mathcal{B}_V)$. See for example, [Hu, p. 156]. It suffices to prove that $\mathcal{C} \subseteq \mathfrak{U}(\mathcal{L})_{\mathbb{Z}}$, and this follows since $-H_\alpha^* = [y_\alpha, y_{-\alpha}]$ for all $\alpha \in \Delta$ (cf. [B, section 3.2]). □

We now apply the sublemma above to the complex vector space U whose complex structure is determined by the linear transformation $J : U \to U$ from 2). Let $\mathcal{C}^*$ be a Chevalley basis of $(\mathfrak{G}^*)^{\mathbb{C}} \subseteq \mathfrak{G}^{\mathbb{C}}$. By the sublemma there exists a $\mathbb{C}$-basis $\mathcal{B}' = \{\xi_1, ..., \xi_m\}$ of U such that each element of $d\rho(\mathcal{C}^*)$ leaves invariant $\mathbb{Z} - \mathrm{span}(\mathcal{B}')$. Let $\mathcal{B} = \{u_1, ..., u_{2m}\}$, where $u_i = \xi_i$ for $1 \le i \le m$ and $u_{m+i} = J\xi_i$ for $1 \le i \le m$. Then $\mathcal{B}$ is an $\mathbb{R}$-basis for U regarded as a real $\mathfrak{G}$-module.

From 2) above it follows immediately that

a) $d\rho(Z_k)$ leaves invariant $\mathbb{Z} - \mathrm{span}(\mathcal{B})$ for $1 \le k \le p$.

By the discussion in (1.3a) the Chevalley basis $\mathcal{C}^* = \{H_\alpha^*, y_\beta : \alpha \in \Delta, \beta \in \Phi\}$ for $(\mathfrak{G}^*)^{\mathbb{C}}$ may be chosen so that $\mathcal{C}_{\mathbb{R}}^* = \{iH_\alpha^*, u_\beta, v_\beta : \alpha \in \Delta, \beta \in \Phi\}$ is a basis for $\mathfrak{G}^*$, where $u_\beta = y_\beta - y_{-\beta}$ and $v_\beta = iy_\beta + iy_{-\beta}$ for all $\beta \in \Phi$. Note that for all $\beta \in \Phi$, $iy_\beta = Jy_\beta$ leaves invariant $\mathbb{Z} - \mathrm{span}(\mathcal{B})$ since J and y_β have this property. We conclude

b) Every element of $d\rho(\mathcal{C}_{\mathbb{R}}^*)$ leaves invariant $\mathbb{Z} - \mathrm{span}(\mathcal{B})$.

Assertion 4) follows immediately from a) and b) since $\mathcal{C}' = \{Z_1, Z_2, ..., Z_p\} \cup \mathcal{C}_{\mathbb{R}}^*$. □

Further sources.

Appendix 2 Lie triple systems

Appendix 3 Clifford algebras and Lie triple systems

Appendices 2 and 3 can be found on the author's website at (www.math.unc.edu).

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