COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 10, Number 5, 1151-1180, 2002

Hamilton's injectivity radius estimate for sequences with almost nonnegative curvature operators

BENNETT CHOW[1] and DAN KNOPF[2] and PENG LU

1. Introduction.

In recent years, one of the focuses in the study of the Ricci Flow on Riemannian manifolds has been on classifying the singularities that form in low dimensions. In particular, Hamilton has obtained partial classification theorems in dimensions three and four. These classifications are in the sense of obtaining and classifying pointed limits, provided they exist, of dilations of a solution to the Ricci Flow about sequences of points and times tending to the singularity time, after passing to a suitable subsequence. In dimension four, Hamilton's classification together with his geometric–topological surgery methods yield a classification of diffeomorphism types of those compact 4-manifolds with positive isotropic curvature that do not admit any incompressible 3-dimensional space form not diffeomorphic to either $\mathbb{S}^3$ or $\mathbb{RP}^3$ [H2]. (Note that [MM] obtained an earlier classification of homeomorphism types of compact simply-connected n-manifolds with positive isotropic curvature by using harmonic maps.) In dimension three, Hamilton's classification of singularities in [H1] plays a major role in his program for approaching Thurston's Geometrization Conjecture [T] by Ricci Flow methods. It is conjectured by Hamilton that for the volume normalized Ricci flow on a compact 3-manifold, after a finite number of geometric–topological surgeries at some finite sequence of times, the solution will exist for all time and the curvature will remain uniformly bounded. If this is so, then the 3-manifold admits a geometric decomposition by [H3].

A fundamental tool used to obtain limits of sequences of solutions to the Ricci Flow is the Gromov-type compactness theorem of Hamilton [H4].

[1]Partially supported by NSF grant DMS-9971891.
[2]Partially supported by NSF grant DMS-0202796.
[3]We would like to thank the referee for suggesting changes to improve the presentation of the paper.

(See [P] for a survey of compactness theorems in Riemannian geometry.) As is usual for compactness theorems, the main assumptions are bounded curvature (which by the Bernstein–Shi estimates [S] and §7 of [H1] implies bounds on all derivatives of curvature for solutions of the Ricci flow) and a lower bound for the injectivity radius at a point (which by [CLY] or [CGT] implies injectivity radii estimates at all points, depending on distance). For the sequences arising from dilations about a singularity, the curvature bound follows from the conditions imposed on the choice of points and times and choice of dilation factor. (See §16 of [H1].) This leaves one with the problem of obtaining an injectivity radius bound. In fact, Hamilton's Little Loop Lemma (§15 of [H1]) asserts that for a solution of the Ricci Flow, a strengthened injectivity radius estimate should follow from a suitable differential Harnack inequality of Li–Yau–Hamilton type. By [H5], a differential Harnack inequality holds for complete solutions of the Ricci flow with nonnegative curvature operator. See [LY] for the seminal result of this type for solutions of the heat equation. For Type I singularities in dimension three, Hamilton proved an isoperimetric estimate that implies an injectivity radius estimate (§23 of [H1]). For Type II singularities in dimension three, Hamilton also conjectures that there is an injectivity radius estimate. In fact, the Little Loop Lemma, which is conjectured to be true for all solutions of the Ricci Flow on compact 3-manifolds, subsumes this conjecture. Similarly, for Kähler manifolds with positive bisectional curvature, there is an injectivity radius estimate useful for the study of the Kähler-Ricci flow (see [CT]).

Although the above conjecture is still open, there is an important case where an injectivity radius estimate should be true. Namely, consider a sequence of complete (in practice, usually compact) solutions to the Ricci flow with bounded curvature on a common time interval such that:

- The diameters are tending to infinity. Note that in the bounded diameter case, either there is an injectivity radius estimate for the sequence or the sequence collapses. In the former case, one can obtain a limit; and in the latter case, when the sequence of underlying topological manifolds is a fixed compact 3-dimensional manifold $\mathcal{M}^3$, Cheeger–Gromov theory proves that $\mathcal{M}^3$ is classified as a graph manifold.
- The curvature operators are tending to nonnegative. (In dimension three, this follows by §24 of [H1] or §4 of [H3] for a sequence of solutions arising from dilations about a singularity.)
- The curvatures at the origin are uniformly bounded from below by a positive constant. (Otherwise we have the split case, which we hope

to address in a future article.)

In this case, Hamilton has claimed an injectivity radius estimate after passing to a suitable subsequence. (See Theorem 25.1 in [H1] and Theorem 2.3, below.) One of the applications of Theorem 25.1 of [H1] is in the proof of the classification of 4-manifolds with positive isotropic curvature given in [H2]. (See all three subsections of the "Recovering the manifold from surgery" section of that paper.)

The purpose of this paper is to give a complete proof of the aforementioned injectivity radius estimate. The reason for giving this new proof is related to the possibility of collapse. In particular, there appears to be a gap in the argument in [H1]. We shall explain this in more detail in later sections. (See the remarks before and after Example 2.4 and also Remark 3.6.) The main overall structure of our proof is the same as Hamilton's. However, our approach makes essential changes in the construction of Busemann-type sublevel sets, and relies on new arguments to establish the crucial fact that they are ultimately bounded. In particular, our proof does not rely on the continuity of the function ℓ_∞ introduced in Lemma 25.3 of [H1]. The main technical innovations of our method are in sections 5 and 6. We summarize them here for the convenience of the reader:

- In §25 of [H1], it is argued that the distances to the cut loci at the origins along any sequence of solutions to the Ricci flow satisfying Definition 2.2 converge to a continuous function $\ell_\infty : \mathcal{S}_1^{n-1} \to [0, \infty]$, where $\mathcal{S}_1^{n-1}$ is the standard $(n-1)$-sphere of radius 1. There are subtle difficulties with this approach relating to the possibility of collapse. (See Example 2.4 and the Remark that follows.) To get around these difficulties, we define an alternate function $\sigma_\infty : \mathcal{S}_1^{n-1} \to [0, \infty]$ in (3.1) by a lim sup, which obviates proving continuity.

- In §25 of [H1], a set $\mathcal{D}$ of *distinguished directions* is defined as $\ell_\infty^{-1}(\infty)$, "those [directions] in which we can go off to infinity without hitting the cut locus." We replace this by a set $\mathcal{R}_\infty \doteqdot \sigma_\infty^{-1}(\infty)$ of *ray-like directions*. The set $\mathcal{R}_\infty$ is nonempty, and there exist arbitrarily long minimizing geodesics in directions arbitrarily close to each of its members. (See Remark 3.6.)

- For robustness under the action of passing to subsequences, we find in $\mathcal{R}_\infty$ for any $\varepsilon > 0$ a finite subset $\{V_\alpha\}$ such that no member of $\mathcal{R}_\infty$ lies more than distance ε away from some V_α, and (most importantly) such that the lim sup in Definition (3.1) is attained as a limit for each V_α.

- In §25 of [H1], sets N_i are defined that act as substitutes for the sublevel sets of a Busemann function, and properties of the function ℓ_∞ are invoked to claim that they are uniformly bounded. We replace the N_i by sets $N_i(L) \equiv N_i(L,1)$ that depend on a length scale $L \gg 1$. (See Definition (5.1) in §5.) We do not show that the sets $N_i(L)$ are uniformly bounded. But the key innovation in our method is the proof of the boundedness property (Proposition 6.1), which states, roughly, that by going far enough out in the sequence, depending on L, one can bound the size of all remaining $N_i(L)$ independently of L.

Acknowledgment This paper was prepared while the authors enjoyed the hospitality provided during the summer of 2001 by the National Center for Theoretical Sciences in Hsinchu, Taiwan. We wish to thank the NCTS for providing partial support and a wonderful research environment.

2. Hamilton's injectivity radius estimate.

We now recall the setup from §25 of [H1]. Consider a sequence

$$\{\mathcal{M}_i^n, g_i(t), O_i, F_i : i \in \mathbb{N}\}$$

of complete solutions of the Ricci flow

$$\frac{\partial}{\partial t} g_i(t) = -2\,\mathrm{Rc}(g_i(t))$$

defined for $t \in (\alpha, \omega)$, where $\alpha < 0 < \omega \leq \infty$. Each solution is marked by an origin O_i and a frame $F_i = \{e_1^i, \dots, e_n^i\}$ at O_i which is orthonormal with respect to $g_i(0)$.

Definition 2.1. We say such a sequence has **uniformly bounded geometry** if there exist constants C_k for all $k \in \mathbb{N} \cup \{0\}$ such that

$$\sup_{i \in \mathbb{N}} \sup_{\mathcal{M}_i \times (\alpha,\omega)} \left| \nabla^k \mathrm{Rm}(g_i) \right|_{g_i} \leq C_k. \tag{2.1}$$

We denote the eigenvalues of the curvature operator

$$\mathrm{Rm}_i(x,t) \doteqdot \mathrm{Rm}(g_i)(x,t) : \Lambda^2 T_x \mathcal{M}_i \to \Lambda^2 T_x \mathcal{M}_i$$

by $\lambda_j(\mathrm{Rm}_i)(x,t)$, where $1 \leq j \leq m \doteqdot \dim \mathfrak{so}(n)$, and $\lambda_1 \leq \cdots \leq \lambda_m$.

Definition 2.2. We call $\{\mathcal{M}_i^n, g_i(t), O_i, F_i : i \in \mathbb{N}\}$ a **sequence with almost nonnegative curvature operators** if it has uniformly bounded geometry and satisfies the following three assumptions:

1. there exists a sequence $\delta_i \searrow 0$ such that
$$-1 \leq -\delta_i \leq \lambda_1(\mathrm{Rm}_i)(x,t)$$
for all $x \in \mathcal{M}_i$ and $t \in (\alpha, \omega)$;

2. the manifolds $(\mathcal{M}_i^n, g_i(0))$ are growing without bound,
$$\lim_{i\to\infty} [\mathrm{diam}(\mathcal{M}_i^n, g_i(0))] = \infty;$$
and

3. there exists $\varepsilon > 0$ such that O_i is an ε-**bumplike point** at $t = 0$, namely
$$\lambda_1(\mathrm{Rm}_i)(O_i, 0) \geq \varepsilon.$$

As stated in the introduction, the objective of this paper is to give a complete proof of:

Theorem 2.3. *For any sequence with almost nonnegative curvature operators and* $\mathrm{sect}(g_i)(x,0) \leq 1$ *for all* $x \in \mathcal{M}_i$ *and* $i \in \mathbb{N}$, *there exists a subsequence*
$$\{\mathcal{M}_i^n, g_i(t), O_i, F_i\}$$
such that for all i,
$$\mathrm{inj}_{g_i(0)}(O_i) \geq 1.$$

This result is equivalent to Theorem 25.1 of [H1]. The basic strategy of our proof is the same as the one employed there, and comprises essentially three steps:

(a) use conditions (1) and (2) to find arbitrarily long minimizing geodesics along which the curvature is arbitrarily close to nonnegative;

(b) use condition (3) and the strong maximum principle to construct large uniform neighborhoods of the origins in which the curvature is uniformly positive; and

(c) rule out short geodesics in these neighborhoods by means of a second-variation argument along the long geodesics found in step (a).

However, our implementation of this strategy is distinct in a number of ways from the methods employed in [H1]. We could not follow one of the steps in the original proof. In particular, let σ_i denote the distance to the cut locus from the origin in $(\mathcal{M}_i^n, g_i(0))$. In the original paper, it is argued that the σ_i converge to a continuous function $\ell_\infty : \mathcal{S}_1^{n-1} \to [0,\infty]$. However there appears to be a gap in the part of the argument in [H1] that deals with the construction of a Jacobi field in a geodesic tube for the case that $\exp_{O_i}(\ell_i V_i) = \exp_{O_i}(\ell_i W_i)$ for a sequence of distinct vectors such that $|V_i - W_i| \to 0$. This is precisely because collapse for the sequence has not yet been ruled out at this point in the argument.

Example 2.4. Consider a sequence $\{\mathcal{T}_i^2 : i = 1, 2, \dots\}$ of collapsing flat tori with fundamental domains

$$[-i, i] \times [-1/i, 1/i] \subset \mathbb{R}^2.$$

Take $O_i = (0,0)$, and define constant-speed geodesics

$$\alpha_i, \beta_i : \left[0, \frac{\sqrt{i^2-1}}{i}\right] \to \mathcal{T}_i^2$$

by

$$\alpha_i(s) = \left(s, \frac{s}{\sqrt{i^2-1}}\right) \qquad \text{and} \qquad \beta_i(s) = \left(s, -\frac{s}{\sqrt{i^2-1}}\right).$$

Then $\operatorname{length} \alpha_i = \operatorname{length} \beta_i = 1$ for all i, but their limit in the universal cover $\mathbb{R}^2$ is just the segment $s \mapsto (s, 0)$ for $0 \le s \le 1$. Since $\mathbb{R}^2$ is flat, there is no nontrivial Jacobi field which vanishes at its endpoints.

Remark 2.5. This example does not contain bumplike points, so it is *not* a counterexample to the claim in [H1]. However, as mentioned above, it does illustrate difficulties that are due to the possibility of collapse (which is an issue before an injectivity radius estimate has been proved). One can also construct 'local counterexamples' with constant positive curvature by removing small neighborhoods of the cone points from $\mathcal{S}^2/\mathbb{Z}_i$ for $i \in \mathbb{N}$, and letting $i \to \infty$. This construction does not produce global counterexamples, since gluing thin infinite cylinders $\mathcal{S}^1_{1/(2i)} \times (0,\infty)$ to both ends and smoothing the metric will not result in metrics of almost nonnegative curvature.

In order to overcome this difficulty, we were forced to make some modifications to both steps (a) and (b) of Hamilton's original proof.

3. Finding ray-like directions.

For each member of the sequence $\{\mathcal{M}_i^n, g_i(t), O_i, F_i\}$, the frame F_i defines a canonical isometry

$$I_i : (\mathbb{R}^n, g_{\text{can}}) \to (T_{O_i}\mathcal{M}_i, g(O_i, 0)).$$

Denote the unit sphere bundle of a Riemannian manifold $(\mathcal{M}^n, g)$ by $\mathcal{S}^{n-1}\mathcal{M}^n$. For each $V \in \mathcal{S}_{O_i}^{n-1}\mathcal{M}_i$, let $\rho_i(V) \in (0, \infty]$ denote the distance from O_i to the cut point of O_i along the geodesic $s \mapsto \exp_{O_i}(sV)$ in the metric $g_i(0)$. Denote by $\left(\mathcal{S}_1^{n-1}, g_{\text{can}}\right)$ the unit sphere in $\mathbb{R}^n$ with its canonical metric, and define

$$\sigma_i \doteqdot \rho_i \circ I_i|_{\mathcal{S}_1^{n-1}} : \mathcal{S}_1^{n-1} \to (0, \infty].$$

The set of directions $V \in \mathcal{S}_1^{n-1}$ for which $\exp_{O_i}(sI_i(V))$ is a ray is given by

$$\sigma_i^{-1}(\infty) = \left\{V \in \mathcal{S}_1^{n-1} : \rho_i(I_i(V)) = \infty\right\}.$$

There is no reason to expect $\sigma_i^{-1}(\infty)$ to be nonempty. Indeed, typical applications of Theorem 2.3 are when $\mathcal{M}_i \equiv \mathcal{M}$ is closed or when $\mathcal{M}_i$ is obtained from a closed manifold $\mathcal{M}$ by finitely many surgeries. In either case we have $\sigma_i^{-1}(\infty) \equiv \emptyset$. Nonetheless, assumption (2) allows us to pick out directions along which there are arbitrarily long minimizing geodesics.

If $V \in \mathcal{S}_1^{n-1}$, let $\mathfrak{S}(V)$ denote the set of all sequences $\{V_i\} \subset \mathcal{S}_1^{n-1}$ such that $\lim_{i\to\infty} |V_i - V|_{g_{\text{can}}} = 0$. Define

$$\sigma_\infty : \mathcal{S}_1^{n-1} \to [0, \infty]$$

for all $V \in \mathcal{S}_1^{n-1}$ by

$$\boxed{\sigma_\infty(V) \doteqdot \sup_{\mathfrak{S}(V)} \left(\limsup_{i\to\infty} \sigma_i(V_i)\right).} \tag{3.1}$$

Remark 3.1. This definition is the point of departure of our proof from the argument in §25 of [H1].

Definition 3.2. If $\{\mathcal{M}_i^n, g_i(0), O_i, F_i\}$ is a sequence of complete manifolds, its set of **ray-like directions** is

$$\mathcal{R}_\infty \doteqdot \sigma_\infty^{-1}(\infty).$$

In contrast with the sets $\sigma_i^{-1}(\infty)$, the set $\sigma_\infty^{-1}(\infty)$ will certainly be nonempty.

Lemma 3.3. *If* $\{\mathcal{M}_i^n, g_i(0), O_i, F_i\}$ *is any sequence such that*

$$\operatorname{diam}(\mathcal{M}_i^n, g_i(0)) \to \infty$$

as $i \to \infty$, *then* $\mathcal{R}_\infty$ *is nonempty.*

Proof It is a standard fact that each σ_i is a continuous function on the compact set $\mathcal{S}_1^{n-1}$. For each $i \in \mathbb{N}$, choose $V_i \in \mathcal{S}_1^{n-1}$ such that

$$\sigma_i(V_i) = \sup_{V \in \mathcal{S}_1^{n-1}} \sigma_i(V).$$

Then because $\operatorname{diam}(\mathcal{M}_i^n, g_i(0)) \to \infty$, we have

$$\lim_{i \to \infty} \sigma_i(V_i) = \infty.$$

A subsequence of V_i converges to some $V_\infty \in \mathcal{S}_1^{n-1}$. Clearly, $\sigma_\infty(V_\infty) = \infty$, and hence $V_\infty \in \mathcal{R}_\infty$. □

Lemma 3.4. *If* $\{\mathcal{M}_i^n, g_i(0), O_i, F_i\}$ *is any sequence such that*

$$\operatorname{diam}(\mathcal{M}_i^n, g_i(0)) \to \infty$$

as $i \to \infty$, *then* $\mathcal{R}_\infty$ *is compact.*

Proof Since $\mathcal{R}_\infty \subseteq \mathcal{S}_1^{n-1}$, it will suffice to show that $\mathcal{R}_\infty$ is closed. So let $\{V^\alpha : \alpha \in \mathbb{N}\}$ be a sequence from $\mathcal{R}_\infty$ such that $\lim_{\alpha\to\infty} V^\alpha = V^\infty \in \mathcal{S}_1^{n-1}$ exists. Then by definition of $\mathcal{R}_\infty$, there is for every α a sequence $\{V_i^\alpha : i \in \mathbb{N}\}$ of unit vectors such that

$$\lim_{i\to\infty} V_i^\alpha = V^\alpha \qquad \text{and} \qquad \lim_{i\to\infty} \sigma_i(V_i^\alpha) = \infty.$$

Observe that we can choose $i(\alpha) \geq \alpha$ for all α such that

$$\left| V_{i(\alpha)}^\alpha - V_\alpha \right|_{g_{\mathrm{can}}} < \frac{1}{\alpha} \qquad \text{and} \qquad \sigma_{i(\alpha)}\left(V_{i(\alpha)}^\alpha\right) > \alpha.$$

Then

$$\lim_{\alpha\to\infty} V_{i(\alpha)}^a = \lim_{\alpha\to\infty} V^\alpha = V^\infty$$

and

$$\lim_{\alpha\to\infty} \sigma_{i(\alpha)}\left(V_{i(\alpha)}^\alpha\right) = \infty.$$

Hence $V^\infty \in \mathcal{R}_\infty$. □

Remark 3.5. The reader may wish to contrast $\mathcal{R}_\infty$ with the set $\mathcal{D}$ of *distinguished directions* defined in [H1] as

$$\mathcal{D} \doteqdot \ell_\infty^{-1}(\infty),$$

where $\ell_\infty : \mathcal{S}_1^{n-1} \to [0,\infty]$ is defined by

$$\ell_\infty(V) \doteqdot \lim_{i\to\infty} \sigma_i(V_i).$$

It is claimed there that this limit is independent of the sequence $V_i \to V$. In contrast, our proof does not rely on such a property of independence of sequence.

Remark 3.6 (Tilting frames). Without the bump-like origins condition in Definition 2.2, it is not necessarily true that the distance with respect to $g_i(0)$ from O_i to the cut locus in a direction $V \in \mathcal{R}_\infty$ can be made arbitrarily large by going far enough out in the sequence. This is illustrated by the following example: for $i = 1, 2, \ldots$, let

$$\left(\mathcal{M}_i^2, g_i\right) = \mathcal{S}_{1/i}^1 \times \mathbb{R},$$

where

$$\mathcal{S}_{1/i}^1 = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1/i^2\right\}.$$

It is not important that the manifolds $\mathcal{M}_i$ are not compact, since once can replace the $\mathcal{M}_i$ by tori of various lengths. Regard each $\mathcal{M}_i$ as a submanifold of $\mathbb{R}^3$ with coordinates (x,y,z), and take the origins to be $O_i = (0, 1/i, 0)$. (Actually, any sequence of points in $\mathcal{M}_i$ will do.) Given any positive real number λ, define the tilted frame $\mathcal{F}_i = \left\{e_1^i, e_2^i\right\}$ at O_i by rotating the standard frame

$$\mathcal{F} = \{e_1 = (0,0,1), e_2 = (1,0,0)\}$$

clockwise by an angle of $\arctan(\pi/i\lambda)$. Using I_i to identify $\mathbb{R}^2$ with $T_{O_i}\mathcal{M}_i$, we have $I_i^{-1}\left(e_1^i\right) \equiv (1,0) \in \mathbb{R}^2$. Hence

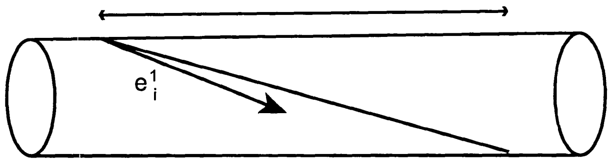
$$\sigma_i((1,0)) = \sigma_i\left(I_i^{-1}\left(e_1^i\right)\right) = \sqrt{\lambda^2 + \pi^2/i^2},$$

so that

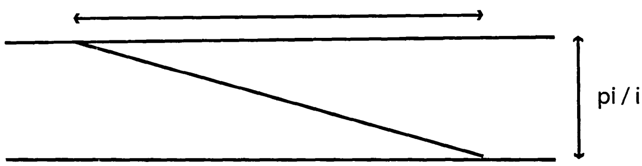
$$\lim_{i\to\infty} \sigma_i((1,0)) = \lambda.$$

On the other hand

$$\sigma_i\left(I_i^{-1}(e_1)\right) = \infty.$$



Tilted frames in collapsing cylinders: the geodesic minimizes up to approximately distance lambda.



The geodesic lifted to the universal cover.

Since $\lim_{i\to\infty} I_i^{-1}(e_1) = (1,0)$, this implies that

$$\sigma_\infty((1,0)) = \infty.$$

Observe in particular that

$$\lim_{i\to\infty} \sigma_i((1,0)) \neq \sigma_\infty((1,0)).$$

Remark 3.7. If $V \in \mathcal{R}_\infty$, it *is* true that we can find arbitrarily long minimizing geodesics in directions arbitrarily close to V. In particular, $V \in \mathcal{S}_1^{n-1}$ belongs to $\mathcal{R}_\infty$ if and only if there is a sequence $\{V_i\}$ from $\mathcal{S}_1^{n-1}$ such that $\lim_{i\to\infty} |V_i - V|_{g_{\text{can}}} = 0$ and $\lim_{i\to\infty} \sigma_i(V_i) = \infty$.

Remark 3.8. *A priori,* our definition allows $\mathcal{R}_\infty$ to become smaller each time we pass to a subsequence. We shall deal with this issue carefully in the sequel.

4. Finding large balls of positive curvature.

Our proof of Theorem 2.3 requires us to show that the curvature can be made positive in arbitrarily large neighborhoods of the origin by going sufficiently far out in the sequence. The key to this result is the strong maximum principle of [H6], which lets us pass from purely local results to results that hold on arbitrarily large sets. Because this part of our argument is essentially the same as the construction in §25 of [H1], we shall omit or merely outline most proofs.

4.1. Preconvergence in geodesic tubes.

To begin, we recall the procedure of taking limits of the pullback metrics in geodesic tubes, where one can avoid the need for an injectivity radius estimate for the sequence $\{\mathcal{M}_i^n, g_i(t), O_i, F_i\}$.

If $\gamma : (-L, L) \to (\mathcal{M}^n, g)$ is a unit-speed geodesic, we denote its normal bundle by

$$N_\gamma \doteqdot \{V \in \gamma^*(T\mathcal{M}) : \langle V, \dot\gamma \rangle = 0\}.$$

Note that N_γ is a rank $n-1$ vector bundle over $(-L, L)$. Define $\Phi : N_\gamma \to \mathcal{M}$ by

$$\Phi(\gamma(s)) \doteqdot \exp_{\gamma(s)}\Big|_{N_\gamma(\gamma(s))}.$$

Now let $F = \{e_1, \dots, e_n\}$ be any orthonormal frame at $\gamma(0)$ with $e_1 = \dot\gamma$. Taking the pullback of F and parallel translating it along γ, we obtain a orthonormal basis in each fiber of N_γ, which we continue to denote by $\{e_2, \dots, e_n\}$. Denoting the open ball of radius $r > 0$ centered at $\vec{0} \in \mathbb{R}^{n-1}$ by $B(\vec{0}, r)$, we define the cylinder $T_{L,r} \doteqdot (-L, L) \times B(\vec{0}, r)$ and a map $\iota_F : T_{L,r} \to N_\gamma$ by

$$\iota_F : (v^1, v^2, \dots, v^n) \mapsto \left(\gamma(v^1), \textstyle\sum_{k=2}^n v^k e_k(v_1)\right),$$

where $(v^1, v^2, \dots, v^n)$ are the natural Euclidean coordinates in $T_{L,r}$.

The following two results are well known; their proofs are virtually identical to those of corresponding results for exponential coordinates.

Lemma 4.1. *If $-1 \le \operatorname{sect}(g) \le 1$, there exists $\rho > 0$ depending only on n such that the map*

$$\Psi_{F,L,2\rho} \doteqdot \Phi \circ \iota_F : T_{L,2\rho} \to \mathcal{M}^n$$

is an immersion for every geodesic $\gamma : (-L, L) \to \mathcal{M}$, every orthonormal frame $F = \{\dot\gamma = e_1, e_2, \dots, e_n\}$ at $\gamma(0)$, and every $L > 0$.

Lemma 4.2. *Suppose $-1 \le \operatorname{sect}(g) \le 1$. If δ denotes the Euclidean metric on $T_{L,2\rho}$ and h is the pullback metric*

$$h \doteqdot \Psi^*_{F,L,2\rho} g,$$

then there exist constants $0 < c < C < \infty$ *depending only on* n *such that*

$$c\,\delta \leq h \leq C\,\delta. \tag{4.1}$$

Moreover, if there are constants C_k *such that*

$$\left|\nabla^k \operatorname{Rm}\right|_g \leq C_k$$

for all $k \in \mathbb{N}$, *then there exist* $C'_k = C'_k(C_0, \ldots, C_k)$ *such that*

$$\left|\frac{\partial^k}{\partial v^{\alpha_1} \cdots \partial v^{\alpha_k}} h\right|_\delta \leq C'_k. \tag{4.2}$$

Now let $\{\mathcal{M}_i^n, g_i(t), O_i, F_i\}$ be a sequence having uniformly bounded geometry. For each $A \in \mathrm{O}(n)$ and $L > 0$, there is a sequence $\left(T_{L,2\rho},\ \Psi^*_{AF_i,L,2\rho}(g_i(t))\right)$ of geodesic tubes extending in the direction determined by the frame AF_i obtained from the natural action of A on F_i. Let $\{A_\alpha : \alpha \in \mathbb{N}\}$ be a countable dense set in the compact Lie group $\mathrm{O}(n)$, and let $\{L_\beta : \beta \in \mathbb{N}\}$ be a sequence of positive real numbers with $L_\beta \nearrow \infty$ as $\beta \to \infty$.

Lemma 4.3. *There exists a subsequence* $\{\mathcal{M}_i^n, g_i(t), O_i, F_i\}$ *such that for all* $\alpha, \beta \in \mathbb{N}$, *the geodesic tube*

$$\left(T_{L_\beta,2\rho},\ \Psi^*_{A_\alpha F_i,L_\beta,2\rho}(g_i(t))\right)$$

converges as $i \to \infty$ *uniformly in each* C^k *norm to a solution*

$$\left(T_{L_\beta,2\rho},\ h^\infty_{A_\alpha,L_\beta,2\rho}(t)\right)$$

of the Ricci flow for $t \in (\alpha, \omega)$.

Proof By Lemma 4.2, $\Psi^*_{A_\alpha F_i,L_\beta,\rho}(g_i(t))$ is a sequence of solutions of the Ricci flow on $T_{L_\beta,\rho}$ such that the pullback metrics satisfy (4.1) and (4.2) uniformly in $i \in \mathbb{N}$ and $t \in (\alpha, \omega)$. Thus the result follows from Arzela–Ascoli by consecutive diagonalization arguments. □

Since $L_\beta \to \infty$, we have as an immediate consequence:

Corollary 4.4. *There exists a subsequence* $\{\mathcal{M}_i^n, g_i(t), O_i, F_i\}$ *such that for all* $\alpha \in \mathbb{N}$ *and all* $L > 0$, *the geodesic tube*

$$\left(T_{L,2\rho},\ \Psi^*_{A_\alpha F_i,L,2\rho}(g_i(t))\right)$$

converges as $i \to \infty$ uniformly in each C^k norm to a solution

$$\left(T_{L,2\rho},\ h^{\infty}_{A_\alpha,L,2\rho}(t)\right)$$

of the Ricci flow for $t \in (\alpha, \omega)$.

From this, one can obtain a subsequence that converges in any geodesic tube of any finite length.

Definition 4.5. A sequence $\{\mathcal{M}_i^n, g_i(t), O_i, F_i\}$ that has uniformly bounded geometry is said to be **preconvergent in geodesic tubes** if for all $A \in \mathrm{O}(n)$ and $L > 0$, the geodesic tube

$$\left(T_{L,\rho},\ \Psi^*_{AF_i,L,\rho}(g_i(t))\right)$$

converges as $i \to \infty$ uniformly in each C^k norm to a solution

$$\left(T_{L,\rho},\ h^{\infty}_{A,L,\rho}(t)\right)$$

of the Ricci flow for $t \in (\alpha, \omega)$.

Lemma 4.6. *Any sequence having uniformly bounded geometry contains a subsequence that is preconvergent in geodesic tubes.*

Proof Given $A \in \mathrm{O}(n)$ and $L > 0$, consider $\Psi_{AF_i,L,\rho} : T_{L,\rho} \to (\mathcal{M}_i, g_i(t))$. Choose a sequence A_α such that $A_\alpha \to A$ as $\alpha \to \infty$ in some (hence any) norm on $\mathrm{O}(n)$. By standard covering-space theory, there exists $\alpha' = \alpha'(L, \rho)$ independent of i such that for all $\alpha \geq \alpha'$ there exists a smooth embedding $\iota_{\alpha,L,\rho} : T_{L,\rho} \to T_{2L,2\rho}$ such that $\Psi_{A_\alpha F_i,2L,2\rho} \circ \iota_{\alpha,L,\rho} = \Psi_{AF_i,L,\rho}$. Note that all derivatives of $\iota_{\alpha,L,\rho}$ are bounded uniformly with respect to α. Note too that as $\alpha \to \infty$, the maps $\iota_{\alpha,L,\rho}$ converge uniformly in each C^k norm to the inclusion map $\iota_{L,\rho} : T_{L,\rho} \to T_{2L,2\rho}$ that is defined so that $\Psi_{AF_i,2L,2\rho} \circ \iota_{L,\rho} = \Psi_{AF_i,L,\rho}$. So as $\alpha \to \infty$, we have $\left(T_{L,\rho},\ \Psi^*_{A_\alpha F_i,L,\rho}(g_i(t))\right) \to \left(T_{L,\rho},\ \Psi^*_{AF_i,L,\rho}(g_i(t))\right)$ uniformly in each C^k norm. Since this convergence is independent of i, a routine diagonalization argument completes the proof. □

Remark 4.7. The solutions $\left(T_{L,\rho},\ h^{\infty}_{A,L,\rho}(t)\right)$ obtained by this construction are not complete.

4.2. Preconvergence in distance.

The distance function in each geodesic tube gives an upper bound for the distance function in the original geometry. This fact ensures that any sequence that is preconvergent in geodesic tubes contains a subsequence that is *preconvergent in distance,* namely a subsequence such that the limit

$$d_\infty (X,Y) \doteqdot \lim_{i\to\infty} \left[d_i \left(\exp_{O_i} (I_i (X)), \exp_{O_i} (I_i (Y))\right)\right]$$

exists for all $X, Y \in \mathbb{R}$, where d_i denotes the distance function induced on the manifold $\mathcal{M}_i^n$ by the Riemannian metric $g_i(0)$. Preconvergence in distance is a stability property; it ensures, for instance, that the distance to the cut locus in a particular direction is not going to infinity along one subsequence while remaining uniformly bounded along another subsequence. However, the methods we develop in § 6 — in particular, covering $\mathcal{R}_\infty$ by a finite ε-net of directions for which the lim sup in Definition (3.1) is attained as a limit — make it unnecessary to use this property. Consequently, we omit the proof.

4.3. Preconvergence to positive curvature.

Notation 4.8. If $x \in \mathcal{M}_i^n$, we denote the open $g_i(0)$-ball of radius $r > 0$ centered at x by

$$B_i (x,r) \doteqdot \{y \in \mathcal{M}_i^n : d_i (x,y) < r\}$$

and the corresponding closed ball by

$$\bar{B}_i (x,r) \doteqdot \{y \in \mathcal{M}_i^n : d_i (x,y) \le r\}.$$

We denote by d_i the distance function induced on $\mathcal{M}_i^n$ by the metric $g_i(0)$.

Definition 4.9. We say a sequence $\{\mathcal{M}_i^n, g_i(t), O_i, F_i\}$ with almost nonnegative curvature operators is **preconverging to positive curvature** if for each $L > 0$ there are some $\eta(L) > 0$ and $\iota(L) \in \mathbb{N}$ such that

$$\lambda_1 (\mathrm{Rm}_i) (x,0) \ge \eta$$

for all $i \ge \iota$ and all $x \in \bar{B}_i (O_i, L)$.

Proposition 4.10. *Any sequence with almost nonnegative curvature operators contains a subsequence that is preconverging to positive curvature.*

Proof Since the result is equivalent to Lemma 25.2 of [H1], we shall merely sketch the proof. By Lemma 4.6, we first pass to a subsequence that is preconvergent in geodesic tubes. If the proposition is false, there is some $L \in (0,\infty)$ such that for all $\eta > 0$ and each $i_0 \in \mathbb{N}$, we have $\lambda_1(\mathrm{Rm}_i)(x,0) < \eta$ for some $i \geq i_0$ and some $x \in \bar{B}_i(O_i, L)$. One can then argue that there exists a further subsequence such that

$$\lambda_1(\mathrm{Rm}_i)\left(\exp_{O_i}(V_i), 0\right) \to 0$$

as $i \to \infty$, where $I_i^{-1}(V_i)$ converges to some $V_\infty \in \mathbb{R}^n$ with $0 < |V_\infty| \leq L$. Now preconvergence in geodesic tubes ensures that the subsequence to which we have passed has the property that

$$\left(T_{2L,\rho},\ \Psi^*_{AF_i,2L,\rho}(g_i(t))\right) \to \left(T_{2L,\rho},\ h^\infty_{A,2L,\rho}(t)\right),$$

where $A \in \mathrm{O}(n)$ is chosen such that the geodesic tube lies in the direction $V_\infty / |V_\infty|$. It follows that

$$\lambda_1\left(\mathrm{Rm}\left(h^\infty_{A,2L,\rho}(0)\right)\right)\left(|V_\infty|, \vec{0}\right) = 0.$$

Because assumption (1) implies that $\mathrm{Rm}\left(h^\infty_{A,2L,\rho}(0)\right) \geq 0$, we may then apply the strong maximum principle in the form proved in [H6] to conclude that

$$\lambda_1\left(\mathrm{Rm}\left(h^\infty_{A,2L,\rho}(0)\right)\right)\left(0, \vec{0}\right) = 0.$$

But this is possible only if $\lambda_1\left(\mathrm{Rm}\left(\Psi^*_{AF_i,2L,\rho}(g_i(0))\right)\right)\left(0,\vec{0}\right) \to 0$ as $i \to \infty$, which contradicts assumption (3), because

$$\lambda_1\left(\mathrm{Rm}\left(\Psi^*_{AF_i,2L,\rho}(g_i(0))\right)\right)\left(0,\vec{0}\right) = \lambda_1(\mathrm{Rm}_i)(O_i, 0) \geq \varepsilon > 0.$$

The contradiction proves the proposition. □

5. Mimicking the sublevel sets of a Busemann function.

If $(\mathcal{M}^n, g)$ is a complete noncompact manifold of positive sectional curvature bounded above by $\kappa < \infty$, then Gromoll and Meyer [GM] proved that its injectivity radius can be bounded below by $\pi/\sqrt{\kappa}$. One way to prove this is to fix an origin $O \in \mathcal{M}^n$, use the rays emanating from O to construct a Busemann function associated to that origin, use that Busemann function to construct a totally convex neighborhood N of O, and then use a second variation argument along rays to rule out short geodesics in the neighborhood N. (See Greene [G] for a survey of noncompact manifolds with nonnegative curvature.)

Following Hamilton, we want to mimic this construction along a sequence $\{\mathcal{M}_i^n, g_i(t), O_i, F_i\}$ that is preconverging to positive curvature. Since we may only have ray-like directions in general, we need a substitute for the Busemann construction.

Notation 5.1. We shall henceforth identify $V \in \mathbb{R}^n$ with $I_i(V) \in T_{O_i}\mathcal{M}_i$ and vice versa, using the canonical isometries

$$I_i : (\mathbb{R}^n, g_{\text{can}}) \to (T_{O_i}\mathcal{M}_i, g(O_i, 0)).$$

As a substitute for the sublevel sets of a Busemann function, we define

(5.1)

$$N_i(L,K) \doteqdot \left\{ \exp_{O_i}(tW) \;\middle|\; \begin{array}{c} W \in \mathcal{S}_1^{n-1} \text{ and } t \in [0, \sigma_i(W)] \setminus \{\infty\} \text{ are such that} \\ \text{for every } V \in \mathcal{S}_1^{n-1} \text{ with } \sigma_i(V) \geq L, \\ \text{all } r \in [0, \sigma_i(V)] \setminus \{\infty\}, \text{ and all } s \in [0,t], \text{ we have} \\ d_i\left(\exp_{O_i}(rV), \exp_{O_i}(sW)\right) \geq r - K \end{array} \right\}.$$

The corresponding sets in §25 of [H1] are constructed using those V for which $\ell_\infty(V) = \infty$. But for our proof, it is important to allow V such that $\sigma_i(V)$ is large but finite.

Notice that each $N_i(L,K)$ is weakly star shaped with respect to O_i: namely, $\exp_{O_i}(tW) \in N_i(L,K)$ implies that $\exp_{O_i}(sW) \in N_i(L,K)$ for all $0 \leq s \leq t$. It will be useful to collect a few more elementary observations about the sets $N_i(L,K)$.

Lemma 5.2. *$N_i(L,K)$ contains the closed ball of radius* $\min\{\pi, K\}$*:*

$$\bar{B}_i(O_i, \min\{\pi, K\}) \subseteq N_i(L,K).$$

Proof By assumption (1), there are no conjugate points of O_i in $B_i(O_i, \pi)$. So if $x \in \bar{B}_i(O_i, \min\{\pi, K\})$, there exist $W \in \mathcal{S}_1^{n-1}$ and $t \in [0, \sigma_i(W)]$ such that $x = \exp_{O_i}(tW)$. Then for any $V \in \mathcal{S}_1^{n-1}$ with $\sigma_i(V) \geq L$, all $r \in [0, \sigma_i(V)] \setminus \{\infty\}$, and all $s \in [0, t]$, the triangle inequality yields

$$\begin{aligned} r &= d_i\left(O_i, \exp_{O_i}(rV)\right) \\ &\leq d_i\left(O_i, \exp_{O_i}(sW)\right) + d_i\left(\exp_{O_i}(sW), \exp_{O_i}(rV)\right) \\ &= s + d_i\left(\exp_{O_i}(sW), \exp_{O_i}(rV)\right) \\ &\leq K + d_i\left(\exp_{O_i}(sW), \exp_{O_i}(rV)\right). \end{aligned}$$

The inequality on the last line holds because $s \leq t = d_i(O_i, x) \leq \min\{\pi, K\}$. □

Lemma 5.3. *If $\exp_{O_i}(tW) \in N_i(L, K)$, then for all $V \in \mathcal{S}_1^{n-1}$ such that $\sigma_i(V) \geq L$, all $r \in [0, \sigma_i(V)] \setminus \{\infty\}$, and all $s \in [0, t]$, we have*

$$d_i\left(O_i, \exp_{O_i}(sW)\right) \leq K + 2 \cdot d_i\left(\exp_{O_i}(sW), \exp_{O_i}(rV)\right).$$

Proof By the triangle inequality,

$$d_i\left(O_i, \exp_{O_i}(rV)\right) \geq d_i\left(O_i, \exp_{O_i}(sW)\right) - d_i\left(\exp_{O_i}(rV), \exp_{O_i}(sW)\right).$$

So because $\exp_{O_i}(tW) \in N_i(L, K)$, we have

$$\begin{aligned} &d_i\left(O_i, \exp_{O_i} sW\right) - d_i\left(\exp_{O_i}(rV), \exp_{O_i}(sW)\right) \\ &\leq d_i\left(O_i, \exp_{O_i}(rV)\right) = r \\ &\leq K + d_i\left(\exp_{O_i}(rV), \exp_{O_i}(sW)\right), \end{aligned}$$

and hence $d_i\left(O_i, \exp_{O_i}(sW)\right) \leq K + 2 \cdot d_i\left(\exp_{O_i}(rV), \exp_{O_i}(sW)\right)$. □

Corollary 5.4. *$N_i(L, K)$ is contained in the closed ball of radius $\max\{L, K\}$:*

$$N_i(L, K) \subseteq \bar{B}_i(O_i, \max\{L, K\}).$$

Proof If $\exp_{O_i}(tW) \in N_i(L, K)$ and $d_i\left(O_i, \exp_{O_i}(tW)\right) > L$, apply Lemma 5.3 with $V = W$ and $r = s = t$ to get $d_i\left(O_i, \exp_{O_i}(tW)\right) \leq K$. □

Lemma 5.5. *Each $N_i(L, K)$ is compact.*

Proof By Corollary 5.4, it suffices to show that each $N_i(L, K)$ is closed. Let

$$\left\{x_\alpha = \exp_{O_i}(t_\alpha W_\alpha) : \alpha \in \mathbb{N}\right\} \subset N_i(L, K)$$

be a sequence such that $\lim_{\alpha\to\infty} x_\alpha \doteqdot x \in \mathcal{M}_i^n$ exists. Then

$$\lim_{\alpha\to\infty} t_\alpha \doteqdot t \in [0, \max\{L, K\}]$$

and

$$\lim_{\alpha\to\infty} W_\alpha \doteqdot W \in \mathcal{S}_1^{n-1}$$

exist also. Since $t_\alpha \in [0, \sigma_i(W_\alpha)]$ for all α, the continuity of σ_i implies that $t \in [0, \sigma_i(W)]$.

Now let $V \in \mathcal{S}_1^{n-1}$ with $\sigma_i(V) > L$, $r \in [0, \sigma_i(V)] \setminus \{\infty\}$, and $s \in [0, t]$ be given. Choose $s_\alpha \in [0, t_\alpha]$ such that $\lim_{\alpha\to\infty} s_\alpha = s$. Then for any $\varepsilon > 0$, there exists $A < \infty$ such that for all $\alpha \geq A$ we have

$$\left| d_i\left(\exp_{O_i}(rV), \exp_{O_i}(sW)\right) - d_i\left(\exp_{O_i}(rV), \exp_{O_i}(s_\alpha W_\alpha)\right)\right| < \varepsilon.$$

Since $\exp_{O_i}(t_\alpha W_\alpha) \in N_i(L, K)$ for each α, this implies that

$$d_i\left(\exp_{O_i}(rV), \exp_{O_i}(sW)\right) \geq r - K - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we see that $x = \exp_{O_i}(tW) \in N_i(L, K)$. □

6. The boundedness property.

What is not obvious is the fact that the $N_i(L, K)$ can be uniformly bounded. That is the content of the following crucial result:

Proposition 6.1 (Boundedness property). *Any sequence preconverging to positive curvature contains a subsequence for which there exists a constant $C < \infty$ depending on K such that for each $L \in (0, \infty)$, there exists $I(L)$ such that for all $i \geq I(L)$, we have*

$$N_i(L, K) \subseteq B_i(O_i, C).$$

The proof of the proposition has two main steps. The first step is to observe that by passing to a subsequence, we can in a sense replace $\mathcal{R}_\infty$ by a finite ε-net of directions for which the lim sup in Definition (3.1) is attained as a limit. We have already observed in Remark 3.8 that $\mathcal{R}_\infty$ can become smaller each time we pass to a subsequence. But the special subsequence we are about to construct has the property that the finitely many directions composing an ε-net in $\mathcal{R}_\infty$ are stable under the action of passing to further subsequences.

Lemma 6.2. *Let $\varepsilon > 0$ be given, and let $\{\mathcal{M}_i^n, g_i(t), O_i, F_i : i \in \mathbb{N}\}$ be a sequence that is preconverging to positive curvature. Then there exists a finite set of directions from $\mathcal{R}_\infty$, say*

$$\left\{V^\alpha \in \mathcal{S}_1^{n-1} : \alpha = 1, \ldots, A\right\} \subseteq \mathcal{R}_\infty,$$

such that

$$\mathcal{R}_\infty \subseteq \bigcup_{\alpha=1}^{A} B_{g_{\mathrm{can}}}(V^\alpha, \varepsilon).$$

And there exists for each $a = 1, \ldots, A$, a particular sequence $\left\{V_j^\alpha : j \in \mathbb{N}\right\}$ with $\lim_{j\to\infty} V_j^\alpha = V^\alpha$ such that along a subsequence

$$\left\{\mathcal{M}_{i_j}^n, g_{i_j}(0), O_{i_j}, F_{i_j} : j \in \mathbb{N}\right\},$$

we have

$$\lim_{j\to\infty} \sigma_{i_j}\left(V_j^\alpha\right) = \sigma_\infty(V^\alpha) = \infty$$

for all $\alpha = 1, \ldots, A$.

Proof Denote by $\mathcal{R}_\infty^0$ the set of ray-like directions corresponding to the original sequence

$$\mathfrak{M}_0 \doteqdot \{\mathcal{M}_i^n, g_i(t), O_i, F_i : i \in \mathbb{N}\}$$

Then $\mathcal{R}_\infty^0 \neq \emptyset$ by Lemma 3.3. Choose $V^1 \in \mathcal{R}_\infty^0$, and pass to a subsequence

$$\mathfrak{M}_1 \doteqdot \left\{\mathcal{M}_{i(1,j)}^n, g_{i(1,j)}(t), O_{i(1,j)}, F_{i(1,j)} : j \in \mathbb{N}\right\}$$

along which $\sigma_\infty\left(V^1\right) = \infty$ is attained as the limit

$$\lim_{i(1,j)\to\infty} \sigma_{i(1,j)} V_j^1 = \sigma_\infty\left(V^1\right) = \infty$$

for some sequence $V_j^1 \to V^1$.

Denote by $\mathcal{R}_\infty^1 \subseteq \mathcal{R}_\infty^0$ the set of ray-like directions corresponding to the subsequence $\mathfrak{M}_1$. Then $\mathcal{R}_\infty^1 \neq \emptyset$ by Lemma 3.3. If

$$\mathcal{R}_\infty^1 \subseteq B_{g_{\mathrm{can}}}\left(V^1, \varepsilon\right)$$

stop. Otherwise choose

$$V^2 \in \mathcal{R}_\infty^1 \backslash B_{g_{\mathrm{can}}}\left(V^1, \varepsilon\right),$$

and pass to a subsequence

$$\mathfrak{M}_2 \doteqdot \left\{\mathcal{M}^n_{i(2,j)}, g_{i(2,j)}(t), O_{i(2,j)}, F_{i(2,j)} : j \in \mathbb{N}\right\}$$

along which $\sigma_\infty\left(V^2\right) = \infty$ is attained as the limit

$$\lim_{i(2,j)\to\infty} \sigma_{i(2,j)} V^2_j = \sigma_\infty\left(V^2\right) = \infty$$

for some sequence $V^2_j \to V^2$.

In general, denote by $\mathcal{R}^\alpha_\infty \subseteq \mathcal{R}^{\alpha-1}_\infty \subseteq \cdots \subseteq \mathcal{R}^0_\infty$ the set of ray-like directions corresponding to the subsequence $\mathfrak{M}_\alpha$. Then $\mathcal{R}^\alpha_\infty \neq \emptyset$ by Lemma 3.3. If

$$\mathcal{R}^\alpha_\infty \subseteq \bigcup_{\beta=1}^{\alpha} B_{g_{\mathrm{can}}}\left(V^\beta, \varepsilon\right)$$

stop. Otherwise choose

$$V^{\alpha+1} \in \mathcal{R}^\alpha_\infty \Bigg\backslash \bigcup_{\beta=1}^{\alpha} B_{g_{\mathrm{can}}}\left(V^\beta, \varepsilon\right),$$

and pass to a subsequence

$$\mathfrak{M}_{\alpha+1} \doteqdot \left\{\mathcal{M}^n_{i(\alpha+1,j)}, g_{i(\alpha+1,j)}(t), O_{i(\alpha+1,j)}, F_{i(\alpha+1,j)} : j \in \mathbb{N}\right\}$$

along which $\sigma_\infty\left(V^{\alpha+1}\right) = \infty$ is attained as the limit

$$\lim_{i(\alpha+1,j)\to\infty} \sigma_{i(\alpha+1,j)} V^{\alpha+1}_j = \sigma_\infty\left(V^{\alpha+1}\right) = \infty$$

for some sequence $V^{\alpha+1}_j \to V^{\alpha+1}$. Since each $\mathcal{R}^\alpha_\infty$ is contained in $\mathcal{R}^0_\infty \subseteq \mathcal{S}^{n-1}_1$ and $\mathcal{S}^{n-1}_1$ is compact, this process must eventually terminate. □

Notation 6.3. Henceforth we shall denote the subsequence whose existence is ensured by Lemma 6.2 simply by

$$\{\mathcal{M}^n_i, g_i(t), O_i, F_i : i \in \mathbb{N}\}.$$

To facilitate the final step of the proof of Proposition 6.1, we fix a length scale Λ at which to compare distance. To motivate our choice, consider an isosceles triangle Δ that is symmetric about an angle $\theta \leq \theta_0 < \pi/3$. If Δ is embedded in Euclidean space and has side lengths k, ℓ, ℓ, then $k \leq$

$\ell\sqrt{2(1-\cos\theta_0)}$. In particular, given $K \in (0,\infty)$, we choose Λ depending only on K to be large enough that

$$\Lambda > \frac{2K}{1-2\sqrt{2\left(1-\cos\frac{\pi}{8}\right)}} > 0. \tag{6.1}$$

Combined with the simple estimate in Lemma 5.3, this somewhat non-intuitive choice will let us argue to a contradiction below.

Proof [Proof of the boundedness property] Suppose the statement is false. Then for every $C_j \equiv j \in \mathbb{N}$, there exists some $L_j \in (0,\infty)$ such that for every $I(j) \equiv I(L_j)$, there exists some $i(j) \geq I(j)$ such that

$$N_{i(j)}(L_j, K) \not\subseteq B_{i(j)}\left(O_{i(j)}, j\right).$$

(Recall that $B_i(x,r)$ denotes the open ball with center $x \in \mathcal{M}_i^n$ and radius r, measured with respect to the metric $g_i(0)$.) In particular, there exists for each j some $W_j \in \mathcal{S}_1^{n-1}$ such that

$$d_{i(j)}\left(O_{i(j)}, \exp_{O_{i(j)}}\left(jW_{i(j)}\right)\right) = j \tag{6.2}$$

and

$$\exp_{O_{i(j)}}(jW_j) \in N_{i(j)}(L_j, K). \tag{6.3}$$

Notice that $W_j \in \mathbb{R}^n$ is being identified with

$$I_{i(j)}(W_j) \in T_{O_{i(j)}}\mathcal{M}^n_{i(j)}.$$

We may assume without loss of generality that

$$L_{j+1} \geq L_j + 1$$

for all $j \in \mathbb{N}$, since if $L_* \leq L^*$ then

$$N_i(L_*, K) \subseteq N_i(L^*, K)$$

for all $i \in \mathbb{N}$ and $K > 0$.

We now show how to choose the $I(j)$. Let $\varepsilon \in (0, \pi/24)$ be given. Then by Lemma 6.2, there exists a finite set of ray-like directions

$$\{V^\alpha : \alpha = 1, \ldots, A\} \subseteq \mathcal{R}_\infty$$

such that

$$\mathcal{R}_\infty \subseteq \bigcup_{\alpha=1}^{A} B_{g_{\mathrm{can}}}\left(V^\alpha, \varepsilon\right);$$

and there exist particular sequences $\{V_i^\alpha\}$ such that for each $\alpha = 1, \ldots, A$ we have

$$\lim_{i\to\infty} V_i^\alpha = V^\alpha \tag{6.4}$$

and

$$\lim_{i\to\infty} \sigma_i\left(V_i^\alpha\right) = \infty. \tag{6.5}$$

For each $j \in \mathbb{N}$, first choose $I'(j)$ so large that if $i \geq I'(j)$, we have

$$\sigma_i\left(V_i^\alpha\right) \geq L_j$$

for all $a = 1, \ldots, A$; then choose $I(j) \geq I'(j)$ so large that if $i \geq I(j)$, we have

$$\operatorname{sect}\left[g_i(x,0)\right] \geq 0$$

for all $x \in \bar{B}_i\left(O_i, 3L_j\right)$. This is possible by Proposition 4.10.

The construction just completed yields a subsequence

$$\left\{\mathcal{M}^n_{i(j)}, g_{i(j)}(0), O_{i(j)}, F_{i(j)} : j \in \mathbb{N}\right\}$$

along which (6.2) and (6.3) are satisfied for each $i(j) \geq I(j)$. We next pass from this to a subsequence

$$\left\{\mathcal{M}^n_{i(j(k))}, g_{i(j(k))}(0), O_{i(j(k))}, F_{i(j(k))} : k \in \mathbb{N}\right\}$$

such that

$$\lim_{k\to\infty} W_{j(k)} \doteqdot W_\infty \in \mathcal{S}_1^{n-1}$$

exists. Denote by $\mathcal{R}'_\infty$ the set of ray-like directions for this subsequence. Then it may be that $\mathcal{R}'_\infty \subsetneq \mathcal{R}_\infty$. But by (6.4) and (6.5), we still have

$$\{V^\alpha : \alpha = 1, \ldots, A\} \subseteq \mathcal{R}'_\infty,$$

and

$$\mathcal{R}'_\infty \subseteq \bigcup_{\alpha=1}^{A} B_{g_{\mathrm{can}}}\left(V^\alpha, \varepsilon\right);$$

moreover, for each $\alpha = 1, \ldots, A$, we also have

$$\lim_{k\to\infty} V^{\alpha}_{i(j(k))} = V^{\alpha}$$

and

$$\lim_{k\to\infty} \sigma_{i(j(k))}\left(V^{\alpha}_{i(j(k))}\right) = \infty.$$

Observe that $W_\infty \in \mathcal{R}'_\infty$, since we have

$$\sigma_{i(j(k))}\left(W_{j(k)}\right) \geq j(k) \to \infty \qquad \text{as} \qquad k \to \infty$$

by (6.2). In particular, there exists some $\alpha \in 1, \ldots, A$ such that

$$\left|W_\infty - V^{\alpha}\right|_{g_{\mathrm{can}}} < \varepsilon.$$

To finish the proof, choose k so large that

$$L_{j(k)} \geq \Lambda$$

and that

$$\left|V^{\alpha}_{i(j(k))} - V^{\alpha}\right|_{g_{\mathrm{can}}} < \varepsilon$$

and that

$$\sigma_{i(j(k))}\left(W_{j(k)}\right) \geq j(k) \geq \Lambda$$

and that

$$\left|W_{j(k)} - W_\infty\right|_{g_{\mathrm{can}}} < \varepsilon.$$

Then we have

$$\left|W_{j(k)} - V^{\alpha}_{i(j(k))}\right|_{g_{\mathrm{can}}} < 3\varepsilon.$$

Since

$$\exp_{O_{i(j(k))}}\left(j(k) \cdot W_{j(k)}\right) \in N_{i(j(k))}\left(L_{j(k)}, K\right)$$

and

$$\sigma_{i(j(k))}\left(V^{\alpha}_{i(j(k))}\right) \geq L_{j(k)} \geq \Lambda,$$

we may apply Lemma 5.3 with $V = V^{\alpha}_{i(j(k))}$ and $r = s = \Lambda$ to obtain the estimate

$$\begin{aligned}\Lambda &= d_{i(j(k))}\left(O_{i(j(k))}, \exp_{O_{i(j(k))}}\left(\Lambda W_{j(k)}\right)\right)\\ &\leq K + 2 \cdot d_{i(j(k))}\left(\exp_{O_{i(j(k))}}\left(\Lambda W_{j(k)}\right), \exp_{O_{i(j(k))}}\left(\Lambda V^{\alpha}_{i(j(k))}\right)\right).\end{aligned}$$

But since

$$d_{i(j(k))}\left(\exp_{O_{i(j(k))}}\left(\Lambda W_{j(k)}\right), \exp_{O_{i(j(k))}}\left(\Lambda V^{\alpha}_{i(j(k))}\right)\right) \leq 2\Lambda \leq 2L_{j(k)},$$

any minimizing geodesic between

$$\exp_{O_{i(j(k))}}\left(\Lambda W_{j(k)}\right) \quad \text{and} \quad \exp_{O_{i(j(k))}}\left(\Lambda V^{\alpha}_{i(j(k))}\right)$$

must lie in $\bar{B}_{i(j(k))}\left(O_{i(j(k))}, 3L_{j(k)}\right)$, where the sectional curvature is nonnegative. Hence the hinge version of the Toponogov comparison theorem (Theorem 2.2 (B) of [CE]) gives the estimate

$$\begin{aligned} &d_{i(j(k))}\left(\exp_{O_{i(j(k))}}\left(\Lambda W_{j(k)}\right), \exp_{O_{i(j(k))}}\left(\Lambda V^{\alpha}_{i(j(k))}\right)\right) \\ &< \Lambda\sqrt{2\left(1-\cos\left(3\varepsilon\right)\right)} \leq \Lambda\sqrt{2\left(1-\cos\frac{\pi}{8}\right)}. \end{aligned}$$

Combining these two estimates yields $\Lambda \leq K + 2\Lambda\sqrt{2\left(1-\cos\pi/8\right)}$, hence

$$0 < \Lambda \leq \frac{K}{1-\sqrt{8\left(1-\cos\frac{\pi}{8}\right)}} < \frac{1}{2}\Lambda$$

by the choice we made in (6.1). This contradiction establishes the proposition. □

7. Proof of the injectivity radius estimate.

The remainder of our proof of Theorem 2.3 proceeds exactly like the analogous part of §25 of [H1]. But because the argument here uses our innovations (the sets $N_i\left(L, K\right)$, for example) in an essential way, we shall give it in detail. To prepare for this, we introduce some notation and recall an important fact.

Let $(\mathcal{M}^n, g)$ be any Riemannian manifold.

Definition 7.1. If $k \in \{1, 2, \dots\}$, a **proper geodesic** k**-gon** is a collection

$$\Gamma = \{\gamma_i : [0, \ell_i] \to \mathcal{M} : i = 1, \dots, k\}$$

of unit-speed geodesic paths between k pairwise distinct vertices $p_i \in \mathcal{M}$ such that $p_i = \gamma_i\left(0\right) = \gamma_{i-1}\left(\ell_{i-1}\right)$ for each i, where all indices are interpreted modulo k. The **length** of a proper geodesic k-gon is $L\left(\Gamma\right) \doteqdot \sum_{i=1}^{k} L\left(\gamma_i\right)$. We say Γ is a **nondegenerate proper geodesic k-gon** if $\measuredangle_{p_i}(-\dot\gamma_{i-1};\dot\gamma_i) \neq 0$ for each $i = 1,\dots,k$; if $k = 1$, we interpret this to mean $L(\Gamma) > 0$. Finally, a **(nondegenerate) geodesic k-gon** is a (nondegenerate) proper geodesic j-gon for some $j = 1,\dots,k$.

Now let $N \subset \mathcal{M}^n$ be a nonempty subset, and let Ω denote the space of unit-speed nondegenerate geodesic 1-gons contained in N. Let $L : \Omega \to [0,\infty)$ denote the length function, and define $A : \Omega \to \mathcal{S}^{n-1}\mathcal{M}\big|_N \times [0,\infty)$ for all unit-speed nondegenerate geodesic 1-gons α by $A(\alpha) \doteqdot (\dot\alpha(0), L(\alpha))$. The map A is injective and induces a topology on Ω from the topology on $\mathcal{S}^{n-1}\mathcal{M}\big|_N \times [0,\infty)$. If N is compact, the set $L^{-1}[0,K] \subseteq \Omega$ is compact for every $K \in (0,\infty)$. If K is large enough so that $L^{-1}[0,K]$ is nonempty, then there exists a nondegenerate geodesic 1-gon $\beta \in L^{-1}[0,K] \subset \Omega$ of minimal length. Clearly, β is of minimal length among all nondegenerate geodesic 1-gons contained in N; in particular, we have $L(\beta) = \inf_{\alpha \in L^{-1}[0,K]} L(\alpha) = \inf_{\alpha\in\Omega} L(\alpha)$.

Proof [Proof of Theorem 2.3] Pass to a subsequence $\{\mathcal{M}_i^n, g_i(t), O_i, F_i : i \in \mathbb{N}\}$ that is preconverging to positive curvature and has the boundedness property guaranteed by Proposition 6.1. Then there exists $C < \infty$ such that for any $L > 2$ to be chosen later, there exists $I'(L)$ such that

$$N_i(L,1) \subseteq B_i(O_i, C)$$

for all $i \geq I'(L)$. By Proposition 4.10, there exist $I(L) \geq I'(L)$ and $\eta > 0$ such that for all $i \geq I(L)$, we have

$$\inf\left\{\operatorname{sect}(g_i(x,0)) : x \in \bar B_i(O_i, C+2)\right\} \geq \eta. \tag{7.1}$$

Suppose the theorem is false. Then there exists i_0 such that $\operatorname{inj}_{g_i(0)}(O_i) < 1$ for all $i \geq i_0$. So there exists for each $i \geq i_0$ a nondegenerate geodesic 2-gon α_i based at O_i and of length < 2, hence contained in $B_i(O_i,1)$. By Lemma 5.2, $\bar B_i(O_i,1) \subseteq N_i(L,1)$. So by a standard shortening argument, there exists for each $i \geq i_0$ a nondegenerate geodesic 1-gon $\tilde\alpha_i$ based at O_i, contained in $N_i(L,1)$, and such that $\operatorname{length}_{g_i(0)} \tilde\alpha_i < \operatorname{length}_{g_i(0)} \alpha_i < 2$. By Lemma 5.5, $N_i(L,1)$ is compact. So there exists for each $i \geq i_0$ a shortest element β_i in the set of all nondegenerate geodesic 1-gons contained in $N_i(L,1)$. Each β_i is smooth except perhaps at its base $\beta_i(0) = \beta_i(\ell_i)$, where

$$\ell_i \doteqdot \operatorname{length}_{g_i(0)} \beta_i \leq \operatorname{length}_{g_i(0)} \tilde\alpha_i < 2.$$

We first consider the (easier) case that there exists a subsequence for which β_i is smooth at $\beta_i(0) = \beta_i(\ell_i)$. By Lemma 3.3, the set $\mathcal{R}_\infty$ for this subsequence is nonempty. Hence definition (3.1) implies that for every L and every $J \in \mathbb{N}$, there exists some $i(I(L), J) \geq \max\{I(L), J\}$ and some $V_i \in \mathcal{S}_1^{n-1}$ such that $\sigma_i(V_i) \geq L$. Let

$$y_i \doteqdot \exp_{O_i}(LV_i),$$

and define

$$S_i \doteqdot d_i(y_i, \beta_i).$$

Since $\beta_i \subset N_i(L,1)$ is compact, there are $W_i \in \mathcal{S}_1^{n-1}$ and $t_i \in [0, \sigma_i(W_i)] \setminus \{\infty\}$ such that

$$x_i \doteqdot \exp_{O_i}(t_i W_i) \in \beta_i$$

is the point on β_i closest to y_i, so that $d_i(x_i, y_y) = S_i$. Since $x_i \in N_i(L,1)$, the definition of $N_i(L,1)$ implies that

$$S_i = d_i(x_i, y_i) \geq L - 1 > 1.$$

Let γ_i be a minimal unit-speed geodesic from x_i to y_i. Note in particular that $\text{length}_{g_i(0)} \gamma_i = S_i > 1$ and that

$$\gamma_i|_{[0,1]} \subset B_i(O_i, C+1). \tag{7.2}$$

Since β_i is smooth, we can apply the first variation formula to conclude that $\dot{\beta}_i \perp \dot{\gamma}_i$ at x_i, where $\dot{\beta}_i, \dot{\gamma}_i$ denote the unit tangent vectors of β_i, γ_i respectively. Let X_i be the unit vector field that results from parallel translation of $\dot{\beta}_i$ along γ_i from x_i, and define the cutoff function

$$f_i(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq 1 \\ (S_i - s)/(S_i - 1) & \text{if } 1 < s \leq S_i \end{cases}.$$

Then the minimality of γ_i implies that the second-variation index form $\mathcal{I}$ in the direction $f_i X_i$ is nonnegative:

$$0 \leq \mathcal{I} \equiv \mathcal{I}(f_i X_i, f_i X_i) \doteqdot \int_{\gamma_i} \left(|\nabla_{\dot{\gamma}_i}(f_i X_i)|^2 - \langle R(\dot{\gamma}_i, f_i X_i)(f_i X_i), \dot{\gamma}_i \rangle \right) ds.$$

But (7.1) and (7.2) imply that all sectional curvatures are bounded below by $\eta > 0$ along $\gamma_i|_{[0,1]}$. And assumption (1) implies that all sectional curvatures are bounded below by $-\delta_i \nearrow 0$ throughout $\mathcal{M}_i^n$. Hence we can estimate

$$\begin{aligned}\mathcal{I} &= -\int_0^1 \langle R(\dot\gamma_i, X_i) X_i, \dot\gamma_i\rangle \, ds + \int_1^{S_i} \left((df_i(\dot\gamma_i))^2 - f_i^2 \langle R(\dot\gamma_i, X_i) X_i, \dot\gamma_i\rangle \right) ds \\ &\leq -\eta + \frac{1}{S_i - 1} + \delta_i \frac{S_i - 1}{3}.\end{aligned}$$

Now choose L so large that $S_i \geq L - 1$ satisfies $1/(S_i - 1) \leq \eta/3$. Then choose J so large that for all $i \geq J$, we have $\delta_i (S - 1) \leq \eta$. Thus for $i = i(I(L), J)$, we get $\mathcal{I} \leq -\eta/3 < 0$. This contradicts the minimality of γ_i and proves the theorem in this case.

Now we consider the case that there exists $i_1 \geq i_0$ such that β_i fails to be smooth at $\beta_i(0) = \beta_i(\ell_i)$ for all $i \geq i_1$. It is a standard fact that for any complete Riemannian manifold $(\mathcal{M}^n, g)$ with sectional curvatures bounded above by $\kappa > 0$, any points $p, q \in \mathcal{M}^n$, any geodesic path γ from p to q of length less than $\pi/\sqrt{\kappa}$, and any points $\tilde p, \tilde q$ sufficiently near p, q respectively, there exists a unique geodesic $\tilde\gamma$ from $\tilde p$ to $\tilde q$ that is close to γ. Consider a variation moving $\beta_i(\ell_i)$ in the direction $\dot\beta_i(0)$ and observe that the first variation in this direction is strictly negative: $\left\langle \dot\beta_i(\ell_i), \dot\beta_i(0)\right\rangle - \left\langle \dot\beta_i(0), \dot\beta_i(0)\right\rangle < 0$. Now we have $\operatorname{sect}(g_i(0)) \leq 1$ by assumption (1), and $\operatorname{length}_{g_i(0)} \beta_i = \ell_i < 1$ by hypothesis. It follows that there exists a nondegenerate geodesic 1-gon $\tilde\beta_i$ with

$$\operatorname{length}_{g_i(0)} \tilde\beta_i < \operatorname{length}_{g_i(0)} \beta_i,$$

and such that

$$\tilde\beta_i(0) \in \beta_i \subset N_i(L, 1)$$

and

$$\tilde\beta_i \subset B_i(O_i, C + 1).$$

$\tilde\beta_i$ may not be smooth at its base $\tilde\beta_i(0)$ either, but it is smooth everywhere else.

By our choice of β_i, it must be that $\tilde\beta_i$ does not lie entirely in $N_i(L, 1)$. Hence there must exist a point z_i on $\tilde\beta_i$ but not in $N_i(L, 1)$. Choose $W_i \in \mathcal{S}_1^{n-1}$ and $t_i \in [0, \sigma_i(W_i)] \setminus \{\infty\}$ such that

$$z_i \doteqdot \exp_{O_i}(t_i W_i) \in \tilde\beta_i \backslash N_i(L, 1).$$

By definition of $N_i(L, 1)$, there exist some $V_i \in \mathcal{S}_1^{n-1}$ with $\sigma_i(V_i) \geq L$, some $r_i \in [0, \sigma_i(V_i)] \setminus \{\infty\}$, and some $s_i \in [0, t_i]$ such that

$$d_i\left(\exp_{O_i}(r_i V_i), \exp_{O_i}(s_i W_i)\right) < r_i - 1. \tag{7.3}$$

Define

$$y_i \doteqdot \exp_{O_i}(LV_i),$$

and let ζ_i denote the geodesic

$$\zeta_i : [0, t_i] \to \mathcal{M}_i^n, \qquad \zeta_i : \tau \mapsto \exp_{O_i}(\tau W_i).$$

We claim that $z_i = \zeta_i(t_i)$ is the point on ζ_i closest to y_i. To see this, first note that the closest point is not $O_i = \zeta_i(0)$, since (7.3) implies that

$$\begin{aligned} d_i(y_i, \zeta_i(s_i)) &\le d_i\left(y_i, \exp_{O_i}(r_i V_i)\right) + d_i\left(\exp_{O_i}(r_i V_i), \zeta_i(s_i)\right) \\ &< (L - r_i) + (r_i - 1) = d_i(y_i, O_i) - 1. \end{aligned} \tag{7.4}$$

If the closest point is an interior point, say $\zeta_i(\tau_i)$ for some $\tau_i \in (0, t_i)$, let ξ_i be a minimal geodesic from $\zeta_i(\tau_i)$ to y_i. Because $z_i \in \tilde{\beta}_i \subset B_i(O_i, C+1)$, and all sectional curvatures are bounded below by $\eta > 0$ in $B_i(O_i, C+2)$, a second variation argument (like the one above) along ξ_i will yield a contraction. This proves that the closest point to y_i on ζ_i cannot be an interior point. Hence the only possibility is that the closest point to y_i along ζ_i is its other endpoint $z_i = \zeta_i(t_i)$. This proves the claim. (Note that applying this argument along segments proves the stronger fact that the function $\tau \mapsto d_i(y_i, \zeta_i(\tau))$ is monotone decreasing for $\tau \in [0, t_i]$.)

By the claim and (7.4), we have

$$d_i(y_i, z_i) \le d_i(y_i, \zeta_i(s_i)) < L - 1.$$

But since $\tilde{\beta}_i(0) \in N_i(L, 1)$, we have

$$d_i\left(y_i, \tilde{\beta}_i(0)\right) \ge L - 1.$$

Hence the closest point to y_i on $\tilde{\beta}_i$ is not its base $\tilde{\beta}_i(0)$. In particular, $\tilde{\beta}_i$ is smooth at its closest point to y_i. So we can construct a length-minimizing geodesic $\tilde{\gamma}_i$ from $\tilde{\beta}_i$ to y_i and apply a second variation argument along it, exactly as in the first case. Because $\tilde{\gamma}_i|_{[0,1]} \subset B_i(O_i, C+2)$, where the sectional curvatures are bounded from below by $\eta > 0$, this argument leads to a contradiction, just as before. This finishes the proof. □

References.

[CE] J. Cheeger and D. Ebin, *Comparison theorems in Riemannian geometry*, North–Holland, Amsterdam, and American Elsevier, New York, 1975.

[CGT] J. Cheeger, M. Gromov, and M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds,* J. Differential Geom. **17** (1982) 15–53.

[CT] X.X. Chen and G. Tian, *Ricci flow on Kähler-Einstein surfaces,* Invent. math. **147** (2002) 487–544.

[CLY] S.-Y. Cheng, P. Li, S.-T. Yau, *On the upper estimate of the heat kernel of a complete Riemannian manifold,* Amer. J. Math. **103** (1981), no. 5, 1021–1063.

[G] R. E. Greene, "A genealogy of noncompact manifolds of nonnegative curvature: history and logic," in *Comparison Geometry,* ed. Grove and Petersen, MSRI Publ. **30** (1997) 99–134.

[GM] D. Gromoll and W. Meyer, *On complete manifolds of positive curvature,* Ann. of Math. **90** (1969) 75–90.

[H1] R. S. Hamilton, "The formation of singularities in the Ricci flow," in *Surveys in Differential Geometry* **2** (1995) International Press, 7–136.

[H2] R. S. Hamilton, *Four-manifolds with positive isotropic curvature,* Comm. Anal. Geom. **5** (1997) 1–92.

[H3] R. S. Hamilton, *Nonsingular solutions of the Ricci Flow on three-manifolds,* Comm. Anal. Geom. **7** (1999) 695–729.

[H4] R. S. Hamilton, *A compactness property for solutions of the Ricci flow,* Amer. J. Math. **117** (1995) 545–572.

[H5] R. S. Hamilton, *The Harnack estimate for the Ricci flow,* J. Differential Geom. **37** (1993) 225–243.

[H6] R. S. Hamilton, *Four-manifolds with positive curvature operator,* J. Differential Geom. **24** (1986) 153–179.

[LY] P. Li and S. T. Yau, *On the parabolic kernel of the Shrödinger operator,* Acta Math. **156** (1986) 153–201.

[MM] M. Micallef and J. D. Moore, *Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes,* Ann. of Math. (2) **127** (1988) 199–227.

[P] P. Petersen, "Convergence theorems in Riemannian geometry," in *Comparison Geometry,* ed. Grove and Petersen, MSRI Publ. **30** (1997) 167–202.

[S] W. X. Shi, *Deforming the metric on complete Riemannian manifolds,* J. Differential Geom. **30** (1989) 223–301; and *Ricci deformation of the metric on complete noncompact Riemannian manifolds,* J. Differential Geom. **30** (1989) 303–394.

[T] W. P. Thurston, *Three-dimensional manifolds, Kleinian groups, and hyperbolic geometry,* Bull. Amer. Math. Soc. **6** (1982) 357–381.

Bennett Chow
University of California, San Diego

Dan Knopf
University of Wisconsin

Peng Lu
McMaster University

Received September 17, 2001
Revised May 1, 2002.