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Transformation Groups of Holomorphic Foliations

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We prove that the self–bimeromorphisms group of a foliation of general type on a projective surface is finite. Along the proof we study the structure of arbitrary codimension foliations on projective varieties invariant by an infinite linear algebraic group.

1. Introduction.

A classical Theorem due to Schwarz says that the group of automorphisms of a compact Riemann surface with genus at least two is finite. Andreotti, in [1], generalized Schwarz's Theorem proving that the group of self bimeromorphisms of an algebraic variety of general type is finite.

In this paper we prove a similar statement for holomorphic foliations on projective surfaces. More precisely,

Theorem 1. *If $\mathcal{F}$ is a holomorphic foliation of general type on a projective surface then* $\mathrm{Bim}(\mathcal{F})$ *is finite.*

We proceed in two steps. First we investigate the structure of arbitrary codimension holomorphic foliations admitting *many* automorphisms. In this direction we obtain:

Theorem 2. *Let $\mathcal{F}$ be a codimension q holomorphic foliation on a projective variety M^m. Suppose that* $\mathrm{Aut}(\mathcal{F})$ *contains an infinite linear algebraic group. Then $\mathcal{F}$ belongs to one of the following classes:*

1. *$\mathcal{F}$ has codimension one and is birationally equivalent to a Riccati foliation;*
2. *there exists a projective variety N and a rational map(possibly with indeterminacy points) $\pi : M \dashrightarrow N$ whose fibers are rational curves and such that $\mathcal{F}$ is the pull–back of a holomorphic foliation $\mathcal{G}$ on N;*
3. *$\mathcal{F}$ has codimension at least 2 and is tangent to a holomorphic foliation $\mathcal{G}$ of codimension $q-1$.*

Recall that a foliation $\mathcal{F}$ on a projective surface M is called Riccati if there exists a rational fibration on M such that $\mathcal{F}$ is transverse to the generic fiber of the fibration. In item 1 of the theorem above we consider a natural generalization of this concept for codimension one foliations on projective varieties. A codimension one foliation $\mathcal{F}$ on a projective variety M is a Riccati foliation if there exists a rational fibration on M whose generic fiber is transversal to $\mathcal{F}$.

Next we use Brunella's minimal model and pluricanonical maps to reduce the study of $\mathrm{Bim}(\mathcal{F})$ to the study of closed subgroups of $\mathrm{Aut}(\mathbb{P}^k_{\mathbb{C}})$. We remark that at this point our proof mimics Matsumura's proof of Andreotti's Theorem, see [7] and [10].

The paper is organized as follows. In section 2 we recall the concepts of Kodaira dimension and minimal models for holomorphic foliations and state some results that will be necessary through the paper. Section 3 contains some basic facts about the group of automorphisms of holomorphic foliations and the proof of Theorem 2. Section 4 is devoted to the pluricanonical maps associated to foliations of general type. In the final section we prove Theorem 1.

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2. Bimeromorphic Theory of Foliations.

2.1. Kodaira Dimension.

A *holomorphic foliation* $\mathcal{F}$ on a compact complex surface S is given by an open covering $\{U_i\}$ and holomorphic vector fields X_i over each U_i such that whenever the intersection of U_i and U_j is non–empty there exists an invertible holomorphic function g_{ij} satisfying $X_i = g_{ij}X_j$. The collection $\{(g_{ij})^{-1}\}$ defines the holomorphic line–bundle $T\mathcal{F}$, called the *tangent bundle* of $\mathcal{F}$. The dual of $T\mathcal{F}$ is the *cotangent bundle* $T^*\mathcal{F}$, also called the *canonical bundle* $K_{\mathcal{F}}$.

Recall that a *reduced foliation* $\mathcal{F}$ is a foliation such that every singularity p is reduced in Seidenberg's sense, i.e., for every vector field X generating $\mathcal{F}$ and every singular point p of X, the eigenvalues of the linear part of X are not both zero and their quotient, when defined, is not a positive rational number.

Definition 1. Let $\mathcal{F}$ be a foliation on the complex surface S, and $\mathcal{G}$ any reduced foliation bimeromorphically equivalent to $\mathcal{F}$. The *Kodaira dimension* of $\mathcal{F}$ is given by

$$\mathrm{kod}(\mathcal{F}) = \limsup_{n\to\infty} \frac{\log h^0(S, K_{\mathcal{G}}^{\otimes n})}{\log n}.$$

It can proved that the Kodaira dimension is well defined and is a bimeromorphic invariant of $\mathcal{F}$, see [6].

The concept of Kodaira dimension for holomorphic foliations have been introduced independently by L. G. Mendes and M. McQuillan. For more information on the subject see [2], [6] and [8].

When the foliation has Kodaira dimension 2 we say that the foliation is of *general type*. This terminology is justified by the classification of the foliations with Kodaira dimension smaller than two. We summarize the classification in table 1, for more details see [8] and [2].

$\mathrm{kod}(\mathcal{F})$	**Description**
$-\infty$	Rational fibration
	Hilbert modular foliation
0	up to ramified coverings and birational morphisms $\mathcal{F}$ is generated by a global holomorphic vector field.
1	Riccati foliation
	Turbulent foliation
	Nonisotrivial elliptic fibration
	Isotrivial fibration of genus ≥ 2
2	General type

Table 1: *Classification of holomorphic foliations on algebraic surfaces*

Recall that a foliation $\mathcal{F}$ on a surface M is a *Riccati* (resp. *turbulent*) foliation, if there exists a rational (resp. elliptic) fibration on M, whose generic fiber is transverse to $\mathcal{F}$.

2.2. Minimal Models.

Brunella, in [3], introduced the concept of minimal model for a holomorphic foliation. This can be understood as the foliated analogue of Zariski's minimal models for algebraic surfaces.

In order to define a minimal model for a holomorphic foliation $\mathcal{F}$, Brunella first introduces the concept of relatively minimal foliation and then when the relatively minimal model is unique(modulo biholomorphisms) he says that it is a minimal model.

It is proved in [3] that the following definition is equivalent to the one sketched above.

Definition 2. *Let $\mathcal{F}$ be a reduced holomorphic foliation on a projective surface S. We say that $\mathcal{F}$ is minimal if, and only if, for any reduced foliation $\mathcal{G}$ on a projective surface M and a bimeromorphic map $\phi : M \dashrightarrow S$ which sends $\mathcal{G}$ to $\mathcal{F}$ is in fact a morphism.*

The foliations that do not admit a minimal model are described, in a very precise way, by the following Theorem due to Brunella.

Theorem 3. *Let $\mathcal{F}$ be a holomorphic foliation on a projective surface S without minimal model. Then $\mathcal{F}$ is bimeromorphically equivalent to a foliation in the following list:*

1. *rational fibrations ;*
2. *nontrivial Riccati foliations ;*
3. *the very special foliation $\mathcal{H}$ described in page 291 of [3].*

Since all the foliations on the Theorem above have Kodaira dimension at most one, we obtain the following.

Corollary 1. *Let $\mathcal{F}$ be a holomorphic foliation of general type on the projective surface S. Then there exists a unique minimal model $\mathcal{G}$ of $\mathcal{F}$ and* $\mathrm{Bim}(\mathcal{F}) \cong \mathrm{Aut}(\mathcal{G})$.

3. Automorphisms of Holomorphic Foliations.

Definition 3. Let $\mathcal{F}$ be a holomorphic foliation on a complex manifold M. The *automorphism group of* $\mathcal{F}$, $\mathrm{Aut}(\mathcal{F})$, is the maximal subgroup of $\mathrm{Aut}(M)$ that preserves $\mathcal{F}$. The *self bimeromorphism group of* $\mathcal{F}$, $\mathrm{Bim}(\mathcal{F})$, is the maximal subgroup of $\mathrm{Bim}(M)$ that preserves $\mathcal{F}$.

In the definition above $\mathrm{Aut}(M)$ denotes the group of biholomorphisms and $\mathrm{Bim}(M)$ denotes the group of self bimeromorphisms of the complex manifold M. A well–known result, due to Bochner–Montgomery (see [5] page 76), says that if M is compact complex manifold then $\mathrm{Aut}(M)$ is a complex Lie transformation group and its Lie algebra consists of global holomorphic vector fields on M.

Proposition 1. *Let $\mathcal{F}$ be a codimension p holomorphic foliation on a compact complex manifold M. Then* $\mathrm{Aut}(\mathcal{F})$ *is a closed Lie subgroup of* $\mathrm{Aut}(M)$.

proof. Take p meromorphic 1–forms $\omega_1, \ldots, \omega_p$ defining $\mathcal{F}$. More precisely, $\omega_1, \ldots, \omega_p$ defines a field of p-planes outside the zero set of $\Omega = \omega_1 \wedge \ldots \wedge \omega_p$. Since

$$\mathrm{Aut}(\mathcal{F}) = \{g \in \mathrm{Aut}(M) | g^* \omega_i \wedge \Omega = 0, i = 1, 2, \ldots, p\}$$

the proposition follows. ∎

Remark 1. Observe that in general Proposition 1 does not imply that $\mathrm{Aut}(\mathcal{F})$ has a finite number of connect components, even if the manifold is projective. This is due to the fact that the automorphism group of a projective manifold can have an infinite number of connected components.

Let $\mathcal{F}$ be a codimension one foliation and X a holomorphic vector field. We will say that X is *transverse* to $\mathcal{F}$ when the generic orbit of X is not contained in any leaf of the foliation. When X is transverse to $\mathcal{F}$ the *tangency locus* of $\mathcal{F}$ and X is the subvariety locally defined by $\omega(X)$, where ω is any holomorphic 1–form locally defining $\mathcal{F}$.

Proposition 2. *Let $\mathcal{F}$ be codimension one holomorphic foliation on a compact complex manifold M. Let X be a holomorphic vector field that belongs to the Lie algebra of* $\mathrm{Aut}(\mathcal{F})$ *and is transverse to $\mathcal{F}$. Then the tangency locus of $\mathcal{F}$ and X is invariant by $\mathcal{F}$, i.e., there exists a finite number of leaves of $\mathcal{F}$ whose closure coincides with the tangency locus of $\mathcal{F}$ and X.*

proof. Let $\{U_i\}$ be an open covering of M and suppose that $\mathcal{F}_{|U_i}$ is defined by $\omega_i = 0$. Here the 1-forms ω_i are integrable and satisfy the relation $\omega_i = f_{ij}\omega_j$, where $f_{ij} \in \mathcal{O}^*(U_i \cap U_j)$. Since X is in the Lie algebra of $\mathrm{Aut}(\mathcal{F})$ we have that

$$L_X(\omega_i) \wedge \omega_i = 0\,,$$

where $L_X := di_X + i_X d$ is the *Lie derivative* . Therefore

$$d\omega_i(X) \wedge \omega_i + \iota_X(d\omega_i) \wedge \omega_i = 0\,.$$

By the integrability of ω_i we obtain

$$\omega_i(X) d\omega_i + (\iota_X d\omega_i) \wedge \omega_i = 0\,.$$

From this last equality we derive that

$$\omega_i \wedge d\omega_i(X) = \omega_i(X) d\omega_i\,, \tag{3.1}$$

thus $\omega_i(X)$ is invariant by ω_i. This is sufficient to assure that the tangency locus of $\mathcal{F}$ and X is invariant by $\mathcal{F}$. ∎

Remark 2. Observe that when X admits a codimension one zero set than the proposition above show that this set is contained on the closure of a finite numbers of leaves of $\mathcal{F}$.

Corollary 2. *Let $\mathcal{F}$ be codimension one holomorphic foliation on a compact complex manifold M. Let X be a holomorphic vector field that belongs to the Lie algebra of* $\mathrm{Aut}(\mathcal{F})$ *and is transverse to $\mathcal{F}$. Then there exists a closed meromorphic* 1*-form defining $\mathcal{F}$.*

proof: From formula (3.1) one can deduce, as in [4] page 35–36, that $\mathcal{F}$ is defined by a closed meromorphic 1–form defined over all M. In fact

$$\frac{\omega_i}{\omega_i(X)} = \frac{\omega_j}{\omega_j(X)}$$

whenever $U_i \cap U_j \neq \emptyset$ and

$$d\left(\frac{\omega_i}{\omega_i(X)}\right) = \frac{\omega_i \wedge d\omega_i(X) - \omega_i(X)d\omega_i}{\omega_i(X)^2} = 0\,.$$

∎

Proof of Theorem 2. Let $G \subset \mathrm{Aut}(\mathcal{F})$ be an infinite linear algebraic group. Since it is infinite it has a non–trivial Lie algebra. Take a global holomorphic vector field X on the Lie algebra of G. If we denote by G_X the 1–parameter subgroup of $\mathrm{Aut}(\mathcal{F})$ induced by X, then its Zariski closure $\overline{G}_X$ will be a closed commutative subgroup of $G \subset \mathrm{Aut}(\mathcal{F})$. Being $\overline{G}_X$ commutative we can find a closed one–parameter subgroup H, i.e., a one–dimensional linear algebraic subgroup of G. Denote by Y an element on the Lie algebra of $\mathrm{Aut}(\mathcal{F})$ which generates H.

Theorem 10 of [9] says that M/H is a quasiprojective variety of dimension $m-1$ and that M is birationally equivalent to $M/H \times \mathbb{C}P(1)$. Hence the morphism

$$\pi : M \to \frac{M}{H}\,,$$

induces a 1–dimensional foliation on M, tangent to Y, such that the closure of every leaf is a rational curve. Since the indeterminacies of π are contained in the singularities of Y, after resolving them we obtain a projective variety together with a global holomorphic vector field, which is tangent to the 1–dimensional rational fibration induced by the resolution of π. Hence we can suppose without loss of generality that π is a fibration.

Suppose that the generic fiber of π is contained in a leaf of $\mathcal{F}$. Let $\sigma : \frac{M}{H} \to M$ be a section of π. Define $\mathcal{G}$ as the pull–back of $\mathcal{F}$ under σ, i.e., $\mathcal{G} \cong \sigma^*(\mathcal{F})$. Hence $\mathcal{F} \cong \pi^*(\mathcal{G}) \cong \pi^*(\sigma^*(\mathcal{F}))$ and $\mathcal{F}$ is in case 2 of the statement.

If the generic fiber of π is not contained in a leaf of $\mathcal{F}$ and the codimension of $\mathcal{F}$ is at least 2, we proceed as follows. For every $p \in M$ regular point of $\mathcal{F}$ we have a neighborhood where $\mathcal{F}$ is generated by a system of $(m-q)$ involutive vector fields, namely, $X_1, X_2, \ldots, X_{m-q}$. Consider now the system of $(m-(q-1))$ vector fields, $X_1, \ldots, X_{m-q}, Y$. Since Y preserves the leaves of $\mathcal{F}$, see figure 1, we have that

$$[X_i, Y] = \sum_{i=1}^{m-q} \lambda_i \cdot X_i\,,$$

for some holomorphic functions λ_i. Hence this system is involutive and defines a holomorphic foliation $\mathcal{G}$ of codimension $q-1$ which contains $\mathcal{F}$. Hence $\mathcal{F}$ is in the case 3 of the statement.

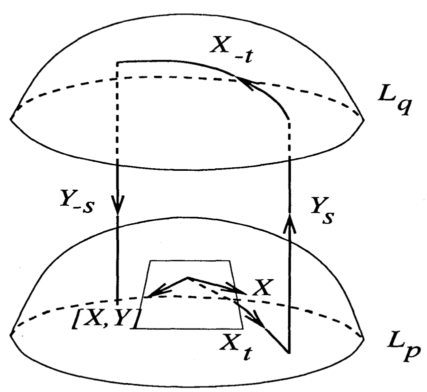

Figure 1: Case 3: the Lie bracket of X_i and Y.

When the generic fiber of π is not contained in a leaf of $\mathcal{F}$ and $\mathcal{F}$ has codimension one follows from Proposition 2 that the tangency locus between $\mathcal{F}$ and π is composed by fibers of π and is invariant by $\mathcal{F}$. In other words $\mathcal{F}$ is a Riccati foliation with respect to π and it is in the case 1 of the statement.

∎

4. Pluricanonical maps.

When $\mathcal{F}$ is a reduced foliation of general type on a surface M we have for a sufficiently large m that the map

$$\begin{array}{rcl}\phi_m : M & \to & \mathbb{C}P(k) \\ p & \mapsto & (s_0(p) : \cdots : s_k(p))\end{array}$$

is a bimeromorphism between M and the closure of the image of ϕ_m, see [10] page 57. Here s_i are sections of $K_{\mathcal{F}}{}^{\otimes m}$ and $k = h^0(M, K_{\mathcal{G}}{}^{\otimes m}) - 1$. The map ϕ_m will be called the *m–th pluricanonical map of $\mathcal{F}$*.

Proposition 3. *Let $\mathcal{F}$ be a holomorphic foliation of general type on the projective surface M. Then* $\mathrm{Bim}(\mathcal{F})$ *is isomorphic to a linear algebraic group.*

proof: By Corollary 1 we can suppose that $\mathcal{F}$ is a minimal foliation and in this case $\mathrm{Bim}(\mathcal{F}) \cong \mathrm{Aut}(\mathcal{F})$. Thus, for a sufficiently large integer m, the m-th pluricanonical map ϕ_m is a bimeromorphism between M and the closure of its image, which we will denote by N.

Observe that $\mathrm{Aut}(\mathcal{F})$ acts naturally on the projectivization of $H^0(M, K_{\mathcal{F}}^{\otimes m})$. If σ is a section of $K_{\mathcal{F}}^{\otimes m}$ and α is an automorphism of $\mathcal{F}$ then the action is given by $\alpha(\sigma) = \alpha^*\sigma$.

Being ϕ_m a bimeromorphism between M and N, the action above induces a monomorphism of groups

$$\psi : \mathrm{Aut}(\mathcal{F}) \to PSL(k, \mathbb{C}),$$

where $k = \dim_{\mathbb{C}} H^0(M, K_{\mathcal{F}}^{\otimes m})$.

Since the image of ψ is precisely the automorphisms of $\mathbb{C}P(k)$ leaving N and $\mathcal{G}$ invariant, we can conclude that $\mathrm{Bim}(\mathcal{F}) \cong \mathrm{Aut}(\mathcal{F})$ is a closed linear algebraic subgroup of $PSL(k, \mathbb{C})$. ∎

5. Proof of Theorem 1.

We can suppose that $\mathcal{F}$ is a minimal foliation and proposition 3 implies that $\mathrm{Bim}(\mathcal{F}) \cong \mathrm{Aut}(\mathcal{F})$ is a linear algebraic group.

Assume, by contradiction, that $\mathrm{Bim}(\mathcal{F})$ is infinite. By Theorem 2 we have that $\mathcal{F}$ is a Riccati foliation or a fibration by rational curves. In the case $\mathcal{F}$ is a Ricatti foliation then $\mathrm{kod}(\mathcal{F}) \leq 1$ and when $\mathcal{F}$ is a rational fibration then $\mathrm{kod}(\mathcal{F}) = -\infty$, see for instance Theorem 3.3.1 in [6]. Since $\mathcal{F}$ is of general type, i.e., $\mathrm{kod}(\mathcal{F}) = 2$, we obtain a contradiction and conclude that $\mathrm{Bim}(\mathcal{F})$ is finite. ∎

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