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Stability of the Ricci flow at Ricci-flat metrics

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If g is a metric whose Ricci flow $g(t)$ converges, one may ask if the same is true for metrics $\tilde{g}$ that are small perturbations of g. We use maximal regularity theory and center manifold analysis to study flat and Ricci-flat metrics. We show that if g is flat, there is a unique exponentially-attractive center manifold at g consisting entirely of equilibria for the flow. Adding a continuity argument, we prove stability for any metric whose Ricci flow converges to a flat metric. We obtain a slightly weaker stability result for a Kähler–Einstein metric on a $K3$ manifold.

1. Introduction.

Since the introduction of the Ricci flow

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Rc}, \qquad g(0) = g_0, \tag{1}$$

as a useful tool [H2] for the study of relationships between manifolds and the Riemannian geometries they admit, there has been considerable progress in our understanding of the behavior of geometries deformed by the Ricci flow. (See for instance [H5], [H6], and the survey [CC].) However, some basic questions of nonlinear analysis concerning this behavior are to date unresolved. One of these is the question of stability of converging Ricci flows. In particular, let g_0 be a geometry whose Ricci flow $g(t)$ converges. Is it true that the Ricci flow $\tilde{g}(t)$ converges for all geometries $\tilde{g}_0$ that are sufficiently close to g_0 in some appropriate topology?

The work of Ye answers this question affirmatively [Ye] if g_0 is a metric of constant nonzero sectional curvature and if one replaces the Ricci flow by the volume-normalized Ricci flow

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Rc} + \frac{2}{n}\left(\oint R\,d\mu\right) g, \qquad g(0) = g_0. \tag{2}$$

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(Here and throughout this paper, we denote the average of a scalar function f on a compact manifold by $\oint f\, d\mu \doteqdot \int f\, d\mu / \int d\mu$.) Among other results, that work shows that for any sufficiently Riemann-pinched Einstein metric g_0 of nonzero scalar curvature, there is a C^2 neighborhood $\mathcal{N}_{g_0}$ of g_0 such that the Ricci flow $\tilde{g}(t)$ of any $\tilde{g}_0 \in \mathcal{N}_{g_0}$ converges to g_0.

Left undetermined by Ye's work is the stability of Ricci flow convergence for metrics near a flat geometry, or more generally, near an Einstein metric g_0 of vanishing scalar curvature. A key feature of such geometries is the existence of zero eigenvalues for the linearization of the flow, regarded as a differential operator on symmetric $(2,0)$-tensors. Note that Ye's result requires a positive spectrum for the operator

$$L[h]_{ij} \doteqdot -\Delta h_{ij} - 2R_{pij}^{q} h_q^p + \left(R_i^q - \frac{1}{n} R\, \delta_i^q \right) h_{qj}.$$

A zero eigenvalue signals the presence of a nontrivial center manifold in the space of metrics near g_0, with corresponding complications in the analysis of the flow of nearby metrics.

The maximal regularity theory developed by Da Prato and Grisvard [DG] and notably applied to quasilinear parabolic reaction-diffusion systems by Simonett [S] enables one to establish stability, long-time existence, and convergence of dynamical flows with nontrivial center manifolds present. We use these methods to prove a convergence stability theorem for the Ricci flow of metrics near a flat geometry. To the best of our knowledge, this is the first time that center-manifold analysis has been applied to the Ricci flow. A secondary purpose of our investigation, therefore, has been to explore how effective such techniques may be for studying this geometric evolution problem. It should be noted that linear stability and instability analysis has been successfully used to study the curve shortening problem; see [AL] and especially [EW].

As detailed in Theorem 3.1, our main result says that for all metrics $\tilde{g}_0$ in a little-Hölder $\|\cdot\|_{2+\rho}$ neighborhood $\mathcal{N}_{g_0}$ of a flat metric g_0 on a torus $\mathcal{T}^n$, the Ricci flow $\tilde{g}(t)$ converges exponentially fast in the $\|\cdot\|_{2+\rho}$ norm to a flat metric $\tilde{g}_\infty$. The limit metric $\tilde{g}_\infty$ is generally not g_0; however, the set of flat metrics forms a $n(n+1)/2$-dimensional submanifold of the space of all metrics on $\mathcal{T}^n$, and the intersection of this submanifold with the neighborhood $\mathcal{N}_g$ comprises the center manifold for the Ricci flow dynamical system near g_0. Although center manifolds in dynamical systems are in general not unique, it is a remarkable consequence of this analysis that the center manifold at a flat metric is unique, consisting entirely of equilibria.

If one has determined the stability of Ricci flow convergence for metrics near a specified flat metric g_0 (Theorem 3.1), then it is relatively straightforward to show that it is stable for metrics near some h_0 whose Ricci flow $h(t)$ converges to g_0 (Corollary 3.7). The basis for this argument is finite-time continuity of the flow, which implies that for any neighborhood $\mathcal{N}_{g_0}$ of g_0, there exists a neighborhood $\mathcal{N}_{h_0}$ of h_0 such that if $k_0 \in \mathcal{N}_{h_0}$, then the Ricci flow $k(t)$ enters $\mathcal{N}_{g_0}$ in finite time. Combining this with stability about g_0, one verifies the stability of Ricci flow convergence about h_0. Applying this result, one can show Ricci flow convergence to a flat metric for any initial metric k_0 sufficiently close to a product geometry on $\mathcal{T}^2 \times \mathcal{S}^1$ ([H4] and §11 of [H5]) or sufficiently close to a polarized Gowdy metric [CIJ].

As noted above, Ye's studies show stability of Ricci flow convergence for Riemann-pinched Einstein metrics of nonzero scalar curvature, but his results leave the case of zero scalar curvature unresolved. In three dimensions, g is Einstein and has vanishing scalar curvature if and only if g is flat, in which case Theorem 3.1 establishes stability. In dimension four and above, there are nonflat Ricci-flat metrics, so we may hope to find further cases for which we can attempt to show stability of Ricci flow convergence using the techniques discussed here. In §4, we discuss such a case: we consider Kähler–Einstein metrics on $K3$ complex surfaces. These are geometries on a certain manifold $\mathcal{M}^4$ of four real dimensions; they are Ricci-flat and therefore fixed points of the Ricci flow, but are not flat. In Theorem 4.3, we show that for any Kähler–Einstein metric g_0 on a $K3$, there is a $\|\cdot\|_{2+\rho}$ neighborhood $\mathcal{N}_{g_0}$ of g_0 in the space of all metrics on $\mathcal{M}^4$ such that the DeTurck flow $\tilde{g}(t)$ of any initial metric $\tilde{g}_0 \in \mathcal{N}_{g_0}$ exponentially approaches a 58-dimensional center manifold containing g_0, for as long as $\tilde{g}(t)$ remains in $\mathcal{N}_{g_0}$. (The DeTurck flow is equivalent by diffeomorphisms to the Ricci flow; see the next paragraph for an introduction and §2.1 for a precise statement of this equivalence.) Note that one result of Cao's paper [C] is that every initially Kähler metric on a $K3$ converges under the Ricci flow to a Ricci-flat Kähler metric. This makes it natural to conjecture that the Ricci flow of any initial metric in $\mathcal{N}_{g_0}$ converges to a unique limit metric in the 58-dimensional space of Ricci-flat Kähler metrics known to exist on a $K3$ surface. Our results thus far support but do not yet prove this conjecture.

While the heart of the proof of Ricci flow convergence stability for both the flat and $K3$ Kähler–Einstein metrics is maximal regularity analysis, a preliminary step is needed in each case. The Ricci flow PDE system is not itself strictly parabolic; it is thus not a system to which one can directly apply the methods of Simonett. However, one can work with an alternative flow whose PDE system is strictly parabolic, and whose solutions are related to solutions of the Ricci flow by a 1-parameter family of diffeomorphisms. In §2, we review this alternative flow (sometimes called the 'DeTurck flow'), establish some notation, discuss the function spaces we will use, and provide a very brief introduction to some ideas of maximal regularity.

Our main result for flat metrics is stated and proved in §3. The proof involves essentially five steps:

1. Compute the linearization of the DeTurck flow and analyze its spectrum at a given Ricci-flat metric g_0.

2. Verify certain characteristics and properties of the flow that are necessary for the application of maximal regularity results.

3. Show the existence of C^r center manifolds and describe their tangent space at the fixed point g_0.

4. In the flat case, prove there is a unique smooth center manifold present.

5. In the flat case, use the fact that the center manifold consists entirely of flat metrics first to show that exponential approach to the center manifold implies convergence of the DeTurck flow to a unique flat metric, and then to prove that the same is true for the Ricci flow.

We carry out each of these steps in §3. We then state as a corollary the stability of Ricci flow convergence for metrics whose Ricci flow converges to a flat metric.

In §4, we discuss the application of the analysis developed here to other metrics. The focus there is on Kähler–Einstein metrics on $K3$ manifolds. We sketch the proof of our results for such metrics in that section, following essentially the first three steps outlined above. As we have already noted, we are not yet able to complete a full stability analysis for the center manifold about a $K3$ Kähler–Einstein metric. In the remainder of §4, we note a few other geometries to which our methods may apply.

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2. Background.

The intent of this section is to establish notation, to introduce the function spaces needed for our study, and to provide a brief introduction to some of the tools we shall need, including the DeTurck flow and maximal regularity theory. We start by fixing some notation.

Given a closed connected smooth manifold $\mathcal{M}$, we denote by $S_2(\mathcal{M})$ the bundle of symmetric covariant 2-tensors over $\mathcal{M}$ and by $S_2^+(\mathcal{M})$ the subset of positive-definite tensors. In this context, a (smooth) Riemannian metric is an element of $C^\infty\left(S_2^+(\mathcal{M})\right)$. For convenience, we shall write $\mathcal{S}_2 \doteqdot C^\infty(S_2(\mathcal{M}))$ and $\mathcal{S}_2^+ \doteqdot C^\infty\left(S_2^+(\mathcal{M})\right)$. If $g \in \mathcal{S}_2^+$, we denote its Riemannian curvature tensor by Rm and write its components as $R_{ijk\ell}$, where $R_{1221} > 0$ on the round sphere. Then $\mathrm{Rc} \in \mathcal{S}_2$ denotes the Ricci tensor of g with components R_{ij}, and R is its scalar curvature.

We denote by $\Lambda^p \doteqdot \Lambda^p(T^*\mathcal{M})$ the bundle of p-forms on $\mathcal{M}$ and by $\Omega^p \doteqdot C^\infty(\Lambda^p)$ the space of differential p-forms. We indicate the de Rham cohomology groups of $\mathcal{M}$ by $H^p \doteqdot H^p_{dR}(\mathcal{M}, \mathbb{R})$ and denote the harmonic p-forms by H^p_Δ.

Given a Riemannian metric g on $\mathcal{M}$ with volume form $d\mu$, we define $\delta = \delta_g : \mathcal{S}_2 \to \Omega^1$ as the map

$$\delta : h \mapsto \delta h = -g^{ij}\nabla_i h_{jk} dx^k \tag{3}$$

whose formal adjoint under the L^2 inner product

$$(\cdot,\cdot) \doteqdot \int_{\mathcal{M}} \langle \cdot,\cdot \rangle \, d\mu \tag{4}$$

is the map $\delta^* = \delta_g^* : \Omega^1 \to \mathcal{S}_2$ given by

$$\delta^* : \omega \mapsto \frac{1}{2}\mathcal{L}_{\omega^\#} g = \frac{1}{2}\left(\nabla_i \omega_j + \nabla_j \omega_i\right) dx^i \otimes dx^j. \tag{5}$$

(Here $\mathcal{L}$ is the Lie derivative and $\omega^\#$ is the vector field metrically isomorphic to ω.) We shall also denote by $\delta = \delta_g$ the map $\Omega^p \to \Omega^{p-1}$ formally adjoint to $d : \Omega^p \to \Omega^{p+1}$; the meaning will be clear from the context.

2.1. The DeTurck equations.

The Ricci flow evolution equation (1) posed on $\mathcal{S}_2^+$ is only weakly parabolic [H2]. In order to obtain a strictly parabolic system, we follow [D] and define

$G : \mathcal{S}_2^+ \times \mathcal{S}_2 \to \mathcal{S}_2$ by

$$(g,u) \mapsto G(g,u) \doteqdot \left(u_{ij} - \frac{1}{2} g^{k\ell} u_{k\ell} g_{ij} \right) dx^i \otimes dx^j. \tag{6}$$

Then (for positive-definite u) we define $P : \mathcal{S}_2^+ \times \mathcal{S}_2^+ \to \mathcal{S}_2$ by

$$(g,u) \mapsto P_u(g) \doteqdot -2\delta_g^* \left(u^{-1} \delta_g (G(g,u)) \right), \tag{7}$$

and consider the evolution equation

$$\frac{\partial}{\partial t} g = -2 \operatorname{Rc}[g] - P_u(g), \qquad g(0) = g_0. \tag{8}$$

It is remarkable that the right hand side of (8), regarded as a quasilinear operator on g, is strongly elliptic for any choice of $u \in \mathcal{S}_2^+$; in fact we have the following:

Theorem 2.1 (DeTurck). *The right hand side of* (8) *has the same symbol as the Laplacian; consequently the evolution equation* (8) *is parabolic for any choice of $u \in \mathcal{S}_2^+$. The unique solution g of* (8) *provides a unique solution $\phi_t^* g$ of the Ricci flow evolution equation* (1) *with initial value g_0, where the diffeomorphisms ϕ_t are generated by integrating the vector field*

$$V^i \doteqdot g^{ij} u_{jk}^{-1} g^{k\ell} g^{pq} \left(\nabla_p u_{q\ell} - \frac{1}{2} \nabla_\ell u_{pq} \right). \tag{9}$$

Note that we shall usually take u to be the particular flat (or Ricci-flat) metric about which we wish to determine stability. (For Ricci solitons, a different approach is needed.)

2.2. The space of Riemannian structures.

As is well known, $\mathcal{S}_2$ with the C^∞ topology is a Frechét space, and $\mathcal{S}_2^+ \subset \mathcal{S}_2$ is an open convex cone. There is a natural right action of the group $\mathcal{D}(\mathcal{M})$ of smooth diffeomorphisms of $\mathcal{M}$ on $\mathcal{S}_2^+$ given by $(h, \phi) \mapsto \phi^* h$. It is easy to check that a metric g is Einstein if and only if $\phi^* g$ is Einstein. So for purposes of studying distinguished metrics on $\mathcal{M}$, one may regard $\mathcal{S}_2^+$ as a union of orbits $\mathcal{O}_g$. The slice theorem of Ebin [E] shows that $\mathcal{S}_2^+$ is 'almost' an infinite-dimensional manifold possessing an exponential map. (More precisely, the theorem states that for any metric g, there is a map $\chi : \mathcal{U} \to \mathcal{D}(\mathcal{M})$ of a neighborhood $\mathcal{U}$ of g in $\mathcal{O}_g$ such that $(\chi(\phi^* g))^* g = \phi^* g$ for all $\phi^* g \in \mathcal{U}$, and there is a submanifold Γ of $\mathcal{S}_2^+$ containing g such that the map $\mathcal{U} \times \Gamma \to \mathcal{S}_2^+$ given by $(\phi^* g, \gamma) \mapsto (\chi(\phi^* g))^* \gamma$ is a diffeomorphism onto a neighborhood of g in $\mathcal{S}_2^+$.) We shall require only the infinitesimal version of the slice theorem, which gives a useful decomposition of $T_g \mathcal{S}_2^+$. For each $g \in \mathcal{S}_2^+$, let δ_g and δ_g^* be the maps defined in (3) and (5), respectively. It is clear that $\left(h, \delta_g^* W\right) = (\delta_g h, W)$, hence that $\ker \delta_g \perp \operatorname{im} \delta_g^*$. With more analysis (see [E] or [BE]) it can be shown that these spaces span: in fact, one has

$$T_g \mathcal{S}_2^+ = \mathcal{H}_g \oplus \mathcal{V}_g, \tag{10}$$

where

$$\mathcal{H}_g \doteqdot \ker \delta_g \qquad \text{and} \qquad \mathcal{V}_g \doteqdot \operatorname{im} \delta_g^*. \tag{11}$$

Our notation is meant to suggest 'horizontal' and 'vertical' subspaces, because $T_g \mathcal{O}_g = \operatorname{im} \delta_g^* = \mathcal{V}_g$. In the remainder of this paper, we shall freely use the following observations, whose proofs are straightforward calculations.

Lemma 2.1. *Given $g \in \mathcal{S}_2^+$ and $h \in \mathcal{S}_2$, define $H \doteqdot \operatorname{tr}_g h = g^{ij} h_{ij}$. Let $\tilde{g} \doteqdot g + \varepsilon h$, and denote the Christoffel symbols, curvature, and volume form of $\tilde{g}$ by $\tilde{\Gamma}$, $\tilde{R}$, and $d\tilde{\mu}$, respectively. Then:*

1. $\left.\frac{\partial}{\partial \varepsilon} \tilde{\Gamma}_{jk}^{\ell}\right|_{\varepsilon=0} = \frac{1}{2}\left(\nabla_j h_k^{\ell} + \nabla_k h_j^{\ell} - \nabla^{\ell} h_{jk}\right).$

2. $\left.\frac{\partial}{\partial \varepsilon} \tilde{R}_{ijk}^{\ell}\right|_{\varepsilon=0} = \frac{1}{2}\left(\begin{array}{c} \nabla_i \nabla_k h_j^{\ell} - \nabla_i \nabla^{\ell} h_{jk} - \nabla_j \nabla_k h_i^{\ell} \\ +\nabla_j \nabla^{\ell} h_{ik} + R_{ijm}^{\ell} h_k^m - R_{ijk}^m h_m^{\ell} \end{array}\right).$

3. $\left.\frac{\partial}{\partial \varepsilon} \tilde{R}_{jk}\right|_{\varepsilon=0} = -\frac{1}{2}\left(\begin{array}{c} \Delta h_{jk} + \nabla_j \nabla_k H - \nabla_j \operatorname{div} h_k - \nabla_k \operatorname{div} h_j \\ -R_{j\ell} h_k^{\ell} - R_{k\ell} h_j^{\ell} + 2R_{jpqk} h^{pq} \end{array}\right).$

4. $\left.\frac{\partial}{\partial \varepsilon} \tilde{R}\right|_{\varepsilon=0} = -\left(\Delta H - \operatorname{div}(\operatorname{div} h) + \langle \operatorname{Rc}, h \rangle\right).$

5. $\frac{\partial}{\partial \varepsilon} d\tilde{\mu}_{\varepsilon=0} = \frac{1}{2} H \, d\mu.$

6. $\left.\frac{\partial}{\partial \varepsilon} \oint \tilde{R} \, d\tilde{\mu}\right|_{\varepsilon=0} = \oint \left(\frac{1}{2}\left(R - \oint R \, d\mu\right) H - \langle \operatorname{Rc}, h \rangle\right) d\mu.$

7. $\left.\frac{\partial}{\partial \varepsilon} (\mathcal{L}_X \tilde{g})_{ij}\right|_{\varepsilon=0} = X^k \nabla_k h_{ij} + h_{ik} \nabla_j X^k + h_{jk} \nabla_i X^k$ *for any vector field* X.

2.3. Maximal regularity and dynamic analysis.

It is often useful to regard an evolution PDE as an ODE posed in an infinite-dimensional space. This viewpoint suggests the utility of a qualitative geometric or dynamic theory for parabolic evolution equations — a framework in which one can decompose the state space of an equation into invariant subspaces and then discuss their stability or lack thereof. Such an approach was developed for semilinear equations in [H]. More recently, the concept has been extended to nonlinear parabolic equations by Da Prato–Lunardi [DL], and refined for quasilinear systems by Simonett [S]. A key ingredient of that theory is some sort of implicit function theorem or fixed-point theorem. For this to work, one needs function spaces in which each linear Cauchy problem of the sort

$$\frac{\partial}{\partial t}\phi = L\phi + \psi(t), \qquad \phi(0) = \phi_0$$

has a unique solution ϕ such that $\partial\phi/\partial t$ and $L\phi$ have the same regularity as ψ. One approach for achieving this is to use the *maximal regularity* theory of Da Prato and Grisvard [DG], which in turn is based on the use of *interpolation spaces*. There are several methods of defining such spaces in the literature; each yields a suitably functorial map taking any Banach couple $\mathcal{Y}_1 \hookrightarrow \mathcal{Y}_0$ to a Banach space $\mathcal{Y}$ such that $\mathcal{Y}_1 \subseteq \mathcal{Y} \subseteq \mathcal{Y}_0$. (For further background, see [CH].)

In §3.3, we lay the groundwork that allows us to apply this hierarchy of theories to the Ricci flow. Our objective there is to apply the following theorem of Simonett. We state it here in a form suited to our purposes; this is an adaptation of more general results derived from Theorem 4.1, Remark 4.2, and Theorem 5.8 of [S]. Roughly speaking, the theorem tells us that if A is a suitable quasilinear differential operator acting on appropriate function spaces, and if its linearization DA at a fixed point has an eigenvalue on the imaginary axis, then the evolution of solutions starting near that fixed point can be characterized by the presence of exponentially attractive center (unstable) manifolds.

Here and in the remainder of this paper, we denote by $B(\mathcal{X}, x, d)$ the open ball of radius d centered at x in the metric space $\mathcal{X}$.

Theorem 2.2 (Simonett). *Let $\mathcal{X}_1 \hookrightarrow \mathcal{X}_0$ be a continuous dense inclusion of Banach spaces, and let $\mathcal{X}_\alpha$ and $\mathcal{X}_\beta$ denote the continuous interpolation* *spaces corresponding to fixed* $0 < \beta < \alpha < 1$. *Let*

$$\frac{\partial}{\partial t} g = A(g)\, g \tag{12}$$

be an autonomous quasilinear parabolic equation posed for $t \geq 0$, *such that* $A(\cdot) \in C^k(\mathcal{G}_\beta, \mathcal{L}(\mathcal{X}_1, \mathcal{X}_0))$ *for some positive integer* k *and some open subset* $\mathcal{G}_\beta \subseteq \mathcal{X}_\beta$. *Assume that there exists a pair* $\mathcal{E}_0 \supseteq \mathcal{E}_1$ *of Banach spaces, that there exist extensions* $\tilde{A}(\cdot)$ *of* $A(\cdot)$ *to domains* $D(\tilde{A}(\cdot))$ *that are dense in* $\mathcal{E}_0$, *and that the following conditions hold for each* $g \in \mathcal{G}_\alpha \doteqdot \mathcal{G}_\beta \cap \mathcal{X}_\alpha$:

- $\tilde{A}(g) \in \mathcal{L}(\mathcal{E}_1, \mathcal{E}_0)$ *generates a strongly continuous analytic semigroup on* $\mathcal{L}(\mathcal{E}_0)$;
- $\mathcal{X}_0 \cong (\mathcal{E}_0, D(\tilde{A}(g)))_\theta$ *and* $\mathcal{X}_1 \cong (\mathcal{E}_0, D(\tilde{A}(g)))_{(1+\theta)}$ *for some* $\theta \in (0,1)$, *where* $(\cdot,\cdot)_\theta$ *denotes the continuous interpolation method of* [DG];
- $A(g)$ *agrees with the restriction of* $\tilde{A}(g)$ *to the dense subset* $D(A) \subseteq \mathcal{X}_0$;
- $\mathcal{E}_1 \hookrightarrow \mathcal{X}_\beta \hookrightarrow \mathcal{E}_0$ *is a continuous and dense inclusion with the property that there are* $C > 0$ *and* $\delta \in (0,1)$ *such that for all* $\eta \in \mathcal{E}_1$, *one has*

$$\|\eta\|_{\mathcal{X}_\beta} \leq C \|\eta\|_{\mathcal{E}_0}^{1-\delta} \|\eta\|_{\mathcal{E}_1}^{\delta}.$$

Let $\hat{g} \in \mathcal{G}_\alpha$ *be a fixed point of* (12). *Suppose that the spectrum* Σ *of the linearized operator* $DA|_{\hat{g}}$ *admits the decomposition* $\Sigma = \Sigma_s \cup \Sigma_{cu}$, *where* $\Sigma_s \subset \{z : \operatorname{Re} z < 0\}$, *and where* $\Sigma_{cu} \subset \{z : \operatorname{Re} z \geq 0\}$ *consists of finitely many eigenvalues of finite multiplicity. Suppose further that* $\Sigma_{cu} \cap i\mathbb{R} \neq \emptyset$. *Then:*

1. *If* $S(\lambda)$ *denotes the algebraic eigenspace of* $\lambda \in \Sigma_{cu}$, *then* $\mathcal{X}_\alpha$ *admits the decomposition* $\mathcal{X}_\alpha = \mathcal{X}_\alpha^s \oplus \mathcal{X}_\alpha^{cu}$ *for all* $\alpha \in [0,1]$, *where* $\mathcal{X}_\alpha^{cu} \equiv \bigoplus_{\lambda \in \Sigma_{cu}} S(\lambda)$.
2. *For each* $r \in \mathbb{N}$, *there exists* $d_r > 0$ *such that for all* $d \in (0, d_r]$, *there is a bounded* C^r *map* $\psi = \psi_d^r : B(\mathcal{X}_1^{cu}, \hat{g}, d) \to \mathcal{X}_1^s$ *with* $\psi(\hat{g}) = 0$ *and* $D\psi(\hat{g}) = 0$. *The image of* ψ *lies in the closed ball* $\bar{B}(\mathcal{X}_1^s, \hat{g}, d)$, *and its graph is a* C^r *manifold*

$$\mathcal{M}_{loc}^{cu} \doteqdot \{(\gamma, \psi(\gamma)) : \gamma \in B(\mathcal{X}_1^{cu}, \hat{g}, d)\} \subset \mathcal{X}_1$$

satisfying $T_{\hat{g}} \mathcal{M}_{loc}^{cu} \cong \mathcal{X}_1^{cu}$. *We call* $\mathcal{M}_{loc}^{cu}$ *a* local center manifold *if* $\Sigma_{cu} \subset i\mathbb{R}$ *and a* local center unstable manifold *otherwise.* $\mathcal{M}_{loc}^{cu}$ *is locally invariant for solutions of* (12) *as long as they remain in* $B(\mathcal{X}_1^{cu}, \hat{g}, d) \times B(\mathcal{X}_1^s, 0, d)$.

3. *For all $\alpha \in (0,1)$, there are constants $C_\alpha > 0$ independent of $\hat{g}$ and constants $\omega > 0$ and $\hat{d} \in (0, d_0]$ such that for each $d \in (0, \hat{d}]$, one has*

$$\|\pi^s g(t) - \psi(\pi^{cu} g(t))\|_{\mathcal{X}_1} \leq \frac{C_\alpha}{t^{1-\alpha}} e^{-\omega t} \|\pi^s g(0) - \psi(\pi^{cu} g(0))\|_{\mathcal{X}_\alpha}$$

for all solutions $g(t)$ with $g(0) \in B(\mathcal{X}_\alpha, \hat{g}, d)$ and all times $t \geq 0$ such that the solution $g(t)$ remains in $B(\mathcal{X}_\alpha, \hat{g}, d)$. Here π^s and π^{cu} denote the projections onto $\mathcal{X}_\alpha^s \cong (\mathcal{X}_1^s, \mathcal{X}_0^s)_\alpha$ and $\mathcal{X}_\alpha^{cu}$ respectively.

Remark 2.2. The C^r local center (unstable) manifolds constructed in statement (2) of the theorem are not in general unique. For instance, it can happen that $d_r \to 0$ as $r \to \infty$.

Remark 2.3. Statement (3) of the theorem implies in particular that solutions whose initial data lie sufficiently near a fixed point in the $\mathcal{X}_\alpha$ topology are attracted at an exponential rate in the $\mathcal{X}_1$ topology to solutions whose initial data belong to one of the finite-dimensional local center (unstable) manifolds. Note however that the theorem does not in general tell us anything about the dynamics *within* a local center (unstable) manifold.

3. Stability of Ricci flow convergence to a flat metric.

In this section, we state and prove our main results, which concern the behavior of the Ricci flow near a flat metric, or near a solution of the Ricci flow that converges to a flat metric.

The key step in obtaining these results involves the application of Theorem 2.2 to the DeTurck flow evolution equation (8). So we shall need to identify appropriate function spaces and study the properties of the DeTurck flow operator and its linearization in order to verify the hypotheses of Theorem 2.2 in our particular case. We do this preliminary analysis in subsections 3.1–3.3. Then after a discussion in subsection 3.4 of the relationship between convergence of the DeTurck flow and of the Ricci flow, we state and prove our main result (Theorem 3.1) in subsection 3.5. Theorem 3.1 pertains to Ricci flows starting near a flat metric; in subsection 3.6, we use the continuity of finite-time evolution together with Theorem 3.1 to verify the stability of Ricci flow convergence near any metric g_0 whose Ricci flow converges to a flat metric (Corollary 3.7).

3.1. The DeTurck operator.

We begin by examining the form of the DeTurck operator — the right hand side of equation (8) — in local coordinates and by noting some of its properties.

We first observe that the DeTurck operator is quasilinear in g. In order to match the notation used in Theorem 2.2, let us write this operator as $\mathcal{A}_u(g)\,g$, so that equation (8) takes the form

$$\frac{\partial}{\partial t}g = \mathcal{A}_u(g)\,g, \qquad g(0) = g_0.$$

Lemma 3.1. *If we express $\mathcal{A}_u(g)\,g$ in terms of first and second derivatives of g in local coordinates, we obtain*

$$\begin{aligned}(\mathcal{A}_u(g)\,g)_{ij} &= a(x,u,g)_{ij}^{k\ell pq}\frac{\partial^2}{\partial x^p\partial x^q}g_{k\ell}\\ &\quad + b(x,u,\partial u,g)_{ij}^{k\ell p}\frac{\partial}{\partial x^p}g_{k\ell} + c(x,u,\partial u)_{ij}^{k\ell}\,g_{k\ell}.\end{aligned} \tag{13}$$

The functions $a(x,\cdot,\cdot)$, $b(x,\cdot,\cdot,\cdot)$, and $c(x,\cdot,\cdot)$ depend smoothly on $x\in\mathcal{M}$ and are analytic functions of their remaining arguments.

Proof. In a smooth chart $\{x^i\}$, it follows from the standard formulas

$$\begin{aligned}\Gamma_{ij}^k &= \frac{1}{2}g^{k\ell}\left(\frac{\partial}{\partial x^i}g_{j\ell} + \frac{\partial}{\partial x^j}g_{i\ell} - \frac{\partial}{\partial x^\ell}g_{ij}\right)\\ R_{ijk}^\ell &= \frac{\partial}{\partial x^i}\Gamma_{jk}^\ell - \frac{\partial}{\partial x^j}\Gamma_{ik}^\ell + \Gamma_{im}^\ell\Gamma_{jk}^m - \Gamma_{ik}^m\Gamma_{jm}^\ell\end{aligned}$$

by straightforward computation that

$$\begin{aligned}-2R_{ij} &= g^{k\ell}\left(\frac{\partial^2}{\partial x^i\partial x^j}g_{k\ell} + \frac{\partial^2}{\partial x^k\partial x^\ell}g_{ij} - \frac{\partial^2}{\partial x^i\partial x^k}g_{j\ell} - \frac{\partial^2}{\partial x^j\partial x^\ell}g_{ik}\right)\\ &\quad + \pi\left(g,g^{-1},\partial g\right),\end{aligned}$$

where $\pi\left(g,g^{-1},\partial g\right)$ is a generic polynomial in g, g^{-1}, and first derivatives of g.

Similarly, one observes that

$$(P_u(g))_{ij} = (Q_u g)_{ij} + (Q_u g)_{ji},$$

where

$$(14)\qquad (Q_u g)_{ij} \doteqdot \nabla_i \left[\left(u^{-1}\right)_{jk} g^{k\ell} g^{pq} \left(\nabla_p u_{q\ell} - \frac{1}{2} \nabla_\ell u_{pq} \right) \right].$$

Since

$$\nabla_p u_{q\ell} - \frac{1}{2} \nabla_\ell u_{pq} = \left(\frac{\partial}{\partial x^p} u_{q\ell} - \Gamma_{pq}^m u_{m\ell} - \Gamma_{p\ell}^m u_{qm} \right) - \frac{1}{2} \left(\frac{\partial}{\partial x^\ell} u_{pq} - \Gamma_{\ell p}^m u_{mq} - \Gamma_{\ell q}^m u_{pm} \right),$$

one has

$$(Q_u g)_{ij} = - \left(u^{-1}\right)_{jk} u_{\ell m} g^{k\ell} g^{pq} \left(\frac{\partial}{\partial x^i} \Gamma_{pq}^m \right) + \pi \left(g, g^{-1}, \partial g, u, u^{-1}, \partial u \right).$$

Since for any matrix M, the components of M^{-1} are analytic functions of the components of M, the result follows. □

3.2. The linearization of the DeTurck operator.

We next study the infinitesimal structure of the DeTurck operator. Given a Riemannian metric g on $\mathcal{M}$, we denote by $\Delta \doteqdot g^{ij} \nabla_i \nabla_j$ the rough Laplacian. The Lichnerowicz Laplacian is then the map $\Delta_\ell : \mathcal{S}_2 \to \mathcal{S}_2$ given by

$$(15)\qquad \Delta_\ell h_{ij} \doteqdot \Delta h_{ij} + 2 R_{ipqj} h^{pq} - R_i^k h_{kj} - R_j^k h_{ik}.$$

While the spectrum of the Lichnerowicz Laplacian is negative semidefinite for a flat metric and for many other geometries, it is not negative semidefinite for all Riemannian metrics [Av]. This is relevant for the linearization of the DeTurck flow (8), because we have the following:

Proposition 3.2. *The linearization of the DeTurck operator $\mathcal{A}_u(g)\, g$ about g_0 is given by*

$$(16)\qquad \left(D\mathcal{A}_u(g)\right)\big|_{g_0} h = \Delta_\ell h - \Psi_u h,$$

where

$$
\begin{aligned}
(\Psi_u h)_{ij} \doteqdot{}& \left(\nabla^k h_{ij} - \nabla_i h_j^k - \nabla_j h_i^k\right) u_{k\ell}^{-1} \left(\nabla^m u_m^\ell - \frac{1}{2}\nabla^\ell U\right) \\
&+ \left(\nabla_i h_\ell^k u_{jk}^{-1} + \nabla_j h_\ell^k u_{ik}^{-1}\right) \left(\frac{1}{2}\nabla^\ell U - \nabla^m u_m^\ell\right) \\
&+ \left(\nabla_i h_m^\ell u_{jk}^{-1} + \nabla_j h_m^\ell u_{ik}^{-1}\right) \left(\frac{1}{2}\nabla^k u_\ell^m - \nabla^m u_\ell^k\right) \\
&+ h_\ell^k \left[\begin{array}{c} \nabla_i \left(u_{jk}^{-1}\left(\frac{1}{2}\nabla^\ell U - \nabla^m u_m^\ell\right)\right) \\ +\nabla_j \left(u_{ik}^{-1}\left(\frac{1}{2}\nabla^\ell U - \nabla^m u_m^\ell\right)\right) \end{array}\right] \\
&+ h_m^\ell \left[\begin{array}{c} \nabla_i \left(u_{jk}^{-1}\left(\frac{1}{2}\nabla^k u_\ell^m - \nabla^m u_\ell^k\right)\right) \\ +\nabla_j \left(u_{ik}^{-1}\left(\frac{1}{2}\nabla^k u_\ell^m - \nabla^m u_\ell^k\right)\right) \end{array}\right]. \qquad (17)
\end{aligned}
$$

Here, all covariant differentiation is done with respect to the Levi-Civita connection of g_0, indices are raised and lowered using g_0, and $U \doteqdot \operatorname{tr}_{g_0} u$.

Proof. By definition, we have

$$
\left(D\mathcal{A}_u(g)\right)\big|_{g_0} h = -2\, \frac{\partial}{\partial \varepsilon} \operatorname{Rc}\left[g_0 + \varepsilon h\right]\bigg|_{\varepsilon=0} - \frac{\partial}{\partial \varepsilon} P_u\left(g_0 + \varepsilon h\right)\bigg|_{\varepsilon=0}.
$$

We calculate $\frac{\partial}{\partial \varepsilon} \operatorname{Rc}[g_0 + \varepsilon h]$ using formula (3) of Lemma 2.1. To calculate $\frac{\partial}{\partial \varepsilon} P_u(g_0 + \varepsilon h)$, we use the identity

$$
(P_u g)_{ij} = (Q_u g)_{ij} + (Q_u g)_{ji},
$$

with $Q_u g$ defined by (14), and then use Lemma 2.1 to compute

$$
\begin{aligned}
\frac{\partial}{\partial \varepsilon}\left(Q_u(\tilde{g})\right)_{ij}\bigg|_{\varepsilon=0} ={}& \frac{1}{2}\nabla_i \nabla_j H - \nabla_i \operatorname{div} h_j \\
&+ \frac{1}{2}\left(\nabla^k h_{ij} - \nabla_i h_j^k - \nabla_j h_i^k\right) u_{k\ell}^{-1} \left(\nabla^m u_m^\ell - \frac{1}{2}\nabla^\ell U\right) \\
&+ \nabla_i h_\ell^k u_{jk}^{-1} \left(\frac{1}{2}\nabla^\ell U - \nabla^m u_m^\ell\right) \\
&+ \nabla_i h_m^\ell u_{jk}^{-1} \left(\frac{1}{2}\nabla^k u_\ell^m - \nabla^m u_\ell^k\right) \\
&+ h_\ell^k \nabla_i \left[u_{jk}^{-1} \left(\frac{1}{2}\nabla^\ell U - \nabla^m u_m^\ell\right)\right] \\
&+ h_m^\ell \nabla_i \left[u_{jk}^{-1} \left(\frac{1}{2}\nabla^k u_\ell^m - \nabla^m u_\ell^k\right)\right].
\end{aligned}
$$

□

3.3. The DeTurck flow in the context of maximal regularity.

We now exhibit appropriate Banach spaces $\mathcal{X}_0$, $\mathcal{X}_1$, $\mathcal{E}_0$, and $\mathcal{E}_1$ such that the DeTurck operator and its linearization satisfy the hypotheses of Theorem 2.2.

The spaces we use are certain *little Hölder spaces*. Recall that if $r \in \mathbb{N}$ and $\rho \in (0,1)$, the ordinary Hölder space $C^{r,\rho}$ is the Banach space of all C^r functions $f : \mathbb{R}^n \to \mathbb{R}$ for which the Hölder norm $\|f\|_{r+\rho}$ is finite. The subspace of C^∞ functions in $C^{r,\rho}$ is not dense; indeed, $C^{r,\rho}$ is isomorphic [Ci] to ℓ_∞, hence is not separable. One defines the little Hölder space $h^{r+\rho}$ to be the closure of the subspace of C^∞ functions with respect to the $\|\cdot\|_{r+\rho}$ norm; one then verifies that $h^{r+\rho}$ is a Banach space and that $h^{r+\rho} \hookrightarrow h^{s+\sigma}$ is a continuous and dense inclusion for $s \leq r$ and $0 < \sigma < \rho < 1$. (Corresponding statements hold for $C^{r,\rho}(\Omega)$ when $\Omega \subset \mathbb{R}^n$ is a bounded C^∞ domain.)

These definitions extend readily [BJ] using a smooth atlas to functions defined on a smooth closed manifold $\mathcal{M}$ and taking values in the bundle $S_2(\mathcal{M})$ of symmetric $(2,0)$-tensors over $\mathcal{M}$. Accordingly, we shall use the notation $\|\cdot\|_{r+\rho}$ to denote the Hölder norm on $C^r(\mathcal{M}, S_2(\mathcal{M}))$, and $h^{r+\rho}$ to denote the little Hölder spaces formed in this fashion. Note in particular that

$$h^{r+\rho} \hookrightarrow h^{s+\sigma} \tag{18}$$

remains a continuous and dense inclusion.

An *exact interpolation method* I_θ of exponent θ takes any pair $B_1 \subseteq B_0$ of Banach spaces to a Banach space $I_\theta(B_0, B_1)$ such that $B_1 \subseteq I_\theta(B_0, B_1) \subseteq B_0$, and such that if $T \in \mathcal{L}(B_0, A_0) \cap \mathcal{L}(B_1, A_1)$ then $T \in \mathcal{L}(I_\theta(B_0, B_1), I_\theta(A_0, A_1))$ and

$$\|T\|_{\mathcal{L}(I_\theta(B_0,B_1),I_\theta(A_0,A_1))} \leq \|T\|^{1-\theta}_{\mathcal{L}(B_0,A_0)} \|T\|^{\theta}_{\mathcal{L}(B_1,A_1)}.$$

To apply Theorem 2.2, we will use the *continuous interpolation spaces* $(B_0, B_1)_\theta$ introduced in [DG]. By [DF], these are equivalent in norm to the *real interpolation spaces* frequently found in the literature; hence we will freely use results originally proved for the latter spaces. The continuous interpolation method is exact and may be defined in a number of equivalent ways. (See [Tr] or [CH].) For instance, one can characterize $(B_0, B_1)_\theta$ as the set of all $x \in B_0$ such that there exist sequences $\{y_n\} \subseteq B_0$ and $\{z_n\} \subseteq B_1$ with $x = y_n + z_n$, where

$$\|y_n\|_{B_0} = o\left(2^{-n\theta}\right) \qquad \text{and} \qquad \|z_n\|_{B_1} = o\left(2^{n(1-\theta)}\right)$$

as $n \to \infty$. The norm on $(B_0, B_1)_\theta$ is equivalent to

$$\inf \left\{ \sup_{n \geq 1} \left(2^{n\theta} \|y_n\|_{B_0}, \; 2^{-n(1-\theta)} \|z_n\|_{B_1} \right) \right\},$$

where the infimum is taken over all such sequences (y_n, z_n).

For our purposes, the key fact [Tr] about the continuous interpolation spaces is that for $s \leq r \in \mathbb{N}$, $0 < \sigma < \rho < 1$, and $0 < \theta < 1$, there is a Banach space isomorphism

$$(19) \qquad \left(h^{s+\sigma}, h^{r+\rho}\right)_\theta \cong h^{(\theta r + (1-\theta)s) + (\theta\rho + (1-\theta)\sigma)},$$

provided that the exponent $\theta(r+\rho) + (1-\theta)(s+\sigma)$ is not an integer. If it is not an integer, then there is $C < \infty$ such that for all $\eta \in h^{r+\rho}$,

$$(20) \qquad \|\eta\|_{(h^{s+\sigma}, h^{r+\rho})_\theta} \leq C \|\eta\|_{h^{s+\sigma}}^{1-\theta} \|\eta\|_{h^{r+\rho}}^{\theta}.$$

Thus for fixed $0 < \sigma < \rho < 1$, we define the following nested spaces:

$$(21) \qquad \begin{array}{ccc} \mathcal{E}_0 & \doteqdot & h^{0+\sigma} \\ \cup & & \\ \mathcal{X}_0 & \doteqdot & h^{0+\rho} \\ \cup & & \\ \mathcal{E}_1 & \doteqdot & h^{2+\sigma} \\ \cup & & \\ \mathcal{X}_1 & \doteqdot & h^{2+\rho}. \end{array}$$

Notice that for $\theta \doteqdot (\rho - \sigma)/2 \in (0,1)$, it follows from (19) that

$$(22) \qquad \mathcal{X}_0 \cong (\mathcal{E}_0, \mathcal{E}_1)_\theta \qquad \text{and} \qquad \mathcal{X}_1 \cong (\mathcal{E}_0, \mathcal{E}_1)_{(1+\theta)}.$$

We now wish to focus on the DeTurck operator $\mathcal{A}_u(g)$ defined by (8) and written in expanded form in (13). Let us fix a smooth metric u and write $\mathcal{A}(g) \doteqdot \mathcal{A}_u(g)$. For fixed $0 < \varepsilon \ll 1$ and $1/2 \leq \beta < \alpha < 1$, we define

$$\mathcal{G}_\beta = \mathcal{G}_\beta(u, \varepsilon) \doteqdot \left\{ g \in (\mathcal{X}_0, \mathcal{X}_1)_\beta : g > \varepsilon u \right\},$$
$$\mathcal{G}_\alpha = \mathcal{G}_\alpha(u, \varepsilon) \doteqdot \mathcal{G}_\beta \cap (\mathcal{X}_0, \mathcal{X}_1)_\alpha,$$

where $g > \varepsilon u$ means $g(X, X) > \varepsilon$ for any vector X such that $|X|_u^2 = 1$. Observe that for each $g \in \mathcal{G}_\beta$, equation (13) allows us to regard $\mathcal{A}(g)$ as a linear operator

$$(\mathcal{A}(g)\gamma)_{ij} = a(x, u, g)_{ij}^{k\ell pq} \frac{\partial^2}{\partial x^p \partial x^q} \gamma_{k\ell} + b(x, u, \partial u, g)_{ij}^{k\ell p} \frac{\partial}{\partial x^p} \gamma_{k\ell} + c(x, u, \partial u)_{ij}^{k\ell} \gamma_{k\ell}$$

on $h^{2+\sigma} = \mathcal{E}_1$. For $g \in \mathcal{G}_\beta$, let us denote by $\mathcal{A}_{\mathcal{E}_1}(g) : \mathcal{E}_1 \subseteq \mathcal{E}_0 \to \mathcal{E}_0$ the unbounded linear operator on $\mathcal{E}_0$ with dense domain $D(\mathcal{A}_{\mathcal{E}_1}(g)) = \mathcal{E}_1$. And let us denote by $\mathcal{A}_{\mathcal{X}_1}(g) : \mathcal{X}_1 \subseteq \mathcal{X}_0 \to \mathcal{X}_0$ the unbounded linear operator on $\mathcal{X}_0$ with dense domain $D(\mathcal{A}_{\mathcal{X}_1}(g)) = \mathcal{X}_1$. We need to establish the following:

Lemma 3.3. *The functions $g \mapsto \mathcal{A}_{\mathcal{X}_1}(g)$ and $g \mapsto \mathcal{A}_{\mathcal{E}_1}(g)$ define analytic maps $\mathcal{G}_\alpha \to \mathcal{L}(\mathcal{X}_1, \mathcal{X}_0)$ and $\mathcal{G}_\beta \to \mathcal{L}(\mathcal{E}_1, \mathcal{E}_0)$ respectively.*

Proof. Analyticity of these maps is an immediate corollary of Lemma 3.1, once we have verified that these two functions map into the correct spaces. So it suffices to show that $\gamma \mapsto \mathcal{A}_{\mathcal{X}_1}(g)\gamma$ is a bounded linear map from $\mathcal{X}_1$ to $\mathcal{X}_0$ for all $g \in \mathcal{G}_\alpha$, and that $\gamma \mapsto \mathcal{A}_{\mathcal{E}_1}(g)\gamma$ is a bounded linear map from $\mathcal{E}_1$ to $\mathcal{E}_0$ for all $g \in \mathcal{G}_\beta$.

Let $\gamma \in \mathcal{X}_1$ and consider the first term: $a(x,u,g)_{ij}^{k\ell pq} \frac{\partial^2}{\partial x^p \partial x^q} \gamma_{k\ell}(x)$. Writing $(F \circ g)(x) = a(x,u,g)$ and suppressing indices for clarity, we wish to estimate the Hölder norm of

$$\begin{aligned}(23)\qquad & (F \circ g)(x) \cdot (\partial^2\gamma)(x) - (F \circ g)(y) \cdot (\partial^2\gamma)(y) \\ &= (F \circ g)(x) \cdot \left[(\partial^2\gamma)(x) - (\partial^2\gamma)(y)\right] \\ &\quad + (\partial^2\gamma)(y) \cdot \left[(F \circ g)(x) - (F \circ g)(y)\right].\end{aligned}$$

Observe that we can estimate $|(F \circ g)(x) - (F \circ g)(y)|$ by integrating the directional derivative along a minimizing u-geodesic p from x to y of length $\operatorname{dist}_u(x,y)$:

$$\begin{aligned}|(F \circ g)(x) - (F \circ g)(y)| &= \left| \int_p D(F \circ g)(p')\, ds \right| \\ &\le \sup |D(F \circ g)| \cdot \operatorname{dist}_u(x,y).\end{aligned}$$

Then recalling from the proof of Lemma 3.1 that $F \circ g$ is a polynomial in u, u^{-1}, g, and g^{-1} of total degree N, and noticing that g^{-1} can be controlled by u^{-1} when $g \in \mathcal{G}_\alpha = \mathcal{G}_\alpha(u,\varepsilon)$, we find that there is a constant C depending only on u, ε, N, n, and $\mathcal{M}$ such that

$$\sup |D(F \circ g)| \le C\left(1 + \|g\|_{h^{0+\rho}}^{N-1}\right) \|g\|_{h^{1+\rho}}.$$

Combining these estimates and noting that

$$\operatorname{dist}_u(x,y) \le (\operatorname{dist}_u(x,y))^\rho \left(1 + (\operatorname{diam}_u \mathcal{M})^{1-\rho}\right),$$

we obtain a constant C_0 such that

$$|(F \circ g)(x) - (F \circ g)(y)| \leq C_0 \left(1 + \|g\|_{h^{0+\rho}}^{N-1}\right) \|g\|_{h^{1+\rho}} (\operatorname{dist}_u(x,y))^\rho .$$

The remaining term in (23) is easily estimated when one recalls that $\mathcal{X}_\alpha = (\mathcal{X}_0, \mathcal{X}_1)_\alpha = h^{2\alpha+\rho} \hookrightarrow h^{1+\rho} \hookrightarrow h^{0+\rho}$. Thus we find there is C_1 such that

$$\left\| a(x,u,g)_{ij}^{k\ell pq} \frac{\partial^2}{\partial x^p \partial x^q} \gamma_{k\ell} \right\|_{h^{0+\rho}} \leq C_1 \left(1 + \|g\|_{\mathcal{X}_\alpha}^N\right) \|\gamma\|_{h^{2+\rho}} .$$

Similarly, one obtains C_2 and C_3 such that

$$\left\| b(x,u,\partial u,g)_{ij}^{k\ell p} \frac{\partial}{\partial x^p} \gamma_{k\ell} \right\|_{h^{0+\rho}} \leq C_2 \left(1 + \|g\|_{\mathcal{X}_\alpha}^N\right) \|\gamma\|_{h^{1+\rho}}$$

$$\left\| c(x,u,\partial u)_{ik}^{k\ell} \gamma_{k\ell} \right\|_{h^{0+\rho}} \leq C_3 \|\gamma\|_{h^{0+\rho}} .$$

Thus we have shown that $\gamma \mapsto \mathcal{A}_{\mathcal{X}_1}(g)\gamma$ is a bounded linear map from $\mathcal{X}_1$ to $\mathcal{X}_0$, and hence that $\mathcal{G}_\alpha \to \mathcal{L}(\mathcal{X}_1, \mathcal{X}_0)$. Replacing α by β and ρ by σ in the argument above proves the assertion for $\mathcal{G}_\beta \to \mathcal{L}(\mathcal{E}_1, \mathcal{E}_0)$. □

Although for every $g \in \mathcal{G}_\beta$, $\mathcal{A}_{\mathcal{E}_1}(g)$ is bounded when regarded as an operator $\mathcal{E}_1 \to \mathcal{E}_0$, it is unbounded when regarded as an operator $\mathcal{E}_0 \to \mathcal{E}_0$, and is in fact only defined on a dense subspace $D(\mathcal{A}_{\mathcal{E}_1}(g)) = \mathcal{E}_1$. Nonetheless, it has the desirable property of generating a strongly continuous analytic semigroup, which is bounded (and hence defined everywhere) as a map $\mathcal{E}_0 \to \mathcal{E}_0$:

Lemma 3.4. *For every $g \in \mathcal{G}_\beta$, $\mathcal{A}_{\mathcal{E}_1}(g)$ is the infinitesimal generator of a strongly continuous analytic semigroup on $\mathcal{L}(\mathcal{E}_0)$.*

Proof. By DeTurck's result (Theorem 2.1), $\mathcal{A}_{\mathcal{E}_1}(g)$ is strongly elliptic for any $g \in \mathcal{G}_\beta$. By classical elliptic theory, its spectrum is discrete, having a limit point only at $-\infty$. (See for instance Theorem 37 in Appendix I of [B].) Hence there is $\lambda_0 > 0$ such that $\lambda I - \mathcal{A}_{\mathcal{E}_1}(g)$ is a topological linear isomorphism from $\mathcal{E}_1$ onto $\mathcal{E}_0$ whenever $\operatorname{Re}\lambda \geq \lambda_0$. In this case, the standard Schauder estimate (Theorem 27 in Appendix H of [B]) applied to the operator $\lambda I - \mathcal{A}_{\mathcal{E}_1}(g)$ yields $C < \infty$ such that

$$\|\eta\|_{\mathcal{E}_1} \doteqdot \|\eta\|_{h^{2+\sigma}} \leq C \|(\lambda I - \mathcal{A}_{\mathcal{E}_1}(g))\eta\|_{h^{0+\sigma}} \doteqdot C \|(\lambda I - \mathcal{A}_{\mathcal{E}_1}(g))\eta\|_{\mathcal{E}_0}$$

for every $\eta \in \mathcal{E}_1 = D(\mathcal{A}_{\mathcal{E}_1}(g))$. By Theorem 1.2.2 and Remark 1.2.1(a) of [A], this suffices to prove the result. □

3.4. Equivalence of DeTurck and Ricci flow convergence.

Our objective here is to show that a solution of the DeTurck flow (8) converges exponentially fast to a unique flat metric only if the corresponding solution of the Ricci flow (1) converges exponentially fast to a unique (though possibly distinct) flat metric.

Lemma 3.5. *Let $V(t)$ be a vector field on a Riemannian manifold $(\mathcal{M}^n, g(t))$, where $0 \leq t < \infty$, and suppose there are constants $0 < c \leq C < \infty$ such that*

$$\sup_{x \in \mathcal{M}} |V(x,t)|_{g(t)} \leq Ce^{-ct}.$$

Then the diffeomorphisms ϕ_t generated by V converge exponentially to a fixed diffeomorphism ϕ_∞ of $\mathcal{M}$.

Proof. Given $x \in \mathcal{M}$, let $\gamma : [0,\infty) \to \mathcal{M}$ be an integral curve for V starting at x. Then γ satisfies

$$\gamma'(t) = V(\gamma(t), t)$$
$$\gamma(0) = x,$$

where we make the standard identification $\gamma' \equiv \gamma_*(d/dt)$. The length $L[\gamma]$ of the integral curve is nondecreasing and bounded above, because

$$L[\gamma](t) = \int_0^t |V(x,\tau)|_{g(\tau)}\, d\tau \leq C \int_0^t e^{-c\tau}\, d\tau = \frac{C}{c}\left(1 - e^{-ct}\right) \leq \frac{C}{c}.$$

This proves that $L[\gamma]$ converges; to see that the convergence is exponential, it suffices to note that

$$\int_t^\infty |V(x,\tau)|_{g(\tau)}\, d\tau \leq C \int_t^\infty e^{-c\tau}\, d\tau = \frac{C}{c} e^{-ct}.$$

Since $\gamma(t) = \phi_t(x)$ and since $x \in \mathcal{M}$ is arbitrary, the result follows. □

Proposition 3.6. *Let g_0 be a flat metric on a manifold $\mathcal{M}$. Suppose there is a neighborhood $\mathcal{O}$ of g_0 in $\mathcal{S}_2^+$ with respect to the $\|\cdot\|_{2+\rho}$ Hölder norm such that for every $\tilde{g}_0 \in \mathcal{O}$, the unique solution $\bar{g}(t)$ of the DeTurck flow*

$$\frac{\partial}{\partial t}\bar{g} = -2\overline{\mathrm{Rc}} - P_{g_0}(\bar{g}), \qquad \bar{g}(0) = \tilde{g}_0$$

converges exponentially fast to a flat metric $\bar{g}_\infty$. *Then the unique solution* $\tilde{g}(t) \doteqdot (\phi_t^* \bar{g})$ *of the Ricci flow (1) with* $\tilde{g}(0) = \tilde{g}_0 \in \mathcal{O}$ *guaranteed by Theorem* 2.1 *converges exponentially fast to a flat metric* $\tilde{g}_\infty$.

Proof. It is clear that $\tilde{g}_\infty$ will be flat if it exists, so all we need do is to show that $\tilde{g}(t)$ converges. But because $\bar{g}_\infty$ and g_0 are both flat, their Levi-Civita connections are each trivial, whence it follows that

$$\nabla_{[\bar{g}_\infty]}(g_0) = \nabla_{[g_0]}(g_0) \equiv 0.$$

Then since $\bar{g}(t) \to \bar{g}_\infty$ exponentially fast, it follows that $\bar{V}(t) \to 0$ exponentially, where $\bar{V}(t)$ is given by

$$\bar{V}^i \doteqdot \bar{g}^{ij} \left(g_0^{-1}\right)_{jk} \bar{g}^{k\ell} \bar{g}^{pq} \left(\bar{\nabla}_p (g_0)_{q\ell} - \frac{1}{2} \bar{\nabla}_\ell (g_0)_{pq} \right).$$

(Here $\bar{\nabla}$ denotes covariant differentiation with respect to the Levi-Civita connection of $\bar{g}(t)$.) Hence by Lemma 3.5, the solution $\tilde{g}(t)$ of the ODE corresponding to $V(t)$ exhibits exponential convergence to some limit $\tilde{g}_\infty$. □

3.5. Main theorem.

Having established the preliminary results of subsections 3.1 –3.4 concerning the DeTurck operator, its linearization, and the relation between convergence of the DeTurck and Ricci flows, we are ready to state and prove our main theorem, which says that the Ricci flow of any metric sufficiently close to a flat metric will necessarily converge exponentially quickly to a flat metric.

Theorem 3.1. *Let* g_0 *be a flat Riemannian metric on a torus* $\mathcal{T}^n$. *For fixed* $\rho \in (0,1)$, *let* $\mathcal{X}$ *denote the closure of* $\mathcal{S}_2 \supset \mathcal{S}_2^+$ *with respect to the* $\|\cdot\|_{2+\rho}$ *Hölder norm. Then:*

1. $T_{g_0}\mathcal{S}_2^+ \cong \mathcal{X}$ *admits the decomposition*

$$\mathcal{T}_{g_0}\mathcal{S}_2^+ = \mathcal{X}^s \oplus \mathcal{X}^c,$$

where $\mathcal{X}^c$ *is the* $n(n+1)/2$*-dimensional space of* $(2,0)$*-tensors parallel with respect to the Levi-Civita connection of* g_0.

2. *There exists $d_0 > 0$ such that for all $d \in (0, d_0]$, there is a bounded C^∞ map $\psi : B(\mathcal{X}^c, g_0, d) \to \mathcal{X}^s$ with $\psi(g_0) = 0$ and $D\psi(g_0) = 0$. The image of ψ lies in the closed ball $\bar{B}(\mathcal{X}^s, g_0, d)$, and its graph*

$$\mathcal{M}^c_{loc} \doteqdot \{(\gamma, \psi(\gamma)) : \gamma \in B(\mathcal{X}^c, g_0, d)\}$$

satisfies $T_{g_0}\mathcal{M}^c_{loc} \cong \mathcal{X}^c$. This unique C^∞ local center manifold $\mathcal{M}^c_{loc}$ is of dimension $n(n+1)/2$ and consists entirely of flat metrics.

3. *There are constants $C > 0$, $\omega > 0$ and $d_* \in (0, d_0]$ such that for each $d \in (0, d_*]$, one has*

$$\|\pi^s \tilde{g}(t) - \psi(\pi^c \tilde{g}(t))\|_{2+\rho} \le Ce^{-\omega t} \|\pi^s \tilde{g}(0) - \psi(\pi^c \tilde{g}(0))\|_{2+\rho}$$

for all solutions $\tilde{g}(t)$ of the Ricci flow with $\tilde{g}(0) \in B(\mathcal{X}, g_0, d)$ and all times $t \ge 0$. Here π^s and π^c denote the projections onto $\mathcal{X}^s$ and $\mathcal{X}^c$ respectively. In particular, any solution $\tilde{g}(t)$ of the Ricci flow with initial data sufficiently near g_0 converges exponentially to a flat metric near g_0.

Proof. For reasons discussed earlier, we work first with the DeTurck flow rather than the Ricci flow. We take the background metric u to be the given flat metric g_0, and thus consider the DeTurck flow

$$\frac{\partial}{\partial t}\bar{g} = -2\overline{\mathrm{Rc}} - P_{g_0}(\bar{g}), \qquad \bar{g}(0) = \bar{g}_0. \tag{24}$$

Note that any flat metric is a stationary solution of this flow, because $P_{g_0}(\bar{g}) = 0$ if $\bar{g}$ is flat. Note also that $\Psi_u h$ in equation (16) vanishes for this choice of u, whence by Proposition 3.2, the linearization of (24) reduces to the basic heat equation:

$$\frac{\partial}{\partial t}h_{ij} = \Delta h_{ij}.$$

It is clear that the rough Laplacian is negative semidefinite on $\mathcal{S}_2$ with kernel consisting exactly of parallel $(2,0)$-tensors, hence of dimension at most $n(n+1)/2$. Recalling the choices made for $\mathcal{X}_0$, $\mathcal{X}_1$, $\mathcal{E}_0$, and $\mathcal{E}_1$, and applying Lemmas 3.3 and 3.4, we thus verify that the DeTurck operator satisfies the hypotheses of Theorem 2.2. This proves that local C^r center manifolds ${}^r\mathcal{M}^c_{loc}$ exist, and that the DeTurck flow of any metric starting near g_0 exponentially approaches ${}^r\mathcal{M}^c_{loc}$.

We now claim that the ${}^r\mathcal{M}^c_{loc}$ are independent of r, and consist entirely of flat metrics. To prove this, we observe that any flat metric $\hat{g}$ sufficiently near g_0 belongs to ${}^r\mathcal{M}^c_{loc}$ for all $r \in \mathbb{N}$: if not, then statement (3) of Theorem 2.2 would imply that $\hat{g}$ converges exponentially to ${}^r\mathcal{M}^c_{loc}$, contradicting the fact that $\hat{g}$ is a fixed point of (24). But it is a standard fact that the space of flat metrics on the torus is a convex $n(n+1)/2$-dimensional set. (See for instance §12.18 of [B].) Since each ${}^r\mathcal{M}^c_{loc}$ is at most $n(n+1)/2$-dimensional, it follows that ${}^r\mathcal{M}^c_{loc}$ consists exactly of flat metrics for all $r \in \mathbb{N}$.

Our argument thus far shows there is a neighborhood $B(\mathcal{X}, g_0, \delta)$ such that the DeTurck flow $\bar{g}(t)$ of any metric $\bar{g}(0) \in B(\mathcal{X}, g_0, \delta)$ becomes flat exponentially fast in the $\|\cdot\|_{2+\rho}$ norm for as long as $\bar{g}(t) \in B(\mathcal{X}, g_0, \delta)$. By (14), we have

$$\begin{aligned}(P_{g_0}(\bar{g}))_{ij} = \bar{\nabla}_i &\left[\left(g_0^{-1}\right)_{jk} \bar{g}^{k\ell}\bar{g}^{pq}\left(\bar{\nabla}_p (g_0)_{q\ell} - \frac{1}{2}\bar{\nabla}_\ell (g_0)_{pq}\right)\right] \\ &+ \bar{\nabla}_j \left[\left(g_0^{-1}\right)_{ik} \bar{g}^{k\ell}\bar{g}^{pq}\left(\bar{\nabla}_p (g_0)_{q\ell} - \frac{1}{2}\bar{\nabla}_\ell (g_0)_{pq}\right)\right],\end{aligned}$$

where $\bar{\nabla}$ denotes covariant differentiation with respect to the Levi-Civita connection of $\bar{g}(t)$. Since $|\bar{R}_{ij}|$, $|\bar{\nabla}_j (g_0)_{k\ell} - \partial_j (g_0)_{k\ell}|$, and $|\bar{\nabla}_i\bar{\nabla}_j (g_0)_{k\ell} - \partial_i\partial_j (g_0)_{k\ell}|$ all decay exponentially fast when $\bar{g}(t) \in B(\mathcal{X}, g_0, \delta)$, there are $\bar{C} = \bar{C}(\delta) < \infty$ and $\bar{\omega} = \bar{\omega}(\delta) > 0$ such that

$$\left|\frac{\partial}{\partial t}\bar{g}\right| = \left|-2\overline{\mathrm{Rc}} - P_g(\bar{g})\right| \leq \bar{C}e^{-\bar{\omega}t}$$

for as long as $\bar{g}(t)$ remains in $B(\mathcal{X}, g_0, \delta)$. Choose $0 < \varepsilon < \delta$ small enough that $\bar{C}(\varepsilon)/\bar{\omega}(\delta) < \delta - \varepsilon$. Then for all solutions $\bar{g}(t)$ with $\bar{g}(0) \in B(\mathcal{X}, g_0, \varepsilon)$, we can estimate

$$|\bar{g}(t) - g_0| \leq |\bar{g}(t) - \bar{g}(0)| + |\bar{g}(0) - g_0| < \delta - \varepsilon + \varepsilon = \delta$$

independently of $t \geq 0$. It follows that $\bar{g}(t)$ remains in $B(\mathcal{X}, g_0, \delta)$ for all time and hence converges to a unique flat metric. By Proposition 3.6, the Ricci flow of any metric starting sufficiently near g_0 also converges to a unique flat metric. □

3.6. Stability of solutions that become flat.

There are various families $\mathcal{F}$ of metrics for which it is known that if $g(t)$ is the solution of the Ricci flow starting at some $g_0 \in \mathcal{F}$, then the flow

$g(t)$ necessarily converges to a flat metric. This is true, for example, for the polarized Gowdy metrics [CIJ], for direct-product metrics $(\mathcal{T}^2, \mu) \times (\mathcal{S}^1, dx^2)$ with μ an arbitrary Riemannian metric on $\mathcal{T}^2$ [H4], and for square torus bundles over $\mathcal{S}^1$ (§11 of [H5]). Note that all of these families are characterized by isometries rather than curvature restrictions.

A straightforward corollary to Theorem 3.1 shows that if the Ricci flow $h(t)$ of a metric starts sufficiently near one of these families, it too must converge to a flat metric. We emphasize that $h(0)$ need not share the isometries of the family $\mathcal{F}$.

Corollary 3.7. *Let $g(t)$ be a solution of the Ricci flow that converges to a flat metric g_∞. Then there is a $\|\cdot\|_{2+\rho}$ neighborhood $\mathcal{O}$ of $g(0)$ in $\mathcal{S}_2^+$ such that every solution $h(t)$ of the Ricci flow with $h(0) \in \mathcal{O}$ converges to a flat metric h_∞ near g_∞.*

Proof. We fix $g(t)$ and its limit g_∞. It follows from Theorem 3.1 that there exists a neighborhood $\mathcal{N}$ of g_∞ such that the Ricci flow of any metric starting in $\mathcal{N}$ converges to a flat metric near g_∞. Since $g(t)$ converges to g_∞, there exists a time $T = T(g(0), \mathcal{N})$ such that $g(t) \in \mathcal{N}$ for $t > T$. Choose $\varepsilon > 0$ small enough that $B(\mathcal{X}, g(2T), \varepsilon) \subseteq \mathcal{N}$. Since Ricci flow for finite time is a continuous map, there exists $\delta > 0$ such that for all $h(0) \in B(\mathcal{X}, g(0), \delta)$, we have $h(2T) \in B(\mathcal{X}, g(2T), \varepsilon)$. It follows that $h(t)$ converges to a flat metric near g_∞. □

4. Other results.

The intent of this section is to stimulate further research by demonstrating that our methods are applicable to other questions of stability regarding the Ricci flow. It should be noted, however, that the results stated below are weaker than our main theorem, either because they are incomplete or because they in some sense rediscover known results.

4.1. Stability at metrics which are Ricci flat but not flat.

In this section, we consider the DeTurck flow

$$\frac{\partial}{\partial t}\bar{g} = -2\overline{\mathrm{Rc}} - P_{g_0}(\bar{g}), \qquad \bar{g}(0) = \bar{g}_0, \tag{25}$$

where $(\mathcal{M}^n, g_0)$ is a given Ricci-flat (but not flat) geometry about which we wish to determine stability. (Such geometries exist only for $n \geq 4$.) Note that $g(t) \equiv g_0$ is a stationary solution of both the Ricci flow (1) and of (25). On the other hand, if $\hat{g}_0$ is another Ricci-flat metric on $\mathcal{M}^n$, then $\hat{g}(t) \equiv \hat{g}_0$ is a stationary solution of the Ricci flow (1) but not necessarily of (25). Nonetheless, in this case (25) reduces to

$$\frac{\partial}{\partial t}\hat{g} = -P_{g_0}(\hat{g}),$$

which is just the Lie derivative of $\hat{g}$. So $\hat{g}(t)$ moves only by diffeomorphisms, and in particular remains Ricci-flat.

If we linearize (25) at the distinguished Ricci-flat metric g_0, the $\Psi_u h$ term in equation (16) vanishes, so that we obtain

$$\frac{\partial}{\partial t}h_{ij} = \Delta_\ell h_{ij} = \Delta h_{ij} + 2R_{ipqj}h^{pq}.$$

Thus in order to understand the stability of the DeTurck flow near g_0, it is necessary to analyze the spectrum of the Lichnerowicz Laplacian on a Ricci-flat manifold. Since Δ_ℓ is elliptic and self adjoint, we know that its spectrum is real, discrete, of finite multiplicity, and has no positive accumulation point. It follows immediately that Theorem 2.2 can be applied at the Ricci-flat metric g_0, with the function spaces $\mathcal{X}_0$, $\mathcal{X}_1$, $\mathcal{E}_0$, and $\mathcal{E}_1$ chosen as in §3.1. Thus there is for each $r \in \mathbb{N}$ a C^r center (unstable) manifold at g_0, and the flow of nearby metrics will approach it. But to obtain useful information about the dynamics of the Ricci flow near g_0, we need to know much more about those center manifolds. To do this, we decompose the tangent space $T_{g_0}\mathcal{S}_2^+$ at a Ricci-flat metric g_0 into a number of subspaces and relate these to the spectrum of Δ_ℓ, in order to describe the tangent space to the center manifolds at g_0. We carry out this analysis in Lemmas 4.1–4.7, and summarize our results in Proposition 4.8. This is a first step toward understanding the dynamics. For the special case that g_0 is a Kähler–Einstein metric on a $K3$ surface, we obtain a stronger result (Theorem 4.3) that falls just short of determining stability, as has been done above for flat metrics.

To simplify notation, let us assume for now that g is a fixed Ricci-flat metric. To start our analysis of the spectrum of its Lichnerowicz Laplacian, we recall that the Hodge–de Rham Laplacian is the map $\Delta_d : \Omega^p \to \Omega^p$ given by

$$\Delta_d = -(d\delta + \delta d). \tag{26}$$

Note that our sign convention is opposite to the standard one, but is more convenient for studying heat flows. It is well known [L] that for any Ricci-parallel manifold (namely, any manifold for which $\nabla \mathrm{Rc} \equiv 0$), one has

$$\Delta_\ell \delta^* = \delta^* \Delta_d \tag{27}$$

and

$$\delta \Delta_\ell = \Delta_d \delta. \tag{28}$$

If moreover $\mathrm{Rc} \equiv 0$, we note that Δ_d on Ω^1 reduces to

$$\Delta_d \omega_i \doteqdot -\left((d\delta + \delta d)\,\omega\right)_i = \Delta\omega_i - R_i^j \omega_j = \Delta\omega_i. \tag{29}$$

We follow [Bu] in defining certain subspaces of $\mathcal{S}_2 \cong T_g \mathcal{S}_2^+$; for ease of notation, we suppress the subscript indicating dependence on g. We set

$$C \doteqdot \left\{\delta^*(\delta\eta) : \eta \in \Omega^2\right\} \subseteq \mathcal{V} \tag{30}$$
$$E \doteqdot \{\nabla\nabla f : f \in C^\infty(\mathcal{M})\} \subseteq \mathcal{V} \tag{31}$$
$$Z \doteqdot \left\{\nabla\omega : \omega \in H_\Delta^1\right\} \subseteq \mathcal{V}, \tag{32}$$

where H_Δ^1 denotes the space of harmonic 1-forms (defined in §2) and

$$N \doteqdot \{h \in \mathcal{S}_2 : \delta h = 0,\ \operatorname{tr} h = 0\} \subseteq \mathcal{H} \tag{33}$$
$$S \doteqdot \{(\Delta f + \alpha)\, g - \nabla\nabla f : f \in C^\infty(\mathcal{M}, \mathbb{R}),\ \alpha \in \mathbb{R}\} \subseteq \mathcal{H} \tag{34}$$
$$G \doteqdot \{\alpha g : \alpha \in \mathbb{R}\} \subset S. \tag{35}$$

Recalling that $\mathcal{V}$ and $\mathcal{H}$ are defined in (11), it is easy to check the indicated inclusions.

We now make a number of claims regarding these subspaces. Many of these claims are similar to those in [Bu]. However, negative scalar curvature is assumed in that paper, whereas we have $\mathrm{Rc} \equiv 0$. Thus (since the proofs are short) we verify the results directly.

Lemma 4.1. *Each of the spaces defined in* (30)–(35) *is an invariant subspace for Δ_ℓ, in the sense that $\Delta_\ell h$ belongs to the space whenever h does.*

Proof. Invariance of $\mathcal{V}$ is a trivial consequence of (27). Invariance of C follows from (27) and the fact that $\Delta_d \delta = \delta \Delta_d$ as maps $\Lambda^2 \to \Lambda^1$ on any manifold. Invariance of E and Z follows from (27) and the fact that $\nabla\omega = \delta^*\omega$ for any closed 1-form ω on any manifold.

Invariance of $\mathcal{H}$ follows from (28). Invariance of N also follows from (28) and the identity $\operatorname{tr}(\Delta_\ell h) = \Delta(\operatorname{tr} h)$. The invariance of G is clear, and that of S when $\mathrm{Rc} \equiv 0$ follows from the computation

$$\Delta_\ell\left[(\Delta f + \alpha)\, g_{ij} - \nabla_i\nabla_j f\right] = (\Delta\Delta f)\, g_{ij} - \nabla_i\nabla_j\Delta f. \tag{36}$$

□

Lemma 4.2. *The spaces C, E, Z, N, and S are pairwise orthogonal with respect to the L^2 inner product $(\cdot,\cdot) \doteqdot \int_{\mathcal{M}} \langle\cdot,\cdot\rangle\, d\mu$.*

Proof. For any closed 1-form θ, we have $\delta\delta^*\theta = -\Delta\theta = -\Delta_d\theta$ by (29). So let $c = \delta^*\delta\eta \in C$, $e = \nabla\nabla f = \delta^* df \in E$, and $z = \nabla\omega = \delta^*\omega \in Z$ be arbitrary. Then $C \perp Z$, because

$$(c, z) = (\delta^*\delta\eta, \delta^*\omega) = -(\delta\eta, \Delta_d\omega) = 0.$$

Similarly, $E \perp Z$, because

$$(e, z) = -(df, \Delta_d\omega) = 0.$$

And $C \perp E$, because

$$(c, e) = (\delta^*\delta\eta, \delta^* df) = -(\delta\eta, \Delta_d df) = (\delta\eta, d\delta df) = \left(\delta^2\eta, \delta df\right) = 0.$$

To finish the proof, it suffices to show that $N \perp S$, because $C, E, Z \subseteq \mathcal{V}$ and $N, S \subseteq \mathcal{H}$. If $h \in N$ and $s = (\Delta f + \alpha)\, g - \nabla\nabla f \in S$, then

$$(h, s) = \int \langle(\Delta f + \alpha)\, g - \nabla\nabla f,\, h\rangle\, d\mu = -\int \langle\nabla\nabla f, h\rangle\, d\mu = -(df, \delta h) = 0.$$

□

Lemma 4.3. $\mathcal{H} = N \oplus S$.

Proof. We already know that $N \oplus S \subseteq \mathcal{H}$. Given $h \in \mathcal{H}$, define $H \doteqdot \operatorname{tr} h$ and $\bar{H} \doteqdot \int H\, d\mu$. Then if $V \doteqdot \operatorname{Vol}(\mathcal{M}, g)$ and $\alpha \doteqdot \bar{H}/nV$, we have

$$\int (H - n\alpha)\, d\mu = 0.$$

So there is a unique solution $f \in C^\infty$ of the Poisson problem

$$\Delta f = \frac{H - n\alpha}{n-1}, \qquad \int (\operatorname{tr} f) = 0.$$

Set $s = (\Delta f + \alpha)\, g - \nabla\nabla f$. Then $s \in S$ and

$$\operatorname{tr} s = g^{ij}\left[(\Delta f + \alpha)\, g_{ij} - \nabla_i \nabla_j f\right] = (n-1)\,\Delta f + n\alpha = H = \operatorname{tr} h$$

Hence $(h - s) \in N$, which completes the proof. □

We are now ready to analyze the spectrum of Δ_ℓ on the spaces (30)–(35).

Lemma 4.4. *Δ_ℓ vanishes on Z, which is at most n-dimensional.*

Proof. If $z = \nabla\omega = \delta^*\omega \in Z$, then $\omega \in H^1_\Delta$ by definition. So $\Delta_\ell \delta^* \omega = \delta^* \Delta_d \omega = 0$ by (27). This proves the first assertion. The second follows from Bochner's theorem, which says that any harmonic 1-form on a closed manifold of non-negative Ricci curvature is parallel:

$$0 = \int \langle \Delta_d \omega, \omega \rangle \, d\mu = -\int \left(|\nabla\omega|^2 + \operatorname{Rc}(\omega, \omega) \right) d\mu \leq -\int |\nabla\omega|^2 \, d\mu.$$

□

Lemma 4.5. *$\Delta_\ell < 0$ on E.*

Proof. Let $e = \nabla\nabla f = \delta^* df \in E$ be arbitrary. Since $\nabla \operatorname{Rc} \equiv 0$, we have

$$\Delta_\ell e_{ij} = \Delta \nabla_i \nabla_j f + 2R_{ipqj} \nabla^p \nabla^q f = \nabla_i \Delta \nabla_j f,$$

and hence

$$(\Delta_\ell e, e) = \int \nabla_i \Delta \nabla_j f \nabla^i \nabla^j f \, d\mu = -\int |\Delta \nabla f|^2 \, d\mu \leq 0.$$

Equality is possible only if $0 \equiv \Delta \nabla f = \Delta_d df$, hence only if $e \in E \cap Z = \{0\}$. □

Lemma 4.6. *$\Delta_\ell < 0$ on C.*

Proof. Let $c = \delta^* \delta \eta \in C$ be arbitrary, and write $\omega \doteqdot \delta n$. Since by (27),

$$\Delta_\ell \delta^* \delta \eta = \delta^* \Delta_d \delta \eta = -\delta^* (d\delta + \delta d)\, \delta \eta = -\delta^* \delta d\omega,$$

we have

$$(\Delta_\ell c, c) = -\left(\delta^* \delta d\omega, \delta^* \omega\right) = -\left(\delta d\omega, \delta\delta^* \omega\right).$$

But since $\mathrm{Rc} \equiv 0$, we get

$$\begin{aligned}(\delta d\omega)_j &= -\nabla^i (\nabla_i \omega_j - \nabla_j \omega_i) = -\Delta\omega_j + \nabla_j \nabla^i \omega_i \\ &= -\Delta\omega_j - \nabla_j \delta\omega = -\Delta\omega_j - \nabla_j (\delta^2 \eta) = -\Delta\omega_j,\end{aligned}$$

and similarly

$$(\delta\delta^* \omega)_j = -\frac{1}{2}\nabla^i (\nabla_i \omega_j + \nabla_j \omega_i) = -\frac{1}{2}\Delta\omega_j.$$

Hence by (29), we have $(\Delta_\ell c, c) = -\int |\Delta_d \omega|^2 \, d\mu \leq 0$, with equality only if $\omega \in H^1_\Delta$, hence only if $c \in C \cap Z = \{0\}$. □

Lemma 4.7. *Δ_ℓ vanishes on the 1-dimensional subspace G, and $\Delta_\ell < 0$ on $S \backslash G$.*

Proof. The first statement is clear. Let $s = [(\Delta f + \alpha) g_{ij} - \nabla_i \nabla_j f] \in S$ be arbitrary. Since $\mathrm{Rc} \equiv 0$, we have $\nabla_i \nabla_j \nabla^i f = \nabla_j \Delta f$. So by (36), we get

$$\begin{aligned}(\Delta_\ell s, s) &= \int [(\Delta\Delta f) g_{ij} - \nabla_i \nabla_j \Delta f] [(\Delta f + \alpha) g_{ij} - \nabla_i \nabla_j f] \, d\mu \\ &= (n-2) \int \Delta\Delta f \Delta f + (n-1)\alpha \int \Delta\Delta f + \int \langle \nabla\nabla\Delta f, \nabla\nabla f \rangle \, d\mu \\ &= -(n-1) \int |\nabla\Delta f|^2 \, d\mu \leq 0.\end{aligned}$$

Equality is possible only if Δf is constant, hence only if $\Delta f \equiv 0$, hence only if f is constant. □

As noted above, since Δ_ℓ is elliptic and self adjoint, we may readily apply Theorem 2.2 and thereby determine that center manifolds exist for the dynamics of the Ricci flow near a Ricci-flat metric. Combining this with the results of Lemmas 4.1 –4.7, we are able to make the following observation:

Proposition 4.8. *Let $(\mathcal{M}^n, g_0)$ be Ricci flat; and for fixed $\rho \in (0,1)$, let $\mathcal{X}$ denote the closure of $\mathcal{S}_2 \supset \mathcal{S}_2^+$ with respect to the $\|\cdot\|_{2+\rho}$ Hölder norm.*

1. *$T_{g_0}\mathcal{S}_2^+ \cong \mathcal{X}$ admits the decomposition $T_{g_0}\mathcal{S}_2^+ = \mathcal{X}^s \oplus \mathcal{X}^{cu}$, where $\mathcal{X}^{cu}$ is finite dimensional. The eigenspace corresponding to the zero eigenvalue of the linearization of the DeTurck flow at g_0 contains the space*

$Z \oplus G$ of dimension $b_1 + 1$, where $b_1 \doteqdot \dim H^1(\mathcal{M}, \mathbb{R})$ is the first Betti number of $\mathcal{M}$, and possibly a subspace of N. If any positive eigenvalues of the linearization exist, their eigenspaces are the closures of finite-dimensional subspaces of N.

2. *For each $r \in \mathbb{N}$, there is a C^r center (unstable) manifold $\mathcal{M}^{cu}_{loc}$ existing in a neighborhood $\mathcal{O}_r$ of g_0 in $\mathcal{X}$. Each center (unstable) manifold $\mathcal{M}^{cu}_{loc}$ is tangential to $\mathcal{X}^{cu}$ and is locally invariant for solutions of* (25) *as long as they remain in $\mathcal{O}_r$.*

3. *There are positive constants C and ω, and neighborhoods $\mathcal{O}'_r$ of g_0 in $\mathcal{X}$ defined for all $r \in \mathbb{N}$, such that*

$$\|\pi^s \tilde{g}(t) - \psi(\pi^{cu}\tilde{g}(t))\|_{\mathcal{X}} \leq Ce^{-\omega t} \|\pi^s \tilde{g}(0) - \psi(\pi^{cu}\tilde{g}(0))\|_{\mathcal{X}}$$

for all solutions $\tilde{g}(t)$ of (25) *and all times $t \geq 0$ such that $\tilde{g}(t) \in \mathcal{O}'_r$. (The projections π^s and π^{cu} here are those defined in Theorem* 2.2.)

We now consider the special case of a Ricci-flat metric on a $K3$ surface.

Definition 4.9. A $K3$ surface is a closed connected smooth complex surface with vanishing first Chern class and no global holomorphic 1-form.

A $K3$ surface is a 2-dimensional complex manifold, hence a 4-dimensional real manifold. In fact, each $K3$ surface is diffeomorphic to a unique simply-connected orientable manifold, namely the quartic hypersurface

$$\mathcal{M}^4 = \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 : \sum_{j=0}^{3} z_j^4 = 0 \right\} \subset \mathbb{CP}^3.$$

Siu has proven [Si] that every $K3$ admits some Kähler metric, and Yau's proof [Y] of the Calabi conjecture shows that each Kähler class of a $K3$ contains a unique Ricci-flat Kähler metric. (For general background, the reader is referred to [P].)

We are interested in fixing a Ricci-flat Kähler metric g_0 on the $K3$ surface $\mathcal{M}^4$, and considering the Ricci flow $\tilde{g}(t)$ of metrics for which $\tilde{g}(0)$ is $\|\cdot\|_{2+\rho}$ close to g_0. Proposition 4.8 applies, but we shall be able to say more about the center manifolds in this special case.

Our first observation is that the kernel of the Lichnerowicz Laplacian is well understood for $K3$ geometries. Indeed, for any Riemannian manifold $(\mathcal{M}^n, g)$, let $\varepsilon(g)$ denote the space of infinitesimal Einstein deformations of g. (See 12.29 of [B].) The usual definition of $\varepsilon(g)$ is equivalent by [BE] to the following characterization, which is most convenient for our purposes:

Definition 4.10. An element $h \in \mathcal{S}_2$ is an *infinitesimal Einstein deformation* of g if and only if $h \in N$ and satisfies

$$\Delta h_{ij} + 2R_{ipqj}h^{pq} = 0.$$

It is clear that $\varepsilon(g)$ coincides with the kernel of $\Delta_\ell | N$ on any Ricci-flat manifold. This space can be described exactly [B1]:

Theorem 4.1. *If $(\mathcal{M}^4, g_0)$ is a Ricci-flat Kähler metric on a $K3$ surface, then $\varepsilon(g_0)$ is isomorphic to the tensor product of the 3-dimensional space of parallel self-dual 2-forms and the 19-dimensional space of harmonic anti-self-dual 2-forms.*

In general, the fact that $\frac{d}{d\varepsilon} \operatorname{Rc}[g + \varepsilon h]\big|_{\varepsilon=0} = 0$ says nothing about $\operatorname{Rc}[g + \varepsilon h]$ for $0 < \varepsilon \ll 1$. It is thus a remarkable fact that the infinitesimal deformations of a Kähler–Einstein metric on a $K3$ surface actually correspond to Ricci-flat metrics [T1], [T2]:

Theorem 4.2. *Let g_0 be a Kähler–Einstein metric on a $K3$ surface $\mathcal{M}^4$. Then there is a submanifold $\mathcal{E} \subset \mathcal{S}_2^+$ of Ricci-flat metrics near g_0 with*

$$T_{g_0}\mathcal{E} = \varepsilon(g_0).$$

Remark 4.11. The theorem implies in particular that $\mathcal{E}' \doteqdot \{\lambda g : \lambda > 0, g \in \mathcal{E}\}$ is a 58-dimensional family of metrics that evolve only by diffeomorphisms under the DeTurck flow (25).

With this understanding of the eigenspace corresponding to the zero eigenvalue of Δ_ℓ, our remaining task is to elucidate the eigenspaces corresponding to positive eigenvalues of Δ_ℓ, should any exist. Proposition 4.8 tells us that any such eigenspaces must be subspaces of N. We now show that for a Ricci-flat $K3$ geometry, no such spaces exist. To do this let us recall some standard facts about 4-dimensional geometries. On any oriented Riemannian manifold $(\mathcal{M}^n, g)$, the Hodge operator $* : \Lambda^p \to \Lambda^{n-p}$ is defined for $0 \leq p \leq n$ by $\alpha \wedge (*\beta) = \langle \alpha, \beta \rangle \mu$, where μ is the volume form of g; it satisfies $*^2 = (-1)^{p(n-p)} \operatorname{id}_{\Lambda^p}$. If $n = 4$, this induces a natural decomposition $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ into the self-dual and anti-self-dual eigenspaces of $*$ corresponding to $+1$ and -1, respectively. If $\{e_i\}$ is an orthonormal moving frame with dual coframe $\{\theta^i\}$ on an open set $\mathcal{U} \subseteq \mathcal{M}$, and $\left\{\eta^k = \eta^k_{ij}\, \theta^i \wedge \theta^j\right\}$

is an orthonormal basis of Λ^2, say

$$\eta^1 = \frac{1}{\sqrt{2}}\theta^1 \wedge \theta^2, \quad \eta^2 = \frac{1}{\sqrt{2}}\theta^1 \wedge \theta^3, \quad \eta^3 = \frac{1}{\sqrt{2}}\theta^1 \wedge \theta^4,$$
$$\eta^4 = \frac{1}{\sqrt{2}}\theta^2 \wedge \theta^3, \quad \eta^5 = \frac{1}{\sqrt{2}}\theta^2 \wedge \theta^4, \quad \eta^6 = \frac{1}{\sqrt{2}}\theta^3 \wedge \theta^4,$$

it is easily checked that $\{\eta^1 \pm \eta^6, \eta^2 \mp \eta^5, \eta^3 \pm \eta^4\}$ is an orthogonal basis of $\Lambda^\pm$. Let Λ_*^+ denote the self-dual 2-forms of norm 1, and let S_2^0 denote the bundle of traceless symmetric $(2,0)$-tensors. Then it is well known [B2] that there is a natural isomorphism $\sigma : \Lambda_*^+ \otimes \Lambda^- \to S_2^0$ given by $\sigma : \alpha \otimes \beta \mapsto \alpha \boxtimes \beta$, where

$$\alpha \boxtimes \beta \doteqdot \sum_{k=1}^{4} (\iota_{e_k}\alpha) \otimes (\iota_{e_k}\beta), \tag{37}$$

and $(\iota_X \eta)(Y) \doteqdot \eta(X,Y)$. The bases $\{\eta^1 \pm \eta^6, \eta^2 \mp \eta^5, \eta^3 \pm \eta^4\}$ induce a block decomposition of any linear map $\Lambda^2 \to \Lambda^2$, in particular of the self-adjoint map $\mathrm{Rm} : \Lambda^2 \to \Lambda^2$ defined by

$$\langle \mathrm{Rm}(\eta^p), \eta^q \rangle = R_{ijk\ell}\eta^p_{ij}\eta^q_{\ell k}, \tag{38}$$

where $R_{ijk\ell}$ denotes a component of the Riemann curvature tensor with respect to the coframe $\{\theta^i\}$. Now in any dimension, one also has the orthogonal decomposition

$$\mathrm{Rm} = \frac{R}{2n(n-1)}(g \odot g) + \frac{1}{n-2}\left(\overset{\circ}{\mathrm{Rc}} \odot g\right) + W \tag{39}$$

that defines the Weyl tensor W, where $\overset{\circ}{\mathrm{Rc}}$ is the trace-free part of the Ricci tensor, and $\odot$ denotes the Kulkarni–Nomizu product of symmetric tensors. If $(\mathcal{M}^4, g)$ is Ricci flat, one may combine these points of view to identify Rm with the block decomposition [ST] of the Weyl tensor,

$$\mathrm{Rm} = W = \begin{pmatrix} W^+ & \\ & W^- \end{pmatrix}, \tag{40}$$

where each block $W^\pm$ is self adjoint and trace free.

Lemma 4.12. *Let $(\mathcal{M}^4, g)$ be a Kähler–Einstein metric on a $K3$ surface. Let $\alpha \in \Lambda_*^+$ and $\beta \in \Lambda^-$. Then*

$$2\,\mathrm{Rm} \circ (\alpha \boxtimes \beta) = \alpha \boxtimes (W^-(\beta)),$$

where $(\mathrm{Rm} \circ S)_{ij} \doteqdot R_{ipqj}S^{pq}$ denotes the natural action of the curvature operator on a symmetric tensor.

Proof. It is well known that any Calabi–Yau metric on a $K3$ satisfies $W^+ \equiv 0$. (See [B1] or 13.17 of [B].) Calculating with respect to the orthonormal basis $\{e_i\}$, we have

$$\left(\alpha \boxtimes W^-(\beta)\right)_{ij} = \alpha_{ki} W^-_{kjqp} \beta_{pq} = \alpha_{ik} W^-_{jkqp} \beta_{pq}.$$

Then using the symmetry of the product between self-dual and anti-self-dual 2-forms, we get

$$(\mathrm{Rm} \circ (\alpha \boxtimes \beta))_{ij} = W^-_{ipqj} \alpha_{kp} \beta_{kq} = \alpha_{ip} W^-_{kpqj} \beta_{kq}.$$

Hence by the first Bianchi identity,

$$\begin{aligned}(\mathrm{Rm} \circ (\alpha \boxtimes \beta))_{ij} &= -\alpha_{ip} \left(W^-_{kqjp} + W^-_{kjpq}\right) \beta_{kq} \\ &= \alpha_{ip} W^-_{jpqk} \beta_{kq} - \alpha_{ip} W^-_{qpkj} \beta_{qk} \\ &= \left(\alpha \boxtimes W^-(\beta)\right)_{ij} - (\mathrm{Rm} \circ (\alpha \boxtimes \beta))_{ij}.\end{aligned}$$

□

We are now able to show that Δ_ℓ has no positive eigenvalues:

Corollary 4.13. *If $(\mathcal{M}^4, g)$ is a Kähler–Einstein metric on a $K3$ surface, then $\Delta_\ell \le 0$ on N and $\Delta_\ell < 0$ on $N \backslash \varepsilon(g)$.*

Proof. Recall the general formula for the Hodge-de Rham Laplacian acting on a 2-form:

$$\Delta_d \eta_{ij} = -\left[(d\delta + \delta d)\,\eta\right]_{ij} = \Delta \eta_{ij} + 2R_{ipqj}\eta_{pq} - R_{ik}\eta_{kj} - R_{jk}\eta_{ik}.$$

Let $\alpha \in \Lambda^+_*$ and $\beta \in \Lambda^-$ on a Kähler–Einstein $K3$. Then

$$\left(W^-(\beta)\right)_{ij} = W^-_{ijpq}\beta_{qp} = -\left(R_{ipqj} + R_{iqjp}\right)\beta_{qp} = 2R_{ipqj}\beta_{pq}.$$

Hence by Lemma 4.12, we obtain the useful identity

$$\begin{aligned}(\Delta_\ell(\alpha \boxtimes \beta))_{ij} &= (\Delta(\alpha \boxtimes \beta))_{ij} + 2\,(\mathrm{Rm} \circ (\alpha \boxtimes \beta))_{ij} \\ &= (\Delta\alpha \boxtimes \beta)_{ij} + (\alpha \boxtimes \Delta_d \beta)_{ij} + 2\nabla_p \alpha_{ki} \nabla_p \beta_{kj}.\end{aligned}$$

Integrating by parts and recalling that $|\alpha| \equiv 1$, we get

$$(\Delta_\ell(\alpha \boxtimes \beta),\, \alpha \boxtimes \beta) = -\int |\nabla\alpha|^2 |\nabla\beta|^2 \, d\mu + \int \langle \Delta_d \beta, \beta \rangle \, d\mu \le 0.$$

Since there are no parallel anti-self-dual forms, equality is possible only if α is parallel and β is harmonic. $\square$

Since $b_1 = 0$ on a $K3$ surface, we have thus proved the following:

Theorem 4.3. *Let $(\mathcal{M}^4, g_0)$ be a Kähler–Einstein metric on a $K3$ surface. For fixed $\rho \in (0,1)$, let $\mathcal{X}$ denote the closure of $\mathcal{S}_2 \supset \mathcal{S}_2^+$ with respect to the $\|\cdot\|_{2+\rho}$ Hölder norm.*

1. *$T_{g_0}\mathcal{S}_2^+ \cong \mathcal{X}$ admits the decomposition $T_{g_0}\mathcal{S}_2^+ = \mathcal{X}^s \oplus \mathcal{X}^c$. The space $\mathcal{X}^c$ is the closure of $\varepsilon(g_0) \oplus G$, where $\varepsilon(g_0)$ is isomorphic to the tensor product of a 3-dimensional space of parallel self-dual 2-forms and a 19-dimensional space of harmonic anti-self-dual 2-forms. $\mathcal{X}^c$ is thus 58-dimensional.*
2. *For each $r \in \mathbb{N}$, there is a C^r center manifold $\mathcal{M}^c_{loc}$ that exists in a neighborhood $\mathcal{O}_r$ of g_0 in $\mathcal{X}$. Each center manifold is tangent to $\mathcal{X}^c$ and is locally invariant for solutions of (25) as long as they remain in $\mathcal{O}_r$.*
3. *There are positive constants C and ω, and neighborhoods $\mathcal{O}'_r$ of g_0 in $\mathcal{X}$ defined for all $r \in \mathbb{N}$, such that*
$$\|\pi^s\tilde{g}(t) - \psi(\pi^c\tilde{g}(t))\|_{\mathcal{X}} \le Ce^{-\omega t}\|\pi^s\tilde{g}(0) - \psi(\pi^c\tilde{g}(0))\|_{\mathcal{X}}$$
for all solutions $\tilde{g}(t)$ of (25) and all times $t \ge 0$ such that $\tilde{g}(t) \in \mathcal{O}'_r$.

Remark 4.14. As mentioned in the introduction, Cao has shown [C] that any Kähler metric on a $K3$ surface converges under the Ricci flow to a Ricci-flat Kähler–Einstein metric. His result does not imply Theorem 4.3 however, because we consider the Ricci flow of all metrics near g_0, not just Kähler metrics.

4.2. Stability at other Einstein metrics.

If $(\mathcal{M}^n, g_0)$ is a Riemannian manifold of constant nonzero sectional curvature, it is not possible to choose u so that g_0 becomes a fixed point of the DeTurck flow. So we modify our method, proceeding in two steps.

First we apply the DeTurck trick to the volume-normalized Ricci flow (2), obtaining

$$\frac{\partial}{\partial t}g = \mathcal{A}_u g + \frac{2}{n}\left(\oint R\,d\mu\right)g, \qquad g(0) = g_0. \tag{41}$$

Clearly, Theorem 2.1 applies to this equation as well. Moreover, every metric g_0 of constant curvature becomes a fixed point of (41) if we again choose $u = g_0$. By straightforward calculation, it follows from Proposition 3.2, Lemma 2.1, and the formula for Rm on a manifold of constant sectional curvature that the linearization of (41) at g takes the form

$$\begin{aligned} \frac{\partial}{\partial t} h_{ij} &= \Delta h_{ij} + 2R_{ipqj}h^{pq} - \frac{2R}{n^2}\left(\frac{\int H\,d\mu}{\int d\mu}\right) g_{ij} \\ &= \Delta h_{ij} + \frac{R}{n\,(n-1)}\,(Hg_{ij} - h_{ij}) - \frac{2R}{n^2}\left(\frac{\int H\,d\mu}{\int d\mu}\right) g_{ij}, \end{aligned} \tag{42}$$

where $H \doteqdot g^{ij}h_{ij}$.

Then we restrict our attention to the space $\mathcal{S}_2^{\mu}$ of metrics on $\mathcal{M}$ which have the same volume element as g_0. This involves no loss of generality, since by [M], any metric in $\mathcal{S}_2^{+}$ can be transformed into an element of $\mathcal{S}_2^{\mu}$ by homothetic rescaling and an action of $\mathcal{D}(\mathcal{M})$. Moreover, $\mathcal{S}_2^{\mu}$ has rather nice geometric properties: Ebin's slice theorem applies to $\mathcal{S}_2^{\mu}$ (see §8 of [E]), implying in particular that $\mathcal{S}_2^{\mu}$ is 'almost' an infinite-dimensional symmetric space whose tangent space $\mathcal{S}_2^{0}$ consists exactly of those elements of $\mathcal{S}_2$ of trace zero. Moreover, the subset $\mathcal{D}_{\mu}(\mathcal{M}) \subset \mathcal{D}(\mathcal{M})$ of diffeomorphisms preserving $d\mu$ is a closed Lie subgroup (Theorem 2.5.3 of [H1]).

Thus on $T\mathcal{S}_2^{\mu}$, we have $H \equiv 0$, whence equation (42) reduces to

$$\frac{\partial}{\partial t} h_{ij} = L h_{ij} \doteqdot \Delta h_{ij} - \frac{R}{n\,(n-1)} h_{ij}. \tag{43}$$

Since when $R > 0$, we have

$$(Lh, h) = -\|\nabla h\|_{L^2}^2 - \frac{R}{n\,(n-1)}\|h\|_{L^2}^2 < 0$$

for any nonzero $h \in \mathcal{S}_2^{0}$, we can apply the construction in §3.3 to $\mathcal{S}_2^{\mu}$ and thereby obtain the following:

Proposition 4.15. *Let $(\mathcal{M}^n, g_0)$ be a metric of constant positive curvature. Then there is a neighborhood $\mathcal{O}$ of g_0 in $\mathcal{S}_2^{\mu}$ with the $\|\cdot\|_{2+\rho}$ Hölder norm such that every $\tilde{g} \in \mathcal{O}$ converges exponentially to g_0 under the flow* (41).

Remark 4.16. We include this result merely as an illustration of the method. It does *not* provide an alternative proof of Hamilton's convergence theorems for $n = 3$ [H2] and $n = 4$ [H3], nor of Huisken's result for $n \geq 4$ [Hu].

4.3. Ricci solitons.

Suppose $(\mathcal{M}^n, g(t))$ is a steady Ricci gradient soliton with $g(0) = g_0$. Then

$$g(x,t) = (\theta_t^* g_0)(x)$$

for some family θ_t of diffeomorphisms generated by vector fields $-X(t)$ whose dual 1-forms are closed. In particular,

$$\mathrm{Rc} = \nabla\nabla f,$$

where $X(t) = \nabla f(t)$ for some 1-parameter family of smooth functions f on $\mathcal{M}$. In dimension $n \geq 3$, the choice

$$u = e^{\frac{2}{n-2}f} g_0$$

makes g_0 a fixed point of the DeTurck flow, because at $t = 0$ one has

$$(P_u g)_{ij} = -2\nabla_i \nabla_j f = -2R_{ij}.$$

The corresponding linearization at g_0 is given by

$$\begin{aligned}\frac{\partial}{\partial t} h_{ij} &= \Delta h_{ij} + \nabla^k h_{ij} \nabla_k f - 2\frac{n-3}{n-2}\left(\nabla_i h_j^k + \nabla_j h_i^k\right)\nabla_k f \\ &\quad - \frac{1}{n-2}\left(\nabla_i H \nabla_j f + \nabla_i f \nabla_j H\right) + 2R_{ipqj}h^{pq} \\ &\quad - 2\frac{n-3}{n-2}\left(\nabla_i \nabla_k f h_j^k + \nabla_j \nabla_k f h_i^k\right) - \frac{2}{n-2} H \nabla_i \nabla_j f.\end{aligned}$$

However, we have not yet extensively studied the spectrum of the operator that results from this construction.

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