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# Schrödinger Flow on Hermitian Locally Symmetric Spaces [1]

PETER Y.H. PANG, HONG-YU WANG AND YOU-DE WANG

In this paper, we show that there exist global (inhomogeneous) Schrödinger flows from the real line $R^1$ as well as the circle $S^1$ into Hermitian locally symmetric spaces. Moreover, the Schrödinger flows obey a conservation law. Via the correspondence between the Schrödinger flow on complex Grassmannians and the matrix nonlinear Schrödinger equation (focusing case) on the real line $R^1$, Terng and Uhlenbeck recently established the existence of global Schrödinger flow from $R^1$ into complex Grassmannian manifolds using methods of complete integrability and inverse scattering. In a particular case, our result provides a geometric analytic approach to this global existence result on $R^1$.

## 1. Introduction.

Let $N$ be a complete Kähler manifold equipped with a Kähler form $\omega$, a complex structure $J$, and the Kähler metric $h(\cdot,\cdot) = \omega(\cdot, J\cdot)$. Then, given a map $u_0$ from a Riemannian manifold $(M, g)$ into $N$, the Schrödinger flow (see [5]) $u(\cdot, t) : M \to N$ for $u_0$ is defined by the Cauchy problem

$$\begin{cases} \partial_t u = J(u)\tau(u), \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

Here, $\tau(u)$ is the tension field of $u$; in local coordinates,

$$\tau^\alpha(u) = \Delta u^\alpha + g^{ij}\Gamma^\alpha_{\beta\gamma}(u)\frac{\partial u^\beta}{\partial x^i}\frac{\partial u^\gamma}{\partial x^j},$$

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where $\Delta$ is the Laplace-Beltrami operator on $M$ with respect to the metric $g$ and $\Gamma^{\alpha}_{\beta\gamma}$ are the Christoffel symbols of the target manifold $N$ (see [7]).

In [20], H.Y. Wang and Y.D. Wang formulated an *inhomogeneous* version of the Schrödinger flow as follows:

Given a scalar-valued, nonnegative function $f(x)$ on $M$, define the inhomogeneous energy $E_f(u)$ of a map $u \in C^1(M, N)$ with respect to the *coupling function* $f(x)$ by

$$E_f(u) = \frac{1}{2}\int_M |du|^2 f(x)\, dM,$$

where $|du|^2 = \mathrm{Trace}_g u^*h$, $u^*h$ being the pull-back of the metric tensor $h$ on $N$ by $u$. With respect to an orthonormal frame $\{e_i\}$ on $M$, the $L^2$-gradient of $E_f$ at $u$, denoted by $\tau_f(u)$, can be expressed as

$$\tau_f(u) = f\tau(u) + \nabla f \cdot du.$$

Here $\nabla f \cdot du = \sum_{i=1}^m (\nabla_{e_i} f) du(e_i)$, $m = \dim M$, and $\nabla_{e_i}$ denotes the covariant differential. In particular, in the case where $M = \Omega$ is a domain in Euclidean space $R^m$,

$$\nabla f \cdot du = \sum_{i=1}^m \frac{\partial f}{\partial x_i} du(\frac{\partial}{\partial x_i}).$$

We will call $\tau_f(u)$ the *inhomogeneous tension field* of $u$ with respect to the coupling function $f$. The inhomogeneous Schrödinger flow is then given by

$$\text{(1.2)} \qquad \begin{cases} \partial_t u = J(u)\tau_f(u) = J(u)\{f(x)\tau(u) + \nabla f(x) \cdot du\}, \\ u(x, 0) = u_0(x). \end{cases}$$

When $N = S^2$, the equation in (1.2) reduces to the inhomogeneous Heisenberg or ferromagnetic spin chain system, also known as the Landau-Lifshitz equation [14]:

$$\partial_t u = f(x)(u \times \Delta u) + \nabla f(x) \cdot (u \times \nabla u),$$

where $u$ takes values in $S^2 \subset R^3$ and $\times$ denotes the cross product in $R^3$. For details, we refer the reader to [4] and the references therein.

Also recall the nonlinear Schrödinger (NLS) equation

$$i\psi_t + \psi_{xx} + 2\kappa|\psi|^2\psi = 0,$$

where $\kappa \neq 0$ is a constant. This equation, which has many applications in physics, has been widely studied, see for example [2, 11, 23]. In particular, the lattice nonlinear Schrödinger equations with $\kappa = \pm 1$ can be written respectively as Hamiltonian equations on $S^2$ and the Lobachevskian plane, and thus represent respectively $SU(2)$ and $SU(1,1)$ magnetic models (see [8] for details).

Zakharov and Takhtajan [24] and Lakshmanan [13] pointed out that the Heisenberg spin chain system defined on $R^1$ is gauge equivalent to the nonlinear Schrödinger equation with $\kappa = 1$ (focusing case), thus establishing a deep relation between these two integrable systems. Fordy and Kulish [9] further observed that these systems have a gauge equivalent geometric formulation. In particular, they studied the integrability and Hamiltonian structure of matrix nonlinear Schrödinger equations associated with Hermitian symmetric spaces.

In [5], W.Y. Ding and Y.D. Wang studied the Schrödinger flow from $M = S^1$ into a general complete Kähler manifold $(N, J, h)$. Via an analytic approach, they proved that the Cauchy problem admits a unique local smooth solution if $u_0$ is smooth. Further, if $N$ is compact with constant sectional curvature $K$, the solution is in fact global. These results have been extended by Pang, Wang and Wang [15, 16, 20] in various directions.

Chang, Shatah and Uhlenbeck [3] considered the Cauchy problem for the Schrödinger flow from $M = R^m$, $m = 1, 2$, into a closed Riemann surface. By a generalized Hasimoto transformation, they showed that, for $m = 1$ and smooth Cauchy data $u_0$, the global smooth Schrödinger flow exists. For $m = 2$, they considered radially symmetric maps, and equivariant maps when the target surface has $S^1$ symmetry, and proved global existence and uniqueness in the small energy case (see [3] for details). Recently Terng and Uhlenbeck [18, 19] showed that the Schrödinger flow from $R^1$ into a complex Grassmannian manifold is gauge equivalent to the Cauchy problem of the following matrix nonlinear Schrödinger (MNLS) equation:

$$B_t = i(B_{xx} + BB^*B), \tag{1.3}$$

where $B$ is a map from $R^1 \times [0, \infty)$ to the space $\mathcal{M}_{k\times(n-k)}$ of $k \times (n-k)$ $(n > k)$ complex matrices, and $B^* = \bar{B}^t$ is the adjoint. This equation was first studied by Fordy and Kulish [9] as a generalization of the nonlinear Schrödinger equation. As a consequence of this correspondence, Terng and Uhlenbeck [18, 19] established the global existence of Schrödinger flow from $M = R^1$ into a complex Grassmannian.

In this paper, we consider the inhomogeneous Schrödinger flow from $M = R^1$ or $S^1$ into a Hermitian locally symmetric space. Examples of such manifolds include bounded symmetric domains, complex Grassmannians, complex hyperbolic spaces $CH^n$ with the Bergmann metric and their compact or noncompact quotients by isometric subgroups. Adopting the approach in [5] we will prove the global existence of Schrödinger flow by means of a conservation (or semi-conservation) law. More specifically, we have the following results (the definition of $\mathcal{H}^{\ell,2}(R^1,N)$ will be given in Section 2):

**Theorem 1.** *Let $(N,J,h)$ be a complete Hermitian locally symmetric space. Suppose the coupling function $f(x) \in C^\infty(R^1)$ satisfies $\inf_{x\in R^1} f(x) > 0$ and $\|\partial_x^k f\|_{C^0(R^1)} \le C$ for any $1 \le k \le \ell - 1$, where $C$ is a universal constant and $\ell \ge 4$. Then, given an initial map $u_0 \in \mathcal{H}^{\ell,2}(R^1,N)$ with bounded image set $u_0(R^1)$, the Cauchy problem (1.2) for the inhomogeneous Schrödinger flow from $R^1$ into $N$ admits a unique global solution $u \in L^\infty_{loc}([0,\infty);\mathcal{H}^{\ell,2}(R^1,N))$.*

As a direct consequence of Theorem 1, we have the following result. This provides a geometric analytic approach to the global existence result on $R^1$ of Terng and Uhlenbeck [19].

**Theorem 2.** *Let $(N,J,h)$ be a complete Hermitian locally symmetric space. Then, given an initial map $u_0 \in \mathcal{H}^{\ell,2}(R^1,N)$, $\ell \ge 4$, with bounded image set $u_0(R^1)$, the Cauchy problem (1.1) for the Schrödinger flow from $R^1$ into $N$ admits a unique global solution $u \in L^\infty_{loc}([0,\infty);\mathcal{H}^{\ell,2}(R^1,N))$.*

The analogues of these results for $M = S^1$ are given below:

**Theorem 3.** *Let $(N,J,h)$ be a complete Hermitian locally symmetric space. Let $f(x) \in C^\infty(S^1)$ be a positive function. Then, given an initial map $u_0 \in W^{\ell,2}(S^1,N)$ where $\ell \ge 4$, the Cauchy problem (1.2) for the inhomogeneous Schrödinger flow from $S^1$ into $N$ admits a unique global solution $u \in L^\infty_{loc}([0,\infty);W^{\ell,2}(S^1,N))$.*

As a direct consequence of Theorem 3, we have

**Theorem 4.** *Let $(N,J,h)$ be a complete Hermitian locally symmetric space. Then, given an initial map $u_0 \in W^{\ell,2}(S^1,N)$ where $\ell \ge 4$, the Cauchy problem (1.1) for the Schrödinger flow from $S^1$ into $N$ admits a unique global solution $u \in L^\infty_{loc}([0,\infty);W^{\ell,2}(S^1,N))$.*

The proofs of Theorems 2 and 4 will be omitted as they are direct consequences of Theorems 1 and 3. In fact, only the proof of Therorem 1 needs to be given as it also covers the proof of Theorem 3. Briefly, the method for establishing Theorem 1 can be summarised as follows: First, we use a family of periodic Schrödinger flows, defined on $[-D_i, D_i]$, $D_i \uparrow \infty$, to approximate the Schrödinger flow from $R^1$. As the approximate equations can be viewed as flows on circles, they have unique local solutions (see [5, 20]). By uniform estimates of the covariant derivatives of the solutions with respect to $i$, we show that the domain on which the local solutions are defined is independent of the parameter $i$. Thus, taking limit, one obtains a local Schrödinger flow $u$. Finally, using the (semi-)conservation laws for the energy $E(u)$ and $\|\tau(u)\|_{L^2}$, we can extend $u$ to a global flow.

This paper is organized as follows: In Section 2, we recall some facts and notations in differential geometry and some relations among Sobolev norms. In Section 3, we establish the local existence of the Schrödinger flow from $R^1$ into a complete Kähler manifold. In Section 4 we prove global existence and uniqueness by exploiting the geometric symmetries to derive some (semi-)conservation laws. The paper ends with a few concluding remarks.

**A note on notation:** We will use $C$ generically to denote constants appearing in the estimates in this paper. Some of these may depend on certain parameters, geometric properties of spaces, or the Cauchy data $u_0$. When we wish to specify this dependence, we will include the relevant spaces or quantities as arguments, e.g., $C(\|\tau(u_0)\|_2, E(u_0))$ means that $C$ depends on the quantities $\|\tau(u_0)\|_2$ and $E(u_0)$ *only*. Unless otherwise specified, $C$ depends on its arguments smoothly.

## 2. Preliminaries.

Let $u : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds. Let $\Gamma(TM)$ denote the space of smooth sections of $TM$. We will use $\nabla$ to denote the covariant differential on $\otimes^p T^*M \otimes u^*TN$ induced by the Riemannian metrics on $M$ and $N$. Thus, for $X \in \Gamma(TM)$, in local coordinates, we have $\nabla_{\partial/\partial x_i} u_*(X) = u^* \nabla^N_{u_*(\partial/\partial x_i)} u_*(X)$. We will use the shorthand notation $\nabla_i$ for $\nabla_{\partial/\partial x_i}$, or, when $\dim M = 1$, $\nabla_x$ for $\nabla_{d/dx}$. Sometimes, to further simply notations, we may also denote $\nabla_x u$ by $u_x$.

We shall denote the bundle-valued Sobolev spaces by $H^{k,r}$, and their norm functions by $\|\cdot\|_{H^{k,r}}$. For example,

$$\|\nabla_x u\|_{H^{k,r}} = \left( \sum_{i=0}^{k} \int |\nabla_x^{i+1} u|^r \, dx \right)^{\frac{1}{r}}.$$

In particular, $\|\cdot\|_{H^{0,r}} = \|\cdot\|_{L^r}$, which is also denoted by $\|\cdot\|_r$.

We may regard the exterior derivative $du$, also denoted by $\nabla u$, as a 1-form with values in the pull-back bundle $u^*TN$, i.e., $du \in \Gamma(T^*M \otimes u^*TN)$. In terms of local orthonormal frames $\{e_i\}$ (with dual frames $\{e_i^*\}$) on $M$ and $\{\bar{e}_\alpha\}$ on $N$,

$$du = (u_* e_i)^\alpha e_i^* \otimes \bar{e}_\alpha.$$

The energy density of $u$ is defined by $e(u) = \frac{1}{2}|du|^2$, which is written in local coordinates as

$$e(u) = \frac{1}{2} g^{ij} h_{\alpha\beta}(u) \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j}.$$

The energy functional is then defined by

$$E(u) = \int_M e(u)\, dx = \frac{1}{2} \int_M |du|^2\, dx. \tag{2.1}$$

Critical points of the energy $E$ as a functional on $C^1(M, N)$ are exactly the harmonic maps cite7 and the $L^2$-gradient of $E$ is just the tension field that was mentioned in Section 1, i.e., $\tau(u) = \nabla_i(u_* e_i)$.

Henceforth, we shall always embed the manifold $N$ into a Euclidean space $R^{\tilde{n}}$. Thus, the map $u$ can be viewed as a mapping from $M$ into $R^{\tilde{n}}$. We will denote the Sobolev norm of $u \in W^{k,p}(M, R^{\tilde{n}})$ by $\|u\|_{k,p}$. Note that $\|\cdot\|_{0,2} = \|\cdot\|_2$.

For any interger $\ell \geq 1$, define

$$\mathcal{H}^{\ell,2}(R^1, N) = \{u \in W^{\ell,2}_{\text{loc}}(R^1, N) : \|\partial_x^i u\|_2 < \infty, \quad i = 1, \dots, \ell\}.$$

We note, by Proposition 2.3 stated later in this section, that $\mathcal{H}^{\ell,2}(R^1, N)$ can also be defined by

$$\mathcal{H}^{\ell,2}(R^1, N) = \{u \in W^{\ell,2}_{\text{loc}}(R^1, N) : \|\nabla_x^i u\|_2 < \infty, \quad i = 1, \dots, \ell\}.$$

We now mention several results concerning Sobolev norms which will be of use later. For a positive number $D$, let $S^1(D) = R^1/D\mathcal{Z}$, where $\mathcal{Z}$ is the set of all integers, denote the circle of length $D$. We remark at this point that a key ingredient in the proof of Theorem 1 will be estimates on $S^1(D)$ that are independent of $D$.

**Proposition 2.1.** *Let $M = S^1(D)$ and let $q, r$ be real numbers satisfying $1 \leq q, r \leq \infty$, and $j, n$ be integers such that $0 \leq j < n$. Then there exists $k$, a constant depending only on $n, j, q, r, a$ and not depending on $D$, such that for all $u \in C^\infty(S^1(D))$ with $\int_{S^1(D)} u\, dx = 0$,*

$$\|\nabla^j u\|_p \leq k \|\nabla^n u\|_r^a \|u\|_q^{1-a},$$

*where*

$$\frac{1}{p} = j + a\left(\frac{1}{r} - n\right) + (1-a)\frac{1}{q},$$

*for all a in the interval* $\frac{j}{n} \le a \le 1$ *for which p is nonnegative. If* $r = \frac{1}{n-j} \neq 1$, *then the above inequality is not valid for* $a = 1$.

The proof of this proposition, which will be omitted, relies on a rescaling argument. We thank Professor W.Y. Ding for pointing this out to us.

The next result concerns integrals of the type

$$G = \int_{S^1(D)} |\nabla_x^k \nabla_x u||\nabla_x^{s_1}\nabla_x u| \cdots |\nabla_x^{s_l}\nabla_x u|\, dx.$$

**Proposition 2.2.** *Let* $D > 1$, $k \ge 2$, $k \ge s_i \ge 0$ *for* $i = 1, \dots, l$, *and* $\sum_{i=1}^{l} s_i \le k$. *Then there exists a positive constant* $C(\|\nabla_x u\|_{H^{k-1,2}})$, *which does not depend on D, such that*

$$G \le C(\|\nabla_x u\|_{H^{k-1,2}}) \left\{ 1 + \int_{S^1(D)} |\nabla_x^{k+1} u|^2\, dx \right\}. \tag{2.2}$$

*Proof.* For any function $g$ defined on $S^1(D)$, set

$$m(g) = \frac{1}{D}\int_{S^1(D)} g\, dx.$$

By applying Proposition 2.1 to $|u_x| - m(|u_x|)$ and noting the Kato inequality

$$\|\nabla_x|\nabla_x u|\|_2 \le \|\nabla_x^2 u\|_2,$$

we have

$$\|\nabla_x u\|_{C^0} - m(|\nabla_x u|) \le C\|\nabla_x^2 u\|_2^{\frac{1}{2}} \||\nabla_x u| - m(|\nabla_x u|)\|_2^{\frac{1}{2}}.$$

As

$$\||\nabla_x u| - m(|\nabla_x u|)\|_2 \le \|\nabla_x u\|_2 \quad \text{and} \quad \frac{1}{D}\int_{S^1(D)} |\nabla_x u|\, dx \le \frac{1}{\sqrt{D}}\|\nabla_x u\|_2,$$

we have

$$\|\nabla_x u\|_{C^0} \le C\|\nabla_x^2 u\|_2^{\frac{1}{2}} \|\nabla_x u\|_2^{\frac{1}{2}} + \|\nabla_x u\|_2. \tag{2.3}$$

Similarly, for $s > 1$,

$$\|\nabla_x^s u\|_{C^0} \le C\|\nabla_x^{s+1} u\|_2^{\frac{1}{2}} \|\nabla_x^s u\|_2^{\frac{1}{2}} + \|\nabla_x^s u\|_2. \tag{2.4}$$

Now, we turn to the integral $G$. Without loss of generality, we may assume that $s_1 \ge s_2 \ge \cdots \ge s_l \ge 0$. First, we consider the special case $k = s_1 > s_2 = \cdots = s_l = 0$. Obviously

$$G \le \|\nabla_x u\|_{C^0}^{l-1} \int_{S^1(D)} |\nabla_x^{k+1} u|^2 \, dx.$$

Thus the desired inequality follows from (2.3).

If $k - 1 \ge s_1 \ge s_2 \ge \cdots \ge s_l \ge 0$, we need to argue the following two cases:

**Case (i):** $k = 2$. In this case $G$ takes either of the following forms:

$$G = \int_{S^1(D)} |\nabla_x^3 u||\nabla_x^2 u|^2 |\nabla_x u|^{l-2} \, dx \qquad \text{or}$$

$$G = \int_{S^1(D)} |\nabla_x^3 u||\nabla_x^2 u||\nabla_x u|^{l-1} \, dx.$$

For the former, by applying Hölder's inequality we derive

$$G \le \|\nabla_x^3 u\|_2 \|\nabla_x^2 u\|_2 \|\nabla_x^2 u\|_{C^0} \|\nabla_x u\|_{C^0}^{l-2}. \tag{2.5}$$

Plugging (2.3) and (2.4) with $s = 2$ into (2.5), we obtain the desired inequality. In view of (2.4), we can also prove that the inequality holds when $G$ is of the latter form.

**Case (ii):** $k \ge 3$. In this case $G$ takes either of the following forms:

$$G = \int_{S^1(D)} |\nabla_x^k \nabla_x u||\nabla_x^{k-1} \nabla_x u||\nabla_x^{s_2} \nabla_x u| \cdots |\nabla_x^{s_l} \nabla_x u| \, dx, \tag{2.6}$$

where $k - 1 > s_2 \ge \cdots \ge s_l \ge 0$; or

$$G = \int_{S^1(D)} |\nabla_x^k \nabla_x u||\nabla_x^{s_1} \nabla_x u| \cdots |\nabla_x^{s_l} \nabla_x u| \, dx, \tag{2.7}$$

where $k - 1 > s_1 \ge \cdots \ge s_l \ge 0$. Applying Hölder's inequality to (2.6), we obtain

$$G \le \|\nabla_x^{k+1} u\|_2 \|\nabla_x^k u\|_2 \|\nabla_x^{s_2+1} u\|_{C^0} \cdots \|\nabla_x^{s_l+1} u\|_{C^0}.$$

Substituting (2.4) with $k - 1 > s_2 \ge \cdots \ge s_l \ge 0$ into the last inequality, we derive the desired inequality (2.2). A similar argument applies to (2.7) and the proof of the proposition is complete.

**Proposition 2.3.** *Let $N$ be a complete Riemannian manifold and $\Sigma$ be a compact subset of $N$. If $u : S^1(D) \to \Sigma \subset N$ is in $W^{k,2}(S^1(D), R^{\tilde{n}})$, $k \geq 0$ and $D > 1$, then*

$$\|\partial_x u\|_{k,2}^2 \leq 2\|\nabla_x^{k+1} u\|_{0,2}^2 + C(k, \Sigma, \|\partial_x u\|_{k-1,2}),$$

*where $C$ does not depend on $D$. In particular, if $u : R^1 \to \Sigma \subset N$ is in $\mathcal{H}^{k,2}(R^1, N)$, $k \geq 1$, then the above inequality holds.*

*Proof.* We note that for $k \geq 1$,

$$\nabla_x^{k+1} u = \partial_x^{k+1} u + \mathcal{P}(u)(\partial_x u, \dots, \partial_x^k u),$$

where $\mathcal{P}$ is a polynomial satisfying

$$|\mathcal{P}(u)(\partial_x u, \dots, \partial_x^k u)| \leq C \sum_{2 \leq l \leq k+1} \sum_{1 \leq j_i \leq k}^{j_1 + \cdots + j_l = k+1} |\partial_x^{j_1} u| \cdots |\partial_x^{j_l} u|.$$

By arguments similar to those in the proof of the above proposition, the desired result follows from Proposition 2.1. For details, see [5].

Now suppose $(N, J, h)$ is a Kähler manifold (hence $\nabla J \equiv 0$) and let $R(\cdot, \cdot, \cdot, \cdot)$ denote its Riemann curvature tensor. Then, we recall that

(i) $R(JX, JY, Z, W) = R(X, Y, JZ, JW) = R(X, Y, Z, W)$;

(ii) $R(\cdot, \cdot) \circ J = J \circ R(\cdot, \cdot)$.

If $N$ is a Hermitian locally symmetric space, by Cartan's theorem, we have the additional property that the curvature is covariant constant, i.e.,

(iii) $$\nabla R \equiv 0.$$

We note that a Hermitian locally symmetric space is the quotient of a Hermitian symmetric space by an isometric subgroup and recall the following facts about Hermitian symmetric spaces: Irreducible Hermitian symmetric spaces are classified into compact and noncompact types. The sectional curvature of a space of noncompact type is nonpositive and bounded from below. The scalar curvature of a Hermitian symmetric space is constant. For further details, we refer the reader to [10, 12]. These facts will be used freely in the remainder of the paper.

To end this section, we recall a local existence result for the smooth inhomogeneous Schrödinger flow from the circle.

**Proposition 2.4 ([20]).** *Let $M = S^1$ and $(N, J, h)$ be a complete Kähler manifold. If $f(x) \in C^\infty(S^1)$ with $\min_{x\in S^1} f(x) > 0$, and $u_0 \in C^\infty(S^1, N)$, then the Cauchy problem (1.2) of the inhomogeneous Schrödinger flow has a unique smooth solution $u \in C^\infty([0,T) \times S^1, N))$ for some $T \in (0,\infty]$. Furthermore, the energy is conserved along the solution, i.e., $E_f(u(x,t)) \equiv E_f(u_0(x))$.*

## 3. Local Existence of Schrödinger Flows from $R^1$.

We now consider the local existence of the (inhomogeneous) Schrödinger flow from $R^1$ into a complete Kähler manifold.

**Theorem 3.1 (Local Existence).** *Let $N$ be a complete Kähler manifold. Suppose the coupling function $f(x) \in C^\infty(R^1)$ satisfies $\inf_{x\in R^1} f(x) > 0$ and $\|\partial_x^i f\|_{C^0(R^1)} \le C$ for any $1 \le i \le \ell - 1$, where $C$ is a universal constant and $\ell \ge 4$. Then, given an initial map $u_0 \in \mathcal{H}^{\ell,2}(R^1, N)$ with bounded image set $u_0(R^1)$, there exists a positive $T = T(N, f, E(u_0), \|\tau(u_0)\|_2)$ such that the Cauchy problem (1.2) for the inhomogeneous Schrödinger flow from $R^1$ into $N$ admits a unique local solution $u \in L^\infty([0,T], \mathcal{H}^{\ell,2}(R^1, N))$.*

In order to prove the local existence theorem, we need to establish the following lemma:

**Lemma 3.2.** *Let $D > 1$ and $N$ be a complete Kähler manifold. Suppose the coupling function $f(x) \in C^\infty(S^1(D))$ satisfies $\min_{x\in S^1(D)} f(x) > 0$. Then, given an initial map $u_0 \in W^{\ell,2}(S^1(D), N)$ where $\ell \ge 4$, then there is a positive $T = T(N, f, E(u_0), \|\tau(u_0)\|_2)$, which does not depend on $D$, such that the Cauchy problem (1.2) for the inhomogeneous Schrödinger flow from $S^1(D)$ into $N$ admits a solution on the interval $[0,T]$ satisfying the following estimates:*

$$\sup_{t\in[0,T]} \|\nabla_x^i \tau(u)\|_2 \le C_i(T, \|\nabla_x u_0\|_{H^{i+1,2}}, \min f, \|f\|_{C^{i+1}}), \quad i = 0, 1, \dots, \ell - 2,$$

*where $C_i$ do not depend on $D$.*

*Proof.* First, suppose $u_0$ is $C^\infty$. Proposition 2.4 (or Theorem 3.1 in [20]) tells us that there exists a $\tilde{T}$ such that the Cauchy problem (1.2) admits a unique smooth (local) solution $u$ on $S^1(D) \times [0, \tilde{T}]$ satisfying

$$E_f(u(x,t)) \equiv E_f(u_0(x)). \tag{3.1}$$

Let $\Omega = \{p \in N : dist_N(p, u_0(S^1(D))) < 1\}$. Then $\Omega$ is an open subset of $N$ with compact closure $\bar{\Omega}$. Let

$$T' = \sup\{t > 0 : u(S^1(D), t) \subset \Omega\}.$$

Then we have

$$\begin{aligned}(3.2)\quad \frac{d}{dt}\int_{S^1(D)} |u_t|^2\,dx &= \int_{S^1(D)} \langle u_t, \nabla_t(J\tau_f(u))\rangle\,dx \\ &= \int_{S^1(D)} f\{\langle u_t, J\nabla_x^2 u_t\rangle + \langle u_t, R(u)(u_x, u_t)Ju_x\rangle\}\,dx \\ &\quad + \int_{S^1(D)} (\partial_x f)\langle u_t, J\nabla_x u_t\rangle\,dx,\end{aligned}$$

where $R$ is the Riemann curvature tensor of $N$.

Integrating by parts on the right hand side of the above equation and noting the antisymmetry and integrability of $J$, we obtain from (3.1) and (3.2) that

$$(3.3)\qquad \frac{d}{dt}\int_{S^1(D)} |u_t|^2\,dx = \int_{S^1(D)} f\langle R(u)(u_x, u_t)Ju_x, u_t\rangle\,dx.$$

Hence, it follows by the Hölder inequality that

$$(3.4)\qquad \frac{d}{dt}\int_{S^1(D)} |u_t|^2\,dx \le C(\Omega, f)\int_{S^1(D)} |u_t|^2|u_x|^2\,dx.$$

It is easy to see that (2.3) implies

$$(3.5)\qquad \|u_x\|_{C^0} \le C(f)(E_f(u_0)^{\frac{1}{2}} + \|\nabla_x u_x\|_2^{\frac{1}{2}} E_f(u_0)^{\frac{1}{4}}),$$

where $C$ does not depend on $D$. Noting $|u_t|^2 = |\tau_f(u)|^2$, it follows from (3.4) and (3.5) that

$$\frac{d}{dt}\|\tau_f(u)\|_2^2 \le C(\Omega, \min f, \|f\|_{C^1}, E_f(u_0))\left\{1 + \|\tau_f(u)\|_2^3\right\},$$

where $C$ does not depend on $D$. It follows from this ordinary differential inequality that for any constant $\mathcal{C} > \|\tau_f(u_0)\|_2$, we can find a positive $T^* = T^*(\mathcal{C}, \Omega, f, E_f(u_0), \|\tau_f(u_0)\|_2)$ such that

$$\sup_{t\in[0,T^*]} \|\tau_f(u)\|_2 \le \mathcal{C}.$$

This also implies that

$$\sup_{t\in[0,T^*]} \|\tau(u)\|_2 \leq C(\mathcal{C}, E_f(u_0), \|f\|_{C^1}, \min f, T^*). \tag{3.6}$$

Next, we compute the derivative with respect to $t$ ($\in [0, T']$) of the integral

$$\int_{S^1(D)} |\nabla_x u_t|^2 f\, dx.$$

Keeping the integrability of the complex structure $J$ in mind, we have

$$\frac{1}{2}\frac{d}{dt}\int_{S^1(D)} |\nabla_x u_t|^2 f\, dx = \int_{S^1(D)} \langle \nabla_x u_t, \nabla_t \nabla_x \{J\tau_f(u)\}\rangle f\, dx \overset{\text{def}}{=} I. \tag{3.7}$$

We now compute $I$.

$$\begin{aligned} I = &\int_{S^1(D)} \langle \nabla_x u_t, J\{f\nabla_t\nabla_x\nabla_x u_x + 2\partial_x f \nabla_t \nabla_x u_x\}\rangle f\, dx \\ &+ \int_{S^1(D)} \langle \nabla_x u_t, \partial_x^2 f J \nabla_x u_t\rangle f\, dx \\ = &\int_{S^1(D)} \langle \nabla_x u_t, J\{f\nabla_t\nabla_x\nabla_x u_x + 2\partial_x f \nabla_t \nabla_x u_x\}\rangle f\, dx. \end{aligned} \tag{3.8}$$

By the definition of the curvature operator,

$$\nabla_t\nabla_x u_x = \nabla_x\nabla_x u_t + R(u_x, u_t)u_x.$$

Hence,

$$\begin{aligned} \nabla_t\nabla_x \tau(u) = &\nabla_x\nabla_x\nabla_x u_t + R(\tau(u), u_t)u_x + R(u_x, \nabla_x u_t)u_x \\ &+ 2R(u_x, u_t)\tau(u) + (\nabla_x R)(u_x, u_t)u_x. \end{aligned} \tag{3.9}$$

Substituting the above curvature identities into the right hand side of (3.8) and integrating by parts, we obtain that

$$\begin{aligned} I = &\int_{S^1(D)} \langle \nabla_x u_t, J\{f\nabla_x\nabla_x\nabla_x u_t + 2\partial_x f \nabla_x \nabla_x u_t\}\rangle f\, dx \\ &+ \int_{S^1(D)} \langle \nabla_x u_t, fJ\{R(\tau(u), u_t)u_x + R(u_x, \nabla_x u_t)u_x\}\rangle f\, dx \end{aligned} \tag{3.10}$$

$$+\int_{S^1(D)} \langle \nabla_x u_t, fJ\{2R(u_x,u_t)\tau(u) + (\nabla_x R)(u_x,u_t)u_x\}\rangle f\,dx$$

$$+\int_{S^1(D)} \langle \nabla_x u_t, J\{2\partial_x f R(u_x,u_t)u_x\}\rangle f\,dx$$

$$=\int_{S^1(D)} \langle \nabla_x u_t, J\{R(\tau(u),u_t)u_x + R(u_x,\nabla_x u_t)u_x\}\rangle f^2\,dx$$

$$+\int_{S^1(D)} \langle \nabla_x u_t, J\{2R(u_x,u_t)\tau(u) + (\nabla_x R)(u_x,u_t)u_x\}\rangle f^2\,dx$$

$$+2\int_{S^1(D)} \langle \nabla_x u_t, JR(u_x,u_t)u_x\rangle (\partial_x f) f\,dx.$$

Hence, it follows that for $t \le T'$,

(3.11)

$$I \le C(\Omega,\mathcal{C},f)\int_{S^1(D)} \{|\nabla_x u_t|^2|u_x|^2 + |\nabla_x u_t||u_x||u_t|(|\tau(u)| + |u_x| + |u_x|^2)\}dx.$$

Applying the Hölder inequality to the right hand side of (3.11), we obtain that, for $t \le \min\{T^*,T'\}$,

(3.12)

$$\frac{1}{2}\frac{d}{dt}\int_{S^1(D)} |\nabla_x u_t|^2 f\,dx \le C(\Omega,\mathcal{C},f)\{\|\nabla_x u_t\|_2^2\|u_x\|_{C^0}^2 + (\|\tau(u)\|_2 + \|u_x\|_{C^0} + \|u_x\|_{C^0}^2)\|u_t\|_{C^0}\|u_x\|_{C^0}\|\nabla_x u_t\|_2\}.$$

From (2.4), we deduce that for $D > 1$,

(3.13)

$$\|u_t\|_{C^0} \le C(\|\nabla_x u_t\|_2^2 + \|u_t\|_2^2)^{\frac{1}{4}}\|u_t\|_2^{\frac{1}{2}} + \|u_t\|_2$$
$$\le C(f)\|u_t\|_2^{\frac{1}{2}}\left\{\int_{S^1(D)} |\nabla_x u_t|^2 f\,dx + \int_{S^1(D)} |\tau(u)|^2 f^2\,dx\right\}^{\frac{1}{4}} + \|u_t\|_2.$$

Plugging (3.1), (3.5), (3.6) and (3.13) into (3.12), we obtain that, for $t \le \min\{T^*,T'\}$,

$$\frac{d}{dt}\int_{S^1(D)} |\nabla_x u_t|^2 f\,dx \le C(\Omega,\mathcal{C},u_0,f)\left(1 + \int_{S^1(D)} |\nabla_x u_t|^2 f\,dx\right).$$

Thus, as $|\nabla_x u_t|^2 = |\nabla_x \tau_f(u)|^2$,

$$\frac{d}{dt}\int_{S^1(D)} |\nabla_x \tau_f(u)|^2 f\,dx \le C(\Omega, \mathcal{C}, u_0, f)\left(1 + \int_{S^1(D)} |\nabla_x \tau_f(u)|^2 f\,dx\right).$$

By the Gronwall inequality, for $t \le \min\{T^*, T'\}$,

$$\|\nabla_x \tau_f(u(t))\|_2^2 \le (1 + \|\nabla_x \tau_f(u_0)\|_2^2)\exp(Ct) - 1, \tag{3.14}$$

where $C = C(\Omega, \mathcal{C}, \|\nabla_x u_0\|_{H^{2,2}}, \min f, \|f\|_{C^2})$ does not depend on $D$. This implies that, for $t \le \min\{T^*, T'\}$,

$$\|\nabla_x \tau(u)\|_2 \le C(\Omega, \mathcal{C}, \|\nabla_x u_0\|_{H^{2,2}}, \min f, \|f\|_{C^2}). \tag{3.15}$$

Note that a positive lower bound for $T'$ can be derived from (3.15). Indeed, it is easy to see from (3.13) that for $t \le \min\{T^*, T'\}$, there exists some $\mathcal{M}$ such that

$$\|u_t\|_{C^0} \le \mathcal{M}.$$

It follows that for $t \le \min\{T^*, T'\}$,

$$\sup_{x\in S^1(D)} dist_N(u(x,t), u_0(x)) \le \mathcal{M}t.$$

If $T' > T^*$, then $T^*$ is a lower bound. So we may assume that $T' \le T^*$. Letting $t \to T'$ in the last inequality, we get $\mathcal{M}T' \ge 1$. Therefore, if we set $T = \min\{\frac{1}{\mathcal{M}}, T^*\}$, then (3.6) and (3.15) hold true for $t \in [0, T]$. We re-iterate that $T = T(\mathcal{C}, \Omega, f, E(u_0), \|\tau(u_0)\|_2)$ depends only on $\mathcal{C}$, $\Omega$, $\min_{x\in S^1(D)} f$, $\|f\|_{C^2}$, $E(u_0)$, and $\|\tau(u_0)\|_2$, and not on $D$.

We proceed with the proof by induction. Assume that for $i = 0, 1, \ldots, k-1$,

$$\sup_{t\in[0,T]} \|\nabla_x^i \tau(u)\|_2 \le C_i(T, \|\nabla_x u_0\|_{H^{i+1,2}}, \min f, \|f\|_{C^{i+1}}), \tag{3.16}$$

where $C_i$ do not depend on $D$. We note also that these estimates imply the following inequalities which will be used later:

$$\sup_{t\in[0,T]} \|\nabla_x^i \tau_f(u)\|_2 \le C_i(T, \|\nabla_x u_0\|_{H^{i+1,2}}, \min f, \|f\|_{C^{i+1}}),$$

$$i = 0, 1, \ldots, k-1.$$

With the estimates (3.16) at hand, we consider the integral

$$\frac{d}{dt}\int_{S^1(D)} |\nabla_x^k u_t|^2 f^k\,dx \overset{\text{def}}{=} I^*.$$

By virtue of the commutation relation of the covariant derivatives, we deduce that

$$\nabla_t \nabla_x^k u_t = P(\nabla_x u, \dots, \nabla_x^k u, u_t, \dots, \nabla_x^{k-1} u_t) + \nabla_x^k \nabla_t u_t,$$

where $P(\cdot, \dots, \cdot)$ is a vector-valued multilinear functional satisfying

$$\begin{aligned}(3.17)\qquad &|P(\nabla_x u, \cdot, \nabla_x^k u, u_t, \dots, \nabla^{k-1} u_t)| \\ &\leq C(\Omega)\left\{\sum_{k_1+k_2+k_3=k}^{k_2,k_3\leq k-1;1\leq k_1} |\nabla_x^{k_1} u||\nabla_x^{k_2} u_t||\nabla^{k_3} u_t|\right\} \\ &\quad + Q(|\nabla_x u|, \dots, |\nabla_x^{k-1} u|, |u_t|, \dots, |\nabla_x^{k-2} u_t|),\end{aligned}$$

where

(3.18)

$$\begin{aligned}&Q(|\nabla_x u|, \dots, |\nabla_x^{k-1} u|, |u_t|, \dots, |\nabla_x^{k-2} u_t|) \\ &\leq C(\Omega)\left\{\sum_{k\geq s\geq 4}\ \sum_{k_1,k_2\leq k-2;\ k_3,\dots,k_s\geq 1}^{k_1+k_2+\cdots+k_s=k} |\nabla_x^{k_1} u_t||\nabla_x^{k_2} u_t||\nabla_x^{k_3} u|\cdots|\nabla_x^{k_s} u|\right\}.\end{aligned}$$

Then, $I^*$ can be written as

$$\begin{aligned}(3.19)\qquad I^* &= \int_{S^1(D)} \langle \nabla_x^k u_t, \nabla_x^k \nabla_t u_t\rangle f^k\, dx \\ &\quad + \int_{S^1(D)} \langle \nabla_x^k u_t, P(\nabla_x u, \dots, \nabla_x^k u, u_t, \dots, \nabla_x^{k-1} u_t)\rangle f^k\, dx \\ &\overset{\text{def}}{=} I_1^* + I_2^*.\end{aligned}$$

Next, we estimate $I_i^*$, $i = 1, 2$. It follows from (1.2) that

$$(3.20)\qquad I_1^* = \int_{S^1(D)} \langle \nabla_x^k u_t, J\nabla_x^k \nabla_t (f\nabla_x u_x + (\partial_x f) u_x)\rangle f^k\, dx.$$

By a tedious but direct computation and applying the commutation relation of the covariant derivatives, we obtain (see [20] for details)

(3.21)

$$I_1^* = \int_{S^1(D)} \langle \nabla_x^k u_t, fJ\nabla_x^k (R(u_x, u_t)u_x) + \partial_x f J\nabla_x^{k-1}(R(u_x, u_t)u_x)\rangle f^k\, dx$$

$$+\int_{S^1(D)}\left\langle \nabla_x^k u_t, J\sum_{i=2}^{k} C_k^i \partial_x^i f \nabla_x^{k-i}\nabla_t\nabla_x u_x\right\rangle f^k\,dx$$

$$+\int_{S^1(D)}\left\langle \nabla_x^k u_t, J\sum_{i=1}^{k} C_k^i \partial_x^{i+1} f \nabla_x^{k-i}\nabla_x u_t\right\rangle f^k\,dx,$$

where $C_k^i = k!/(k-i)!i!$.

By a direct computation, it is not difficult to see that the right hand side of (3.21) can be bounded by integrals of the following form:

$$\int_{S^1(D)} |\nabla_x^{k+1}\nabla_x u||\nabla_x^{s_1}\nabla_x u|\cdots|\nabla_x^{s_l}\nabla_x u|\,dx,$$

where $s_i \geq 0$, $1 \leq i \leq l$, and $\sum_{i=1}^{l} s_i \leq k+1$. By applying Proposition 2.2 to the above integrals, we can deduce from (3.21) and (3.16) that

$$I_1^* \leq C(T, u_0, f)\{\|\nabla_x^k \tau_f(u)\|_2^2 + 1\}. \tag{3.22}$$

Similarly, in view of (3.16)-(3.18), we may also apply Proposition 2.2 to $I_2^*$ to derive

$$I_2^* \leq C(T, u_0, f)\{\|\nabla_x^k \tau_f(u)\|_2^2 + 1\}.$$

Noting that

$$I^* = \int_{S^1(D)} |\nabla_x^k \tau_f(u)|^2\,dx,$$

it follows from (3.22) and the last inequality that

$$\frac{d}{dt}\int_{S^1(D)} |\nabla_x^k \tau_f(u)|^2\,dx \leq C(T, u_0, f)\left\{\int_{S^1(D)} |\nabla_x^k \tau_f(u)|^2\,dx + 1\right\}.$$

This implies that

$$\sup_{t\in[0,T]} \|\nabla_x^k \tau(u(t))\|_2 \leq C_k(T, \|\nabla_x u_0\|_{H^{k+1,2}}, \min f, \|f\|_{C^{k+1}}),$$

where $C_k$ does not depend on $D$, but only on the geometry of $\Omega$, $T$, $\|\nabla_x u_0\|_{H^{k+1,2}}$, $\min_{x\in S^1(D)} f$ and $\|f\|_{C^{k+1}}$. This completes the induction.

With these estimates, we argue that the solution must exist on the time interval $[0,T]$; otherwise, we may always extend the time interval of existence to cover the interval $[0,T]$.

Finally, when $u_0 \in W^{\ell,2}(S^1(D), N)$, $\ell \geq 4$, but not $C^\infty$, we may choose a sequence of $C^\infty$ maps $u_{i0} : S^1(D) \to N$ such that

$$\|u_{i0} - u\|_{\ell,2} \to 0 \quad \text{as} \quad i \to \infty.$$

Using $u_{i0}$ as the initial data of the Cauchy problem (1.2), for each $i$ we get a solution $u_i$, defined on $[0, T_i]$. The above arguments, however, show that there is a uniform lower bound for $T_i$. Furthermore, denoting this lower bound by $T$, the unique local solution $u$ to the Cauchy problem (1.2) with initial data $u_0$ exists on $[0, T]$ and is given by the limit of $\{u_i\}$ by sending $i$ to $\infty$. Obviously, all the desired estimates on $u$ also hold true. This finishes the proof of the lemma.

*Proof of Theorem* 3.1. First, assume that $N$ is compact. Our strategy is to construct a sequence of periodic inhomogeneous Schrödinger flows with periodic initial maps to approximate the Cauchy problem defined on $R^1$. Since $u_0 \in \mathcal{H}^{\ell,2}(R^1, N)$, one can approximate $u_0$ by a sequence of maps $\{u_{i0}\}$, where $u_{i0} \in \mathcal{H}^{\ell,2}([-D_i, D_i], N)$ for some $D_i > 2$ and $D_i \uparrow \infty$. More precisely, choose $C^\infty$ cut-off functions $\lambda_i$ satisfying $|\partial^j \lambda_i| \leq C_\lambda$, $j = 0, 1, \ldots, \ell$, where $C_\lambda$ does not depend on $i$, and

$$\lambda_i(s) = \begin{cases} 1, & s \in [-D_i + 1, D_i - 1], \\ 0, & s \notin [-D_i, D_i]. \end{cases}$$

Define

$$y(x) = \int_0^x \lambda_i(s)\, ds$$

and let

$$u_{i0}(x) = u_0(y(x))\Big|_{[-D_i, D_i]}, \qquad f_i(x) = f(y(x))\Big|_{[-D_i, D_i]}.$$

We can extend $u_{i0}$ and $f_i$ to $[-2D_i, 2D_i]$ as follows:

$$\tilde{u}_{i0}(x) = \begin{cases} u_{i0}(-x - 2D_i), & x \in [-2D_i, -D_i], \\ u_{i0}(x), & x \in [-D_i, D_i], \\ u_{i0}(-x + 2D_i), & x \in [D_i, 2D_i]; \end{cases}$$

$$\tilde{f}_i(x) = \begin{cases} f_i(-x - 2D_i), & x \in [-2D_i, -D_i], \\ f_i(x), & x \in [-D_i, D_i], \\ f_i(-x + 2D_i), & x \in [D_i, 2D_i]. \end{cases}$$

As $u_{i0} \in W^{\ell,2}([-D_i, D_i], N)$, we may regard $\tilde{u}_{i0} \in W^{\ell,2}(S^1(4D_i), N)$. Similarly, we may regard $\tilde{f}_i$ as a periodic function with respect to $x$.

We consider the following periodic Cauchy problem on $R^1 \times [0,\infty)$:

$$\begin{cases} \dfrac{\partial \tilde{u}_i}{\partial t} = J(\tilde{u}_i)\tau_{\tilde{f}_i}(\tilde{u}_i), \\ \tilde{u}_i(x,0) = \tilde{u}_{i0}(x), \\ \tilde{u}_i(x+4D_i) = \tilde{u}_i(x). \end{cases} \tag{3.23}$$

By Lemma 3.2, for each $i$, the above Cauchy problem admits a unique local smooth periodic solution $\tilde{u}_i$ with period $4D_i$, defined on $S^1(4D_i) \times [0, \tilde{T}_i]$. Furthermore, there exist constants

$$\tilde{C}_i = \tilde{C}_i(\tilde{T}_i, k, f, \|\nabla_x \tilde{u}_{i0}\|_{H^{k,2}([-2D_i,2D_i])})$$

which do not depend on $D_i$, such that for $k = 1, \ldots, \ell$,

$$\sup_{t\in[0,\tilde{T}_i]} \|\nabla_x^k \tilde{u}_i\|_{L^2([-2D_i,2D_i])} \le \tilde{C}_i.$$

By the construction of $\tilde{u}_{i0}$, it is not difficult to find that

$$\begin{aligned} \int_{-2D_i}^{2D_i} |\tau(\tilde{u}_{i0})|^2\,dx = &\ 2\int_{-D_i}^{D_i} |\tau(u_{i0})|^2\,dx \\ \le &\ C(N)(\|\tau(u_0)\|^2_{L^2(R^1)} + E(u_0)), \end{aligned}$$

$$\begin{aligned} E\left(\tilde{u}_{i0}\big|_{[-2D_i,2D_i]}\right) = &\ \int_{-2D_i}^{2D_i} |\nabla_x u_{i0}|^2\,dx \\ = &\ 2\int_{-D_i}^{D_i} |\nabla_x u_{i0}|^2\,dx \le C(N)E(u_0), \end{aligned}$$

and that for $1 \le k \le \ell$,

$$\int_{-2D_i}^{2D_i} |\nabla_x^k \tilde{u}_{i0}|^2\,dx = 2\int_{-D_i}^{D_i} |\nabla_x^k u_{i0}|^2\,dx \le C(N)\left\{\sum_{s=1}^{k} \|\nabla_x^s u_0\|^2_{L^2(R^1)}\right\},$$

where $C$ is independent of $i$. Similarly, we have

$$\|\tilde{f}_i\|_{C^\ell([-2D_i,2D_i])} \le C(N)\|f\|_{C^\ell(R^1)}.$$

Hence, from Lemma 3.2 it follows that there exist a uniform lower bound $T$ of $\tilde{T}_i$ with respect to $i$ and a uniform upper bound $C = C(T,k,u_0,f)$ of $\tilde{C}_i$, such that for $0 \le k \le \ell - 1$ and all $i$,

$$\sup_{t\in[0,T]} \|\nabla_x \tilde{u}_i\|_{H^{k,2}([-2D_i,2D_i])} \le C(T,k,u_0,f). \tag{3.24}$$

We emphasize that $C$ depends only on the geometry of $N$, $T$, $k$, $\inf_{x\in R^1} f$, $\|f\|_{C^k(R^1)}$ and $\|\nabla_x u_0\|_{H^{k,2}(R^1)}$.

Restricting to $[-D_i, D_i]$, we have $u_i = \tilde{u}_i\Big|_{[-D_i,D_i]}$ and $f_i = \tilde{f}_i\Big|_{[-D_i,D_i]}$. Then, obviously, $u_i$ satisfies the following Cauchy problem on $[-D_i, D_i] \times [0,T]$:

$$\begin{cases} \partial_t u_i = J(u_i)\tau_{f_i}(u_i), \\ u_i(x,0) = u_{i0}(x). \end{cases} \tag{3.25}$$

By virtue of the estimate (3.24) and Proposition 2.3, there exists a subsequence, denoted again by $\{u_i\}$, such that

$$u_i \longrightarrow u \qquad [\text{weakly*}] \quad \text{in} \quad L^\infty([0,T]; \mathcal{H}^{\ell,2}(R^1, N)).$$

It is easy to see that the limit $u \in L^\infty([0,T]; \mathcal{H}^{\ell,2}(R^1,N))$ is a solution to (1.2) on $R^1$.

When $N$ is a noncompact complete manifold, we need to modify the above arguments slightly. According to the hypothesis of the theorem, $u_0(R^1)$ is contained in some compact set. Let $\overline{u_0(R^1)}$ denote the closure of $u_0(R^1)$, $\mathcal{S} = \{p : dist_N(p, \overline{u_0(R^1)}) < 1\}$ and $\Omega_i = \{p : dist_N(p, \tilde{u}_{i0}([-2D_i, 2D_i])) < 1\}$. Obviously, $\Omega_i \subset \mathcal{S}$ for $i = 1, 2, \ldots$. Now we consider the Cauchy problem (3.23) for each $i$. From Lemma 3.2 it follows that for each $i$ a unique solution to (3.23) exists on $S^1(4D_i) \times [0, \tilde{T}_i]$, where $\tilde{T}_i$ depends on $\Omega_i$. It is not difficult to see from the proof of Lemma 3.2 that, in the discussion of the local well-posedness of (3.23), if we replace $\Omega_i$ with $\mathcal{S}$, then there is a positive real number

$$\tilde{T}_i' = \tilde{T}_i'(\mathcal{C}, \mathcal{S}, f, E(\tilde{u}_{i0}), \|\tau(\tilde{u}_{i0})\|_2)$$

such that for each $i$, a unique solution to (3.23) exists on $S^1(4D_i) \times [0, \tilde{T}_i']$. We can then argue that there exists a uniform lower bound $T$ of $\tilde{T}_i'$. The arguments for the compact case now apply, and the proof is complete.

## 4. Global Schrödinger Flow.

In this section, we first establish a semi-conservation law. It will then be used to establish the existence of global Schrödinger flows from $R^1$ and $S^1$, *viz.* Theorems 1-4.

**Lemma 4.1.** *Let $N$ be a Hermitian locally symmetric space and $f \in C^3(R^1)$ be a positive function. Let $u : R^1 \times [0, T_{\max}) \to N$, with $u(\cdot, t) \in \mathcal{H}^{4,2}(R^1, N)$*

*for any $t \in [0, T]$, be a sufficiently smooth solution to the Cauchy problem (1.2) of the inhomogeneous Schrödinger flow from $R^1$ into $N$. Then, for $T < T_{\max}$,*

$$\sup_{0 \le t \le T} \int_{R^1} \left\{ |\tau(u)|^2 - \frac{1}{4} R(u_x, Ju_x, u_x, Ju_x) \right\} f^2 \, dx \le C(T, u_0, f),$$

*where*

$$C(T, u_0, f) = C(T, \|\nabla_x u_0\|_{H^{1,2}}, \inf f, \|f\|_{C^3(R^1)})$$

*is finite when $T < \infty$.*

*Proof.* Since $u(\cdot, t) \in \mathcal{H}^{4,2}(R^1, N)$ for any $t \in [0, T]$, as in the proof of Lemma 4.2 in [20], we obtain

$$\frac{d}{dt} \int_{S^1} |\tau(u)|^2 f^2 \, dx = -2 \int_{R^1} \{ R(u_x, u_t, u_x, Ju_t) f + \langle u_x, u_t \rangle f \partial_x^3 f \} \, dx, \tag{4.1}$$

where $R$ denotes the Riemann curvature tensor of $N$.

As $N$ is a Hermitian locally symmetric space, $\nabla J \equiv 0$ and $\nabla R \equiv 0$. Thus,

$$\begin{aligned}
&\frac{d}{dt} \int_{R^1} R(u_x, Ju_x, u_x, Ju_x) f^2 \, dx \\
&= \int_{R^1} \Big\{ R(J \nabla_t u_x, u_x, Ju_x, u_x) + R(Ju_x, \nabla_t u_x, Ju_x, u_x) \\
&\quad + R(Ju_x, u_x, J \nabla_t u_x, u_x) + R(Ju_x, u_x, Ju_x, \nabla_t u_x) \Big\} f^2 \, dx \\
&= 4 \int_{R^1} R(Ju_x, u_x, J \nabla_x u_t, u_x) f^2 \, dx \\
&= 4 \int_{R^1} R(Ju_x, u_x, \nabla_x (Ju_t), u_x) f^2 \, dx \\
&= 4 \int_{R^1} \Big\{ \nabla_x (R(Ju_x, u_x, Ju_t, u_x)) - R(J \nabla_x u_x, u_x, Ju_t, u_x) \\
&\quad - R(Ju_x, \nabla_x u_x, Ju_t, u_x) - R(Ju_x, u_x, Ju_t, \nabla_x u_x) \Big\} f^2 \, dx.
\end{aligned} \tag{4.2}$$

Integrating by parts, we get

(4.3)

$$\frac{d}{dt} \int_{S^1} R(u_x, Ju_x, u_x, Ju_x) f^2 \, dx$$

$$= -4\int_{R^1}\Big\{R(J\nabla_x u_x, u_x, Ju_t, u_x) + R(Ju_x, \nabla_x u_x, Ju_t, u_x)$$
$$+ R(Ju_x, u_x, Ju_t, \nabla_x u_x)\Big\} f^2\, dx - 8\int_{R^1} R(Ju_x, u_x, Ju_t, u_x) f\partial_x f\, dx.$$

As

$$f\nabla_x u_x + (\partial_x f)u_x = -Ju_t,$$

it follows that

$$(4.4)\quad \frac{d}{dt}\int_{S^1} R(u_x, Ju_x, u_x, Ju_x) f^2\, dx$$
$$= -4\int_{R^1}\Big\{R(u_t, u_x, Ju_t, u_x) + R(Ju_x, -Ju_t, Ju_t, u_x)$$
$$+ R(Ju_x, u_x, Ju_t, -Ju_t)\Big\} f\, dx + 4\int_{R^1}\Big\{R(J(\partial_x f)u_x, u_x, Ju_t, u_x)$$
$$+ R(Ju_x, \partial_x f u_x, Ju_t, u_x) + R(Ju_x, u_x, Ju_t, (\partial_x f)u_x)\Big\} f\, dx$$
$$- 8\int_{R^1} R(Ju_x, u_x, Ju_t, u_x) f\partial_x f\, dx$$
$$= 4\int_{R^1}\Big\{R(J(\partial_x f u_x), u_x, Ju_t, u_x) + R(Ju_x, \partial_x f u_x, Ju_t, u_x)$$
$$+ R(Ju_x, u_x, Ju_t, \partial_x f u_x)\Big\} f\, dx - 8\int_{R^1} R(u_t, u_x, Ju_t, u_x) f\, dx$$
$$- 8\int_{R^1} R(Ju_x, u_x, Ju_t, u_x)) f\partial_x f\, dx$$
$$= -8\int_{R^1} R(u_t, u_x, Ju_t, u_x) f\, dx + 4\int_{R^1} R(Ju_x, u_x, Ju_t, u_x)) f\partial_x f\, dx.$$

Combining (4.1) and (4.4), we obtain

$$(4.5)\quad \frac{d}{dt}\int_{R^1}\left\{|\tau(u)|^2 - \frac{1}{4}R(u_x, Ju_x, u_x, Ju_x)\right\} f^2\, dx$$
$$= -2\int_{R^1}\langle u_x, u_t\rangle(\partial_x^3 f) f\, dx - \int_{R^1} R(Ju_x, u_x, Ju_t, u_x)) f\partial_x f\, dx.$$

Note that by the Hölder inequality, we have

$$(4.6)\quad \int_{R^1}\langle u_x, J\nabla_x u_x\rangle(\partial_x^3 f) f^2\, dx$$

$$\le C(T, E(u_0), \inf_{x\in R^1} f, \|\partial_x f\|_{C^3(R^1)}) \left\{ \int_{R^1} |\tau(u)|^2 f^2\, dx + 1 \right\}.$$

Also, by the interpolation inequality, the Kato inequality and (3.1), we have

$$\begin{aligned} \text{(4.7)} \qquad \int_{R^1} |u_x|^6\, dx &\le C \left( \int_{R^1} |u_x|^2\, dx \right)^2 \int_{R^1} |\nabla_x u_x|^2\, dx \\ &\le C(T, \|f\|_{C^0})(E(u_0))^2 \int_{R^1} |\tau(u)|^2\, dx. \end{aligned}$$

Let us look at the second term on the right hand side of (4.5). It follows from the Hölder inequality and (4.7) that, for $t \in [0, T]$,

$$\begin{aligned} \text{(4.8)} \qquad & \int_{R^1} |R(Ju_x, u_x, Ju_t, u_x) f \partial_x f|\, dx \\ &\le C(\|f\|_{C^1}) \int_{S^1(D)} |u_x|^3 |\tau(u)|\, dx \\ &\le C(T, \|f\|_{C^1}) \left\{ \int_{R^1} |u_x|^6\, dx + \int_{R^1} |\tau(u)|^2\, dx \right\} \\ &\le C(T, \|f\|_{C^1}, E(u_0)) \int_{R^1} |\tau(u)|^2\, dx. \end{aligned}$$

Plugging (4.6) and (4.8) into (4.5), it follows that, for $t \in [0, T]$,

$$\begin{aligned} \text{(4.9)} \qquad & \frac{d}{dt} \int_{R^1} \left\{ |\tau(u)|^2 - \frac{1}{4} R(u_x, Ju_x, u_x, Ju_x) \right\} f^2\, dx \\ &\le C(T, N, E(u_0), f) \left\{ \int_{R^1} |\tau(u)|^2 f^2\, dx + 1 \right\}. \end{aligned}$$

Considering the geometry of Hermitian locally symmetric spaces, we need to discuss the following two cases according to the sectional curvature $K_N$ of $N$:

**Case (i):** Let $-B_1 < K_N \le 0$, where $B_1$ is a positive constant. Then (4.9) implies that on $[0, T]$,

(4.10)

$$\begin{aligned} & \frac{d}{dt} \int_{R^1} \left\{ |\tau(u)|^2 - \frac{1}{4} R(u_x, Ju_x, u_x, Ju_x) \right\} f^2\, dx \\ &\le C(T, K_N, E(u_0), f) \left\{ \int_{R^1} \{|\tau(u)|^2 - \frac{1}{4} R(u_x, Ju_x, u_x, Ju_x)\} f^2\, dx + 1 \right\}. \end{aligned}$$

By the Gronwall inequality, we conclude that for any $T > 0$,

$$\sup_{t\in[0,T]} \int_{R^1} \left\{ |\tau(u)|^2 - \frac{1}{4} R(u_x, Ju_x, u_x, Ju_x) \right\} f^2\, dx \\ \le C(T, K_N, \|\nabla_x u_0\|_{H^{1,2}}, f),$$

where $C(T, K_N, \|\nabla_x u_0\|_{H^{1,2}}, f)$ is finite when $T < \infty$. This is the desired result.

**Case (ii):** Let $B_2 > K_N \ge 0$, where $B_2$ is a positive constant. (Thus $N$ is compact.) In this case we need to modify the previous argument slightly. First we note that the interpolation inequality and the Kato inequality imply that

$$\text{(4.11)} \quad \int_{R^1} |u_x|^4\, dx \le C(S^1) \left( \int_{R^1} |u_x|^2\, dx \right)^{\frac{3}{2}} \left( \int_{R^1} |\nabla_x |u_x||^2\, dx \right)^{\frac{1}{2}} \\ \le C \left( \int_{R^1} |u_x|^2\, dx \right)^{\frac{3}{2}} \left( \int_{R^1} |\tau(u)|^2\, dx \right)^{\frac{1}{2}}.$$

Since $\inf_{x\in R^1} f > 0$ by assumption, it follows that on $[0, T]$,

(4.12)

$$\int_{R^1} |u_x|^4 f^2\, dx \le C(T, f) \left\{ \left( \int_{R^1} |u_x|^2 f^2\, dx \right)^{\frac{3}{2}} \left( \int_{R^1} |\tau(u)|^2 f^2\, dx \right)^{\frac{1}{2}} \right\} \\ \le C(T, f) \left\{ \left( \int_{R^1} |u_x|^2 f^2\, dx \right)^3 + \frac{1}{2} \int_{R^1} |\tau(u)|^2 f^2\, dx \right\}.$$

By integrating both sides of the inequality (4.9), it follows that

$$\text{(4.13)} \quad \int_{R^1} \left\{ |\tau(u)|^2 - \frac{1}{4} R(u_x, Ju_x, u_x, Ju_x) \right\} f^2\, dx \\ \le \int_{R^1} \left\{ |\tau(u_0)|^2 - \frac{1}{4} R(u_{0x}, Ju_{0x}, u_{0x}, Ju_{0x}) \right\} f^2\, dx \\ + C(T, E(u_0), f) \int_0^t dt \int_{R^1} |\tau(u)|^2 f^2\, dx + tC(T, E(u_0), f).$$

By applying again (4.12) to control the second term of the left hand side of the above inequality, it follows that, for $t \in [0, T]$,

$$\int_{R^1} |\tau(u)|^2 f^2\, dx \le 2 \int_{R^1} \left\{ |\tau(u_0)|^2 - \frac{1}{4} R(u_{0x}, Ju_{0x}, u_{0x}, Ju_{0x}) \right\} f^2\, dx$$

$$+ C(T, K_N, E(u_0), f) \int_0^t dt \int_{R^1} |\tau(u)|^2 f^2\, dx$$
$$+ tC(T, E(u_0), f) + C(T, K_N, E(u_0), f).$$

Applying the Gronwall inequality, as in [20], we deduce from the last inequality that there exists a constant $C(T, K_N, \|\nabla_x u_0\|_{H^{1,2}}, \inf f, \|f\|_{C^3(R^1)})$, which is finite when $T < \infty$, such that

$$\sup_{t\in[0,T]} \int_{R^1} |\tau(u)|^2 f^2\, dx \le C(T, K_N, \|\nabla_x u_0\|_{H^{1,2}}, \inf f, \|f\|_{C^3(R^1)}). \tag{4.14}$$

Immediately, the desired estimate follows. This completes the proof of the Lemma.

*Proof of Theorem* 1. It suffices to consider the following two cases:

**Case I:** Let $N$ be an irreducible Hermitian symmetric space of noncompact type.

In this case, we first note that the holomorphic sectional curvature is bounded from below, i.e., there exists a positive constant $K_0$ such that

$$-K_0|u'|^4 \le R(u', Ju', u', Ju') \le 0. \tag{4.15}$$

Now, invoking Theorem 3.1, let $u$ be a maximal local smooth solution defined on $R^1 \times [0, T_{\max})$ for the Cauchy problem (1.2) and consider the quantity

$$d_{\max} = \sup_{x\in R^1} \left\{ \int_0^{T_{\max}} |u_t|\, dt \right\}.$$

Noting the semi-conservation law in Lemma 4.1 and keeping $\nabla R \equiv 0$ and $\nabla J \equiv 0$ in mind, we can see from (3.7)-(3.13) that

$$\begin{aligned} \frac{d}{dt}\int_{R^1} |\nabla_x u_t|^2 f\, dx &= \frac{d}{dt}\int_{R^1} |\nabla_x \tau_f(u)|^2 f\, dx \\ &\le C(N, \inf f, \|f\|_{C^3}, \|\tau(u_0)\|_2, E(u_0)) \left\{ \int_{R^1} |\nabla_x \tau_f(u)|^2 f\, dx + 1 \right\} \\ &= C(N, \inf f, \|f\|_{C^3}, \|\tau(u_0)\|_2, E(u_0)) \left\{ \int_{R^1} |\nabla_x u_t|^2\, dx + 1 \right\}. \end{aligned} \tag{4.16}$$

By the Gronwall inequality,

$$\int_{R^1} |\nabla_x u_t(t)|^2\, ds \le (1 + \|\nabla \tau(u_0)\|_2^2) \exp(Ct) - 1, \tag{4.17}$$

where $C = C(N, \inf f, \|f\|_{C^3}, \|\tau(u_0)\|_2, E(u_0))$ depends only on the Sobolev constant of $R^1$, the upper bound $K_0$ of the absolute value of the holomorphic sectional curvature of $N$, $f$ and $u_0$. From (4.17), it follows that

$$\|u_t\|_{C^0(R^1)} \leq C\{(1 + \|\nabla\tau(u_0)\|_2^2)\exp(Ct) - 1\}. \tag{4.18}$$

for any $t \in [0, T_{\max})$.

Now suppose $T_{\max} < \infty$. Then, it follows from (4.16) and the assumption that $u_0(R^1)$ is contained in a compact set of $N$ that $d_{\max} < \infty$. This indicates that the image set of $u$ is contained in some compact subset $\Omega \subset N$. In this case, for a small $\sigma > 0$, consider the Cauchy problem

$$\begin{cases} \partial_t v = J(v)\tau_f(v), \\ v(x, 0) = u(x, T_{\max} - \sigma). \end{cases} \tag{4.19}$$

By repeating the arguments in Theorem 3.1, we can show that there exists a positive real number $T_0$, which depends on $\Omega$ but not on $\sigma$, such that (4.19) admits a local smooth solution $v$ on $R^1 \times [0, T_0)$. Thus $u$ can be extended to $R^1 \times [0, T_{\max} + T_0 - \sigma)$. Since the uniqueness theorem given in [5, 20] is also valid for the case at hand, we know that $v(x,t) = u(x, T_{\max} - \sigma + t)$ for any $t \in [0, T_0)$ so the extended $u$ is still the solution for the Cauchy problem of the inhomogeneous Schrödinger flow. Choosing $\sigma$ small enough so that

$$T_{\max} + T_0 - \sigma > T_{\max}$$

provides a contradiction to the fact that $T_{\max}$ is maximal. Thus $T_{\max}$ must be $\infty$.

**Case II:** Let $N$ be a compact Hermitian locally symmetric space.

With inequality (4.14) (Lemma 4.1) at hand, the proof proceeds similarly as above. The issue of uniqueness can also be addressed as in [5, 20]. This finishes the proof of Theorem 1.

The proof of Theorem 3 follows directly from Lemma 3.2 and Lemma 4.1 (see also [20]), so we shall omit it.

**Remark 4.1.** If $f(x) \equiv 1$ (the homogeneous case), then (4.5) implies the following conservation law:

$$\frac{d}{dt}\int_{S^1} \left\{ |\tau(u)|^2 - \frac{1}{4}R(u_x, Ju_x, u_x, Ju_x) \right\} dx \equiv 0. \tag{4.20}$$

We end this section with a comparison between the conservation laws for the Schrödinger flow from $R^1$ into a complex Grassmannian and MNLS (focusing case) on $R^1$ using the correspondence given in [18, 19].

First we note that MNLS is an infinite-dimensional Hamiltonian system [7] with Hamiltonian functional

$$H(B) = \int_{R^1} \{\mathrm{tr}(B_x B_x^*) - \mathrm{tr}(BB^*BB^*)\}\, dx \tag{4.21}$$

on the space $\mathcal{S}(R^1, \mathcal{M}_{k\times(n-k)})$ of smooth maps of Schwartz class from $R^1$ to $\mathcal{M}_{k\times(n-k)}$ with the symplectic form

$$\omega(B^1, B^2) = \int_{R^1} \langle -iB^1, B^2\rangle\, dx \tag{4.22}$$

defined using the Hermitian inner product

$$\langle B^1, B^2\rangle = \mathrm{Re}\ \mathrm{tr}(B^1 B^{2*}), \qquad B^1, B^2 \in \mathcal{M}_{k\times(n-k)}.$$

Thus, MNLS has conservation laws provided by the $L^2$-norm and the Hamiltonian along the solutions, namely, if $B$ is a solution of (1.3), then

$$\frac{d}{dt}\int_{R^1} |B|^2\, dx \equiv 0 \tag{4.23}$$

and

$$\frac{d}{dt}\int_{R^1} \{\mathrm{tr}(B_x {B_x}^*) - \mathrm{tr}(BB^*BB^*)\}\, dx \equiv 0. \tag{4.24}$$

Now let us recall the correspondence of Terng-Uhlenbeck [18, 19]: Let $B \in C^\infty([0,\infty), \mathcal{S}(R^1, \mathcal{M}_{k\times(n-k)}))$ be a solution of (1.3). Let

$$v = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} -iBB^* & iB_x \\ iB_x^* & iB^*B \end{pmatrix} \tag{4.25}$$

and

$$a = \frac{1}{2}\begin{pmatrix} iI_k & 0 \\ 0 & -iI_{n-k} \end{pmatrix}. \tag{4.26}$$

Then there exists a gauge transformation $g \in C^\infty(R^1 \times [0,\infty), U(n))$, satisfying

$$\begin{cases} g^{-1}g_x = v, \\ g^{-1}g_t = Q_2, \end{cases} \tag{4.27}$$

such that $u = gag^{-1}$ is a solution of (1.1). It is easy to see that $u \in C^\infty(R^1 \times [0,\infty), \mathrm{Gr}(k, C^n))$ and $u'(\cdot, t) \in \mathcal{S}(R^1, T\mathrm{Gr}(k, C^n))$, where $\mathrm{Gr}(k, C^n)$ denotes the complex Grassmannian manifold. As a Hermitian symmetric space,

$$\mathrm{Gr}(k, C^n) \cong \frac{U(n)}{U(k) \times U(n-k)},$$

and has a canonical complex structure given by ad $a$ where $a$ is given by (4.26). It follows that the Schrödinger flow on $\mathrm{Gr}(k, C^n)$ is given by

$$u_t = [u, u_{xx}] \tag{4.28}$$

where $[\cdot,\cdot]$ denotes the Lie bracket. In terms of the above correspondence, the following proposition can be verified by direct calculation:

**Proposition 4.2.** *Let $B$ be a solution of the focusing MNLS on $R^1$ and let $u$ be the corresponding Schrödinger flow on* $\mathrm{Gr}(k, C^n)$ *defined on $R^1$. Then, the conservation laws* (4.23), (4.24) *for MNLS correspond, respectively, to those for the energy functional $E$ defined by* (2.1) *and* (4.20) *for the Schrödinger flow.*

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Department of Mathematics
National University of Singapore
2 Science Drive 2, Republic of Singapore 117543
*E-mail address*: matpyh@nus.edu.sg

Department of Mathematics
Yangzhou University, Yangzhou, Jiangsu 225009
People's Republic of China
*E-mail address*: hywang@yzu.edu.cn

Institute of Mathematics
Academy of Mathematics and System Sciences
Chinese Academy of Sciences
Beijing, People's Republic of China
*E-mail address*: wyd@math03.math.ac.cn

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