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A New Proof of Lee's Theorem on the Spectrum of Conformally Compact Einstein Manifolds

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Let $\overline{M}$ be a compact $n+1$-dimensional manifold with boundary Σ. A Riemannian metric g on the interior M is conformally compact if for any defining function r of the boundary $\overline{g} = r^2 g$ extends to a C^3 metric on $\overline{M}$. The conformal class of the metric $\overline{g}|_\Sigma$ is invariantly defined and is called the conformal infinity of g. We also assume that $|dr|^2_{\overline{g}} = 1$ on Σ. This condition is invariantly defined and is satisfied if g is conformally compact Einstein, i.e., $\mathrm{Ric}\,(g) = -ng$.

It was proved by Mazzeo [7] that the continuous spectrum of the Laplacian on such a Riemannian manifold consists of the ray $[n^2/4, \infty)$ with no embedded eigenvalues; however, in general there may be finitely many eigenvalues in the interval $(0, n^2/4)$. If the metric is conformally compact Einstein, one might expect a relationship between its spectrum and global conformal invariants of its conformal infinity. This is justified by the following beautiful theorem proved by Lee [6].

Theorem 0.1. *Let (M, g) be an $(n+1)$-dimensional conformally compact Einstein manifold. If its conformal infinity has non-negative Yamabe invariant, then $\lambda_0(g) = n^2/4$.*

In this short note by using ideas in Witten-Yau [8] and Cai-Galloway [2] we give a simple proof of Lee's theorem. This new proof avoids the delicate analysis in Lee's original proof and is quite robust. It is possible to adapt the idea to other similar situations of interest.

Proof. By Mazzeo's result, it suffices to prove $\lambda_0(g) \geq n^2/4$. We first deal with the case that the conformal infinity has positive Yamabe invariant. We choose a metric h in the conformal class such that its scalar curvature s is positive. Then there is a unique defining function (see [4]) r in a collar neighborhood of Σ such that

$$g = r^{-2}(dr^2 + h_r),$$

where h_r is an r-dependent family of metrics on Σ and $h_r|_{r=0} = h$. Moreover we have the following expansion (see, e.g., [5])

$$h_r = h - \frac{r^2}{n-2}\left(\mathrm{Ric}\,(h) - \frac{s}{2(n-1)}h\right) + o(r^2). \tag{1}$$

Near infinity we have the unit vector field $v = r\frac{\partial}{\partial r}$ which is the outer unit normal of the level sets of r. Define

$$\Sigma^\epsilon = \{x \in M | r(x) = \epsilon\}.$$

For ϵ small, this is a compact hypersurface in M isotopic to Σ. Its mean curvature is easy to calculate and we find, using (1)

$$\begin{aligned} H &= \operatorname{div} v \\ &= n - \frac{\epsilon}{2}\mathrm{Tr}\,({h_r}^{-1}\dot{h}_r)|_{r=\epsilon} \\ &= n + s\epsilon^2/2(n-1) + o(\epsilon^2). \end{aligned} \tag{2}$$

As the scalar curvature $s > 0$ on Σ, for ϵ small enough there exists a constant $c > 0$ such that Σ^ϵ has mean curvature

$$H = \operatorname{div} v > n + c\epsilon^2. \tag{3}$$

Let S be an (oriented) compact hypersurface in M. For any constant $\delta \in \mathbb{R}$ following Witten-Yau [8] we consider the functional

$$L_\delta(S) = A(S) - (n+\delta)\int_S \Lambda, \tag{4}$$

where $A(S)$ is the area of S and Λ is an n-form such that $\Theta = d\Lambda$ is the volume form of M. Note that if S is the boundary of a domain Ω, we have

$$L_\delta(S) = A(S) - (n+\delta)V(\Omega), \tag{5}$$

where $V(\Omega)$ is the volume of Ω. We prove by contradiction that $L_0(S) \geq 0$ for any compact hypersurface S in M. Suppose there exists a compact hypersurface S_1 with $L_0(S_1) < 0$. Choose $\bar{\epsilon} > 0$ small enough such that S_1 is enclosed by $\Sigma^{\bar{\epsilon}}$ and the mean curvature of $\Sigma^{\bar{\epsilon}}$ satisfies (3). Then choose $0 < \delta < c\bar{\epsilon}^2$ such that $L_\delta(S_1) < 0$. We minimize the functional L_δ on the compact manifold $M^{\bar{\epsilon}} \triangleq \{x \in M | r(x) \leq \bar{\epsilon}\}$ in its homology class represented by S_1. By geometric measure theory a minimizer S_0 exists. By (3) and divergence theorem S_0 does not touch $\Sigma^{\bar{\epsilon}}$ (for detail see [8]). Since

$L_\delta(S_0) \le L_\delta(S_1) < 0$, the hypersurface S_0 is nontrivial. But as shown in [8] this leads to a contradiction with the second variation formula. Therefore $L_0(S) \ge 0$ for any compact hypersurface S in M. This implies that the isoperimetric constant

$$I_1 = \inf \frac{A(\partial\Omega)}{V(\Omega)} \ge n,$$

where the inf is taken over all compact domains $\Omega \subset M$. The theorem then follows from the well known fact (see e.g., [3]) that

$$\lambda_0(g) \ge \frac{I_1^2}{4}.$$

If the conformal infinity has zero Yamabe invariant we take the metric h to have scalar curvature $s \equiv 0$. Then by (2) the mean curvature H of Σ^ϵ satisfies

$$H - n = o(\epsilon^2).$$

Fix $\bar{o} \in M$. By the formula for g near infinity it is easy to see that

$$d(\bar{o}, \Sigma^\epsilon) \le -2\log\epsilon + C.$$

Let $\epsilon_k \to 0$ be a sequence and denote $\Sigma_k = \Sigma^{\epsilon_k}$. By the above two estimates the mean curvature H_k of Σ_k satisfies

$$\lim_{k\to\infty} (H_k - n)e^{2d(\bar{o},\Sigma_k)} = 0. \tag{6}$$

Consider the functions $\beta_k = d(\bar{o}, \Sigma_k) - d(x, \Sigma_k)$. By a simple and beautiful comparison argument, Cai-Galloway [2] prove that $\Delta\beta_k \ge n_k$ in the support sense and $\lim_{k\to\infty} n_k = n$. Let $\Omega \subset M$ be a bounded domain. We consider its first eigenfunction

$$\begin{cases} -\Delta f = \lambda f, f > 0 \text{ in } \Omega, \\ f|_{\partial\Omega} = 0. \end{cases}$$

Suppose $fe^{n\beta_k/2}$ achieves its maximum at $p \in \Omega$. Let ϕ_δ be a C^2 lower support function for β_k at p, i.e.,

$$\phi_\delta \le \beta_k \text{ in a neighborhood } U \text{ of } p, \tag{7}$$

$$\phi_\delta(p) = \beta_k(p), \Delta\phi_\delta(p) \ge n_k - \delta. \tag{8}$$

As β_k is Lipschitz with Lipschitz constant ≤ 1, it is easy to prove

$$|\nabla \phi_\delta|(p) \leq 1. \tag{9}$$

The C^2 function $fe^{n\phi_\delta/2}$ on U achieves its maximum at p, so we have

$$\nabla f(p) = -\frac{n}{2} f(p) \nabla \phi_\delta(p), \tag{10}$$

$$\triangle \left(fe^{n\phi_\delta/2}\right)(p) \leq 0. \tag{11}$$

We calculate, using (8) (9) and (10)

$$\begin{aligned}
&\triangle \left(fe^{n\phi_\delta/2}\right)(p) \\
&= e^{n\phi_\delta(p)/2}\Big(\triangle f(p) + n\nabla f(p) \cdot \nabla \phi_\delta(p) \\
&\qquad + n^2 f(p)|\nabla \phi_\delta|^2(p)/4 + nf(p)\triangle \phi_\delta(p)/2\Big) \\
&= e^{n\phi_\delta(p)/2} f(p) \left(-\lambda - n^2 |\nabla \phi_\delta|^2(p)/4 + n\triangle \phi_\delta(p)/2\right) \\
&\geq e^{n\phi_\delta(p)/2} f(p) \left(-\lambda - n^2/4 + n(n_k - \delta)/2\right)
\end{aligned}$$

Therefore $\lambda \geq n(n_k - \delta)/2 - n^2/4$. Let $\delta \to 0$ we get $\lambda \geq nn_k/2 - n^2/4$. Let $k \to \infty$ we get $\lambda \geq n^2/4$. As this is true for any bounded domain Ω, $\lambda_0(g) \geq n^2/4$. □

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