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Heat kernels and Green's functions on limit spaces

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In this paper, we study the behavior of the Laplacian on a sequence of manifolds $\{M_i^n\}$ with a lower bound in Ricci curvature that converges to a metric-measure space M_∞. We prove that the heat kernels and Green's functions on M_i^n will converge to some integral kernels on M_∞ which can be interpreted, in different cases, as the heat kernel and Green's function on M_∞. We also study the Laplacian on noncollapsed metric cones; these provide a unified treatment of the asymptotic behavior of heat kernels and Green's functions on noncompact manifolds with nonnegative Ricci curvature and Euclidean volume growth. In particular, we get a unified proof of the asymptotic formulae of Colding-Minicozzi, Li and Li-Tam-Wang.

0. Introduction.

Assume M^n is an n dimensional Riemannian manifold with a lower bound in Ricci curvature,

$$\mathrm{Ric}_{M^n} \geq -(n-1)\Lambda, \tag{0.1}$$

where $\Lambda \geq 0$. By the Bishop-Gromov inequality, we have a *uniform* volume doubling condition,

$$\mathrm{Vol}(B_{2R}(p)) \leq 2^\kappa \,\mathrm{Vol}(B_R(p)), \tag{0.2}$$

here we can take $\kappa = n$ if $\Lambda = 0$; if $\Lambda > 0$, we require that R is bounded from above, say, $R < D$ for some $D > 0$. Moreover, there is a *uniform* Poincare inequality

$$\frac{1}{\mathrm{Vol}(B_R(p))}\int_{B_R(p)} |f - f_{p,R}| \leq \tau R\Big(\frac{1}{\mathrm{Vol}(B_R(p))}\int_{B_R(p)} |df|^2\Big)^{\frac{1}{2}}, \tag{0.3}$$

where $f_{p,R}$ is the average of f on $B_R(p)$; see [Bu], [Ch3], [HaKo] and the references therein.

We assume, throughout this paper, $\{M_i^n\}$ is a sequence of complete Riemannian manifolds with (0.1) that converges in the pointed measured

Gromov-Hausdorff sense, to a metric space M_∞; we write $M_i^n \xrightarrow{d_{GH}} M_\infty$, d_{GH} is the Gromov-Hausdorff distance. In particular, (0.2) holds on M_∞. One can show that (0.3) and the *segment inequality*, which is stronger than (0.3), hold on M_∞ as well; note on M_∞, the role of $|du|$ in (0.3) is played by g_f, the *minimal generalized upper gradient*, see [Ch3], [ChCo4].

In Cheeger's paper [Ch3], a significant portion of analysis on smooth manifolds was extended to metric-measure spaces satisfying (0.2), (0.3). In [Ch3], [ChCo4] Cheeger and Colding defined a self-adjoint Laplacian operator Δ on M_∞. By convention Δ is *positive.* They proved that the eigenvalues and eigenfunctions of the Laplacian Δ_i over M_i^n converge to those on M_∞, thereby establishing Fukaya's conjecture [Fu]. So if we consider $\mathcal{RIC}$, the completion under measured Gromov-Hausdorff convergence of the set of smooth manifolds with (0.1), it is natural to expect that quantities associated to Δ should behave continuously.

In this paper, we will study this phenomenon in detail. Our main goal is to prove, in various cases, that the heat kernel H_i and Green's function G_i on M_i^n converge uniformly to the heat kernel H_∞ and Green's function G_∞ on M_∞. We will make precise the definition of these convergences in Section 1. For results concerning heat kernels, a lower bound in Ricci curvature (0.1) is enough; for Green's functions, we require that $\mathrm{Ric}_{M_i^n} \geq 0$; compare with (1.16), (1.18), (1.19).

Moreover, in the noncompact case, we also study the asymptotics of the heat kernel and Green's function on a manifold M^n with $\mathrm{Ric}_{M^n} \geq 0$ and a Euclidean volume growth condition:

$$\mathrm{Vol}(B_R(p)) > v_0 R^n. \tag{0.4}$$

According to [ChCo1], any tangent cone at infinity of a manifold M^n with $\mathrm{Ric}_{M^n} \geq 0$ and (0.4) is a metric cone $C(X)$. So viewed from a sufficiently large scale, M^n appears to be close to some $C(X)$. Combined with the appropriate convergence theorems mentioned above, at a sufficiently large scale the heat kernel and Green's function on M^n are close to those on $C(X)$.

On the other hand, we show that the classical analysis on cones, [Ch1], [Ch2], [ChTa1], can be generalized to $C(X)$. In particular, we have explicit expression of heat kernels and Green's functions on $C(X)$; see (6.21), (4.23). In this way we get a unified treatment for the asymptotic formulae of these integral kernels on M^n. In particular, we get new proofs of the Colding-Minicozzi asymptotic formula for Green's functions, [CoMi1] (compare with [LiTW]), the asymptotic formulae for heat kernels of Li [Li1] and Li-Tam-Wang [LiTW].

The organization of this paper is as follows:

Section 1 reviews some background material that we need in the sequel.

In Section 2, in the *compact* case we prove, $H_i(\cdot,\cdot,t) \to H_\infty(\cdot,\cdot,t)$ uniformly (assuming (0.1)), and $G_i \to G_\infty$ uniformly, off the diagonal (assuming $\mathrm{Ric}_{M^n} \geq 0$). It's well known that there is an eigenfunction expansion for heat kernels, so our results follows easily from the work of Cheeger-Colding [ChCo4], [Ch3], by estimating the remainders of the eigenvalue expansions. We remark, previously in [KK1], [KK2] it was proved that a *subsequence* of H_i converges to *some* kernel on the compact metric space M_∞.

By the Dirichlet's principle and the transplantation theorem of Cheeger [Ch3], we show in Section 3 that the uniform limit of solutions for Poisson equations is a solution for a Poisson equation, see also [Ch3], [ChCo4]. In particular, if $\{M_i^n\}$ are noncompact, $\mathrm{Ric}_{M_i^n} \geq 0$, satisfy (0.4) uniformly, then $G_i \to G_\infty$ uniformly off the diagonal (Theorem 3.21).

We treat the heat kernels on noncompact spaces in Section 5. We assume (0.1). First, some *subsequence* of the Dirichlet heat kernels $H_{R,i}$ on $B_R(p_i) \subset M_i^n$ will converge to some function $H_{R,\infty}$ on $B_R(p_\infty) \subset M_\infty$. However, at present it is not clear if $H_{R,i}$ will converge. On the other hand by a generalized maximum principle, any two $H_{R,\infty}$ (from two different subsequences) can not be too different from each other, see (5.46). Letting $R \to \infty$, we prove that $H_i(\cdot,\cdot,t) \to H_\infty(\cdot,\cdot,t)$ in L^1.

In Theorem 5.59, when the noncollapsed limit M_∞ is a manifold, we prove H_∞ is the heat kernel over M_∞, i.e. the integral kernel of the semigroup $e^{-t\Delta}$. For general M_∞, the picture is not yet clear; however, it is true when $M_\infty = C(X)$ is a noncollapsed tangent cone that is the limit of a sequence of manifolds with nonnegative Ricci curvature, see Theorem 6.1.

In Section 4 and Section 6 we study the Laplacian on $C(X)$. We prove that in this case, one can still separate variables. We use these to study the structure of G_∞ and H_∞ on $C(X)$, and the asymptotic behavior of Green's function and heat kernel on a manifold M^n with $\mathrm{Ric}_{M^n} \geq 0$ and (0.4).

In Section 7 we study the asymptotic behavior of the eigenvalues $\lambda_{j,\infty}$ on a compact metric space M_∞ which is the limit of a sequence of manifolds $\{M_i^n\}$ with (0.1). We will prove in the noncollapsed case, the Weyl asymptotic formula is true on M_∞; see Theorem 7.3. In the collapsed case, we get some link between the behavior of eigenvalues and $\dim_{Mink}(M_\infty)$, the Minkowski dimension of M_∞.

All of the estimates in this paper are *uniform*, i.e. the constants are valid for the whole family of manifolds we are considering (for example all compact manifolds M^n with $\mathrm{Ric}_{M^n} \geq -(n-1)\Lambda$ and $\mathrm{Diam}\, M^n \leq D$).

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1. Background and notation.

Suppose $(M_i^n, \mathrm{Vol}_i) \xrightarrow{d_{GH}} (M_\infty, \mu_\infty)$ in the measured Gromov-Hausdorff sense, i.e. the sequence $\{M_i^n\}$ converges in the Gromov-Hausdorff sense to M_∞, and for any $x_i \to x_\infty$, $(x_i \in M_i^n)$ and $R > 0$, we have $\mathrm{Vol}_i(B_R(x_i)) \to \mu_\infty(B_R(x_\infty))$; here μ_∞ is Borel regular. In fact, for any sequences of manifolds with Ricci curvature bounded from below, after possible renormalization of the measures when $\{M_i^n\}$ is collapsing, one can alway find a subsequence converges in the measured Gromov-Hausdorff sense, see [ChCo2]. In the following we usually let Vol denote the (renormalized if $\{M_i^n\}$ is collapsing) measure on M_i^n, $i = 1, 2, ..., \infty$; on M_∞ sometimes we also write μ_∞ for Vol_∞. We refer to [Ch3], [ChCo2], [Gr] for general background on measured Gromov-Hausdorff convergence.

Definition 1.1. Suppose $K_i \subset M_i^n \xrightarrow{d_{GH}} K_\infty \subset M_\infty$ in the measured Gromov-Hausdorff sense. f_i is a function on M_i^n, $i = 1, 2, ...$; f_∞ is a continuous function on M_∞. Assume $\Phi_i : K_\infty \to K_i$ are ϵ_i-Gromov-Hausdorff approximations, $\epsilon_i \to 0$. If $f_i \circ \Phi_i$ converge to f_∞ uniformly, we say that $f_i \to f_\infty$ *uniformly* over $K_i \xrightarrow{d_{GH}} K_\infty$.

For simplicity, in the above context, we also say that $f_i \to f_\infty$ uniformly on K; when we write $f_i(x) \to f_\infty(x)$, we mean that $f_i \to f_\infty$ uniformly and $f_i(x_i) \to f_\infty(x_\infty)$, where $x_i \to x_\infty$, $x_i \in M_i$.

In many applications, the family $\{f_i\}$ is actually equicontinuous. We remark, the Arzela-Ascoli theorem can be generalized to the case where the functions live on different spaces: when $M_i^n \xrightarrow{d_{GH}} M_\infty$, for any bounded, equicontinuous sequence $\{f_i\}$ (f_i is a function on M_i^n), there is a subsequence that converges uniformly to some continuous function f_∞ on M_∞. The proof is straightforward.

We also introduce the notion of L^p *convergence* $(1 \le p \le \infty)$.

Definition 1.2. We say $f_i \to f_\infty$ in L^p, if for all $\epsilon > 0$, one can write $f_i = \phi_i + \eta_i$ such that $\phi_i \to \phi_\infty$ uniformly and $\limsup_{i \to \infty} \|\eta_i\|_{L^p} \le \epsilon$, $\|\eta_\infty\|_{L^p} \le \epsilon$.

The following is a generalization of the Rellich-Kondrakov theorem:

Lemma 1.3 (Rellich). *Assume $B_1(p_i) \subset M_i^n \xrightarrow{d_{GH}} B_1(p) \subset M_\infty$ in the measured Gromov-Hausdorff sense. u_i is a function on M_i^n, $i = 1, 2, \ldots$. Assume*

$$\int_{B_1(p_i)} (u_i)^2 + |\nabla u_i|^2 \le N. \tag{1.4}$$

Then there is a subsequence of $\{u_i\}$ that converges in L^2 over any compact subset of the open balls B_1.

The proof depends only on a *weak Poincare inequality* (use a bigger ball on the right of (0.3)) and (0.2). One can divide the ball $B_R(p)$ ($R < 1$) into small subsets and approximate f by functions that are constant over each of these small subsets, then one easily finds a convergent subsequence by standard diagonal arguments. Compare [Ch3], especially [CoMi3].

We use subscript i and write $f_i, H_{R,i}, p_i$, etc. to denote functions, points, etc on M_i^n. To simplify notation, when we write an equation with some function or other objects with no subscription (for example, f), it should be understood that the equation is valid for some suitable convergent sequence of functions or other objects (for example, $\{f_i\}$; f_i is defined on M_i^n, $i = 1, 2, \ldots$), according to the context.

In [Ch3], Cheeger defined a Sobolev space $H_{1,2}$ on metric-measure spaces (Z, μ) satisfying (0.2), (0.3), and proved that Lipschitz functions are dense in $H_{1,2}$. Denote by $\overset{\circ}{H}_{1,2}(\Omega)$ the closure in $H_{1,2}$ of Lipschitz functions supported in an open set Ω. Recall in [Ch3], one has a natural finite dimensional cotangent bundle T^*Z. We use du to denote the *differential* of u, see Section 4 of [Ch3]. One can put a norm $|\cdot|$ on T^*Z by assigning $|df| = g_f = \mathrm{Lip}\, f$ for f Lipschitz. Here as in [Ch3],

$$\mathrm{Lip}\, f(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(y,x)}. \tag{1.5}$$

Note we use **Lip**f to denote the Lipschitz constant of f. Clearly, on smooth manifold $|\cdot|$ agrees with the standard norm $|du| = |\nabla u|$. It was proved in [Ch3] that $|\cdot|$ is equivalent to a uniformly convex norm, in particular, an inner product.

If we have the stronger assumption $Z = M_\infty$ with $M_i^n \xrightarrow{d_{GH}} M_\infty$, in [ChCo4] Cheeger and Colding proved that M_∞ is μ_∞-rectifiable, and, as a corollary, the norm $|\cdot|$ actually comes from an inner product $\langle\cdot,\cdot\rangle$. So $H_{1,2}$ is made into a Hilbert space:

$$\langle u,v\rangle_{H_{1,2}} = \int_{M_\infty} uv + \int_{M_\infty} \langle du, dv\rangle, \tag{1.6}$$

and for Lipschitz functions f, one has

$$\|f\|^2_{H_{1,2}} = \|f\|^2_{L^2} + \int_{M_\infty} |\operatorname{Lip} f|^2. \tag{1.7}$$

Now by the standard theory of Dirichlet forms, one gets a *positive* self-adjoint Laplacian Δ on M_∞, see [Ch3], [ChCo4] for the details of this theory.

Recall one form of the transplantation theorem of Cheeger (for a proof, see Lemma 10.7 of [Ch3]):

Lemma 1.8. *Assume $M_i^n \xrightarrow{d_{GH}} M_\infty$. f_∞ is a Lipschitz function on $B_R(x_\infty) \subset M_\infty$, $x_i \to x_\infty$. Then there is a sequence of Lipschitz functions $\{f_i\}$ that converges uniformly to f_∞, here f_i is defined on $B_R(x_i) \subset M_i^n$. Moreover, one can require that*

$$\limsup_{i\to\infty} \mathbf{Lip} f_i \le \mathbf{Lip} f_\infty, \tag{1.9}$$

$$\limsup_{i\to\infty} \|\operatorname{Lip} f_i\|_{L^2} \le \|\operatorname{Lip} f_\infty\|_{L^2}. \tag{1.10}$$

By [J], [HaKo], on length spaces satisfying (0.2), a weak Poincare inequality implies a uniform Poincare-Sobolev inequality (i.e, put $L^{\chi p}$ norm on the left side of (0.3) for some $\chi > 1$). In particular, we have a Dirichlet-Sobolev inequality for $u \in \overset{\circ}{H}_{1,2}(B_R(p))$. So we have

Lemma 1.11 (Moser iteration). *If for all ϕ with compact support in $B_2(p)$,*

$$\int \nabla u \nabla \phi = \int cu\phi + f\phi, \tag{1.12}$$

then for $q > C(\kappa)$, we have,

$$\|u\|_{L^\infty(B_1(p))} \le C(n,q)(1+|c|)^{N(\tau,\kappa)}(\|u\|_{L^2(B_2(p))} + \|f\|_{L^q(B_2(p))}). \tag{1.13}$$

For proof, see chapter 4 of [Lin]. Note here we need to renormalize the measure.

Recall, on smooth manifolds we have

Lemma 1.14 (Gradient estimate). *If $\Delta u - cu = f$, $\|u\|_{L^2} < \infty$ on $B_{2R}(p)$ and f is a C^2 function with Lipschitz constant* $\mathbf{Lip}f$, *c is a constant, then on $B_R(p)$ we have a gradient estimate:*

$$|\nabla u| \le C(\|f\|_{L^\infty}, \|u\|_{L^2}, \mathbf{Lip}f)(1+|c|)^{N(\tau,\kappa)}. \tag{1.15}$$

The proof follows a standard argument of Cheng-Yau, see [CY1], [LiY1], compare also with [Lin], [Li2], [SY]. Then use the Moser iteration to replace $\|u\|_{L^\infty}$ with $\|u\|_{L^2}$.

Finally recall the Li-Yau estimates [LiY2], [SY]: If M^n satisfies (0.1), then its heat kernel H satisfies

$$H(x,y,t) \le C(n)\operatorname{Vol}(B_{\sqrt{t}}(x))^{-1/2}\operatorname{Vol}(B_{\sqrt{t}}(y))^{-1/2}e^{-d^2(x,y)/5t}e^{C\Lambda t}. \tag{1.16}$$

If $t < T$, by volume comparison (1.16) simplifies to

$$H(x,y,t) \le C(n,\Lambda,T)\operatorname{Vol}(B_{\sqrt{t}}(x))^{-1}e^{-d^2(x,y)/5t}e^{C\Lambda t}t^{-C(n)}e^{C(n,\Lambda,T)d(x,y)}. \tag{1.17}$$

If we assume $\operatorname{Ric}_{M^n} \ge 0$, then

$$\frac{C^{-1}(n)}{\operatorname{Vol}(B_{\sqrt{t}}(x))}e^{-d^2(x,y)/3t} \le H(x,y,t) \le \frac{C(n)}{\operatorname{Vol}(B_{\sqrt{t}}(x))}e^{-d^2(x,y)/5t}. \tag{1.18}$$

Assume M^n is noncompact, write G for the minimal positive Green's function on M^n. If $n \ge 3$, M^n satisfies (0.4) and $\operatorname{Ric}_{M^n} \ge 0$, then G exists ([LiY2], [LiT1], [LiT2], [SY]) and satisfies

$$C^{-1}(v_0)d(x,y)^{2-n} \le G(x,y) \le C(v_0)d(x,y)^{2-n}. \tag{1.19}$$

For proofs of the above estimates, see [LiY2], [SY].

2. The compact case.

In this section, we assume M_∞ is compact. First we study the heat kernel H.

It's well known that if $\mathrm{Ric}_{M_i^n} \geq -(n-1)\Lambda$, there is a lower bound for the kth eigenvalue $\lambda_{k,i}$ of the Laplacian Δ_i over M_i^n:

$$\lambda_{k,i} \geq C(n, \Lambda, D) k^{\frac{2}{n}}, \tag{2.1}$$

here $\mathrm{Diam}\, M_i^n \leq D$. The proof uses only (0.2) and (0.3); see [Gr] and Theorem 4.8 of [Ch3]. Hence over M_i^n we have

$$H_i(x, y, t) = \sum_{j=0}^{\infty} e^{-\lambda_{j,i} t} \phi_{j,i}(x) \phi_{j,i}(y), \tag{2.2}$$

here $\phi_{j,i}$ is the eigenfunction of the jth eigenvalue $\lambda_{j,i}$. By the Cheeger-Colding spectral convergence theorem [ChCo4], for each j, $\lambda_{j,i} \to \lambda_{j,\infty}$, and $\phi_{j,i} \to \phi_{j,\infty}$ uniformly when $i \to \infty$, here $\lambda_{j,\infty}$ and $\phi_{j,\infty}$ are the j-th eigenvalue and (renormalized) eigenfunction of Δ on M_∞. So (2.1) is also true for $\lambda_{j,\infty}$. Moreover,

$$\|\phi_{j,i}\|_{L^\infty} \leq C_1(n, \Lambda, D)(1 + \lambda_{j,i})^{C(n)} \|\phi_{j,i}\|_{L^2}, \tag{2.3}$$

$$\|\nabla \phi_{j,i}\|_{L^\infty} \leq C_0(n, \Lambda, D)(1 + \lambda_{j,i})^{C(n)} \|\phi_{j,i}\|_{L^\infty}. \tag{2.4}$$

These are implied by (the proof of) the Moser iteration and the gradient estimate, see [LiY1]. By [ChCo4], (2.3), (2.4) can pass to M_∞ (on M_∞ (2.4) becomes an estimate for $\mathbf{Lip}\phi_{j,\infty}$). So it makes sense to write

$$H_\infty(x, y, t) = \sum_{j=0}^{\infty} e^{-\lambda_{j,\infty} t} \phi_{j,\infty}(x) \phi_{j,\infty}(y). \tag{2.5}$$

By (2.1), H_∞ is the heat kernel over M_∞.

Apply (2.3), (2.4) to $\sum_{j=k}^{\infty} e^{-\lambda_{j,i} t} \phi_{j,i}(x) \phi_{j,i}(y)$, the tail of (2.2), one easily get

Theorem 2.6. *Assume $M_i^n \xrightarrow{d_{GH}} M_\infty$, $\mathrm{Ric}_{M_i^n} \geq -(n-1)\Lambda$. When $t > 0$ fixed, H_i converges to the heat kernel H_∞ over M_∞ uniformly. H_∞ is continuous in t, x, y; when t fixed, it is Lipschitz in x, y.*

Corollary 2.7. *For H_∞ on M_∞, the Li-Yau estimate* (1.18) *is true if $\mathrm{Ric}_{M_i^n} \geq 0$; if $\mathrm{Ric}_{M_i^n} \geq -(n-1)\Lambda$ then* (1.16) *is true.*

Next we study the Green's functions. Assume $\mathrm{Ric}_{M_i^n} \geq 0$. Recall,

$$G_i(x,y) = \int_0^\infty h_i(x,y,t)dt, \tag{2.8}$$

here $h_i(x,y,t) = H_i(x,y,t) - \phi_{0,i}(x)\phi_{0,i}(y) = H_i(x,y,t) - 1$. Note, since the sequence $\{M_i^n\}$ might collapse, we have to renormalize the measures and eigenvalues such that $\mathrm{Vol}(M_i^n) = 1$ and $\{\phi_{j,i}\}_j$ is orthonormal, e.g. $\lambda_{0,i} = 0$, $\phi_{0,i} = 1$. So

$$G_i(x,y) = \int_0^\epsilon h_i(x,y,t)dt + \sum_{j=1}^\infty e^{-\epsilon\lambda_{j,i}/2}\int_{\frac{\epsilon}{2}}^\infty e^{-\lambda_{j,i}t}\phi_{j,i}(x)\phi_{j,i}(y)dt. \tag{2.9}$$

When x is fixed, by (2.3),

(2.10)

$$\sum_{j=k}^\infty \left\| e^{-\epsilon\lambda_{j,i}/2}\int_{\frac{\epsilon}{2}}^\infty e^{-\lambda_{j,i}t}\phi_{j,i}(x)\phi_{j,i}(y)dt \right\|_{L^\infty} \leq \sum_{j=k}^\infty \frac{e^{-\epsilon\lambda_{j,i}}}{\lambda_{j,i}} C_1(1+\lambda_{j,i})^{C_3}.$$

This goes to 0 uniformly in i as $k \to \infty$, by (2.1). On the other hand, clearly (1.18) holds after renormalization; when $\mathrm{Ric} \geq 0$, $R < 1/8$ we have the (rescaled) volume bound

$$C(n,D)R^n \leq \mathrm{Vol}(B_R(x)) \leq \sqrt{R}C(n)\,\mathrm{Vol}(B_{\sqrt{R}}(x)) \leq C'(n)R. \tag{2.11}$$

So when $d(x,y) \geq \delta > 0$, by (1.18),

$$\int_0^\epsilon |h_i(x,y,t)|dt \leq \epsilon + \int_0^\epsilon |H_i(x,y,t)|dt \leq \epsilon + C\int_0^\epsilon t^{-\frac{n}{2}}e^{-\frac{\delta^2}{5t}}dt. \tag{2.12}$$

So in particular, by choosing ϵ small, we get a function $G_\infty(x,y)$ on M_∞, such that $G_i \to G_\infty$ in L^∞ on compact subsets, off the diagonal.

Finally, we want to check G_∞ is the Green's function over M_∞. We now establish an L^1 bound for $G(x,y)$ over the ball $B_R(x)$. Note

(2.13)

$$\int_{B_R(x)} |G(x,y)|dy \leq \int_{B_R(x)}\int_0^1 |h(x,y,t)|dtdy + \int_{B_R(x)}\int_1^\infty |h(x,y,t)|dtdy.$$

Since $\|\phi(y)\|_{L^2} = 1$, by (2.1), (2.3), (2.11) and the Schwartz inequality,

$$\begin{aligned}(2.14)\qquad &\int_{B_R(x)}\int_1^\infty |h(x,y,t)|dtdy \\ &\le \sum_{j=1}^\infty \int_{B_R(x)} e^{-\lambda_j/2}|\phi_j(x)| \int_{\frac{1}{2}}^\infty e^{-\lambda_j t}|\phi_j(y)|dtdy \\ &\le C(n)\sum_{j=1}^\infty e^{-\lambda_j/2}(1+\lambda_j)^{C_2(n)}\sqrt{R}\int_{\frac{1}{2}}^\infty e^{-\lambda_j t}dt \le C'(n)\sqrt{R}.\end{aligned}$$

Now we focus on the first term on the right hand side of (2.13). Since $H - h = 1$, and we have (2.11), it's enough to estimate the integral of H. Put $R < 1/8$, by (1.18), (2.11) we have

$$\begin{aligned}(2.15)\qquad &\int_0^1\int_{B_R(x)} H(x,y,t)dtdy \\ &\le \int_0^1\int_{B_R(x)} C(n)\,\mathrm{Vol}^{-1}(B_{\sqrt{t}}(x))e^{-\frac{d(x,y)^2}{5t}}dtdy \\ &\le \left(\int_0^R\int_{B_R(x)} C(n)\,\mathrm{Vol}^{-1}(B_{\sqrt{t}}(x))e^{-\frac{d(x,y)^2}{5t}}dtdy\right) + C'(n)\sqrt{R}.\end{aligned}$$

Next,

$$(2.16)\qquad \int_0^R \frac{\int_{B_R(x)} e^{-d(x,y)^2/5t}dy}{\mathrm{Vol}(B_{\sqrt{t}}(x))}dt = \int_0^R \frac{\int_0^R e^{-r^2/5t}A(r)r^{n-1}dr}{\int_0^{\sqrt{t}} A(r)r^{n-1}dr}dt,$$

here $A(r)r^{n-1}$ is the surface area element of $\partial B_r(x)$. Since $\mathrm{Ric}_{M_i^n} \ge 0$, $A(r)$ is non-increasing. The right hand side of (2.16) can be bounded by

$$\begin{aligned}(2.17)\qquad &\int_0^R \left(\sum_{s=0}^{[R/\sqrt{t}]+1}\int_{s\sqrt{t}}^{(s+1)\sqrt{t}} e^{-r^2/5t}A(r)r^{n-1}dr \Big/ \int_0^{\sqrt{t}} A(r)r^{n-1}dr\right)dt \\ &\le C_1(n)R + \int_0^R \sum_{s=1}^{[R/\sqrt{t}]+1} e^{-s^2/5}((s+1)^n - s^n)dt \le C'(n)R.\end{aligned}$$

So combine (2.14), (2.15) and (2.11) we get

$$(2.18)\qquad \int_{B_R(x)} |G(x,y)|dy \le C'(n)\sqrt{R}.$$

Since $G_i \to G_\infty$ uniformly off the diagonal, use the Cheeger-Colding theorem on the convergence of eigenfunctions [ChCo4] and (2.3), (2.4), we get, for all x,

$$\begin{aligned}\phi_{j,\infty}(x) = \lim_{i\to\infty}\phi_{j,i}(x) &= \lim_{i\to\infty}\int_{M_i} G_i(x,y)\lambda_{j,i}\phi_{j,i}(y)dy \\ &= \int_{M_\infty} G_\infty(x,y)\lambda_{j,\infty}\phi_{j,\infty}(y)dy.\end{aligned} \tag{2.19}$$

So G_∞ is the Green's function over M_∞. Moreover, by (2.10), (2.12) and Lemma 1.14, G_∞ is Lipschitz continuous off the diagonal. It is harmonic off the diagonal by Lemma 3.17. So we have proved

Theorem 2.20. *Assume $M_i^n \xrightarrow{d_{GH}} M_\infty$, $\mathrm{Ric}_{M_i^n} \geq 0$. Then the Green's function G_∞ on M_∞ exists. On any compact subsets K off the diagonal, G_∞ is Lipschitz and harmonic, $G_i \to G_\infty$ uniformly on K.*

3. The Green's functions on noncompact spaces.

Recall how on a manifold, one solves the the Poisson equation,

$$\Delta u_R = f,\ u_R|_{\partial B_R(p)} = h, \tag{3.1}$$

for Lipschitz functions f, h on the closed ball $B_R(p)$. By the Dirichlet's principle, u_R is the unique minimizer of the functional

$$I(u) = \int_{B_R(p)} (|du|^2 - fu). \tag{3.2}$$

within the space $\mathcal{E} = h + \overset{\circ}{H}_{1,2}(B_R(p))$, note Δ is positive by convention.

Assume $M_i^n \xrightarrow{d_{GH}} M_\infty$ in the measured Gromov-Hausdorff sense, $\mathrm{Ric}_{M_i^n} \geq -(n-1)\Lambda$. Recall, by [Ch3], [ChCo4], Δ is linear on M_∞. So the above variational method is valid also on M_∞.

Lemma 3.3 (Lower semicontinuity of energy). *Suppose u_i, f_i are C^2 functions over M_i^n, $\Delta u_i = f_i$, $u_i \to u_\infty$, $f_i \to f_\infty$ uniformly over the sequence of converging balls $B_{2R}(p_i) \to B_{2R}(p_\infty)$, and there is a uniform gradient estimate for u_i and f_i:*

$$|\nabla u_i|, |\nabla f_i| < L. \tag{3.4}$$

Then we have

$$I(u_\infty) \le \liminf_{i\to\infty} I(u_i). \tag{3.5}$$

Proof. As in [ChCo1], we can get an integral bound for the Hessian of f_i on the ball $B_1(p_i)$: recall the Bochner formula

$$\frac{1}{2}\Delta(|\nabla f_i|^2) = |\mathrm{Hess}_{f_i}|^2 + < \nabla\Delta f_i, \nabla f_i > + \mathrm{Ric}(\nabla f_i, \nabla f_i). \tag{3.6}$$

Multiply by a cut-off function ϕ with supp$\phi \subset B_r \subset B_1(q_i)$, $\phi|_{B_{r/2}} = 1$, $|\nabla\phi| \le c(n,r)$, $|\Delta\phi| \le c(n,r)$; see Theorem 6.33 of [ChCo1]. Since f_i is harmonic,

$$\frac{1}{2}\phi\Delta(|\nabla f_i|^2) = \phi|\mathrm{Hess}_{f_i}|^2 + \phi\,\mathrm{Ric}(\nabla f_i, \nabla f_i). \tag{3.7}$$

Integrate by parts,

$$\int_{B_r} \frac{1}{2}(|\nabla f_i|^2)\Delta\phi = \int_{B_r} \phi|\mathrm{Hess}_{f_i}|^2 + \int_{B_r} \phi\,\mathrm{Ric}(\nabla f_i, \nabla f_i). \tag{3.8}$$

By assumption, there is a definite lower bound for the last term in (3.8). Note $|\Delta\phi|$ is uniformly bounded by construction, we have a uniform upper bound for $\int_{B_r} \phi|\mathrm{Hess}_{f_i}|^2$. So by Lemma 1.3 we can assume some subsequence of $|\nabla f_i|$ converges to a function Γ on $B_R(p_\infty) \subset M_\infty$ in L^2. Assume, $x \in \mathcal{R}_k \subset M_\infty$ for some k (all tangent cone at x is $\mathbf{R}^k$), there is some subset $A(x) \subset M_\infty$ such that and Γ is continuous on $A(x)$, $x \in A(x)$ is a density point of $A(x)$. By Luzin's theorem and the results in [ChCo2], these properties hold for almost all $x \in M_\infty$. For such x, we prove

$$|\mathrm{Lip}\ f_\infty(x)| \le \Gamma(x). \tag{3.9}$$

Clearly, (3.9) implies our lemma.

To prove (3.9), it's enough to prove, for all $\psi > 0$, if $l = d(x,y)$ is sufficiently small, then

$$|f_\infty(x) - f_\infty(y)| \le d(y,x)(\Gamma(x) + 6\psi). \tag{3.10}$$

By the gradient estimate of f_i (so of f_∞), if (3.10) is not true for some y_0, then for all $y \in B_{l\psi/L}(y_0)$,

$$|f_\infty(x) - f_\infty(y)| > d(y,x)(\Gamma(x) + 5\psi). \tag{3.11}$$

Pick $x_i, y_{0,i} \in M_i^n$, $x_i \to x$, $y_{0,i} \to y_0$, $d(x_i, y_{0,i}) = l$. Then for i big enough, for all $y_i \in B_{l\psi/L}(y_{0,i})$ and all minimal geodesic γ_i connecting x_i and y_i,

$$\int_{\gamma_i} |\nabla f_i| \geq d(x_i, y_i)(\Gamma(x) + 4\psi). \tag{3.12}$$

First of all, since $|\nabla f_i|$ is uniformly bounded by L, a simple computation shows along every γ_i we must have

$$|\nabla f_i| > \Gamma(x) + 2\psi, \tag{3.13}$$

on a subset of γ_i which has 1-Hausdorff measure at least $2\psi l/(L - \Gamma(x))$. Put

$$T_i = \{v \in T_{x_i} | v = \gamma'(0) \text{ for some minimal geodesic } \gamma \text{ from } x_i \text{ to } y_i \in B_{l\psi/L}(y_{0,i})\}. \tag{3.14}$$

We must have

$$H^{n-1}(T_i) > C(n, L, \psi) H^{n-1}(\partial B_1(0)), \tag{3.15}$$

where H^{n-1} is the $(n-1)$-Hausdorff measure over the unit sphere $\partial B_1(0)$ in the tangent space T_{x_i}. Combine this with (3.13), when M_∞ is *noncollapsed*, if l is small enough, by the proof of the Bishop-Gromov inequality, for sufficiently big i,

$$\frac{\text{Vol } (\{z_i \in B_l(x_i) | |\nabla f_i(z_i)| > \Gamma(x) + 2\psi\})}{\text{Vol } (B_l(x_i))} \geq C(x, n, L, \psi) > 0. \tag{3.16}$$

Now $|\nabla f_i|$ converge to Γ in L^2, so (3.16) is also true if we substitute $|\nabla f_i|$ in (3.16) by Γ, x_i by x. We get a contradiction to the choice of x.

The proof is the same when M_∞ is *collapsed*. We just use the *segment inequality* ([ChCo1], [ChCo4]) to get (3.16) from (3.11), (3.12) and (3.13). □

Lemma 3.17. *Let u_∞, f_∞ be as in the previous lemma. Then*

$$\Delta u_\infty = f_\infty. \tag{3.18}$$

Proof. Assume this is not true over a ball $B_\lambda(p^*) \subset\subset B_1(0)$. By solving the Dirichlet problem on $B_\lambda(p^*)$ we can find v_∞ with the same boundary value as u_∞ over ∂B_λ, but with smaller energy, say

$$(3.19) \quad I(v_\infty) = \int_{B_\lambda(p^*)} |dv_\infty|^2 - f_\infty v_\infty < \int_{B_\lambda(p^*)} |du_\infty|^2 - f_\infty u_\infty - 2\Psi.$$

By obvious density properties, we can change v_∞ slightly so that v_∞ agrees with u_∞ on a neighborhood of $\partial B_\lambda(p^*)$. By Lemma 3.3, for i big enough,

$$(3.20) \qquad I(v_\infty) \le I(u_i) - \Psi.$$

So by (the proof of) Lemma 1.8 (see Section 10 of [Ch3]), for i big enough we can find a function v_i with the same boundary value on ∂B_i as u_i but with smaller energy I. That contradicts the fact that $\Delta u_i = f_i$. □

The solution of (3.1) is unique on M_∞ because the maximum principle holds, see Section 7 of [Ch3].

We now study the Green's functions. Assume $(M_i^n, p_i, \mathrm{Vol}_i) \xrightarrow{d_{GH}} (M_\infty, p, \mu_\infty)$ in the pointed measured Gromov-Hausdorff sense ([Gr], [ChCo2]), where $\mathrm{Ric}_{M_i^n} \ge 0$, M_i^n is complete, noncompact, $n \ge 3$.

Theorem 3.21. *Assume, M_i^n also satisfies* $\mathrm{Vol}(B_R(p_i)) > v_0 R^n$. *Then on M_∞ there is a Green's function G_∞, $G_i \to G_\infty$ uniformly on any compact subsets of $M \times M$ that does not intersects with the diagonal.*

Proof. Since $n \ge 3$, the Euclidean volume growth condition (0.4) implies that the minimal positive Green's function G_i exists on M_i^n (Δ is *positive*). Moreover, G_i satisfies the Li-Yau estimate (1.19). So by the Cheng-Yau gradient estimate and the Arzela-Ascoli theorem, for any fixed x, for some subsequence (still denoted by G_i), we have

$$(3.22) \qquad G_i(x,y) \to G_\infty(x,y),$$

uniformly over any compact set in $M \setminus \{x\}$. Clearly, G_∞ satisfies (1.19). We will show G_∞ is in fact well defined, and $G_i \to G_\infty$ as stated in the above theorem.

Assume f_∞ is any Lipschitz function supported in $B_K(p_\infty) \subset M_\infty$, $\mathbf{Lip} f_\infty \le L$. By Lemma 1.8 and approximation, there is a sequence of C^2 functions $\{f_i\}$ with $f_i \to f_\infty$ uniformly, $\mathbf{Lip} f_i \le 2L$, $\mathrm{supp} f_i \subset B_{2K}(p_i) \subset M_i^n$, $i = 1, 2, \ldots, \infty$.

Recall on each manifold M_i^n with maximal volume growth condition, the function,

$$u_i(x) = \int_{M_i^n} G_i(x,y) f_i(y) dy, \tag{3.23}$$

solves the Poisson equation

$$\Delta u_i = f_i, \quad \lim_{x \to \infty} u_i(x) = 0. \tag{3.24}$$

Now by the Li-Yau estimate (1.19) and the Euclidean volume growth condition (0.4), G_i is locally integrable, so u_i is uniformly bounded. The gradient estimate Lemma 1.14 shows that u_i are uniformly Lipschitz:

$$\mathbf{Lip} u_i \leq C(L, K, n). \tag{3.25}$$

So we can find a subsequence of $\{u_i\}$ that converges to some Lipschitz function u_∞ on M_∞. Note that by the Li-Yau estimate, (1.19),

$$|u_i(x)| \leq C'(L, K, n) d(x, p_i)^{2-n}, \quad (i = 1, 2, \ldots, \infty). \tag{3.26}$$

So by Lemma 3.17, $\Delta u_\infty = f_\infty$ on M_∞. Using the fact Laplacian is linear, by (3.26) and the maximum principle (Section 7 of [Ch3]), it is clear that u_∞ is well defined and $u_i \to u_\infty$ uniformly.

Notice, by (1.19),

$$u_\infty(x) = \int_{M_\infty} G_\infty(x,y) f_\infty(y) dy. \tag{3.27}$$

Since we can choose arbitrary K, f_∞, clearly G_∞ is also well defined, $G_i \to G_\infty$ uniformly, off the diagonal. By (3.27) and Lemma 3.17, G_∞ can be interpreted as the minimal positive Green's function on M_∞. □

4. Separation of variables on tangent cones.

Assume M_i^n is complete noncompact, $\mathrm{Ric}_{M_i^n} \geq 0$ and satisfies (0.4) uniformly, $M_i^n \xrightarrow{d_{GH}} M_\infty$. Recall that by [ChCo1], [ChCo2], every tangent cone of M_∞ is a metric cone. We denote such a cone by $C(X) = \mathbf{R}_+ \times_r X$, here (X, dx^2) is a compact length space with $\mathrm{Diam}\, X \leq \pi$, [ChCo1]. The metric on $C(X)$ is

$$d\rho^2 = dr^2 + r^2 dx^2. \tag{4.1}$$

Here we write r for the distance from the pole $p_\infty = (0, X)$.

The measure μ_∞ on $C(X)$ is just the n-Hausdorff measure, [ChCo2]. Since we can rescale $C(X)$, μ_∞ induced a natural measure μ_X on X that obviously satisfies a doubling condition (0.2) (with some different κ). Moreover, X satisfies the rectifiability properties as stated in Section 5 of [ChCo4].

Also recall from [Ch3], for $f, g \in H_{1,2}$,

$$d(fg) = f \cdot dg + g \cdot df. \tag{4.2}$$

Moreover, from [ChCo4] and [Ch3], if f is a function depending only on r and g is a function independent of r, then by the polar identity, one gets $\langle df, dg \rangle = 0$.

Lemma 4.3 (Weak Poincare inequality). *For $B_R \subset X$, $3R < \frac{1}{5}$, $f \in H_{1,2}(X)$,*

$$\int_{B_R(x)} |f - f_{x,R}|^2 \le \tau_X R^2 \int_{B_{3R(x)}} |df|^2. \tag{4.4}$$

Proof. Define, for $x \in X$,

$$\mathrm{Box}((1,x), a, b) = \{(t,y) \in C(X) \,|\, |t-1| < a, d_X(x,y) < b\}. \tag{4.5}$$

Put

$$\mathrm{Box}_1 = \mathrm{Box}((1,x), R, R),\ \mathrm{Box}_2 = \mathrm{Box}((1,x), 3R, 3R) \subset C(X) \tag{4.6}$$

So $\mathrm{Box}_1 \subset B_{2R}((1,x)) \subset \mathrm{Box}_2$. We extend f to be a $H_{1,2}$ function independent of r on $C(X)$. Assume f_{Ball} is the average of f on the ball $B_{2R}((1,x)) \subset C(X)$,

(4.7)
$$\begin{aligned}
\int_{B_R(x)} |f - f_{x,R}|^2 &= C(n)R^{-1} \int_{\mathrm{Box}_1} |f - f_{x,R}|^2 \le C(n)R^{-1} \int_{\mathrm{Box}_1} |f - f_{\mathrm{Ball}}|^2 \\
&\le C(n)R^{-1} \int_{B_{2R}((1,x))} |f - f_{\mathrm{Ball}}|^2 \le C(n)\tau R \int_{B_{2R}((1,x))} |df|^2 \\
&\le C(n)\tau R \int_{\mathrm{Box}_2} |df|^2 = \tau_X R^2 \int_{B_{3R}(x)} |df|^2.
\end{aligned}$$

The first and last identity come from the Fubini theorem. Note $f_{x,R}$ is also the average of f over Box_1, and we used the Poincare inequality on $C(X)$ in the middle inequality. ☐

We remark, a weak Poincare inequality is already enough for many purposes. Since X is a length space, by [HaKo] one has (0.3) on X. As in [Ch3], [ChCo4], we define a *positive* operator Δ_X on X. Note by (0.2), (0.3) the compact embedding lemma 1.3 is true on X. So by the standard elliptic theory, on X we have a basis $\{\phi_j\}_{j=0}^{\infty}$ for $L^2(X)$ and a sequence $\mu_j \to \infty$ such that $\Delta_X \phi_j = \mu_j \phi_j$, compare [Ch3], [ChCo4]. Moreover, one can do Moser iteration on X, so ϕ_i is Hölder continuous; see [Lin], [GT]. These have applications in Section 6.

Next we show, even the cross section X may not be a manifold, there is still a separation of variables formula for Δ on $C(X)$. See [Ch1] for the classical case.

Recall that $\langle \cdot, \cdot \rangle$ is the inner product on $T^* M_\infty$ as in [Ch3], [ChCo4].

Lemma 4.8.

$$\Delta(fg) = f\Delta g + g\Delta f - 2\langle df, dg\rangle. \tag{4.9}$$

Proof. Since $d(fg) = f \cdot dg + g \cdot df$, for any Lipschitz (or $H_{1,2}$) function ϕ with compact support, we have (recall Δ is *positive*)

$$\int \langle df, g \cdot d\phi + \phi \cdot dg\rangle - \int g\phi\Delta f = 0. \tag{4.10}$$

Exchange the role of f and g, we get

$$\int \langle d(fg), d\phi\rangle - \int \phi(f\Delta g + g\Delta f - 2\langle df, dg\rangle) = 0. \tag{4.11}$$

□

Similarly, by $d(f \circ g) = f'(g)dg$, we get

$$\Delta f \circ g = -f''(g)|dg|^2 + f'(g)\Delta g. \tag{4.12}$$

Lemma 4.13. *On $C(X)$, r^{2-n} is harmonic away from the pole.*

Proof. By the results in Section 4 of [ChCo1], r^{2-n} is the uniform limit of a sequence of harmonic functions $\mathcal{G}$. So by the proof of Lemma 3.17, r^{2-n} is harmonic. □

By the maximum principle on X (Section 7 in [Ch3]), we have

Lemma 4.14. *If X is compact, and $\Delta_X f = 0$, then f is a constant.*

Theorem 4.15. *Assume u lies in the ring generated by functions of the form $u = fg$ where f depends only on r and g depends only on x. Then on $C(X) \setminus \{p_\infty\}$,*

$$\Delta u = -\frac{\partial^2 u}{\partial r^2} - \frac{n-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Delta_X u. \tag{4.16}$$

Proof. Compare [ChTa1]. By (4.12) and Lemma 4.13, on the cone $C(X)$ we have

$$\Delta f(r) = -f''(r) - f'\frac{n-1}{r}. \tag{4.17}$$

Next we apply Lemma 4.8, recall $\langle df, dg\rangle = 0$. We pick a test function ϕ of the form $\phi = a(r)b(x)$. By scaling we see, $\Delta g(R, x) = R^{-2}\Delta g(1, x)$. Assume a is supported over the interval $[\alpha, \beta]$,

$$\int_{C(X)} \langle dg, d\phi\rangle = \int_\alpha^\beta (t^{1-n} \int_X t^{-2}\langle dg, a(t)db\rangle dx)dt, \tag{4.18}$$

here in the second integral we view g and b as functions on the cross section $X = (1, X)$. So we compute

$$\int_{C(X)} \langle dg, d\phi\rangle = \int_\alpha^\beta \left(t^{-n-1} \int_X a(t)b(x)\Delta_X g dx\right) dt = \int \phi r^{-2}\Delta_X g. \tag{4.19}$$

Since we can choose arbitrary a, b, and $\Delta g(R, x) = R^{-2}\Delta g(1, x)$, we get

$$\Delta g(R, x) = R^{-2}\Delta_X g. \tag{4.20}$$

This suffices to complete the proof. □

Using transformation $D_R : (r, x) \mapsto (Rr, x)$, we deduce from the existence and uniqueness of G_∞ that

$$G_\infty(D_R x, D_R y) = R^{2-n} G_\infty(x, y). \tag{4.21}$$

So $G_\infty(p_\infty, x) = d(p_\infty, x)^{2-n} g(x)$ for some Lipschitz function g. By (4.16) and Lemmas 4.8, 4.14, $g = C$ is a constant.

Corollary 4.22.

$$G_\infty(p_\infty, x) = (n-2)^{-1}\mu_X(X)^{-1}d^{2-n}(p_\infty, x). \tag{4.23}$$

Proof. We know $G_\infty(p_\infty, x) = Cd^{2-n}(p_\infty, x)$. We construct a test function $\phi = \phi(r)$ such that ϕ is a smooth function of r, $\phi = 1$ for r small and $\phi = 0$ for $r \geq 1$. So

(4.24)

$$1 = \int G(p_\infty, y)\Delta\phi(y) = \int_0^1 \left(-\phi'' - \frac{n-1}{r}\phi'\right) Cr^{2-n}r^{n-1}\mu_X(X)dr$$
$$= -C\mu_X(X)\int_0^1 (n-2)\phi' dr = (n-2)C\mu_X(X).$$

□

Corollary 4.25. *Assume $\Delta u = f$ on $B_R(p_\infty) \setminus \{p_\infty\}$, $f \in L^\infty$, and*

$$\lim_{x\to p} |u(x)|d(x, p_\infty)^{n-2} = 0. \tag{4.26}$$

Then $\Delta u = f$ on $B_R(p_\infty)$.

Proof. By the De Giorgi-Nash-Moser theorem, u is bounded and Hölder continuous. In our case, $G_\infty(p_\infty, x) = Cd(p_\infty, x)^{2-n}$, so the proof goes exactly like the $\mathbf{R}^n$ case (where the maximum principle is used). For details see [Lin]. □

Relation (4.23) implies the Colding-Minicozzi asymptotic formula, [CoMi1], compare [LiTW]. In fact, we rescale the manifold M^n to get a sequence of manifolds that converges to $C(X)$, a tangent cone at infinity, see [ChCo2]. By Theorem 3.21, the new (rescaled) Green's functions converge to the Green's function on $C(X)$.

Theorem 4.27 (Colding-Minicozzi). *On a noncompact manifold M^n with $\mathrm{Ric}_{M^n} \geq 0$ and (0.4) we have*

$$\lim_{d(x,p)\to\infty} d(x,p)^{n-2}G(p,x) = (n-2)^{-1}(n \lim_{R\to\infty} R^{-n}\,\mathrm{Vol}(B_R(p)))^{-1}. \tag{4.28}$$

Note the tangent cones may not be unique; in collapsing case, a tangent cone might not be a metric cone, [ChCo2], [Per].

5. Heat kernels on noncompact spaces.

We assume in this section, all the manifolds M^n are noncompact, satisfying (0.1). On M^n, write $H(x, y, t)$ for the heat kernel; we denote by $H_R(x, y, t)$ the Dirichlet heat kernel on the metric ball $B_R(p)$, put $H_R = 0$ outside $B_R(p)$.

One technical issue is, the boundary $\partial B_R(p) = d^{-1}(R)$ may not be smooth, here $d = d(p, \cdot)$. However, we can approximate d by a Morse function d_ϵ, see [Hir], and (assuming) R is not a critical value, etc. So in the sequel we always assume the boundary are smooth.

Lemma 5.1. *Assume* $\mathrm{Ric}_{M^n} \geq -(n-1)\Lambda$. *Then there is a function* $\epsilon(t, \Lambda, R)$ *with* $\lim_{R\to\infty} \epsilon(t, \Lambda, R) = 0$ *for* $t > 0$, *and*

$$\int_{M - B_R(x)} H(x, y, t)dy \leq \epsilon(t, \Lambda, R). \tag{5.2}$$

Proof. By the Bishop-Gromov inequality, it's easy to see

$$\mathrm{Vol}(B_{\sqrt{t}}(x)) \leq C_1(n, \Lambda, t)e^{C_2(n,\Lambda,t)d(x,y)}\,\mathrm{Vol}(B_{\sqrt{t}}(y)). \tag{5.3}$$

Put $s_\Lambda(r) = (1/\sqrt{\Lambda}) \sinh \sqrt{\Lambda} r$. We now use the Li-Yau estimate (1.16):
(5.4)

$$\begin{aligned}
&\int_{M - B_R(x)} H(x, y, t)dy \\
&\qquad \leq C'(n, \Lambda, t) \int_{M - B_R(x)} \mathrm{Vol}^{-1}(B_{\sqrt{t}}(x))e^{-d(x,y)^2/5t}e^{C_2(n,\Lambda,t)d(x,y)} \\
&\qquad = C'(n, \Lambda, t) \int_R^\infty e^{-r^2/5t}e^{C_2 r}A(r)s_\Lambda^{n-1}(r)dr \Big/ \int_0^{\sqrt{t}} A(r)s_\Lambda^{n-1}(r)dr \\
&\qquad \leq C' \int_R^\infty e^{-r^2/5t}e^{C_2 r}s_\Lambda^{n-1}(r)dr \Big/ \int_0^{\sqrt{t}} s_\Lambda^{n-1}(r)dr = \epsilon(t, \Lambda, R).
\end{aligned}$$

Here $A(r)s_\Lambda^{n-1}(r)$ is the surface area element of $\partial B_R(x)$. We used the fact $A(r)$ is non-increasing (Bishop-Gromov inequality) and assumed, without lose of generality, $R > \sqrt{t}$. □

Lemma 5.5. *Let* (M^n, p) *be a noncompact complete manifold. Then*

$$\lim_{R\to\infty} H_R(x, \cdot, t) = H(x, \cdot, t). \tag{5.6}$$

The convergence is uniform, and uniformly in L^1, on any finite interval $t \in [0, T]$.

Proof. Assume $R > \max\{T, 2d(x,p)\}$. Put

$$M(R) = \sup\{H(x,y,t) | y \in \partial B_R(x), 0 < t \leq T\}, \tag{5.7}$$

by (1.17) and volume comparison we have

$$M(R) \leq \sup_{0<t\leq T} C(n,\Lambda,T) t^{-C_1(n)} e^{-R^2/5t} e^{C_2(n,\Lambda,T)R} \operatorname{Vol}(B_R(p))^{-1}, \tag{5.8}$$

so $\lim_{R\to\infty} M(R) \operatorname{Vol}(B_R(p)) = 0$. By the maximum principle,

$$H(x,y,t) - M(R) \leq H_R(x,y,t) \leq H(x,y,t). \tag{5.9}$$

Combining this with Lemma 5.1, we have

$$\|H_R(x,\cdot,t) - H(x,\cdot,t)\|_{L^1} < \epsilon(n,\Lambda,T,R), \tag{5.10}$$

and $\lim_{R\to\infty} \epsilon(n,\Lambda,T,R) = 0$. □

Assume λ_j is the j-th Dirichlet eigenvalue of the Laplacian on $B_R(p)$, ϕ_j is the corresponding eigenfunction, $\|\phi_j\|_{L^2(B_R(p))} = 1$.

Lemma 5.11. *There exists a constant $C(n,\Lambda,R)$ such that*

$$C(n,\Lambda,R)^{-1} R^{-2} k^{\frac{2}{n}} \leq \lambda_k \leq C(n,\Lambda,R) R^{-2} k^2. \tag{5.12}$$

Proof. Since R fixed, we have $\operatorname{Vol}(B_r(x)) \geq C_0(n,\Lambda,R) r^n \operatorname{Vol}(B_R(p))$, for $r < 2R$ and $B_r(x)$ with nonempty intersection with $B_R(p)$. Then since $H_R \leq H$, we can follow the heat kernel argument as in page 178 of [SY] to get the lower bound of λ_k.

The upper bound follows from an argument of Cheng, see page 105 of [SY]. □

Lemma 5.13. *For any $N > 0$, there is a function $\epsilon(N,\Lambda,R,\delta)$ such that for any fixed R, $\lim_{\delta\to 0} \epsilon(N,\Lambda,R,\delta) = 0$, and for k such that $\lambda_k < N$,*

$$\int_{A(p,R-\delta,R)} |\phi_k|^2 \leq \epsilon(N,\Lambda,R,\delta). \tag{5.14}$$

Here $A(p,R-\delta,R)$ is the annulus $\{z | R-\delta \leq d(p,z) \leq R\}$.

Proof. By (1.16) and the Bishop-Gromov inequality, when $t = 1$,

$$\begin{aligned}\int_{A(p,R-\delta,R)} |\phi_k|^2 &\le e^{\lambda_k} \int_{A(p,R-\delta,R)} H(x,x,1)dx \\ &\le e^N \int_{A(p,R-\delta,R)} \frac{C(n,\Lambda,R)}{\mathrm{Vol}(B_R(p))} dx \le \epsilon(N,\Lambda,R,\delta).\end{aligned} \tag{5.15}$$

□

As before, assume $M_i^n \overset{d_{GH}}{\longrightarrow} M_\infty$ in the pointed measured Gromov-Hausdorff sense, M_i is noncompact, satisfies (0.1). Write $\lambda_{j,i}$ for the j-th Dirichlet eigenvalue over $B_R(p_i) \subset M_i^n$. $\phi_{j,i}$ is the corresponding eigenvalue:

$$\Delta\phi_{j,i} = \lambda_{j,i}\phi_{j,i}; \quad \int_{B_R(p_i)} \phi_{j,i}\phi_{k,i} = \delta_{jk}. \tag{5.16}$$

Lemma 5.17. *For fixed $j, k > 0$, assume (for a subsequence of the eigenvalues), $\lambda_{j,i} \to \lambda_{j,\infty}$, $\lambda_{k,i} \to \lambda_{k,\infty}$. Then there is a subsequence (denoted also by $\phi_{j,i}, \phi_{k,i}$) that converges uniformly on compact subsets of B_R, and also in $L^2(B_R)$, to two locally Lipschitz functions $\phi_{j,\infty}, \phi_{k,\infty}$. Moreover,*

$$\Delta\phi_{j,\infty} = \lambda_{j,\infty}\phi_{j,\infty}, \ \Delta\phi_{k,\infty} = \lambda_{k,\infty}\phi_{k,\infty}, \ \int_{B_R(p)} \phi_{j,\infty}\phi_{k,\infty} = \delta_{jk}. \tag{5.18}$$

Proof. The results is clear in view of Lemma 5.11, Lemma 1.14 and Lemma 3.17. The L^2 convergence and the orthonormal property for the limit functions are implied by locally uniform convergence and Lemma 5.13. □

By Lemma 5.11, we can assume, after passing to a subsequence, that *every* eigenvalue and eigenfunction converge:

$$\lim_{i\to\infty} \lambda_{j,i} = \lambda_{j,\infty}, \ \lim_{i\to\infty} \phi_{j,i} = \phi_{j,\infty}. \tag{5.19}$$

Write

$$H_{R,\infty} = \sum_{j=1}^{\infty} e^{-\lambda_{j,\infty}t}\phi_{j,\infty}(x)\phi_{j,\infty}(y). \tag{5.20}$$

For all fixed t, x, by Lemma 5.11 and Lemma 1.11, Lemma 1.14,

$$H_{R,i}(x,\cdot,t) \to H_{R,\infty}(x,\cdot,t). \tag{5.21}$$

The convergence is in L^2, and is locally uniform. Note we don't know if $H_{R,\infty}$ (and $\phi_{j,\infty}$, $\lambda_{j,\infty}$) is well defined. For the moment (before Lemma 5.40), we fix, by a diagonal argument, *one* sequence $R_k \to \infty$, and *one* subsequence $\{M_{i_v}^n\}$ of $\{M_i^n\}$ such that for each k, $H_{R_k,i} \to H_{R_k,\infty}$. For simplicity, we just write $\{M_i^n\}$ for this subsequence of manifolds. So by the results on smooth manifolds, for $R_j < R_k$,

(5.22)

$$0 \le H_{R_j,\infty}(x,y,t) \le H_{R_k,\infty}(x,y,t) \le \frac{C(n,\Lambda)e^{-d^2(x,y)/5t}e^{C\Lambda t}}{\mathrm{Vol}_\infty^{1/2}(B_{\sqrt{t}}(x))\,\mathrm{Vol}_\infty^{1/2}(B_{\sqrt{t}}(y))}.$$

Thus we can also assume that the nondecreasing sequence $H_{R_j,\infty}$ converges pointwise to some function H_∞. We will prove that H_∞ is well defined.

By (5.9) and the locally uniform convergence of $H_{R,i}$ to $H_{R,\infty}$ (5.21), the Li-Yau estimate (1.16) is also true for H_∞:

$$0 \le H_\infty(x,y,t) \le \frac{C(n,\Lambda)e^{-d^2(x,y)/5t}e^{C\Lambda t}}{\mathrm{Vol}_\infty^{1/2}(B_{\sqrt{t}}(x))\,\mathrm{Vol}_\infty^{1/2}(B_{\sqrt{t}}(y))}. \tag{5.23}$$

Note we need to renormalize the measures whenever $\{M_i^n\}$ is collapsing. Clearly, when $\mathrm{Ric}_{M_i^n} \ge 0$, we also have a lower bound of H_∞ as in (1.18).

Corollary 5.24.

$$\int_{M_\infty} H_\infty(x,z,s)H_\infty(z,y,t-s)dz = H_\infty(x,y,t). \tag{5.25}$$

Proof. By (5.21), (5.25) is true for $H_{R,\infty}$. Write $H_\infty(x,z,s) = H_{R,\infty}(x,z,s) + \epsilon_R^1(z)$, similarly $H_\infty(z,y,t-s) = H_{R,\infty}(z,y,t-s) + \epsilon_R^2(z)$, here $H_{R,\infty} = 0$ outside $B_R(p_\infty)$, $\epsilon_R^1, \epsilon_R^2 \ge 0$ are two functions. In view of Lemmas 5.1, 5.5, (5.21) and (5.23),

$$\begin{aligned} &\limsup_{R\to\infty}(\|\epsilon_R^1(z)\|_{L^1} + \|\epsilon_R^2(z)\|_{L^1}) = 0, \\ &\|\epsilon_R^1(z)\|_{L^\infty} + \|\epsilon_R^2(z)\|_{L^\infty} < C(t,s,M_\infty). \end{aligned} \tag{5.26}$$

Now (5.25) is clear. □

Corollary 5.27.

$$\int_{M_\infty} H_\infty(x,y,t)dy = 1. \tag{5.28}$$

Proof. By (5.21), Lemmas 5.1 and 5.5. □

Lemma 5.29. *For any Lipschitz function f with compact support,*

$$\left| \int_{M_\infty} H_{R,\infty}(x,y,t) f(y) dy - f(x) \right| \leq \epsilon(t, \|f\|_{L^\infty}, \mathbf{Lip} f). \tag{5.30}$$

Here for any $F, L > 0$, $\lim_{t \to 0} \epsilon(t, F, L) = 0$. The conclusion is also true for H_∞.

Proof. By an argument similar to those given in Lemma 5.1 and Lemma 5.5. Note on smooth manifolds, when $t \to 0$, the integral of H_R is smaller than, but almost equal to 1, and tends to concentrate on smaller and smaller balls centered at x. In view of (5.21) and the Li-Yau estimate (5.23), we easily get (5.30). □

Let the Sobolev space $\overset{\circ}{H}_{1,2}(B_R(p_\infty))$ be defined as in [Ch3], i.e., the $H_{1,2}$ closure of the set of Lipschitz functions supported in the interior of $B_R(p_\infty)$,

Lemma 5.31. *The space $\overset{\circ}{H}_{1,2}(B_R(p_\infty))$ is contained in Φ, the L^2-linear span of functions $\phi_{j,\infty}$. In particular, any Lipschitz function with support in $B_{R-\delta}$ lies in Φ.*

Proof. If not, by approximation, we have a Lipschitz function f_∞ with compact support and an $\epsilon > 0$ such that

$$\sum_{j=1}^{\infty} \left(\int_{B_R(p_\infty)} f_\infty \phi_{j,\infty} \right)^2 < (1 - 3\epsilon) \|f_\infty\|_{L^2}^2. \tag{5.32}$$

Using Lemma 1.8, we can transplant f_∞ back to a Lipschitz function, f_i, on M_i^n, with compact support which is close to f_∞ in L^∞, such that the energy of f_i is close to that of f_∞. Write

$$f_i = \sum_{j=1}^{N} a_{j,i} \phi_{j,i} + R_{N,i}, \ R_{N,i} = \sum_{j=N+1}^{\infty} a_{j,i} \phi_{j,i}. \tag{5.33}$$

Notice,

$$\lim_{i \to \infty} a_{j,i} = \int_{M_\infty} f_\infty \phi_{j,\infty}. \tag{5.34}$$

So by the min-max principle and Lemma 5.11, $\lim_{i\to\infty} \|\nabla f_i\|_{L^2} = \infty$, we get a contradiction to the construction of f_i, Lemma 1.8. $\square$

Remark, it is not clear if we have $\phi_{j,\infty} \in \overset{\circ}{H}_{1,2}$.

Now for Lipschitz functions f_i with compact support in $B_R(p_i) \subset M_i^n$ $(i = 1, 2, ..., \infty)$, $f_i \to f_\infty$ uniformly, we have

$$f_i = \sum_{j=1}^{\infty} a_{j,i}\phi_{j,i}, \qquad a_{j,i} = \int_{M_i^n} f_i \phi_{j,i}. \tag{5.35}$$

So $a_{j,i} \to a_{j,\infty}$. Clearly,

$$\int_{B_R(p_i)} H_{R,i}(x, y, t) f_i(y) dy = \sum_{j=1}^{\infty} e^{-\lambda_{j,i}t} a_{j,i}\phi_{j,i}(x). \tag{5.36}$$

We say $h(x,t)$ is a *locally strong* solution, if h continuous, Lipschitz in x, $\frac{\partial h}{\partial t}$ exists, continuous on $M \times \mathbf{R}^+$, and when t fixed, $-\Delta h = \frac{\partial h}{\partial t}$, i.e.

$$\int_\Omega \psi \frac{\partial h}{\partial t} + \int_\Omega \langle d_x h, d_x \psi\rangle = 0, \tag{5.37}$$

for all Lipschitz functions ψ with compact support.

By Lemma 5.11, Lemma 1.11 and 1.14,

$$\lim_{k\to\infty} \sum_{j=k} |e^{-\lambda_j t}\phi_j(x) d_y \phi_j(y)| = 0. \tag{5.38}$$

So $H_{R,i}$ is a locally strong solution of the heat equation. Similarly the function,

$$h_i(x,t) = \int_{B_R} H_{R,i}(x, y, t) f_i(y) dy \quad (i = 1, 2, \ldots, \infty), \tag{5.39}$$

is also a locally strong solution. Note for the case $i = \infty$ we used also Lemma 5.17.

For locally strong solutions on M_∞, there is also a weak maximum principle:

Lemma 5.40. *Assume h is a locally strong solution on $B_{2R} \times [0, T+1]$, then if*

$$h|_{B_R \times \{0\}} \leq 0, \ \ h|_{\partial B_R \times [0,T]} \leq 0. \tag{5.41}$$

Then $h \leq 0$ on $B_R \times [0, T]$.

Proof. Define

$$m(s) = \sup\{h(x,s) | x \in B_R, \frac{\partial h}{\partial t}(x,s) \leq 0\}. \tag{5.42}$$

Since h, $\frac{\partial h}{\partial t}(x,s)$ are continuous functions, it's easy to show that m is non-increasing and $m(0) = 0$ implies $m(s) \leq 0$ for all $s > 0$. Now by the weak maximum principle for Poisson equations (see [GT], [Ch3] or (5.37)), we have, when s fixed,

$$\sup\{h(x,s) | x \in B_R\} = m(s) \leq 0. \tag{5.43}$$

□

Now we can address the uniqueness of H_∞. Recall that $(M_i, p_i) \to (M^\infty, p_\infty)$. Assume for $R > 0$, we got two limits $H^{(1)}_{4R,\infty}$, $H^{(2)}_{4R,\infty}$ through different subsequences of manifolds.

Theorem 5.44. *For $x, y \in B_R(p_\infty)$, $t < T$, there is an $\epsilon(R) > 0$ such that*

$$\lim_{R\to\infty} \epsilon(R) = 0, \tag{5.45}$$

$$H^{(1)}_{4R,\infty}(x,y,t) < H^{(2)}_{4R,\infty}(x,y,t) + \epsilon(R). \tag{5.46}$$

Proof. We can assume $R > T^2 > t^2$ and $R > 4$. Assume (5.46) is not true, then there is a point $a \in B_R(p_\infty)$ and $0 < r < 1$ such that

$$H^{(1)}_{4R,\infty}(x,y,t) \geq H^{(2)}_{4R,\infty}(x,y,t) + \epsilon(R), \tag{5.47}$$

for $y \in B_{2r}(a)$. We then construct a test function $f \geq 0$ such that, f Lipschitz, supported in $B_r(a)$,

$$2\int_{B_r(a)} f \geq \mathrm{Vol}(B_r(a)) \sup_{B_r(a)} f. \tag{5.48}$$

Clearly, for $R < \infty$, the functions,

$$F_k(z,s) = \int_{B_r(a)} H^{(k)}_{4R,\infty}(z,y,s) f(y)dy, \ (k = 1, 2), \tag{5.49}$$

are locally strong solutions of the heat equation, and (by the construction of f),

$$(5.50)\quad F_1(x,t) \geq F_2(x,t) + \epsilon \int_{B_r(a)} f \geq F_2(x,t) + \frac{\epsilon(R)}{2} \operatorname{Vol}(B_r(a)) \sup_{B_r(a)} f.$$

For a point z near $\partial B_{2R}(p_\infty)$, say $d(z,p) = 2R$, $d(a,z) \geq R$,

$$(5.51)\quad F_k(z,s) < \operatorname{Vol}(B_r(a)) \frac{C(n)}{\operatorname{Vol}(B_{\sqrt{s}}(z))} e^{-R^2/5s} e^{CR} \sup_{B_r(a)} f, \ (k = 1, 2).$$

By a standard argument of the Bishop-Gromov inequality,

$$(5.52)\quad F_k(z,s) < C_1(n) \frac{\operatorname{Vol}(B_r(a))}{\operatorname{Vol}(B_1(p))} s^{-\frac{n}{2}} e^{-R^2/5s} e^{C'R} \sup_{B_r(a)} f, \ (k = 1, 2).$$

Next we consider the case that s is small. Since f is fixed, by (5.30), $F_k \to f$ uniformly on $B_{2R}(p)$ when $s \to 0$.

In view of the weak maximum principle on $B_{2R}(p_\infty) \times [0, T]$ (Lemma 5.40), clearly we should choose $\epsilon(R)$ such that that for $0 \leq s \leq T$,

$$(5.53)\quad \frac{C_1(n)}{\operatorname{Vol}(B_1(p))} s^{-\frac{n}{2}} e^{-R^2/5s} e^{C'R} < \frac{\epsilon(R)}{4},$$

by the maximum principle we got a contradiction to (5.50). □

Theorem 5.54. *H_∞ is well defined. For fixed $t > 0$, $x_i \to x_\infty$, we have $H_i(x_i, \cdot, t) \to H_\infty(x_\infty, \cdot, t)$ in L^1. When H_∞ is continuous, this convergence is also uniform.*

Proof. By the previous theorem and the construction of H_∞ (compare (5.9)), we see H_∞ is independent of the choice of subsequences, so well defined.

We already know, by (5.9), (5.21), (5.23), that locally $H_i \to H_\infty$ in L^1. The proof of global L^1 convergence is similar with Lemma 5.1, Lemma 5.5, using (1.18), (5.23).

Recall (see [SY] Chapter 4), there is a Harnack inequality

(5.55)
$$H_i(x, y_1, t_1) \leq H_i(x, y_2, t_2) \left(\frac{t_2}{t_1}\right)^n \exp\left(\frac{d^2(y_1, y_2)}{4(t_2 - t_1)} + C(n, \Lambda)(t_2 - t_1)\right),$$

for $0 < t_1 < t_2$. If H_∞ is continuous, then locally H_∞ is uniformly continuous (especially, with respect to t), clearly by (5.55) the convergence $H_i \to H_\infty$ must be uniform, compare with (5.23). □

We now want to interpret the meaning of H_∞. Recall from [ChCo4] and [Ch3], Δ is a *positive self-adjoint operator.* So $-\Delta$ generates a semigroup $e^{-t\Delta}$.

Assume f_i is supported in $B_K(p_i) \subset B_R(p_i)$. Use the notation in (5.35), define

$$W_{R,i}(t)f_i(x) = \sum_{j=1}^{\infty} a_{j,i}\cos(\sqrt{\lambda_{j,i}}t)\phi_{j,i}. \tag{5.56}$$

By the finite speed of propagation (see [Ta]), when t is fixed and $R > K+t$, $W_{R,i}(t)f$ is independent of R. We write $W_i(t)f$ for $W_{R,i}(t)f$ with R big. For $i < \infty$,

$$e^{-t\Delta}f_i(x) = \int_{M_i^n} H_i(x,y,t)f_i(y)dy = \int_{-\infty}^{\infty} e^{-s^2/4t}W_i(s)f_i(x)ds, \tag{5.57}$$

see [CGT], [Ta]. Define

$$W_{R,\infty}(t)f_\infty(x) = \sum_{j=1}^{\infty} a_{j,\infty}\cos(\sqrt{\lambda_{j,\infty}}t)\phi_{j,\infty}. \tag{5.58}$$

We notice that $W_{R,i}$ ($i = 1, 2, ..., \infty$) does not increase L^2 norm, and we should use Lemma 1.8 and approximation to construct C^2 functions f_i on M_i^n that converges to f_∞. Clearly, $W_{R,i}f_i \to W_{R,\infty}f_\infty$ in L^2. We remark that generally, we don't know if $W_{R,\infty}$ is well defined.

Theorem 5.59. *If the limit M_∞ is a smooth manifold, and the limit measure is the canonical measure on M_∞, then H_∞ is the heat kernel on M_∞.*

Proof. In the noncollapsing case, by Colding's theorem [Co], the limit measure is the canonical measure on M_∞; when $M_\infty = \mathbf{R}^k$ for some k, the limit measure is also a multiple of the standard Lebesgue measure on $\mathbf{R}^k$, see [ChCo2]. In these cases, the Laplacian we defined on M_∞ is the same one from the original smooth structure of M_∞.

Pick any C_0^∞ function f supported in B_R, So

$$\int_{M_\infty} (\Delta^k)f\phi_{j,\infty} = (\lambda_{j,\infty})^k \int_{M_\infty} f\phi_{j,\infty} = (\lambda_{j,\infty})^k a_{j,\infty}. \tag{5.60}$$

Since $(\Delta^k)f \in C_0^\infty$, we have for all k, $\lim_{j\to\infty}(\lambda_{j,\infty})^k a_{j,\infty} = 0$. By Lemma 5.11, we have for all k, $\lim_{j\to\infty} j^k a_{j,\infty} = 0$. So $W_{R,\infty}(t)f$ is a classical solution of the wave equation, when R is big enough, $W_{R,\infty}(t)f = W_\infty(t)f$ is independent of R. Since M_∞ is a smooth manifold,

$$(5.61) \qquad e^{-t\Delta}f(x) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-s^2/4t}W_\infty(s)f(x)ds.$$

In view of (5.57), combined with the fact W_∞ does not increase L^2 norm and $H_i(x,y,t)$ converges uniformly to $H_\infty(x,y,t)$, we have

$$(5.62) \qquad e^{-t\Delta}f(x) = \int_{M_\infty} H_\infty(x,y,t)f(y)dy.$$

That concludes the proof. □

6. Laplacian on metric cones.

In this section, we assume $M_i^n \xrightarrow{d_{GH}} C(X)$ where $C(X)$ is a metric cone; $\mathrm{Ric}_{M_i^n} \geq 0$, M_i^n is complete noncompact and satisfies (0.4) uniformly, $n \geq 3$.

Write p_∞ for the pole of $C(X)$, define $r(x) = d(x, p_\infty)$.

Theorem 6.1. *If $M_\infty = C(X)$, then H_∞ is the integral kernel of the semigroup $e^{-t\Delta}$.*

Proof. In view of (5.23), (5.25) and the Young's inequality, one can define a semigroup $E(t)$ on $L^2(M_\infty)$ by

$$(6.2) \qquad E(t)f(x) = \int_{M_\infty} H_\infty(x,y,t)f(y)dy.$$

We want to compare $E(t)$ with $e^{-t\Delta}$. First, by Theorem 3.21, (1.18) and (5.23), one easily get

$$(6.3) \qquad G_\infty(x,y) = \int_0^\infty H_\infty(x,y,t)dt.$$

Pick any L^2 function f with compact support. Write

$$(6.4) \qquad F(x) = \int_{M_\infty} G_\infty(x,y)f(y)dy.$$

We compute

$$\begin{aligned}\frac{E(t)F-F}{t} &= \int_{M_\infty}\left(\frac{H_\infty(x,y,t)}{t}\right)\int_{M_\infty}\int_0^\infty H_\infty(y,z,s)f(z)dsdzdy \\ &\quad -\frac{1}{t}\int_{M_\infty}\int_0^\infty H_\infty(x,z,s)f(z)dsdz \\ &= -\frac{1}{t}\int_0^t\int_{M_\infty} H_\infty(x,z,s)f(z)dzds.\end{aligned} \tag{6.5}$$

So by (0.4), (5.23), (5.30) and the Young's inequality we have

$$\lim_{t\to 0}\frac{E(t)F-F}{t}\to -f. \tag{6.6}$$

in L^2 and L^1.

Now we use the assumption that $M_\infty = C(X)$ is a noncollapsed cone. Recall the results in Section 4, we can construction a function $\phi = \phi(r)$ such that ϕ is a smooth function of r, where $r(x) = d(p_\infty, x)$ is the distance from the pole, and

$$\phi(r) = 1 \text{ if } r < R,\ \phi(r) = 0 \text{ if } r \geq R+2,\ \nabla\phi \leq C_0\sqrt{\phi}. \tag{6.7}$$

So on $M_\infty = C(X)$ we have $\Delta\phi = -\phi'' - (n-1)\phi'/r$. This function can serve as a cut off function.

We prove, if $F, f = \Delta F \in L^2$ have compact support, then

$$F = \int_{C(X)} G_\infty(x,y)f(y)dy. \tag{6.8}$$

In fact, assume $\{f_k\}$ is a sequence of Lipschitz functions, $f_k \to f$ in L^2, and all f_k together with f, F are supported in the ball $B_K(p_\infty)$. So the function

$$F_k = \int_{C(X)} G_\infty(x,y)f_k(y)dy, \tag{6.9}$$

satisfies $\Delta F_k = f_k$ by the discussion in Section 3. Consider the equation $\Delta(F_k - F) = f_k - f$, i.e.

$$\int_{C(X)}\langle dF_k - dF, du\rangle - \int_{C(X)}(f_k - f)u = 0, \tag{6.10}$$

for any $u \in \mathring{H}_{1,2}$. We set $u = \phi(F_k - F)$, so $du = d\phi(F_k - F) + \phi(dF_k - dF)$. By the Schwartz inequality,

(6.11)

$$\begin{aligned}&\|\sqrt{\phi}d(F_k - F)\|_{L^2}^2 - C_0\|(F_k - F)|_{A(R,R+2)}\|_{L^2}\|\sqrt{\phi}d(F_k - F)|_{A(R,R+2)}\|_{L^2} \\ &\qquad - \|(f_k - f)|_{B_K}\|_{L^2}\|(F_k - F)|_{B_K}\|_{L^2} \leq 0,\end{aligned}$$

here $A(R, R+2)$ is the annulus $\{x|R \leq r(x) \leq R+2\}$. Note we have a definite bound for $\|F_k|_{B_K}\|_{L^2}$ by (1.19) and the Young's inequality. Note also by (1.19) we get, for $R > K$,

(6.12)

$$\|(F_k - F)|_{A(R,R+2)}\|_{L^2} = \|F_k|_{A(R,R+2)}\|_{L^2} < C(n, \|f\|_{L^1})(R^{4-2n}R^{n-1})^{1/2} = C(n, \|f\|_{L^1})R^{(3-n)/2} < C(n, \|f\|_{L^1}),$$

since $n \geq 3$. So first, we get that $\|d(F_k - F)\|_{L^2} < \infty$ by letting $R \to \infty$. Then by letting $k \to \infty$, we have $\|d(F_k - F)\|_{L^2} \to 0$, since we can choose R in (6.11) such that $\|\sqrt{\phi} d(F_k - F)|_{A(R,R+2)}\|_{L^2}$ small.

Now by the $(2,2)$-Poincare inequality, (0.4), (1.19) and Young's inequality, $F_k \to F$ in L^2 on compact sets. Also notice, on any compact sets, the right hand side of (6.9) converges to the right hand side of (6.8) in L^2, by the Young inequality (however, in view of (1.19), these convergences might not be globally L^2). That's enough to imply (6.8).

Next we compute, for $f = \Delta F$, $F, f \in L^2$,

(6.13)

$$\|\Delta(\phi F) - f\|_{L^2} \leq \|F\Delta\phi\|_{L^2} + \|(\phi - 1)f\|_{L^2} + 2\| < d\phi, dF > \|_{L^2} \leq C(n)\|F|_{A(R,R+2)}\|_{L^2} + \|f|_{A(R,R+2)}\|_{L^2} + C_0\|dF|_{A(R,R+2)}\|_{L^2}.$$

Similar to (6.11), one shows $\|dF\|_{L^2} < \infty$. So if $R \to \infty$, we have $\phi F \to F$ and $\Delta(\phi F) \to f = \Delta F$ in L^2. Moreover, by (6.8),

$$\phi F(x) = \int_{C(X)} G_\infty(x, y)\Delta(\phi F)(y)dy. \tag{6.14}$$

So the computation (6.5) is valid for the functions ϕF and $\Delta(\phi F)$:

$$\lim_{t\to 0} \frac{E(t)\phi F - \phi F}{t} = -\Delta(\phi F) \tag{6.15}$$

in L^2. We already know $E(t)$ in (6.2) is a semigroup, its infinitesimal generator is a *closed* operator (see [Ta]). So by the above computations, this infinitesimal generator must be the self-adjoint operator $-\Delta$ on $C(X)$. □

By the discussion in the beginning of Section 4, we have an eigenfunction expansion of Laplacian on the unit cross section X. We denote by ϕ_j ($j = 0, 1, 2, \ldots$) the renormalized eigenfunctions with eigenvalues $\mu_j > 0$, note $\phi_0 = \mathrm{Vol}(X)^{-1/2}$. $\mu_j \to \infty$ when $j \to \infty$.

Put $d = \mathrm{Diam}\, X$. Using an argument of Gromov (see [Gr], and Theorem 4.8 of [Ch3]), we have a more precise estimate of μ_j:

$$\mu_j > C(\tau, \kappa)^{-1} d^{-2} j^{\frac{2}{\kappa}}. \tag{6.16}$$

On the other hand, on each ball $B_r(x_k)$ of radius $r = d/2(j+2)$ on X, we define a Lipschitz function ψ_k supported in $B_r(x_k)$ using MacShane's lemma ([Ch3], [ChCo3]):

$$\psi_k(x_k) = r, \ \psi_k(\partial B_r(x_k)) = 0, \ \mathbf{Lip}\psi_k = 1, \tag{6.17}$$

so we can follow the argument of Cheng (see p.105 of [SY]), and get

$$\mu_j \leq C(\kappa) j^2 d^{-2}. \tag{6.18}$$

Now we can use Moser iteration, $|\phi_j|$ is bounded by a definite power of j:

$$|\phi_j| \leq C(d, \kappa, \tau) j^{N(\tau,\kappa)}. \tag{6.19}$$

Moreover, ϕ_j is Hölder continuous, see [GT], [Lin].

Write $\nu_j = \sqrt{\mu_j + \alpha^2}$, here $m = n-1$, $\alpha = (1-m)/2$. We write x, y in polar coordinates, $x = (r_1, x_1), y = (r_2, x_2)$.

Theorem 6.20.

$$H_\infty = (r_1 r_2)^\alpha \sum_{j=0}^{\infty} \left(\frac{1}{2t}\right) e^{-(r_1^2+r_2^2)/4t} I_{\nu_j}\left(\frac{r_1 r_2}{2t}\right) \phi_j(x_1) \otimes \phi_j(x_2). \tag{6.21}$$

Here I_{ν_j} are the modified Bessel functions:

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{2k}. \tag{6.22}$$

In our case Δ is a self-adjoint operator on the whole cone $C(X)$, namely, including the pole p_∞. By Corollary 4.25, the separation of variable formula (4.16) works for $u = f(r)g(x)$ on the whole $C(X)$ if u and Δu are bounded on $C(X) \setminus \{p_\infty\}$. So the heat kernel on M_∞ has the expression as on the right hand side of (6.21); the proof goes exactly like the classical case, see [Ch1], [Ch2] page 592, [ChTa1] and [Ta] chapter 8, we omit the details. By Theorem 6.1, we have (6.21).

By Stirling's formula, (6.16) and (6.18), we see the series (6.21) converges uniformly, when t is bounded away from 0 and r_1, r_2 stay bounded. In particular, H_∞ is continuous, so by Theorem 5.54 we have $H_i \to H_\infty$ uniformly.

If one of the two points x and y, say, y, is the pole p_∞, then there is only one term in (6.21). Note $\nu_0 = -\alpha = (m-1)/2$, $m = n-1$,

$$H_\infty(p_\infty, x, t) = \left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} e^{-(r^2)/4t} \frac{2\pi^{n/2}}{\Gamma(n/2)} (\mathrm{Vol}(X))^{-1}. \tag{6.23}$$

As a corollary, we get a new proof of Li's asymptotic formula for heat kernels [Li1]:

Corollary 6.24 (Li). *Assume M^n is a complete noncompact manifold satisfying* (0.4), $\mathrm{Ric}_{M^n} \geq 0$. *Then*

$$\lim_{t\to\infty} \mathrm{Vol}(B_{\sqrt{t}}(p)) H(p, y, t) = (4\pi)^{-n/2}\omega_n. \tag{6.25}$$

ω_n is the volume of the unit ball in $\mathbf{R}^n$.

Proof. Notice,

$$\lim_{t\to\infty} \mathrm{Vol}(B_{\sqrt{t}}(p)) t^{-n/2} = v_0 = n^{-1}\,\mathrm{Vol}(X). \tag{6.26}$$

So we need to show,

$$\lim_{t\to\infty} t^{n/2}\,\mathrm{Vol}(X) H(p, y, t) = (4\pi)^{-n/2} n\omega_n. \tag{6.27}$$

Assume $t_i \to \infty$, $M_i^n = (M^n, p, t_i^{-1}dx^2) \overset{d_{GH}}{\longrightarrow} C(X)$ for some metric cone $C(X)$; see [ChCo1]. The heat kernel $H_i(p, x, t)$ on M_i^n is

$$H_i(p, y, 1) = t^{n/2} H(p, y, t). \tag{6.28}$$

Here we identify $p, x \in M_i^n$ with $p, x \in M$, however, $d_{M_i^n}(p, x) = t_i^{-1/2} d_M(p, x)$, $d_{M_i^n}$ is the distance on M_i^n. In particular, $d_{M_i^n}(p, x) \to 0$ as $i \to \infty$. Since $M_i^n \overset{d_{GH}}{\longrightarrow} C(X)$, by Theorem 5.54 and (6.23) we have

(6.29)
$$\begin{aligned}\lim_{t\to\infty} t^{n/2}\,\mathrm{Vol}(X) H(p, y, t) &= \mathrm{Vol}(X) \lim_{i\to\infty} H_i(p, x, 1) \\ &= \mathrm{Vol}(X) H_\infty(p_\infty, p_\infty, 1) = (4\pi)^{-n/2} \frac{2\pi^{n/2}}{\Gamma(n/2)}.\end{aligned}$$

We just need to recall $n\omega_n = 2\pi^{n/2}(\Gamma(n/2))^{-1}$ (see [Ta] Chapter 3).

Finally in view of the almost rigidity theorem [ChCo1], we see the above results holds for *all* sequences $t_i \to \infty$. This suffices to complete the proof. □

Similarly, we get the asymptotic formula for heat kernels in [LiTW]:

Corollary 6.30 (Li-Tam-Wang). *Assume M^n is a complete noncompact manifold satisfying* (0.4), $\mathrm{Ric}_{M^n} \geq 0$. *Then for $p \in M^n$, and any $R, T > 0$,*
(6.31)
$$\lim_{d(p,x)\to\infty} \mathrm{Vol}(B_{R^{-1}d(p,x)}(p))H(p,x,Td(p,x)^2R^{-2}) = \frac{\omega_n}{(4\pi T)^{n/2}e^{R^2/4T}}.$$

Proof. We use the same argument as in Corollary 6.24. For x_i with $d(p,x_i) \to \infty$, we study the heat kernels on the sequence $M_i^n = (M^n, p, R^2 d(p,x_i)^{-2}dx^2)$. □

We can similarly get a *local* asymptotic formula for H_∞.

7. Eigenvalues on compact limit spaces.

We assume $M_i^n \xrightarrow{d_{GH}} M_\infty$, with $\mathrm{Ric}_{M_i^n} \geq -(n-1)\Lambda$, M_∞ compact. A point $x \in M_\infty$ is said to be regular, $x \in \mathcal{R}_k$, if all tangent cones at x equal to $\mathbf{R}^k$; see [ChCo2].

Lemma 7.1. *If $x \in \mathcal{R}_n \subset M_\infty$, then*

$$\lim_{t\to 0} H_\infty(x,x,t)t^{\frac{n}{2}} = (4\pi)^{-\frac{n}{2}}. \tag{7.2}$$

Proof. Use a similar argument as the one in Corollary 6.24. □

Theorem 7.3. *Assume $M_i^n \xrightarrow{d_{GH}} M_\infty$, $\mathrm{Ric}_{M_i^n} \geq -(n-1)\Lambda$, and for some $v_0 > 0$, $\mathrm{Vol}(M_i^n) \geq v_0$. Then*

$$\lim_{j\to\infty} j^{-\frac{2}{n}}\lambda_{j,\infty} = 4\pi\Gamma(\frac{n}{2}+1)^{\frac{2}{n}}\mu_\infty(M_\infty)^{-\frac{2}{n}}. \tag{7.4}$$

Proof. In this case we don't need to renormalize the volume on M_i^n (see [ChCo2]). Note for some D we have $\mathrm{Diam}\, M_i^n \leq D$, $i = 1, 2, \ldots, \infty$, by the Bishop-Gromov inequality and (1.16), we get

$$t^{\frac{n}{2}} H_\infty(x,x,t) \leq C(n,\Lambda,D,v_0). \tag{7.5}$$

Moreover, almost every point of M_∞ is in $\mathcal{R}_n$. Now by Corollary 7.1, for $x \in \mathcal{R}_n$, $t^{\frac{n}{2}} H_\infty(x,x,t) \to (4\pi)^{-n/2}$ when $t \to 0$. By the dominated convergence theorem,

$$\lim_{t\to 0} t^{\frac{n}{2}} \int_{M_\infty} H_\infty(x,x,t)dx = (4\pi)^{-\frac{n}{2}} \mu_\infty(M_\infty). \tag{7.6}$$

Finally, by applying the Karamata Tauberian theorem (see [Ta] Chapter 8), we have

$$\lim_{\lambda\to\infty} \lambda^{-\frac{n}{2}} N(\lambda) = \mu_\infty(M_\infty)\Gamma(\frac{n}{2}+1)^{-1}(4\pi)^{-\frac{n}{2}}, \tag{7.7}$$

where $N(\lambda)$ is the number of eigenvalues smaller than λ. Clearly this implies the Weyl asymptotic formula (7.4). □

When the limit space M_∞ is collapsed, at present our results are less satisfactory. Recall the notion of *Minkowski dimensions*; see [Ma]. Assume Z is a metric space. For $d > 0$, let $N(Z,\epsilon) \in \mathbf{Z}$ be the minimal integer such that Z can be covered by $N(Z,\epsilon)$ many balls of radius ϵ. Put

$$v_d^-(Z) = \liminf_{\epsilon\to 0} \epsilon^d N(Z,\epsilon), \tag{7.8}$$

$$v_d^+(Z) = \limsup_{\epsilon\to 0} \epsilon^d N(Z,\epsilon). \tag{7.9}$$

Here $v_d^\pm(M_\infty)$ can be ∞. The upper (lower) Minkowski dimension is defined by

$$\overline{\dim}_{Mink}(Z)\ (\underline{\dim}_{Mink}(Z)) = \inf\{d | v_d^+(Z) = 0\ (v_d^-(Z) = 0)\}. \tag{7.10}$$

Lemma 7.11. *There exist $E_1(n), E_2(n) > 0$ such that for any $d > 0$,*

$$\limsup_{t\to 0} t^{\frac{d}{2}} \int_{M_\infty} H_\infty(x,x,t)dx \le E_2 v_d^+(M_\infty), \tag{7.12}$$

and if, in addition, $\mathrm{Ric}_{M_i^n} \ge 0$, *then*

$$E_1 v_d^-(M_\infty) \le \liminf_{t\to 0} t^{\frac{d}{2}} \int_{M_\infty} H_\infty(x,x,t)dx. \tag{7.13}$$

Proof. Let $\cup_{1\le j\le N(M_\infty,\sqrt{t})} B_{\sqrt{t}}(x_j)$ be a covering of M_∞ by a minimal set of balls of radius $\sqrt{t}$. We add up the integrals of H_∞ on these ball an use Corollary 2.7 to get the estimates (7.12), (7.13). □

Lemma 7.14. *If $v_d^+(M_\infty) < c < \infty$, then there exist C such that*

$$\lambda_{j,\infty} > C j^{\frac{2}{d}}. \tag{7.15}$$

Proof. We can follow an argument of Gromov (see [Gr] or Theorem 4.8 in [Ch3]). Here we use the assumption $v_d^+(M_\infty) < c < \infty$ to estimate the number of balls that is needed to cover M^∞. □

Lemma 7.16. *If $v_d^-(M_\infty) > c > 0$, then there exist C depending on n, c, such that*

$$\lambda_{j,\infty} \le C j^{\frac{2}{d}}. \tag{7.17}$$

If k is the maximal integer such that $\mathcal{R}_k \subset M_\infty$ is not empty, then

$$\lambda_{j,\infty} < C(M_\infty)(j)^{\frac{2}{k}}. \tag{7.18}$$

Proof. For $r > 0$, M_i^n contains $j = C(n,c)r^{-d}$ many disjoint balls of radius r for i big enough. The result follows by a well known argument of Cheng [Cheng]; see page 105 of [SY].

If k is the maximal integer such that $\mathcal{R}_k \subset M_\infty$ is not empty, then the k-Hausdorff measure of M_∞ is positive (see [ChCo3] or [Ch3]). So $v_k^-(X) > 0$. By (7.17) we get (7.18). □

If one can also prove for any $d > k$,

$$\lim_{t\to 0} t^{\frac{d}{2}} \sum_{j=0}^{\infty} e^{-\lambda_{j,\infty} t} = \lim_{t\to 0} t^{\frac{d}{2}} \int_{M_\infty} H_\infty(x,x,t)dx = 0, \tag{7.19}$$

then by Lemma 7.11, $d_M(M_\infty)$, the Minkowski dimension of M_∞ is no more than k. Combine with the results in [ChCo3] and [Ch3], $d_M(M_\infty) = k$. However, at present we don't know how to get (7.19). One related question is,

Question. Is there an $\epsilon(n) > 0$, such that for any M^n with $\mathrm{Ric}_{M^n} \geq 0$, any eigenfunction ϕ of Δ and any set E with $\mathrm{Vol}(E) < \epsilon \, \mathrm{Vol}(M)$, we have

$$\int_{M^n - E} \phi^2 > \frac{1}{2} \int_{M^n} \phi^2 \ ? \tag{7.20}$$

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