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Existence and Uniform Decay of Solutions of a Parabolic-Hyperbolic Equation with Nonlinear Boundary Damping and Boundary Source Term[1]

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Existence and uniform decay of solutions of a mixed problem based on the degenerate equation

$$K_1(x,t)y_{tt} + K_2(x,t)y_t - \Delta_x y = 0$$

are studied. Under the assumptions that we have a nonlinear boundary damping $(1+\alpha(t)\,|y_t|^\rho)y_t$ and a boundary source term of type $\alpha(t)|y|^\gamma\, y$, we establish the global existence theorem provided $\rho \geq \gamma$ and we obtain the uniform decay of strong and weak solutions considering $\rho = \gamma$ and the coefficient $\alpha(t)$ producing a damping effect.

1. Introduction.

Throughout, Ω will be a bounded domain of R^n with C^2 boundary Γ, $\Gamma = \Gamma_0 \cup \Gamma_1$, with both Γ_0 and Γ_1 having positive measure. With this geometry, we shall consider here the following problem

$$(1.1)\quad \begin{cases} K_1(x,t)y_{tt} + K_2(x,t)y_t - \Delta_x y = 0 & \text{in } Q = \Omega \times (0,\infty) \\ y = 0 & \text{on } \Sigma_1 = \Gamma_1 \times (0,\infty) \\ \frac{\partial y}{\partial \nu} + y_t + \alpha(t)\left(|y_t|^\rho\, y_t - |y|^\gamma\, y\right) = 0 & \text{on } \Sigma_0 = \Gamma_0 \times (0,\infty) \\ y(x,0) = y^0(x) \quad \text{and} \quad y_t(x,0) = y^1(x) & \text{for } x \in \Omega \end{cases}$$

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where

$$0 < \gamma, \rho \leq \frac{1}{n-2} \quad \text{if} \quad n \geq 3 \quad \text{or} \quad \gamma, \rho > 0 \quad \text{if} \quad n = 1, 2 \tag{1.2}$$

and ν denotes the unit outward normal vector to the boundary.

The main goal of this paper is to study the existence and uniform decay of solutions to (1.1), assuming that $K_1(x,t)$ can vanish on Q. When $\alpha(t)$ acts as a damping mechanism and $\rho \geq \gamma$, we prove existence of strong and weak solutions to (1.1), when $\rho = \gamma$ the uniform decay of the energy

$$e(t) = \frac{1}{2} \int_\Omega K_1(x,t) \left| y_t(x,t) \right|^2 dx + \int_\Omega \left| \nabla y(x,t) \right|^2 dx \tag{1.3}$$

is obtained.

This kind of problem is specially related to the study of transonic gas dynamics, see e.g., Lar'kin [8]. Nondegenerate evolution equations with nonlinear feedbaks acting on the boundary have received considerable attention and in this direction we refer the works of Lagnese and Leugering [7], Lasiecka and Tataru [10], Zuazua [13] and references therein. Concerning nonlinear damping and source terms acting on the domain we refer the work of Georgiev and Todorova [5]. The existence and boundary stabilization of solutions to degenerate evolution equations were early considered in Literature (see Cavalcanti et al. [1, 2]). The present problem deals with degenerate evolution equations and nonlinear boundary feedback combined with a nonlinear boundary source term. This was not previously considered in Literature and brings up new difficulties.

The existence of solutions is obtained from the Faedo-Galerkin method (see Lions [11]) and the uniform stabilization is proved by using the perturbed energy method (see Zuazua [13]).

Our paper is organized as follows. In section 2 we give some notations and state our main result. In section 3 we obtain existence of strong solutions to problem (1.1) and in section 4 we obtain the uniform decay of the energy.

2. Assumptions and Main Result.

We define

$$V = \left\{ u \in H^1(\Omega); u = 0 \text{ on } \Gamma_1 \right\}, \tag{2.1}$$

$$(u,v) = \int_\Omega u(x)v(x)\, dx, \quad (u,v)_{\Gamma_0} = \int_{\Gamma_0} u(x)v(x)\, d\Gamma,$$

$$|u|^2 = \int_\Omega |u(x)|^2 \, dx; \ |u|^2_{\Gamma_0} = \int_{\Gamma_0} |u(x)|^2 \, d\Gamma,$$

$$||u||_{p,\Gamma_0} = \left(\int_{\Gamma_0} |u(x)|^p \, d\Gamma \right)^{1/p}, \ ||u||_\infty = \text{ess} \sup_{t \geq 0} ||u(t)||_{L^\infty(\Omega)}.$$

The variational formulation associated with problem (1.1) is given by

$$\begin{aligned}(2.2) \qquad & (K_1(t) y_{tt}(t), w) + (K_2(t) y_t(t), w) + (\nabla y(t), \nabla w) + (y_t(t), w)_{\Gamma_0} \\ & + \alpha(t) \left(|y_t(t)|^\rho \, y_t(t), w \right)_{\Gamma_0} \\ & = \alpha(t) \left(|y(t)|^\gamma \, y(t), w \right)_{\Gamma_0}, \forall w \in V.\end{aligned}$$

In order to obtain the existence of solutions we consider $w = y_t(t)$. Concerning strong solutions, an additional estimate is needed, that is, the one obtained by derivating the variational formulation (2.2) with respect to t. In view of the surface integrals, it is not suitable the use of a special basis, for instance, those formed by eigenfunctions. But the presence of the term $|y_{tt}(0)|$ leads us to technical problems. To solve this question we assume that

$$\text{(H.1)} \qquad K_1(x,0) \geq d > 0 \ \text{ a.e in } \ Q$$

and we make the following compatibility hypotheses upon the initial data.

(A.1) Assumptions on the Initial Data.

Let us consider

$$y^0, y^1 \in V \cap H^2(\Omega)$$

verifying the compatibility condition

$$\text{(H.2)} \qquad \frac{\partial y^0}{\partial \nu} + y^1 + \alpha(0) \left(|y^1|^\rho \, y^1 - |y^0|^\gamma \, y^0 \right) = 0 \ \text{ on } \Gamma_0.$$

We observe that even in the linear case, it is not clear that hypothesis (H.1) and (H.2) imply the boundness of $|y_{tt}(0)|$. In fact, in order to notice it let us transform problem (1.1) into an equivalent one with null initial data. More precisely, defining

$$(2.3) \qquad \phi(x,t) = y^0(x) + t y^1(x); \quad (x,t) \in \Omega \times (0, \infty)$$

and

$$(2.4) \qquad v(x,t) = y(x,t) - \phi(x,t)$$

we obtain the equivalent problem for v

$$(2.5)\quad \begin{cases} K_1 v_{tt} + K_2 v_t - \Delta v = F & \text{in } Q \\ v = 0 & \text{on } \Sigma_1 \\ \frac{\partial v}{\partial \nu} + v_t + \alpha(t)\left(|v_t + \phi_t|^\rho (v_t + \phi_t) - |v + \phi|^\gamma (v + \phi)\right) = G & \text{on } \Sigma_0 \\ v(0) = v_t(0) = 0 \end{cases}$$

where

$$(2.6)\qquad F = -K_2 \phi_t + \Delta \phi \quad \text{and} \quad G = -\frac{\partial \phi}{\partial \nu} - \phi_t.$$

The new variational formulation associated with (2.5) is given by

$$(2.7)\qquad \begin{aligned} &(K_1(t) v_{tt}(t), w) + (K_2(t) v_t(t), w) + (\nabla v(t), \nabla w) + (v_t(t), w)_{\Gamma_0} \\ &\quad + \alpha(t)\left(|v_t(t) + \phi_t(t)|^\rho (v_t(t) + \phi_t(t)), w\right)_{\Gamma_0} \\ &= \alpha(t)\left(|v(t) + \phi(t)|^\gamma (v(t) + \phi(t)), w\right)_{\Gamma_0} \\ &\quad + (F(t), w) + (G(t), w)_{\Gamma_0}. \end{aligned}$$

Considering $w = v_{tt}(0)$ in equation (2.7) from (H.1), (H.2), (2.6) and taking into account that $v(0) = v'(0) = 0$ we conclude that there exists $C > 0$ such that

$$(2.8)\qquad |v_{tt}(0)|^2 \leq C.$$

Next, we are going to consider

(A.2) Assumptions on the Coefficients.

Let us assume that

$$(\text{H.3})\qquad K_1, K_2 \in W^{1,\infty}(0, \infty; L^\infty(\Omega)),$$

$$(\text{H.4})\qquad K_2 - \frac{1}{2}|K_{1,t}| \geq \delta > 0 \text{ a.e. in } Q.$$

The hypothesis (H.4) was widely used in degenerate problems. We refer the reader to the works of the authors Lar'kin et al. [8] and Cavalcanti et al. [2].

(A.3) Assumptions on the Coefficient α.

Let us consider

(H.5) $$\alpha \in W^{1,\infty}(0,\infty) \cap L^1(0,\infty), \quad \alpha \geq 0,$$

verifying

(H.6) $$-m_0\alpha(t) \leq \alpha_t(t) \leq -m_1\alpha(t) \quad \text{for all} \quad t \geq 0$$

for some $m_0, m_1 > 0$.

Now we are in position to state our main result.

Theorem 2.1. *Under the assumptions* (A.1), (A.2), (A.3) *and assuming that* γ, ρ *satisfy the hypothesis* (1.2) *with* $\rho \geq \gamma$, *problem* (1.1) *has a unique strong solution* $y : \Omega \to \mathbf{R}$ *verifying*

(2.9) $$y \in L^\infty(0,\infty; V) \quad \textit{and} \quad y' \in L^\infty(0,\infty; V),$$

(2.10) $$\sqrt{K_1}y'' \in L^\infty(0,\infty; L^2(\Omega)) \quad \textit{and} \quad y'' \in L^2(0,\infty; L^2(\Omega)),$$

$$K_1 y'' + K_2 y' - \Delta y = 0 \quad \textit{in} \quad Q,$$
$$y = 0 \quad \textit{on} \quad \Sigma_1,$$
$$\frac{\partial y}{\partial \nu} + y' + \alpha(t)\left(|y'|^\rho y' - |y|^\gamma y\right) = 0 \quad \textit{on} \quad \Sigma_0,$$
$$y(0) = y^0 \quad \textit{and} \quad y'(0) = y^1 \quad \textit{on} \quad \Omega.$$

Moreover, if $\rho = \gamma$ *and* m_1 *is large enough, there exists a positive constant* ε_0 *such that*

(2.11) $$E(t) \leq 3\exp\left(-\frac{\varepsilon}{2}t\right), \quad \forall t \geq 0 \quad \textit{and} \quad \forall \varepsilon \in (0, \varepsilon_0].$$

Theorem 2.2. *Assume that assumptions* (H.1), (A.2) *and* (H.5) *hold; consider* $\alpha(0) = 0$ *and that* (H.6) *holds for all* $t \in (t_0, +\infty)$. *Then, given* $\{y^0, y^1\} \in V \times L^2(\Omega)$, *problem* (1.1) *possesses at least a solution in the class*

(2.12) $$y \in C^0([0,\infty); V) \cap C^1([0,\infty); L^2(\Omega)).$$

In addition, we obtain the same uniform decay rates given in (2.11) *for the weak solution and for all* $t \geq t_0$.

3. Existence and Uniqueness of Solutions.

In this section we are going to obtain existence and uniqueness of strong and weak solutions to problem (1.1) using the Faedo-Galerkin method. For this end we represent by $\{\omega_j\}_{j\in\mathbf{N}}$ a basis in $V \cap H^2(\Omega)$ which is orthonormal in $L^2(\Omega)$, by V_m the subspace of V generate by the first m vectors $\{w_1, \cdots, w_n\}$ and we define for each $\varepsilon > 0$

$$K_{1,\varepsilon} = K_1 + \varepsilon \quad \text{and} \quad v_{\varepsilon m}(t) = \sum_{j=1}^{m} g_{\varepsilon jm}(t)\omega_j, \tag{3.1}$$

where $v_{\varepsilon m}(t)$ is the solution of the following Cauchy problem

(3.2)

$$\begin{aligned}
&\left(K_{1,\varepsilon}(t)v''_{\varepsilon m}(t), w\right) + \left(K_2(t)v'_{\varepsilon m}(t), w\right) + \left(\nabla v_{\varepsilon m}(t), \nabla w\right) + \left(v'_{\varepsilon m}(t), w\right)_{\Gamma_0} \\
&\quad + \alpha(t)\left(\left|v'_{\varepsilon m}(t) + \phi'(t)\right|^{\rho}\left(v'_{\varepsilon m}(t) + \phi'(t)\right), w\right)_{\Gamma_0} \\
&= \alpha(t)\left(\left|v_{\varepsilon m}(t) + \phi(t)\right|^{\gamma}\left(v_{\varepsilon m}(t) + \phi(t)\right), w\right)_{\Gamma_0} \\
&\quad + (F(t), w) + (G(t), w)_{\Gamma_0}, \quad \forall w \in V_m
\end{aligned}$$

$$v_{\varepsilon m}(0) = v'_{\varepsilon m}(0) = 0. \tag{3.3}$$

The above approximate system is a normal one of differential equations which has solution in $[0, T_{\varepsilon m}[$. The extension of these solutions to the whole interval [0,T] is a consequence of the first estimate which we are going to prove below.

A Priori Estimates.

The First Estimate.

Replacing w by $v'_{\varepsilon m}(t)$ in (3.2) we obtain

(3.4)

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\left|\sqrt{K_{1,\varepsilon}(t)}v'_{\varepsilon m}(t)\right|^2 + \frac{1}{2}\left|\nabla v_{\varepsilon m}(t)\right|^2 + \frac{\alpha(t)}{\gamma+2}\left|\left|v_{\varepsilon m}(t) + \phi(t)\right|\right|_{\gamma+2,\Gamma_0}^{\gamma+2}\right\} \\
&\quad + \left(K_2 - \frac{1}{2}K'_1(t), v'^2_{\varepsilon m}(t)\right) + \alpha(t)\left|\left|v'_{\varepsilon m}(t) + \phi'(t)\right|\right|_{\rho+2,\Gamma_0}^{\rho+2} + \left|v'_{\varepsilon m}(t)\right|_{\Gamma_0}^2 \\
&= 2\alpha(t)\left(\left|v_{\varepsilon m}(t) + \phi(t)\right|^{\gamma}\left(v_{\varepsilon m}(t) + \phi(t)\right), \left(v'_{\varepsilon m}(t) + \phi'(t)\right)\right)_{\Gamma_0}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\gamma+2}\alpha'(t)\,\|v_{\varepsilon m}(t)+\phi(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}\\
&+\alpha(t)\left(\left|v'_{\varepsilon m}(t)+\phi'(t)\right|^{\rho}\left(v'_{\varepsilon m}(t)+\phi'(t)\right),\phi'(t)\right)_{\Gamma_0}\\
&-\alpha(t)\left(|v_{\varepsilon m}(t)+\phi(t)|^{\rho}\left(v_{\varepsilon m}(t)+\phi(t)\right),\phi'(t)\right)_{\Gamma_0}\\
&+\left(F(t),v'_{\varepsilon m}(t)\right)+\left(G(t),v'_{\varepsilon m}(t)\right)_{\Gamma_0}.
\end{aligned}
$$

Making use of Young's inequality $ab \le C(\eta)a^p + \eta b^q$, where $\frac{1}{p}+\frac{1}{q}=1$ and η is an arbitrary positive number, considering the assumptions (H.4) and (H.6), from (3.4) we infer

(3.5)

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\left|\sqrt{K_{1,\varepsilon}(t)}v'_{\varepsilon m}(t)\right|^2+\frac{1}{2}|\nabla v_{\varepsilon m}(t)|^2+\frac{\alpha(t)}{\gamma+2}\|v_{\varepsilon m}(t)+\phi(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}\right\}\\
&+(\delta-\eta)\left|v'_{\varepsilon m}(t)\right|^2+(1-\eta)\left|v'_{\varepsilon m}(t)\right|_{\Gamma_0}^2+(1-\eta)\alpha(t)\left\|v'_{\varepsilon m}(t)+\phi'(t)\right\|_{\rho+2,\Gamma_0}^{\rho+2}\\
&\le\frac{1}{4\eta}|F(t)|^2+\frac{1}{4\eta}|G(t)|_{\Gamma_0}^2+\eta\alpha(t)\left\|v'_{\varepsilon m}(t)+\phi'(t)\right\|_{\gamma+2,\Gamma_0}^{\gamma+2}\\
&+C_1(\eta)\alpha(t)\,\|v_{\varepsilon m}(t)+\phi(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}+C_2(\eta,\|\alpha\|)\left\|y^1\right\|_{\rho+2,\Gamma_0}^{\rho+2}\\
&+\alpha(t)\,\|v_{\varepsilon m}(t)+\phi(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}+C_3(\|\alpha\|)\left\|y^1\right\|_{\gamma+2,\Gamma_0}^{\gamma+2}.
\end{aligned}
$$

Estimate for $I := \eta\alpha(t)\,\|v'_{\varepsilon m}(t)+\phi'(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}$.

Since $\rho+2 \ge \gamma+2$ then $L^{\rho+2}(\Gamma_0) \hookrightarrow L^{\gamma+2}(\Gamma_0)$ and therefore we can write

$$
|I| \le \eta C_4\alpha(t)+\eta\alpha(t)C_4\left\|v'_{\varepsilon m}(t)+\phi'(t)\right\|_{\rho+2,\Gamma_0}^{\rho+2}, \tag{3.6}
$$

where C_4 is a positive constant independent of ε and m.

Combining (3.5) and (3.6), integrating the obtained result over $[0,t]$ taking (3.3) into account, employing Gronwall's lemma and choosing $\eta > 0$ sufficiently small we obtain the first estimate

$$
\begin{aligned}
&\left|\sqrt{K_{1,\varepsilon}(t)}v'_{\varepsilon m}(t)\right|^2+|\nabla v_{\varepsilon m}(t)|^2+\alpha(t)\,\|v_{\varepsilon m}(t)+\phi(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}\\
&+\int_0^t\left|v'_{\varepsilon m}(s)\right|^2ds+\int_0^t\left|v'_{\varepsilon m}(s)\right|_{\Gamma_0}^2ds\\
&+\int_0^t\alpha(s)\left\|v'_{\varepsilon m}(s)+\phi'(s)\right\|_{\rho+2,\Gamma_0}^{\rho+2}ds\le L_1,
\end{aligned}
\tag{3.7}
$$

where $L_1 > 0$ is independent of ε and m.

The Second Estimate.

Differentiating (3.2) and substituing w by $v''_{\varepsilon m}(t)$, we have

(3.8)
$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\left|\sqrt{K_{1,\varepsilon}(t)}v''_{\varepsilon m}(t)\right|^2+\frac{1}{2}\left|\nabla v'_{\varepsilon m}(t)\right|^2\right\}+\left(K_2(t)+\frac{1}{2}K'_1,v''^2_{\varepsilon m}(t)\right)\\
&\quad+\left(K_2(t)v'_{\varepsilon m}(t),v''_{\varepsilon m}(t)\right)+\left|v''_{\varepsilon m}(t)\right|^2_{\Gamma_0}\\
&\quad+\alpha'(t)\left(\left|v'_{\varepsilon m}(t)+\phi'(t)\right|^\rho\left(v'_{\varepsilon m}(t)+\phi'(t)\right),v''_{\varepsilon m}(t)\right)_{\Gamma_0}\\
&\quad+(\rho+1)\alpha(t)\left(\left|v'_{\varepsilon m}(t)+\phi'(t)\right|^\rho,v''^2_{\varepsilon m}(t)\right)_{\Gamma_0}\\
&=\alpha'(t)\left(\left|v_{\varepsilon m}(t)+\phi(t)\right|^\gamma\left(v_{\varepsilon m}(t)+\phi(t)\right),v''_{\varepsilon m}(t)\right)_{\Gamma_0}\\
&\quad+(\gamma+1)\alpha(t)\left(\left|v_{\varepsilon m}(t)+\phi(t)\right|^\gamma\left(v'_{\varepsilon m}(t)+\phi'(t)\right),v''_{\varepsilon m}(t)\right)_{\Gamma_0}\\
&\quad+\left(F'(t),v''_{\varepsilon m}(t)\right)+\left(G'(t),v''_{\varepsilon m}(t)\right)_{\Gamma_0}.
\end{aligned}$$

Estimate for $I_1 := \alpha'(t)\left(|v'_{\varepsilon m}(t)+\phi'(t)|^\rho\left(v'_{\varepsilon m}(t)+\phi'(t)\right),v''_{\varepsilon m}(t)\right)_{\Gamma_0}$.

From assumption (H.6) and using the inequality $ab \le \frac{1}{4\eta}a^2+\eta b^2, \eta>0$, we conclude

(3.9)
$$\begin{aligned}
|I_1| \le \frac{m_0\alpha(t)}{4\eta}\left\|v'_{\varepsilon m}(t)+\phi'(t)\right\|^{\rho+2}_{\rho+2,\Gamma_0}\\
+m_0\eta\alpha(t)\left(\left|v'_{\varepsilon m}(t)+\phi(t)\right|^\rho,v''^2_{\varepsilon m}(t)\right).
\end{aligned}$$

Estimate for $I_2 := \alpha'(t)\left(|v_{\varepsilon m}(t)+\phi(t)|^\gamma\left(v_{\varepsilon m}(t)+\phi(t)\right),v''_{\varepsilon m}(t)\right)_{\Gamma_0}$.

Taking into account that $\frac{\gamma}{2\gamma+2}+\frac{1}{2\gamma+2}+\frac{1}{2}=1$, using the generalized Hölder inequality, the continuity of the trace operator $\gamma_0 : H^1(\Omega)\to L^q(\Gamma)$, for $1\le q\le \frac{2n-2}{n-2}$, and the first estimate, it follows that

(3.10)
$$\begin{aligned}
|I_2| &\le C_5\left\|v_{\varepsilon m}(t)+\phi(t)\right\|^\gamma_{\gamma+2,\Gamma_0}\left\|v_{\varepsilon m}(t)+\phi(t)\right\|_{2\gamma+2,\Gamma_0}\left|v''_{\varepsilon m}(t)\right|_{\Gamma_0}\\
&\le C_6(T,\eta)\left|\nabla v_{\varepsilon m}(t)\right|^2+\eta\left|v''_{\varepsilon m}(t)\right|^2\\
&\le C_7(T,\eta)+\eta\left|v''_{\varepsilon m}(t)\right|^2.
\end{aligned}$$

Estimate for $I_3 = (\gamma+1)\alpha(t)\left(|v_{\varepsilon m}(t)+\phi(t)|^\gamma\left(v'_{\varepsilon m}(t)+\phi'(t)\right),v''_{\varepsilon m}(t)\right)_{\Gamma_0}$.

Considering the same arguments used in (3.10) we obtain

(3.11)
$$|I_3| \le C_8(T,\eta)\left|\nabla v'_{\varepsilon m}(t)\right|^2+\eta\left|v''_{\varepsilon m}(t)\right|^2.$$

Making use of assumption (H.4), combining equations (3.8)-(3.10), integrating over [0,t] the obtained result taking equation (2.7) into account, employing Gronwall's lemma and choosing η small enough we obtain the second estimate

(3.12)
$$\left|\sqrt{K_{1,\varepsilon}(t)}v''_{\varepsilon m}(t)\right|^2 + \left|\nabla v'_{\varepsilon m}(t)\right|^2 + \int_0^t \left|v''_{\varepsilon m}(s)\right|^2 ds + \int_0^t \left|v''_{\varepsilon m}(s)\right|^2_{\Gamma_0} ds + \int_0^t \alpha(t)\left(\left|v'_{\varepsilon m}(s)+\phi'(s)\right|^\rho, v''^2_{\varepsilon m}(s)\right)_{\Gamma_0} ds \leq L_2$$

where $L_2 > 0$ is independent of ε and m.

Analysis of the Nonlinear Terms.

From the above estimates we deduce

$$\{v_{\varepsilon m}\} \text{ is bounded in } L^2(0,T;H^{1/2}(\Gamma_0)), \tag{3.13}$$

$$\{v'_{\varepsilon m}\} \text{ is bounded in } L^2(0,T;H^{1/2}(\Gamma_0)), \tag{3.14}$$

$$\{v''_{\varepsilon m}\} \text{ is bounded in } L^2(0,T;L^2(\Gamma_0)). \tag{3.15}$$

From (3.13)-(3.15), observing that the imersion $H^{1/2}(\Gamma_0) \hookrightarrow L^2(\Gamma_0)$ is continuous and compact, and making use of Aubin-Lions theorem, we can extract a subsequence $\{v_{\varepsilon\mu}\}$ of $\{v_{\varepsilon m}\}$ such that

$$v_{\varepsilon\mu} \to v_\varepsilon \text{ and } v'_{\varepsilon\mu} \to v'_\varepsilon \text{ a.e. on } \Sigma_{0,T} = \Gamma_0 \times (0,T). \tag{3.16}$$

Therefore, from (3.16) it follows that

$$|v_{\varepsilon\mu}|^\gamma v_{\varepsilon\mu} \to |v_\varepsilon|^\gamma v_\varepsilon \text{ and } \left|v'_{\varepsilon\mu}\right|^\rho v'_{\varepsilon\mu} \to \left|v'_\varepsilon\right|^\rho v'_\varepsilon \text{ a.e. on } \Sigma_{0,T}. \tag{3.17}$$

On the other hand, from the first and second estimates we obtain

$$\{|v_{\varepsilon\mu}|^\gamma v_{\varepsilon\mu}\} \text{ is bounded in } L^2(\Sigma_{0,T}), \tag{3.18}$$

$$\{|v'_{\varepsilon\mu}|^\rho v'_{\varepsilon\mu}\} \text{ is bounded in } L^2(\Sigma_{0,T}). \tag{3.19}$$

Thus, combining (3.17)-(3.19), we deduce from Lions' lemma

$$|v_{\varepsilon\mu}|^\gamma v_{\varepsilon\mu} \rightharpoonup |v_\varepsilon|^\gamma v_\varepsilon \text{ weakly in } L^2(\Sigma_{0,T}),$$

$$\left|v'_{\varepsilon\mu}\right|^{\rho} v'_{\varepsilon\mu} \rightharpoonup \left|v'_{\varepsilon}\right|^{\rho} v'_{\varepsilon} \text{ weakly in } L^2(\Sigma_{0,T}).$$

The above convergences are sufficient to pass to the limit in the nonlinear terms of (3.2) using standard arguments. From this and taking (2.4) into account we obatin

$$K_1 y'' + K_2 y' - \Delta y = 0 \text{ in } L^2_{\text{loc}}(0,\infty; L^2(\Omega)). \tag{3.20}$$

Moreover, from the generalized Green's formula we infer

$$\frac{\partial y}{\partial \nu} + y_t + \alpha(t)\left(|y_t|^{\rho} y_t - |y|^{\gamma} y\right) = 0 \text{ in } L^2_{\text{loc}}(0,\infty; L^2(\Gamma_0)). \tag{3.21}$$

Uniqueness.

Let y_1 and y_2 be strong solutions to problem (1.1). Defining $z = y_1 - y_2$, we deduce from (3.20) and (3.21)

$$\begin{aligned}(3.22)\qquad &\left(K_1 z''(t), w\right) + \left(K_2 z'(t).w\right) + (\nabla z(t), \nabla w) + \left(z'(t), w\right)_{\Gamma_0} \\ &+ \alpha(t)\left(\left|y'_1\right|^{\rho} y'_1 - \left|y'_2\right|^{\rho} y'_2, w\right)_{\Gamma_0} \\ &= \alpha(t)\left(|y_2|^{\gamma} y_2 - |y_1|^{\gamma} y_1, w\right)_{\Gamma_0},\end{aligned}$$

for all $w \in V$.

Substituing $w = z'(t)$ in (3.22), we obtain from (H.4)

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\left|\sqrt{K_1(t)} z'(t)\right|^2 + \frac{1}{2}\left|\nabla z(t)\right|^2\right\} + \delta\left|z'(t)\right|^2 + \left|z'(t)\right|^2_{\Gamma_0} \\ &\le \alpha(t)\left(|y_2|^{\gamma} y_2 - |y_1|^{\gamma} y_1, z'(t)\right)_{\Gamma_0} \\ &\le C(\gamma)\int_{\Gamma_0}\left(|y_2|^{\gamma} + |y_1|^{\gamma}\right)|z(t)|\left|z'(t)\right| d\Gamma.\end{aligned}$$

Integrating the last inequality over (0,t), using analogous considerations made in the second estimate (see estimate for I_2 term) and employing Gronwall's lemma, we obtain $|z'(t)| = |\nabla z(t)| = 0$. This concludes the proof of uniqueness for strong solutions.

Existence of Weak Solutions.

Let us consider

$$\left\{y^0, y^1\right\} \in V \times L^2(\Omega).$$

Since

$$D(-\Delta) = \left\{u \in V \cap H^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_0\right\}$$

is dense in V and $H_0^1(\Omega) \cap H^2(\Omega)$ is dense in $L^2(\Omega)$, there exist $\{y_\mu^0\} \subset D(-\Delta)$ and $\{y_\mu^1\} \subset H_0^1(\Omega) \cap H^2(\Omega)$ such that

$$y_\mu^0 \to y^0 \text{ strongly in } V, \tag{3.23}$$

$$y_\mu^1 \to y^1 \text{ strongly in } L^2(\Omega), \tag{3.24}$$

and, since $\alpha(0) = 0$, the compatibility conditions given in (H.2) are verified, that is, for each $\mu \in \mathbf{N}$, one has

$$\frac{\partial y_\mu^0}{\partial \nu} + y_\mu^1 = 0 \text{ on } \Gamma_0.$$

Then, repeating the same arguments used in the first estimate and in the uniqueness of strong solutions, we deduce that there exist $\{y_\mu\}$ a sequence of strong solutions of problem (1.1) and also $y : Q \to \mathbf{R}$ such that

$$y_\mu \to y \text{ strongly in } C^0([0,T]; V), \tag{3.25}$$

$$y_\mu' \to y' \text{ strongly in } C^0([0,T]; L^2(\Omega)), \tag{3.26}$$

and

$$\begin{cases} K_1 y'' + K_2 y' - \Delta y = 0 \text{ in } L^2(0,T;V') \\ y(0) = y^0; \; y'(0) = y^1. \end{cases} \tag{3.27}$$

From now on we are going to define a weak solution to problem (1.1), a function y which verifies (3.27).

4. Uniform Decay.

The derivative of the energy defined in (1.3) is given by

$$\begin{aligned} e'(t) = &- \left(K_2(t) - \frac{1}{2} K_1'(t), y'^2(t) \right) - |y'(t)|_{\Gamma_0}^2 \\ &- \alpha(t) \, ||y'(t)||_{\rho+2,\Gamma_0}^{\rho+2} + \alpha(t) \left(|y(t)|^\gamma y(t), y'(t) \right)_{\Gamma_0}. \end{aligned} \tag{4.1}$$

Defining the modified energy by

$$E(t) = e(t) + \frac{1}{\gamma+2} \alpha(t) \, ||y(t)||_{\gamma+2,\Gamma_0}^{\gamma+2} \tag{4.2}$$

we obtain from the assumptions (H.4), (H.6), (4.1) and from (4.2)

$$(4.3)\quad E'(t) \leq -\delta |y'(t)|^2 - |y'(t)|^2_{\Gamma_0} - \alpha(t) \, ||y'(t)||^{\rho+2}_{\rho+2,\Gamma_0} - \frac{1}{\gamma+2} m_1 \alpha(t) \, ||y(t)||^{\gamma+2}_{\gamma+2,\Gamma_0} + 2\alpha(t) \left(|y(t)|^\gamma y(t), y'(t) \right)_{\Gamma_0}.$$

Considering the Young's inequality $ab \leq \eta a^p + C(\eta) b^q$ with $p = \gamma + 2$, $q = \frac{\gamma+2}{\gamma+1}$ and $C(\eta) = \eta^{-\frac{1}{\gamma+1}}$ and supposing that $\gamma = \rho$, we deduce

$$(4.4)\quad E'(t) \leq -\delta |y'(t)|^2 - |y'(t)|^2_{\Gamma_0} - \alpha(t)(1 - 2\eta) \, ||y'(t)||^{\gamma+2}_{\gamma+2,\Gamma_0} -\alpha(t) \left(\frac{\gamma+1}{\gamma+2} - \eta^{-\frac{1}{\gamma+1}} \right) ||y(t)||^{\gamma+2}_{\gamma+2,\Gamma_0}$$

Choosing $\eta = 4^{-(\gamma+1)}$, we have $2 \left[4^{-(\gamma+1)} \right] < \frac{1}{2}$ and consequently from (4.4) it follows

$$(4.5)\quad E'(t) \leq -\delta |y'(t)|^2 - |y'(t)|^2_{\Gamma_0} - \frac{1}{2} \alpha(t) \, ||y'(t)||^{\gamma+2}_{\gamma+2,\Gamma_0} - \beta \alpha(t) \, ||y(t)||^{\gamma+2}_{\gamma+2,\Gamma_0}.$$

where

$$\beta = \frac{m_1}{\gamma+2} - 8 > 0.$$

For every $\varepsilon > 0$ we define the perturbed modified energy

$$(4.6)\quad E_\varepsilon(t) = E(t) + \varepsilon \psi(t),$$

where

$$(4.7)\quad \psi(t) = \int_\Omega K_1 y' y \, dx.$$

In what follows let $\lambda > 0$ be a positive constant such that

$$(4.8)\quad |v|^2 \leq \lambda \, |\nabla v| \, ; \; \forall v \in V.$$

Proposition 4.1. *There exists $C_1 > 0$ such that*

$$|E_\varepsilon(t) - E(t)| \leq \varepsilon C_1 E(t), \quad \forall t \geq 0 \quad \text{and} \quad \forall \varepsilon > 0.$$

Proof. From (4.7), (4.8) and using Schwarz inequality we infer

$$|\psi(t)| \leq ||K_1||_\infty^{1/2} \lambda^{1/2} e(t) \leq ||K_1||_\infty^{1/2} \lambda^{1/2} E(t)$$

and from (4.6) we conclude the desired inequality with $C_1 = ||K_1||_\infty^{1/2} \lambda^{1/2}$. □

Proposition 4.2. *There exist* $C_2 > 0$ and $\varepsilon_1 > 0$ *such that*

$$E'_\varepsilon(t) \leq -\varepsilon C_2 E(t); \quad \forall t \geq 0 \quad and \quad \varepsilon \in (0, \varepsilon_1].$$

Proof. Differentiating $\psi(t)$ with respect to t and replacing $K_1 y''$ by $-K_2 y' + \Delta y$ in the obtained result, it follows that

$$\psi'(t) = \int_\Omega K_1' y' y \, dx - \int_\Omega K_2 y' y \, dx + \int_\Omega \Delta y y \, dx + \int_\Omega K_1 |y'|^2 dx. \tag{4.9}$$

Now, using the generalized Green formula and taking into account that

$$\frac{\partial y}{\partial \nu} = -y' - \alpha(t)|y'|^\rho y' + \alpha(t)|y|^\gamma y$$

we deduce from (4.9)

$$\begin{aligned} \psi'(t) = &\int_\Omega K_1' y' y \, dx - \int_\Omega K_2 y' y \, dx - \int_\Omega |\nabla y|^2 dx + \int_\Omega K_1 |y'|^2 dx \\ &- \int_{\Gamma_0} y' y \, d\Gamma - \alpha(t) \int_{\Gamma_0} |y'|^\gamma y' y \, d\Gamma + \alpha(t) \int_{\Gamma_0} |y|^{\gamma+2} d\Gamma. \end{aligned} \tag{4.10}$$

Adding and subtracting the terms $\int_{\Gamma_0} K_1 |y'|^2 d\Gamma$ and $\alpha(t) \int_{\Gamma_0} |y|^{\gamma+2} d\Gamma$ from (4.10), we obtain the following inequality

(4.11)

$$\begin{aligned} \psi'(t) \leq &-E(t) + 2\int_\Omega K_1 |y'|^2 dx + 2\alpha(t) \int_{\Gamma_0} |y|^{\gamma+2} d\Gamma \\ &+ \int_\Omega K_1' y' y \, dx - \int_\Omega K_2 y' y \, dx - \int_{\Gamma_0} y' y \, d\Gamma - \alpha(t) \int_{\Gamma_0} |y'|^\gamma y' y \, d\Gamma. \end{aligned}$$

Making use of the inequalities $ab \leq \frac{1}{4\eta} a^2 + \eta b^2$ and $ab \leq \theta(\eta) a^p + \eta b^q$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $\eta > 0$ is arbitrary, and considering (4.8), we conclude from (4.11)

(4.12)

$$\psi'(t) \leq -\left[1 - (8+\gamma)\eta\right] E(t) + 2\alpha(t) \, ||y(t)||_{\gamma+2,\Gamma_0}^{\gamma+2}$$

$$+ M_1(\eta)\left|y'(t)\right|^2 + M_2(\eta)\left|y'(t)\right|_{\Gamma_0}^2 + M_3(\eta)\alpha(t)\left|\left|y'(t)\right|\right|_{\gamma+2,\Gamma_0}^{\gamma+2},$$

where

$$M_1(\eta) = 2\left|\left|K_1\right|\right|_\infty + \frac{\left|\left|K_1'\right|\right|_\infty^2 \lambda^2}{4\eta} + \frac{\left|\left|K_2\right|\right|_\infty^2 \lambda^2}{4\eta},$$

$$M_2(\eta) = \frac{C_0^2}{4\eta}\left|y'(t)\right| \quad \textit{and} \quad M_3(\eta) = \theta_1(\eta)$$

and $C_0 > 0$ is such that $|y|_{\Gamma_0} \leq C_0|\nabla y|$.

Choosing $\eta > 0$ so that $C_2 = 1 - (8+\gamma)\eta > 0$ from (4.12) we have

$$\begin{aligned}\psi'(t) \leq {} & -C_2E(t) + 2\alpha(t)\left|\left|y(t)\right|\right|_{\gamma+2,\Gamma_0}^{\gamma+2} \\ & + M_1\left|y'(t)\right|^2 + M_2\left|y'(t)\right|_{\Gamma_0}^2 + M_3\alpha(t)\left|\left|y'(t)\right|\right|_{\gamma+2,\Gamma_0}^{\gamma+2}.\end{aligned} \tag{4.13}$$

Taking the derivative in (4.6) with respect to t, combining (4.5) and (4.13), it follows that

$$\begin{aligned}E_\varepsilon'(t) \leq {} & -(\delta - \varepsilon M_1)\left|y'(t)\right|^2 - (1-\varepsilon M_2)\left|y'(t)\right|_{\Gamma_0}^2 \\ & - \alpha(t)\left(\frac{1}{2} - \varepsilon M_3\right)\left|\left|y'(t)\right|\right|_{\gamma+2,\Gamma_0}^{\gamma+2} \\ & - (\beta - 2\varepsilon)\,\alpha(t)\left|\left|y(t)\right|\right|_{\gamma+2,\Gamma_0}^{\gamma+2} - \varepsilon C_2E(t).\end{aligned} \tag{4.14}$$

Defining

$$\varepsilon_1 = \min\left\{\frac{\delta}{M_1}, \frac{1}{M_2}, \frac{1}{2M_3}\right\},$$

then, for all $\varepsilon \in (0, \varepsilon_1]$, we obtain from (4.14) the desired result and, consequently, the Proposition 4.2 is proved. □

Proof of the Uniform Decay.

Let

$$\varepsilon_0 = \min\left\{1/2C_1, \varepsilon_1\right\},$$

where $C_1 > 0$ is given in Proposition 1, and let us consider $\varepsilon \in (0, \varepsilon_0]$. As we have $\varepsilon < 1/2C_1$, we conclude from Proposition 4.1

$$\frac{1}{2}E(t) \leq E_\varepsilon(t) \leq \frac{3}{2}E(t) \leq 2E(t); \quad \forall t \geq 0. \tag{4.15}$$

Consequently $-\varepsilon C_2E(t) \leq -\frac{\varepsilon}{2}C_2E_\varepsilon(t)$ and it follows from Proposition 2

$$E_\varepsilon'(t) \leq -\frac{\varepsilon}{2}C_2E_\varepsilon(t).$$

Therefore

$$\frac{d}{dt}\left(E_\varepsilon(t)\exp\left(\frac{\varepsilon}{2}t\right)\right) \leq 0,$$

which implies in view of (4.15) that

$$E(t) \leq 3E(0)\exp\left(-\frac{\varepsilon}{2}t\right).$$

This concludes the proof of Theorem 2.1. □

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