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Non-convergence and instability in the asymptotic behaviour of curves evolving by curvature

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We consider curvature-driven evolution equations for curves in the plane, and prove that the isoperimetric ratios of the evolving curves generically approach infinity if the speed of motion is proportional to curvature to a power less than 1/3.

1. Introduction.

Many authors have considered the motion of curves in the plane by speeds depending on curvature and normal direction: If γ_0 is a convex closed curve given by an embedding $x_0 : \mathcal{C} \to \mathbb{R}^2$, this motion is described by an equation of the form

$$(1) \qquad \begin{aligned} \frac{\partial}{\partial t} x(\xi, t) &= -\psi(\mathbf{n}(\xi, t))(\kappa(\xi, t))^\alpha \mathbf{n}(\xi, t), \\ x(\xi, 0) &= x_0(\xi), \end{aligned}$$

for all ξ in $\mathcal{C}$ and $t \in [0, T)$, where $x_t = x(., t)$ is a smooth embedding for each $t \in [0, T)$, $\mathbf{n}(\xi, t)$ is the outward unit normal vector to the curve $\gamma_t = x_t(\mathcal{C})$ at the point $x(\xi, t)$, $\kappa(\xi, t)$ is the curvature of γ_t at $x(\xi, t)$, and $\psi : S^1 \to \mathbb{R}$ is a smooth positive function. We write $A[\gamma_t]$ for the area enclosed by the curve γ_t.

A well-known example of such an evolution equation is the curve-shortening flow, in which $\alpha = 1$ and $\psi \equiv 1$. Gage [14, 15] and Gage and Hamilton [18] proved that convex embedded curves become circular while contracting to points, and Grayson [19] extended this to arbitrary embedded closed curves. The case $\alpha = 1/3$, $\psi \equiv 1$ is natural in affine geometry, and has been applied to image processing and related problems. It has been considered both in the convex case ([25], [3]), where solutions become elliptical in shape as they contract to points, and in the non-convex case [9], where closed embedded curves eventually become convex. Anisotropic evolutions (with ψ non-constant) arise naturally in Finsler or Minkowski geometry on the plane [16, 17], and in physical interface problems (see [6] and

[12]). These have also been considered for convex curves ([16, 17], [5], [13]) and more generally ([7, 8], [23], [27]). Equation (1) assumes that the speed is a homogeneous function of the curvature. Non-homogeneous speeds have also been considered – see for example [11] and sections I3 and II4 of [5].

The paper [5] investigated convex solutions of (1) with α not too small:

Theorem (Theorem II1.11 [5]). *There exists a unique, smooth solution $x : C \times [0,T) \to \mathbb{R}^2$ of (1). The curves γ_t converge to $p \in \mathbb{R}^2$ as $t \to T$. If $\alpha > \frac{1}{3}$ then there exists $\{t_k\} \to T$ such that the rescaled curves $\tilde{\gamma}_{t_k} = \sqrt{\frac{\pi}{A[\gamma_{t_k}]}}(\gamma_{t_k} - p)$ converge in $C^{2+\ell,\beta}$ to a limit Σ satisfying $\psi\kappa^\alpha = c\langle x, \mathbf{n}\rangle$ for some $c > 0$. Here $\ell + \beta = \frac{1}{\max\{\alpha-1,0\}}$.*

The critical case $\alpha = \frac{1}{3}$ with $\psi \equiv 1$ is the flow by affine normal:

Theorem ([3], [25]). *If $\alpha = 1/3$ and $\psi \equiv 1$, then the rescaled curves $\tilde{\gamma}_t$ converge in C^∞ to an ellipse centred at the origin.*

In this paper we consider the main cases not covered by the above results, namely flows of the form (1) with $\alpha < \frac{1}{3}$, or $\alpha = \frac{1}{3}$ if ψ is non-constant. The results of [2] imply that there exist solutions of the isotropic flows (those with ψ constant) which do not converge to circles, if $\alpha < \frac{1}{3}$. In [4] it was conjectured that solutions for isotropic flows become circular for $\alpha > \frac{1}{3}$ but generically do not for smaller α. The results presented here confirm the non-convergence part of the claim; a full discussion of the isotropic case will be presented elsewhere.

The curves $\tilde{\gamma}_t$ satisfy a modified equation: $\tilde{\gamma}_t = \tilde{x}(C, \tau)$, where

$$\frac{\partial}{\partial \tau}\tilde{x}(\xi, \tau) = -\psi\tilde{\kappa}^\alpha \mathbf{n} + \frac{\int_{\tilde{\gamma}} \psi\tilde{\kappa}^\alpha d\tilde{s}}{2\pi}\tilde{x}, \tag{2}$$

and $\tau = \int_0^t \left(\frac{\pi}{A[\gamma_u]}\right)^{\frac{1+\alpha}{2}} du$ (see Section 2). The following partial result is known:

Theorem (Theorem II1.12 of [5]). *Let $\alpha \in (0, \frac{1}{3}]$, and suppose $\tilde{x} : C \times (0,T) \to \mathbb{R}^2$ is a solution of Eq. (2), maximally extended in time. Then either the isoperimetric ratios of $\tilde{\gamma}_t = x_t(C)$ approach infinity as $t \to T$, or $T = \infty$ and there exists a subsequence $t_k \to \infty$ such that $\tilde{\gamma}_{t_k}$ converges in C^∞ to a strictly convex limit Σ which evolves under (2) only by changing parametrisation.*

The condition that Σ evolve solely by reparametrisation is the following:

$$0 = -\psi\kappa^\alpha \mathbf{n} + \frac{\int_\Sigma \psi\kappa^\alpha\, ds}{2\pi} x + V\mathbf{T} \tag{3}$$

for some $V : \Sigma \to \mathbb{R}$, where $\mathbf{T}$ is the unit tangent vector to Σ, and $x : \Sigma \to \mathbb{R}^2$ is the inclusion map. Such a curve will be called a *stationary curve* for (2). The result above can be improved slightly using the following:

Theorem (Theorem 2 of [4]). *If $\alpha > 0$ and $\tilde{x} : \mathcal{C} \times (0, \infty) \to \mathbb{R}^2$ is a solution of (2) such that $\tilde{\gamma}_{t_k} = \tilde{x}_{t_k}(\mathcal{C})$ converges in Hausdorff distance to a strictly convex stationary curve Σ for a subsequence $t_k \to \infty$, then $\tilde{\gamma}_t$ converges to Σ in C^∞ as $t \to \infty$.*

In this paper we work in the space of symmetric convex curves (those invariant under the involution $x \mapsto -x$ of $\mathbb{R}^2$), and prove that all stationary curves are unstable:

Theorem 1. *Suppose $0 < \alpha \leq \frac{1}{3}$, and $\psi : S^1 \to \mathbb{R}$ (non-constant if $\alpha = \frac{1}{3}$) is invariant under the involution $z \mapsto -z$ of S^1. Let Σ be any symmetric stationary curve for (2). Then there exists a smooth, symmetric solution $x : \mathcal{C} \times (-\infty, T) \to \mathbb{R}^2$ of equation (2) such that $\gamma_\tau = x_\tau(\mathcal{C})$ converges in C^∞ to Σ as τ approaches $-\infty$, and the Hausdorff distance from γ_τ to Σ is at least $\min\{e^{\omega\tau}, C\}$ for some positive constants C and ω.*

We will also deduce the following global instability result:

Theorem 2. *Let ψ be symmetric, and $0 < \alpha \leq 1/3$, with ψ non-constant if $\alpha = \frac{1}{3}$. Then for $k \geq 0$ and $\beta \in (0, 1]$ there is a generic subset $U^{k,\beta}$ of the space $\mathcal{K}^{k,\beta}$ of symmetric convex open sets with $C^{k,\beta}$ boundary in $\mathbb{R}^2$, such that if $x_0(\mathcal{C}) = \partial\Omega$ for $\Omega \in U^{k,\beta}$, and $\{x_t\}$ satisfies (1), then the isoperimetric ratios of the curves $\gamma_t = x_t(\mathcal{C})$ approach infinity as $t \to T$.*

The crucial step in the proof of Theorem 1 is to show that every symmetric stationary curve is linearly unstable, by estimating eigenvalues of the linearised equation. The method of proof is similar to that used by Hersch [20], Yang and Yau [26] and Li and Yau [22] in estimating the first eigenvalue of a surface.

The symmetry assumption cannot be removed — an explicit counterexample is given in Proposition 29. However, it is shown in Proposition 30 that the instability result of Theorem 1 still holds in cases where the stationary curve is an off-centre symmetric curve.

2. Notation and preliminary results.

We begin by recalling some special features of the geometry of convex curves in the plane, the details of which can be found in [1].

Let γ be a smooth convex closed curve in the plane. The support function of γ is the function $s : S^1 \to \mathbb{R}$ defined by

$$s(\theta) = \sup_{y\in\gamma}\langle y, e^{i\theta}\rangle \tag{4}$$

for each $\theta \in S^1$. The support function determines γ according to the identity $\gamma = \bar{x}(S^1)$ where $\bar{x} : S^1 \to \mathbb{R}^2 \simeq \mathbb{C}$ is given by

$$\bar{x}(\theta) = \left(s(\theta) + i\frac{\partial s}{\partial\theta}(\theta)\right) e^{i\theta}.$$

The map $\bar{x}$ is the inverse of the Gauss map $\mathbf{n} : \gamma \to S^1$. The curvature of γ can also be recovered from s: For each $\theta \in S^1$ the radius of curvature of γ at $\bar{x}(\theta)$ is given by $\mathfrak{r}[s] = s_{\theta\theta} + s$, where subscripts denote derivatives. The curvature is then $\kappa = \mathfrak{r}^{-1}$. The length L of γ is equal to $\int_{S^1} \mathfrak{r}\, d\theta$, and the enclosed area A is $\frac{1}{2}\int_{S^1} s\mathfrak{r}\, d\theta$.

The equation (1) can be rewritten as an evolution equation for the function s: If $x(\xi, t)$ is a family of embeddings evolving by equation (1), then $\bar{x} = x \circ \mathbf{n}^{-1}$, and

$$\begin{aligned}\frac{\partial}{\partial t}s(\theta,t) &= \frac{\partial}{\partial t}\left\langle x\circ\mathbf{n}^{-1}, e^{i\theta}\right\rangle \\ &= \left\langle -\psi(\theta)\left(\kappa\circ\mathbf{n}^{-1}\right)^{\alpha} e^{i\theta} + Dx_t\left(\frac{\partial}{\partial t}\mathbf{n}^{-1}\right), e^{i\theta}\right\rangle \\ &= -\psi(\theta)\mathfrak{r}[s]^{-\alpha}.\end{aligned} \tag{5}$$

Conversely, it was shown in [5] that any solution of the scalar equation (5) can be used to reconstruct a solution of (1).

Solutions of equation (1) contract to points in finite time. The analysis of the limiting shapes of solutions as this final time is approached will be carried out by normalising the solution curves to keep their enclosed area constant. This requires a choice of centre about which to rescale. A convenient choice is to require

$$\int_{S^1} \psi^{1/\alpha} s^{-1/\alpha}\cos(\theta-\theta_0)\, d\theta = 0 \tag{6}$$

for all θ_0 and at each time. Let $\tilde{\gamma}_t$ be the curve obtained by rescaling γ_t to have fixed enclosed area π and translating to satisfy (6). The support function $\tilde{s}$ of $\tilde{\gamma}$ satisfies

$$\tilde{s} = \sqrt{\frac{\pi}{A}}\, s + c_1 \cos\theta + c_2 \sin\theta$$

for some c_1 and c_2 depending on t, where $A = A[\gamma_t]$. Differentiation gives:

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{s} &= -\sqrt{\frac{\pi}{A}}\psi\mathfrak{r}[s]^{-\alpha} + \frac{s\sqrt{\pi}}{2A^{3/2}}\int_{S^1}\psi\mathfrak{r}^{1-\alpha}\,d\theta + \dot{c}_1\cos\theta + \dot{c}_2\sin\theta \\ &= \left(\frac{\pi}{A}\right)^{\frac{1+\alpha}{2}}\left(-\psi\tilde{\mathfrak{r}}^{-\alpha} + \frac{\tilde{s}}{2\pi}\int_{S^1}\psi\tilde{\mathfrak{r}}^{1-\alpha}\,d\theta\right) + \dot{c}_1\cos\theta + \dot{c}_2\sin\theta. \end{aligned}$$

where $\dot{c}_i = \frac{d}{dt}c_i$. This equation is simplified by the introduction of a new time variable τ according to the definition

$$\tau = \int_0^t \left(\frac{\pi}{A[\gamma_u]}\right)^{\frac{1+\alpha}{2}} du, \tag{7}$$

so that

$$\frac{\partial}{\partial\tau}\tilde{s} = -\psi\tilde{\mathfrak{r}}^{-\alpha} + \frac{\tilde{s}}{2\pi}\int_{S^1}\psi\tilde{\mathfrak{r}}^{1-\alpha}\,d\theta + c_1'\cos\theta + c_2'\sin\theta, \tag{8}$$

where $c_i' = \frac{d}{d\tau}c_i$. The constants c_1' and c_2' can be determined at each time by differentiating the identities (6):

$$\frac{d}{d\tau}\int_{S^1}\left(\frac{\psi}{\tilde{s}}\right)^{\frac{1}{\alpha}}\cos\theta\,d\theta = \frac{d}{d\tau}\int_{S^1}\left(\frac{\psi}{\tilde{s}}\right)^{\frac{1}{\alpha}}\sin\theta\,d\theta = 0.$$

Substitution of the above expression for $\frac{d}{d\tau}\tilde{s}$ yields the following:

$$\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = M^{-1}\begin{bmatrix} \int_{S^1}\left(\frac{\psi}{\tilde{s}}\right)^{1+\frac{1}{\alpha}}\tilde{\mathfrak{r}}^{-\alpha}\cos\theta\,d\theta - \frac{1}{2\pi}\int_{S^1}\psi\mathfrak{r}^{1-\alpha}\,d\theta\int_{S^1}\left(\frac{\psi}{\tilde{s}}\right)^{\frac{1}{\alpha}}\cos\theta\,d\theta \\ \int_{S^1}\left(\frac{\psi}{\tilde{s}}\right)^{1+\frac{1}{\alpha}}\mathfrak{r}^{-\alpha}\sin\theta\,d\theta - \frac{1}{2\pi}\int_{S^1}\psi\mathfrak{r}^{1-\alpha}\,d\theta\int_{S^1}\left(\frac{\psi}{\tilde{s}}\right)^{\frac{1}{\alpha}}\sin\theta\,d\theta \end{bmatrix} \tag{9}$$

where M is the matrix

$$\begin{bmatrix} \int_{S^1}\frac{1}{\tilde{s}}\left(\frac{\psi}{\tilde{s}}\right)^{\frac{1}{\alpha}}\cos^2\theta\,d\theta & \int_{S^1}\frac{1}{\tilde{s}}\left(\frac{\psi}{\tilde{s}}\right)^{\frac{1}{\alpha}}\cos\theta\sin\theta\,d\theta \\ \int_{S^1}\frac{1}{\tilde{s}}\left(\frac{\psi}{\tilde{s}}\right)^{\frac{1}{\alpha}}\cos\theta\sin\theta\,d\theta & \int_{S^1}\frac{1}{\tilde{s}}\left(\frac{\psi}{\tilde{s}}\right)^{\frac{1}{\alpha}}\sin^2\theta\,d\theta \end{bmatrix}.$$

Note that M is invertible: The Cauchy-Schwartz inequality gives

$$\left(\int_{S^1} \frac{1}{\tilde{s}}\left(\frac{\psi}{\tilde{s}}\right)^{\frac{1}{\alpha}} \cos\theta \sin\theta \, d\theta\right)^2$$
$$< \int_{S^1} \frac{1}{\tilde{s}}\left(\frac{\psi}{\tilde{s}}\right)^{\frac{1}{\alpha}} \cos^2\theta \, d\theta \cdot \int_{S^1} \frac{1}{\tilde{s}}\left(\frac{\psi}{\tilde{s}}\right)^{\frac{1}{\alpha}} \sin^2\theta \, d\theta,$$

so the determinant of M is positive.

The following result was proved in [2]:

Proposition 3. *If $\tilde{s} : S^1 \times [0,T) \to \mathbb{R}$ satisfies (6), then $\frac{d}{dt}\mathcal{Z} \leq 0$, where*

$$\mathcal{Z} = \begin{cases} \left(\frac{1}{2\pi}\int_{S^1} \psi \tilde{\mathfrak{r}}^{1-\alpha} d\theta\right)^{1/(\alpha-1)}, & \alpha \neq 1 \\ \exp\left\{-\frac{1}{2\pi}\int_{S^1} \psi \log \tilde{\mathfrak{r}} \, d\theta\right\}, & \alpha = 1. \end{cases}$$

The inequality is strict unless the equation

$$\psi \tilde{\mathfrak{r}}^{-\alpha} = C\tilde{s} \tag{10}$$

is satisfied for some positive constant C.

For convenience we write $\tilde{\mathcal{Z}} = \mathcal{Z}^{\alpha-1} = \frac{1}{2\pi}\int \psi \mathfrak{r}^{1-\alpha} \, d\theta$. Note that (10) follows from (3) by taking the inner product with the normal $\mathbf{n}$.

In the remainder of the paper we will work entirely with the support function s and the evolution equation (8), suppressing the curves and their embeddings. In particular we will write $A[s]$ and $\mathcal{I}[s]$ for the enclosed area and isoperimetric ratio of the curve with support function s. Explicitly, $A[s] = \frac{1}{2}\int_{S^1} s\mathfrak{r}[s] d\theta$ and

$$\mathcal{I}[s] = \frac{\left(\int_{S^1} s \, d\theta\right)^2}{2\pi \int_{S^1} s\mathfrak{r}[s] \, d\theta}. \tag{11}$$

3. Linearisation about a stationary solution.

Let σ be a stationary solution of equation (8), scaled so that $A[\sigma] = \pi$. Then equation (10) holds, and integration against $\mathfrak{r}$ gives the identity $\psi = \sigma\mathfrak{r}[\sigma]^\alpha \tilde{\mathcal{Z}}$.

Proposition 4. *Let $s_\varepsilon(\theta)$ be a family of functions with $A[s_\varepsilon] = \pi$, converging smoothly to σ as $\varepsilon \to 0$. Let $\tilde{s}_\varepsilon(\theta, \tau)$ satisfy equation (8) with initial* *condition* $s_\varepsilon(\theta)$ *for each* $\varepsilon > 0$. *Let* $\eta(\theta,\tau) = \frac{\partial}{\partial\varepsilon}\tilde{s}_\varepsilon(\theta,\tau)\big|_{\varepsilon=0}$. *Then* η *satisfies the linear equation*

$$(12)\qquad \frac{1}{\tilde{\mathcal{Z}}}\frac{\partial}{\partial\tau}\eta = \alpha\frac{\sigma}{\mathfrak{r}[\sigma]}\mathfrak{r}[\eta] + \eta - \alpha\frac{\sigma}{2\pi}\int_{S^1}\mathfrak{r}[\sigma]\eta\,d\theta - \begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix} M^{-1}\begin{bmatrix}\int_{S^1}\frac{\mathfrak{r}[\sigma]}{\sigma}\eta\cos\theta\,d\theta \\ \int_{S^1}\frac{\mathfrak{r}[\sigma]}{\sigma}\eta\sin\theta\,d\theta\end{bmatrix}.$$

where M *is the matrix given by*

$$\begin{bmatrix}\int_{S^1}\frac{\mathfrak{r}[\sigma]}{\sigma}\cos^2\theta\,d\theta & \int_{S^1}\frac{\mathfrak{r}[\sigma]}{\sigma}\cos\theta\sin\theta\,d\theta \\ \int_{S^1}\frac{\mathfrak{r}[\sigma]}{\sigma}\cos\theta\sin\theta\,d\theta & \int_{S^1}\frac{\mathfrak{r}[\sigma]}{\sigma}\sin^2\theta\,d\theta\end{bmatrix}.$$

Proof. Differentiate the normalised equation (8) with respect to ε. In differentiating the first term, note that

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\left(\psi\tilde{\mathfrak{r}}^{-\alpha}\right) &= -\alpha\psi\mathfrak{r}[\sigma]^{-(1+\alpha)}\mathfrak{r}[\eta] \\ &= -\alpha\sigma\mathfrak{r}[\sigma]^{\alpha}\mathfrak{r}[\sigma]^{-(1+\alpha)}\mathfrak{r}[\eta] \\ &= -\alpha\frac{\sigma}{\mathfrak{r}[\sigma]}\mathfrak{r}[\eta]\tilde{\mathcal{Z}}\end{aligned}$$

by the identity (10). The integral in the second term satisfies

$$\begin{aligned}\frac{d}{d\varepsilon}\int_{S^1}\psi\tilde{\mathfrak{r}}^{1-\alpha}\,d\theta &= (1-\alpha)\int_{S^1}\psi\mathfrak{r}[\sigma]^{-\alpha}\mathfrak{r}[\eta]\,d\theta \\ &= (1-\alpha)\tilde{\mathcal{Z}}\int_{S^1}\sigma\mathfrak{r}[\eta]\,d\theta \\ &= (1-\alpha)\tilde{\mathcal{Z}}\int_{S^1}\mathfrak{r}[\sigma]\eta\,d\theta.\end{aligned}$$

In differentiating the final term, note that when $\varepsilon = 0$,

$$\begin{aligned}\int_{S^1}\left(\frac{\psi}{\tilde{s}}\right)^{1+\frac{1}{\alpha}}\tilde{\mathfrak{r}}^{-\alpha}\cos\theta\,d\theta &= \tilde{\mathcal{Z}}^{1+1/\alpha}\int_{S^1}\mathfrak{r}[\sigma]^{\alpha+1}\mathfrak{r}[\sigma]^{-\alpha}\cos\theta\,d\theta \\ &= \tilde{\mathcal{Z}}^{1+1/\alpha}\int_{S^1}\mathfrak{r}[\sigma]\cos\theta\,d\theta \\ &= \tilde{\mathcal{Z}}^{1+1/\alpha}\int_{S^1}\sigma\mathfrak{r}[\cos\theta]\,d\theta \\ &= 0,\end{aligned}$$

and similarly

$$\int_{S^1}\left(\frac{\psi}{\tilde{s}}\right)^{1+\frac{1}{\alpha}}\tilde{r}^{-\alpha}\sin\theta\,d\theta=0.$$

The identity (10) implies two further useful identities:

$$\int_{S^1}\left(\frac{\psi}{\tilde{s}}\right)^{\frac{1}{\alpha}}\cos\theta\,d\theta=\tilde{\mathcal{Z}}^{1/\alpha}\int_{S^1}\mathfrak{r}[\sigma]\cos\theta\,d\theta=0,$$

and the same with $\cos\theta$ replaced by $\sin\theta$. Therefore the only non-vanishing terms occur when these integrals are differentiated with respect to ε. In that case,

$$\begin{aligned}
&\frac{d}{d\varepsilon}\int_{S^1}\left(\frac{\psi}{\tilde{s}}\right)^{1+\frac{1}{\alpha}}\tilde{r}^{-\alpha}\cos\theta\,d\theta\\
&=-\left(1+\frac{1}{\alpha}\right)\int_{S^1}\frac{1}{\sigma}\left(\frac{\psi}{\sigma}\right)^{1+\frac{1}{\alpha}}\eta\mathfrak{r}[\sigma]^{-\alpha}\,d\theta\\
&\quad-\alpha\int_{S^1}\left(\frac{\psi}{\sigma}\right)^{1+\frac{1}{\alpha}}\mathfrak{r}[\sigma]^{-(1+\alpha)}\mathfrak{r}[\eta]\cos\theta\,d\theta\\
&=-\left(1+\frac{1}{\alpha}\right)\tilde{\mathcal{Z}}^{1+1/\alpha}\int_{S^1}\frac{1}{\sigma}\mathfrak{r}[\sigma]^{1+\alpha}\mathfrak{r}[\sigma]^{-\alpha}\eta\cos\theta\,d\theta\\
&\quad-\alpha\tilde{\mathcal{Z}}^{1+1/\alpha}\int_{S^1}\mathfrak{r}[\sigma]^{1+\alpha}\mathfrak{r}[\sigma]^{-(1+\alpha)}\mathfrak{r}[\eta]\cos\theta\,d\theta\\
&=-\left(1+\frac{1}{\alpha}\right)\tilde{\mathcal{Z}}^{1+1/\alpha}\int_{S^1}\frac{\mathfrak{r}[\sigma]}{\sigma}\eta\cos\theta\,d\theta\\
&\quad-\alpha\tilde{\mathcal{Z}}^{1+1/\alpha}\int_{S^1}\mathfrak{r}[\eta]\cos\theta\,d\theta\\
&=-\left(1+\frac{1}{\alpha}\right)\tilde{\mathcal{Z}}^{1+1/\alpha}\int_{S^1}\frac{\mathfrak{r}[\sigma]}{\sigma}\eta\cos\theta\,d\theta,
\end{aligned}$$

and similarly with $\cos\theta$ replaced by $\sin\theta$. Further,

$$\begin{aligned}
\frac{d}{d\varepsilon}\int_{S^1}\left(\frac{\psi}{\tilde{s}}\right)^{\frac{1}{\alpha}}\cos\theta\,d\theta&=-\frac{1}{\alpha}\int_{S^1}\frac{1}{\sigma}\left(\frac{\psi}{\sigma}\right)^{\frac{1}{\alpha}}\eta\cos\theta\,d\theta\\
&=-\frac{1}{\alpha}\tilde{\mathcal{Z}}^{1/\alpha}\int_{S^1}\frac{\mathfrak{r}[\sigma]}{\sigma}\eta\cos\theta\,d\theta,
\end{aligned}$$

and similarly with $\cos\theta$ replaced by $\sin\theta$. □

The structure of this linearised equation is simply described in terms of the differential operator $\mathcal{L}$ which acts on functions $f \in C^\infty(S^1)$ by

$$\mathcal{L}f = \frac{\sigma}{\mathfrak{r}[\sigma]}\mathfrak{r}[f]. \tag{13}$$

This is a self-adjoint operator on the Hilbert space L^2_σ, the space of all functions f on S^1 for which

$$\int_{S^1} \frac{\mathfrak{r}[\sigma]}{\sigma} f^2 \, d\theta < \infty,$$

with the inner product

$$\langle f_1, f_2 \rangle = \int_{S^1} \frac{\mathfrak{r}[\sigma]}{\sigma} f_1 f_2 \, d\theta. \tag{14}$$

The functions σ, $\cos\theta$, and $\sin\theta$ are eigenfunctions of $\mathcal{L}$, with eigenvalues -1, 0, and 0 respectively.

Lemma 5. *There are no other eigenfunctions of $\mathcal{L}$ with eigenvalue less than or equal to zero.*

Proof. The Brunn-Minkowski inequality for convex sets in $\mathbb{R}^2$ states that the square root of the area functional is concave with respect to Minkowski addition: Precisely, if Ω_0 and Ω_1 are convex sets, and $\Omega_t = \{ty + (1-t)x : y \in \Omega_1, x \in \Omega_0\}$, then

$$\frac{d^2}{dt^2} A[\Omega_t]^{\frac{1}{2}} \leq 0,$$

with equality if and only if $\Omega_1 = c\Omega_0 + e$ for some $c > 0$ and some point $e \in \mathbb{R}^2$, so that Ω_1 and Ω_2 are scaled translates of each other. (See [24], Theorem 6.1.1.)

Consider this inequality in the particular case where Ω_0 is the region enclosed by the curve Σ with support function σ, and Ω_1 has support function $\sigma + \delta f$, where f is any eigenfunction of $\mathcal{L}$ and δ is a small positive number. Let σ_t be the support function of Ω_t. Then

$$\begin{aligned}
\sigma_t(\theta) &= \sup_{y \in \Omega_t} \langle y, e^{i\theta} \rangle \\
&= \sup_{x \in \Omega_0, y \in \Omega_1} \langle (1-t)x + ty, e^{i\theta} \rangle \\
&= (1-t) \sup_{x \in \Omega_0} \langle x, e^{i\theta} \rangle + t \sup_{y \in \Omega_1} \langle y, e^{i\theta} \rangle
\end{aligned}$$

$$\begin{aligned}&= (1-t)\sigma(\theta) + t(\sigma(\theta) + \delta f(\theta))\\ &= \sigma + t\delta f(\theta).\end{aligned}$$

Therefore the area of Ω_t is given by

$$\begin{aligned}A[\Omega_t] &= \frac{1}{2}\int_{S^1} \sigma_t \mathfrak{r}[\sigma_t]\, d\theta\\ &= \frac{1}{2}\int_{S^1} (\sigma + t\delta f)\mathfrak{r}[\sigma + t\delta f]\, d\theta\\ &= A + t\delta \int_{S^1} f\mathfrak{r}[\sigma]\, d\theta + \frac{1}{2}t^2\delta^2 \int_{S^1} f\mathfrak{r}[f]\, d\theta.\end{aligned}$$

A direct calculation gives

$$\left.\frac{d^2}{dt^2}A[\Omega_t]^{\frac{1}{2}}\right|_{t=0} = \frac{\delta^2}{2\sqrt{A}}\left(\int_{S^1} fr[f]\, d\theta - \frac{\left(\int_{S^1} fr[\sigma]\, d\theta\right)^2}{2A}\right),$$

so the quantity in the bracket is nonpositive, and strictly negative unless $\sigma + \delta f = c\sigma + e_1\cos\theta + e_2\sin\theta$, in which case f is a linear combination of σ, $\cos\theta$ and $\sin\theta$.

This result may be expressed in terms of $\mathcal{L}$ and L^2_σ:

$$\int_{S^1} f\mathfrak{r}[f]\, d\theta - \frac{\left(\int_{S^1} f\mathfrak{r}[\sigma]\, d\theta\right)^2}{2A} = \langle \mathcal{L}f, f\rangle - \frac{\langle f,\sigma\rangle^2}{\|\sigma\|^2} \le 0.$$

If f is an eigenfunction of $\mathcal{L}$ orthogonal to σ, $\cos\theta$ and $\sin\theta$, then $\mathcal{L}f + \lambda f = 0$ and $\langle f,\sigma\rangle = 0$, so the inequality becomes $-\lambda\|f\|^2 < 0$. □

The linearised equation (9) can be written in terms of the operator $\mathcal{L}$:

$$\frac{\partial}{\partial\tau}\eta = \tilde{\mathcal{Z}}\left(\alpha\mathcal{L} + 1\right)\left(\eta - \pi_{E_{-1}\oplus E_0}\eta\right)$$

where E_λ is the eigenspace of $\mathcal{L}$ in L^2_σ with eigenvalue λ, and π is the orthogonal projection.

Corollary 6. *A stationary solution σ of (8) is linearly unstable in the space of support functions of convex curves of area π if $\lambda < \frac{1}{\alpha}$, where*

$$\lambda = \inf\left\{\left.\frac{\int_{S^1}\varphi_\theta^2 - \varphi^2\, d\theta}{\int_{S^1}\frac{\mathfrak{r}[\sigma]}{\sigma}\varphi^2\, d\theta}\ \right|\ \varphi\in C^\infty(S^1),\ \int_{S^1}\mathfrak{r}[\sigma]\varphi\, d\theta = 0,\ \int_{S^1}\frac{\mathfrak{r}[\sigma]}{\sigma}\varphi z\, d\theta = 0\right\}$$

and $z = (\cos\theta, \sin\theta)$. *A symmetric stationary solution is linearly unstable in the space of support functions of symmetric convex curves of area* π *if* $\lambda_{\text{sym}} < \frac{1}{\alpha}$, *where*

$$\lambda_{\text{sym}} = \inf\left\{ \frac{\int_{S^1} \varphi_\theta^2 - \varphi^2 \, d\theta}{\int_{S^1} \frac{\mathfrak{r}[\sigma]}{\sigma}\varphi^2 \, d\theta} \,\middle|\, \varphi \in C^\infty_{\text{sym}}(S^1), \int_{S^1} \mathfrak{r}[\sigma]\varphi \, d\theta = 0 \right\}.$$

Here $C^\infty_{\text{sym}}(S^1)$ *is the space of* C^∞ *functions on* S^1 *invariant under the involution* $\theta \mapsto \theta + \pi$.

Proof. λ is the first eigenvalue of $\mathcal{L}$ in the subspace orthogonal to σ, $\cos(\theta)$ and $\sin(\theta)$, and λ_{sym} is the first eigenvalue in the space of symmetric functions orthogonal to σ. If η is proportional to the corresponding eigenfunction, then

$$\frac{\partial}{\partial\tau}\eta = \tilde{\mathcal{Z}}[\sigma](1 - \alpha\lambda)\eta.$$

This implies instability if $1 - \alpha\lambda > 0$. □

4. Linear instability of stationary solutions.

Theorem 7. $\lambda_{\text{sym}} \leq 3$ *for any symmetric stationary solution* σ. *Equality holds if and only if* σ *is the support function of an ellipse centred at the origin.*

Proof. The proof of this result is similar to that of the theorem of Hersch [20] of the fact that the constant curvature 2-sphere has the largest first eigenvalue for the Laplacian amongst all metrics on the 2-sphere with the same area. The idea is to use the eigenfunctions for the case $\sigma \equiv 1$, namely $\varphi_1 = \cos(2\theta)$ and $\varphi_2 = \sin(2\theta)$, as test functions in the integral quotient in Corollary 1. If these functions are admissible, then

$$\int_{S^1} (\varphi_i)_\theta^2 - \varphi_i^2 \, d\theta = 3\pi,$$

and

$$\sum_{i=1}^{2} \int_{S^1} \frac{\mathfrak{r}[\sigma]}{\sigma}\varphi_i^2 \, d\theta = \int_{S^1} \frac{\mathfrak{r}[\sigma]}{\sigma} \, d\theta = \int_{S^1} \left(1 + \left(\frac{\sigma_\theta}{\sigma}\right)^2\right) d\theta \geq 2\pi,$$

with equality if and only if σ is constant. This would imply $\max_i \int_{S^1} \frac{\mathfrak{r}[\sigma]}{\sigma}\varphi_i^2 \, d\theta \geq \pi$, and therefore $\lambda_{\text{sym}} \leq 3$.

The problem is that in general these test functions are not admissible, so some further work is required. Let us denote by Σ the convex curve with support function σ. The key to finding admissible test functions is the following property of the operator $\mathcal{L}$:

Proposition 8. *The spectrum of $\mathcal{L}$ is invariant under the change $\sigma \to \sigma_T$, where σ_T is the support function of the curve $T \circ \Sigma$, for any $T \in SL(2,\mathbb{R})$. In particular, λ_{sym} and λ are invariant under such transformations.*

Proof of proposition. The proof requires some properties of the action of the special affine and linear groups on convex sets, and the induced action on $C^\infty(S^1)$. An element of the special affine group consists of a matrix $A \in SL(2,\mathbb{R})$ together with an element $e \in \mathbb{R}^2$, acting on $\mathbb{R}^2$ by taking $x \in \mathbb{R}^2$ to $Ax + e$. This produces an action on the space of convex sets in $\mathbb{R}^2$, taking a set Ω to $A\Omega + e = \{Ax + e : x \in \Omega\}$ which is a convex set with the same area as Ω.

Let $f \in C^\infty(S^1)$. For sufficiently large C, $g = f + C$ satisfies the condition $\mathfrak{r}[g] = g_{\theta\theta} + g > 0$, and hence g is the support function of a convex body in $\mathbb{R}^2$, namely $\Omega(g) = \bigcap_{\theta\in S^1}\{y \in \mathbb{R}^2 : \langle y, e^{i\theta}\rangle \le g(\theta)\}$. Given (A,e) in the special affine group, define $(A,e)f(\theta) = \sup_{y\in A\Omega(g)+e}\langle y, e^{i\theta}\rangle - C\sup_{y\in AB_1(0)}\langle y, e^{i\theta}\rangle$. This is independent of C for C sufficiently large, and defines an action of the special affine group on $C^\infty(S^1)$.

We now proceed to describe the generators of this action explicitly: The Lie algebra $\mathfrak{sl}(2)$ of $SL(2)$ is the three-dimensional space of trace-free 2×2 matrices, which is the product of a one-dimensional subspace generating the rotations, and an orthogonal two-dimensional subspace consisting of matrices of the form

$$X = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

for any real a and b. Given a convex curve with support function s, consider the change in s caused by applying a continuous family of $SL(2)$ transformations in the direction of a Lie algebra element X:

$$\begin{aligned}
Xs(\theta) &= X\langle \bar{x}(\theta), e^{i\theta}\rangle \\
&= \langle Xx, e^{i\theta}\rangle \\
&= \left\langle \begin{bmatrix} a & b \\ b & -a \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} s \\ s_\theta \end{bmatrix}, \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\right\rangle \\
&= \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}\begin{bmatrix} s \\ s_\theta \end{bmatrix}
\end{aligned}$$

$$= R\left(\cos 2(\theta-\theta_0)s - \sin 2(\theta-\theta_0)s_\theta\right)$$

where

$$R = \sqrt{a^2+b^2}, \quad \cos 2\theta_0 = \frac{a}{\sqrt{a^2+b^2}}, \quad \text{and} \quad \sin 2\theta_0 = \frac{b}{\sqrt{a^2+b^2}}.$$

The induced change in the radius of curvature function $\mathfrak{r}[s]$ is then:

$$\begin{aligned} X\mathfrak{r}[s] &= \mathfrak{r}[Xs] \\ &= R\mathfrak{r}\left[\cos 2(\theta-\theta_0)s - \sin 2(\theta-\theta_0)s_\theta\right] \\ &= -R\left(3\mathfrak{r}[s]\cos 2(\theta-\theta_0) + \mathfrak{r}[s]_\theta \sin 2(\theta-\theta_0)\right). \end{aligned}$$

More generally, if s is any function in $C^\infty(S^1)$, not necessarily a support function, the action is given by the same formulae as above. □

Lemma 9. *Let $X \in \mathfrak{sl}(2)$ act on $C^\infty(S^1)$ as described above. Let f_1 and f_2 be arbitrary functions in $C^\infty(S^1)$, and let s be a C^∞ positive support function for some convex curve in $\mathbb{R}^2$. Then*

$$X\int_{S^1} f_1 \mathfrak{r}[f_2]\,d\theta = 0 \qquad \textit{and} \qquad X\int_{S^1} \frac{\mathfrak{r}[s]}{s} f_1 f_2\,d\theta = 0.$$

Proof of Lemma. Note that

$$\left(X + R\sin 2(\theta-\theta_0)\frac{\partial}{\partial\theta}\right)s = R\cos 2(\theta-\theta_0)s$$

and

$$\left(X + R\sin 2(\theta-\theta_0)\frac{\partial}{\partial\theta}\right)\mathfrak{r}[s] = -3R\cos 2(\theta-\theta_0)\mathfrak{r}[s],$$

so that

$$\left(X + R\sin 2(\theta-\theta_0)\frac{\partial}{\partial\theta}\right)s\mathfrak{r}[s] = -2R\cos 2(\theta-\theta_0)s\mathfrak{r}[s]$$

and

$$\begin{aligned} X(s\mathfrak{r}[s]) &= -2R\cos 2(\theta-\theta_0)s\mathfrak{r}[s] - R\sin 2(\theta-\theta_0)(s\mathfrak{r}[s])_\theta \\ &= -\left(R\sin 2(\theta-\theta_0)s\mathfrak{r}[s]\right)_\theta. \end{aligned}$$

Integrating gives $X\int s\mathfrak{r}[s]d\theta = 0$. The same holds replacing s by any other function f, and by the polarisation identity $X\int f_1\mathfrak{r}[f_2]d\theta = 0$. Similarly,

$$\left(X + R\sin 2(\theta-\theta_0)\frac{\partial}{\partial\theta}\right) f_i = R\cos 2(\theta-\theta_0) f_i$$

for $i = 1, 2$. Combining these with the change in s gives:

$$\left(X + R\sin 2(\theta-\theta_0)\frac{\partial}{\partial\theta}\right)\left(\frac{f_i}{s}\right) = \frac{R}{s}\cos 2(\theta-\theta_0) f_i - R\frac{f_i}{s^2}\cos 2(\theta-\theta_0)s = 0,$$

for each i, so

$$\begin{aligned}
&\left(X + R\sin 2(\theta-\theta_0)\frac{\partial}{\partial\theta}\right)\left(\frac{\mathfrak{r}[s]}{s} f_1 f_2\right)\\
&= \left(X + R\sin 2(\theta-\theta_0)\frac{\partial}{\partial\theta}\right)\left(s\mathfrak{r}[s]\frac{f_1}{s}\frac{f_2}{s}\right)\\
&= -2R\cos 2(\theta-\theta_0)\left(\frac{\mathfrak{r}[s]}{s} f_1 f_2\right).
\end{aligned}$$

As before, this implies

$$X\left(\frac{\mathfrak{r}[s]}{s} f_1 f_2\right) = -\left(R\sin 2(\theta-\theta_0)\frac{\mathfrak{r}[s]}{s} f_1 f_2\right)_\theta$$

and integrating over S^1 gives $X\int \frac{\mathfrak{r}[s]}{s} f_1 f_2 d\theta = 0$. □

Proof of Proposition 8, continued. By Lemma 9, the inner products $\langle f_1, f_2\rangle$ and $\langle f_1, \mathcal{L}f_2\rangle$ do not change under the deformation. Furthermore, the deformation leaves symmetric functions symmetric. This implies by the min-max characterisation that the spectrum of $\mathcal{L}$, and in particular λ and λ_{sym}, are invariant. □

The significance of Proposition 8 is that λ_{sym} can be estimated at any point of the orbit of the action of $SL(2)$. The calculation given above therefore proves the result provided there is some element T of $SL(2)$ such that $\int_{S^1}\mathfrak{r}[s_T]\varphi_i\, d\theta = 0$ for $i = 1, 2$. Since $\mathfrak{r}[\varphi_i] = -3\varphi_i$, it suffices to check that $\int_{S^1} s_T\varphi_i\, d\theta = 0$ for each i.

Lemma 10. *There exists some $T \in SL(2,\mathbb{R})$ such that $\int_{S^1}\sigma_T\varphi_i\, d\theta = 0$, $i = 1, 2$.*

Proof. Given a convex curve with support function σ, consider the function L on $SL(2)/SO(2)$ given by $L(T) = \int_{S^1} \sigma_T \, d\theta$, which is just the length of the image of the curve under T (this is well-defined since the length is invariant under rotations). The derivative of L in the direction of a Lie algebra element X is given by

$$\begin{aligned}
D_X L &= \int_{S^1} X\sigma \, d\theta \\
&= \int_{S^1} R\cos 2(\theta - \theta_0)\sigma - R\sin 2(\theta - \theta_0)\sigma_\theta \, d\theta \\
&= 3R \int_{S^1} \sigma \cos 2(\theta - \theta_0) \, d\theta.
\end{aligned}$$

So at a critical point of L, $\int_{S^1} \sigma\varphi_i \, d\theta = 0$ for each i. Also

$$\begin{aligned}
D_X D_X L &= 3R^2 \int_{S^1} \cos 2(\theta - \theta_0) \left(\cos 2(\theta - \theta_0)\sigma - \sin 2(\theta - \theta_0)\sigma_\theta\right) \, d\theta \\
&= \frac{3}{2}R^2 L + \frac{15}{2}R^2 \int_{S^1} \sigma \cos 4(\theta - \theta_0) \, d\theta \\
&= \frac{3}{2}R^2 L - \frac{1}{2}R^2 \int_{S^1} \mathfrak{r}[\sigma] \cos 4(\theta - \theta_0) \, d\theta \\
&\geq R^2 L \\
&= |X|^2 L
\end{aligned}$$

since $|\cos 4(\theta - \theta_0)| \leq 1$ and $\int_{S^1} \mathfrak{r}[\sigma] = \int_{S^1} \sigma = L$. Therefore L is uniformly convex on $SL(2)/SO(2)$, hence proper, and so has a unique critical point. □

This completes the proof of Theorem 7. □

5. Non-converging solutions.

In this section we use the linear instability result of Theorem 7 (and Corollary 6) to complete the proofs of Theorems 1 and 2. Although we prove results here only for the symmetric case, the same methods can easily be modified to include non-symmetric cases if linear instability is known.

Denote by $\mathcal{K}^0$ the set of support functions of symmetric bounded open convex sets of area π in $\mathbb{R}^2$. $\mathcal{K}^0$ is equipped with the C^0 norm (equivalent to the Hausdorff distance). Let $\mathcal{K}^{\ell,\beta} = C^{\ell,\beta} \cap \mathcal{K}^0$ for any integer $\ell \geq 0$ and $\beta \in (0, 1]$. For $\ell \geq 2$ we also write $\mathcal{K}_+^{\ell,\beta} = \{f \in \mathcal{K}^{\ell,\beta} : \mathfrak{r}[f] > 0\}$. By

Theorem II2.8 of [5], every stationary solution of (8) with $\alpha \leq 1$ is in $\mathcal{K}_+^\infty$ for $t > 0$.

For any $\tau > 0$ we denote by $\Theta_\tau : \mathcal{K}^0 \to \mathcal{K}_+^\infty$ the operator which takes $s_0 \in \mathcal{K}^0$ to the function s_τ given by the solution of equation (8) at time τ with initial condition s_0, where this exists. By Theorem II2.8 of [5], Θ is well-defined on $\{(s_0, \tau) : \ s_0 \in \mathcal{K}^0, \ 0 \leq \tau < T(s_0)\}$ where $T(s_0) > 0$ for every $s_0 \in \mathcal{K}^0$.

Proposition 11. *Θ is a continuous map from an open set of $\mathcal{K}^0 \times (0, \infty)$ to $\mathcal{K}_+^\infty$.*

Proof. Since the solutions of Eq. (8) are obtained by solving Eq. (5) and then rescaling, it will suffice to show that the evolution operator $\hat{\Theta}_t$ for Eq. (5) is continuous.

Fix a curve Γ bounding an open convex region Ω of area π, and let C_1 be such that $C_1^{-1} \leq \mathrm{inrad}(\Omega) \leq \mathrm{circumrad}(\Omega) \leq C_1$. Let s be the support function of Γ. By Theorem II2.8 in [5] there exists a unique solution $\hat{\Theta}_t s$ of Eq. (5) on a maximal time interval $(0, T)$ which converges to s in $\mathcal{K}^0$ as $t \to 0$.

In the following we fix $\varepsilon > 0$, an integer $k > 0$, and $t \in (0, T)$. It will suffice to show that there exists $\delta > 0$ such that $\hat{\Theta}_{t'}\bar{s}$ exists and $\left|\hat{\Theta}_{t'}\bar{s} - \hat{\Theta}_t s\right|_{C^{k-1,1}} < \varepsilon$ whenever $\bar{s} \in \mathcal{K}^0$ satisfies $|\bar{s} - s|_{C^0} < \delta$ and $|t' - t| < \delta$.

Lemma 12. *For any $\varepsilon' > 0$ there exists $\delta_1 > 0$ such that if $|t' - t| < \delta_1$, and $|\bar{s} - s|_{C^0} < \delta_1$, then $|\hat{\Theta}_{t'}\bar{s} - \hat{\Theta}_t s|_{C^0} < \varepsilon'$.*

Proof. The proof uses scaled copies of s as barriers. The homogeneity of equation (5) implies that $\hat{\Theta}_t(\mu s) = \mu \hat{\Theta}_{t\mu^{-(1+\alpha)}}(s)$ for any $\mu > 0$.

The solution $\hat{\Theta}_t(s)$ is regular for positive times, and in particular for any $\varepsilon' > 0$ there exists $\delta_4(\varepsilon') > 0$ such that

$$\hat{\Theta}_t(s) - \varepsilon' < \hat{\Theta}_{t(1+\delta_4)}(s) < \hat{\Theta}_{t(1-\delta_4)}(s) < \hat{\Theta}_t(s) + \varepsilon'.$$

For given $\varepsilon' > 0$, choose $\delta_1(\varepsilon') > 0$ sufficiently small to ensure that

$$\max\left\{1 - (1 - \delta_1/t)\left(1 + C_1\delta_1\right)^{-1-\alpha}, (1 + \delta_1/t)\left(1 - C_1\delta_1\right)^{-1-\alpha} - 1\right\} < \delta_4\left(\frac{\varepsilon'}{3}\right)$$

and $\delta_1 < \min\left\{\frac{1}{2C_1}, \frac{\varepsilon'}{2C_1^2}\right\}$. Clearly $|\bar{s} - s|_{C^0} < \delta_1$ implies

$$s(1 - C_1\delta_1) < s - \delta_1 < \bar{s} < s + \delta_1 < s(1 + C_1\delta_1)$$

and so by the comparison principle and the homogeneity relation,

$$\begin{aligned}
&(1 - C_1\delta_1)\hat{\Theta}_{t'(1-C_1\delta_1)^{-1-\alpha}}(s) \\
&= \hat{\Theta}_{t'}\left(s(1 - C_1\delta_1)\right) \\
&< \hat{\Theta}_{t'}(\bar{s}) \\
&< \hat{\Theta}_{t'}\left(s(1 + C_1\delta_1)\right) = (1 + C_1\delta_1)\hat{\Theta}_{t'(1+C_1\delta_1)^{-1-\alpha}}(s).
\end{aligned}$$

The restriction $|t' - t| < \delta_1$ then implies (since $\hat{\Theta}_t(s)$ is decreasing in t) that

$$\begin{aligned}
(1 - C_1\delta_1)\hat{\Theta}_{t(1+\delta_1/t)(1-C_1\delta_1)^{-1-\alpha}}(s) &< \hat{\Theta}_{t'}(\bar{s}) \\
&< (1 + C_1\delta_1)\hat{\Theta}_{t(1-\delta_1/t)(1+C_1\delta_1)^{-1-\alpha}}(s).
\end{aligned}$$

Hence

$$\begin{aligned}
\hat{\Theta}_{t'}(\bar{s}) - \hat{\Theta}_t(s) &< (1 + C_1\delta_1)\hat{\Theta}_{t(1-t^{-1}\delta_1)(1+C_1\delta_1)^{-1-\alpha}}(s) - \hat{\Theta}_t(s) \\
&< C_1\delta_1\hat{\Theta}_{t(1-t^{-1}\delta_1)(1+C_1\delta_1)^{-1-\alpha}}(s) \\
&\quad + (1 + C_1\delta_1)\left(\hat{\Theta}_{t(1-t^{-1}\delta_1)(1+C_1\delta_1)^{-1-\alpha}}(s) - \hat{\Theta}_t(s)\right) \\
&< C_1^2\delta_1 + \frac{3}{2}\left(\hat{\Theta}_{t(1-t^{-1}\delta_1)(1+C_1\delta_1)^{-1-\alpha}}(s) - \hat{\Theta}_t(s)\right) \\
&< \frac{1}{2}\varepsilon' + \frac{3}{2}\frac{\varepsilon'}{3} \\
&= \varepsilon'
\end{aligned}$$

by the choice of δ_1. Similarly,

$$\begin{aligned}
&\hat{\Theta}_t(s) - \hat{\Theta}_{t'}(\bar{s}) \\
&< \hat{\Theta}_t(s) - (1 - C_1\delta_1)\hat{\Theta}_{t(1+t^{-1}\delta_1)(1-C_1\delta_1)^{-1-\alpha}}(s) \\
&< C_1\delta_1\hat{\Theta}_t(s) + (1 - C_1\delta_1)\left(\hat{\Theta}_t(s) - \hat{\Theta}_{t(1+t^{-1}\delta_1)(1-C_1\delta_1)^{-1-\alpha}}(s)\right) \\
&< C_1^2\delta + \left(\hat{\Theta}_t(s) - \hat{\Theta}_{t(1+t^{-1}\delta_1)(1-C_1\delta_1)^{-1-\alpha}}(s)\right) \\
&< \frac{1}{2}\varepsilon' + \frac{1}{3}\varepsilon' \\
&< \varepsilon'.
\end{aligned}$$

□

Lemma 13. *There exists $\delta_2 > 0$ and a constant C_2 such that if $|t' - t| < \delta_2$ and $|\bar{s} - s|_{C^0} < \delta_2$ then $|\hat{\Theta}_{t'}\bar{s} - \hat{\Theta}_T s|_{C^{2k-1,1}} < C_2$.*

Proof. By Lemma 12 we can choose δ_2 to ensure that the isoperimetic ratios remain bounded:

$$\max\left\{\mathcal{I}\left[\hat{\Theta}_{t'}\bar{s}\right], \mathcal{I}\left[\hat{\Theta}_t s\right]\right\} < C.$$

Then Corollary II2.6 of [5] gives the required bound. □

Lemma 14. *There exists $\delta_3 > 0$ such that if s_1, s_2 satisfy $|s_1 - s_2|_{C^0} < \delta_3$ and $|s_1 - s_2|_{C^{2k-1,1}} < C_2$, then $|s_1 - s_2|_{C^{k-1,1}} < \varepsilon$.*

Proof. A standard interpolation inequality between Hölder spaces (see for example Proposition 4.2 of [21]) gives

$$|s_1 - s_2|_{C^{k-1,1}} < C_3 |s_1 - s_2|_{C^0}^{1/2} |s_1 - s_2|_{C^{2k-1,1}}^{1/2}$$

for some constant C_3. Hence $|s_1 - s_2|_{C^{k-1,1}} < C_3 C_2^{1/2} \delta_3^{1/2}$, and it suffices to take $\delta_3 = \frac{\varepsilon^2}{C_2 C_3^2}$. □

Taking $\delta = \min\{\delta_2, \delta_1(\delta_3)\}$, we find by Lemma 13 that $|\hat{\Theta}_{t'}\bar{s} - \hat{\Theta}_t s|_{C^{2k-1,1}} < C_2$, and by Lemma 12 that $|\hat{\Theta}_{t'}\bar{s} - \hat{\Theta}_t s|_{C^0} < \delta_3$, so that by Lemma 14 $|\hat{\Theta}_{t'}\bar{s} - \hat{\Theta}_t s|_{C^{k-1,1}} < \varepsilon$ as required.

Before proceeding to our first main result on the instability of stationary solutions, we will establish some results on the differentiability of the evolution operators which will be useful later in the paper.

Proposition 15. *For any $t > 0$, $\ell \geq 2$, and $\beta \in (0,1)$, the map Θ_t is a C^∞_{loc} map from $\{s \in \mathcal{K}_+^{\ell,\beta} : T(s) > t\}$ to $\mathcal{K}_+^{\ell,\beta}$.*

Proof. We begin by working with the unrescaled evolution operator $\hat{\Theta}_t$. Define $\mathcal{K}^{\ell,\beta}_{M,T,r,R} = \{s \in \mathcal{K}_+^{\ell,\beta} : |s|_{C^{\ell,\beta}} \leq M;\ r \leq \hat{\Theta}_t s \leq R,\ \forall t \in [0,T]\}$. This is an open set in $C^{\ell,\beta}(S^1)$, and we equip it with the $C^{\ell,\beta}$ metric. We will show that $\hat{\Theta}_t$ is C^k for every k from $\mathcal{K}^{\ell,\beta}_{M,T,r,R}$ into $\mathcal{K}_+^{\ell,\beta}$. In proving this it is convenient to define spaces which encapsulate the natural regularity properties of solutions of parabolic equations: Let $P^{\ell,\beta} = \{f : S^1 \times [0,t] \to$

$\mathbb{R} : \ |f|_{P^{\ell,\beta}} < \infty\}$, where

$$
(15) \qquad |f|_{P^{\ell,\beta}} = \sum_{m+2j\le \ell} \sup_{S^1\times[0,T]} |\partial_\theta^m \partial_t^j f| + \sum_{m+2j=\ell-1} \sup_{\theta\in S^1, t_2\neq t_1} \frac{|\partial_\theta^m \partial_t^j f(\theta_2,t_2) - \partial_\theta^m \partial_t^j f(\theta_2,t_2)|}{|t_2-t_1|^{(1+\beta)/2}} + \sum_{m+2j=\ell} \sup_{(\theta_1,t_1)\neq(\theta_2,t_2)} \frac{|\partial_\theta^m \partial_t^j f(\theta_2,t_2) - \partial_\theta^m \partial_t^j f(\theta_2,t_2)|}{(|\theta_2-\theta_1| + |t_2-t_1|^{1/2})^\beta}
$$

This defines a Banach space, and the global Schauder estimates for linear parabolic equations are conveniently stated in terms of these norms:

Proposition 16. *Let $\ell \geq 2$ and $\beta \in (0,1]$. Suppose a, b, c and f are in $P^{\ell-2,\beta}$, with $0 < \lambda \leq a \leq \Lambda < \infty$. Then any $P^{\ell,\beta}$ solution u of*

$$
\frac{\partial u}{\partial t} = a u_{\theta\theta} + b u_\theta + cu + f
$$
$$
u(\theta, 0) = u_0(\theta)
$$

satisfies

$$
|u|_{P^{\ell,\beta}} \leq C\left(|u_0|_{C^{\ell,\beta}} + |f|_{P^{\ell-2,\beta}}\right),
$$

where $C = C\left(\lambda, \Lambda, T, |a|_{P^{\ell-2,\beta}}, |b|_{P^{\ell-2,\beta}}, |c|_{P^{\ell-2,\beta}}\right)$.

See for example [21], Theorem 4.28 for the proof of this result.

Lemma 17. *There exists a constant C such that $|\hat{\Theta}s|_{P^{\ell,\beta}} \leq C$ for all $s \in \mathcal{K}^{\ell,\beta}_{M,T,r,R}$.*

Proof. Bounds above and below on $\mathfrak{k}[\hat{\Theta}_t s]$ follow from Theorems II1.1 and II1.2 of [5]. Bounds in $P^{2,\beta}$ follow from Theorem 14.18 of [21], and bounds in $P^{j,\beta}$ follow by differentiating Eq. (5) and applying Proposition 16. □

We define candidates for the derivatives of all orders of $\hat{\Theta}$: For $k = 1$ and $s \in \mathcal{K}^{\ell,\beta}_{M,T,r,R}$ define $D\hat{\Theta}(s;\varphi)$ to be the solution of the linear parabolic initial value problem

$$
(16) \qquad \frac{\partial}{\partial t}\left(D\hat{\Theta}(s;\varphi)\right) = \alpha\psi\mathfrak{k}[\hat{\Theta}s]^{-(1+\alpha)}\mathfrak{k}[D\hat{\Theta}(s;\varphi)]
$$
$$
D\hat{\Theta}(s;\varphi)(\theta,0) = \varphi(\theta)
$$

for any $\varphi \in C^{\ell,\beta}$. Then we define inductively $D^k\hat{\Theta}$ for $k > 1$ by

$$(17)\quad \frac{\partial}{\partial t}D^k\hat{\Theta}(s;\varphi_1,\ldots,\varphi_k) = \alpha\psi\mathfrak{r}[\hat{\Theta}(s)]^{-(1+\alpha)}\mathfrak{r}[D^k\hat{\Theta}(s;\varphi_1,\ldots,\varphi_k)] + \psi\sum_{n=2}^{k}(-1)^{n-1}\mathfrak{r}[\hat{\Theta}(s)]^{-(\alpha+n)}\prod_{j=0}^{n-1}(\alpha+j) \times \sum_{\{I_1,\ldots,I_n\}\in Z_{k,n}}\prod_{j=1}^{n}\mathfrak{r}[D^{|I_j|}\hat{\Theta}(s;\varphi_{I_j})]$$

$$D^k\hat{\Theta}(s;\varphi_1,\ldots,\varphi_k)(\theta,0) = 0,$$

where $Z_{k,n}$ is the set of partitions of $\{1,\ldots,k\}$ into n non-empty subsets, and $\varphi_I = (\varphi_{i_1},\ldots,\varphi_{i_j})$ if $I = (i_1,\ldots,i_j)$.

Lemma 18. *$D^k\hat{\Theta}(s;.) : \otimes^k C^{\ell,\beta} \to P^{\ell,\beta}$ is a bounded linear map, with norm depending on r, R, α, ψ, and $|s|_{C^{\ell,\beta}}$.*

Proof. Existence follows from Theorem 5.13 of [21].

For $k = 1$, Proposition 16 applied to Eq. (16) implies $|D\hat{\Theta}(s;\varphi)|_{P^{\ell,\beta}} \leq C|\varphi|_{C^{\ell,\beta}}$ for all $\varphi \in C^{\ell,\beta}$, and hence $\|D\hat{\Theta}(s;,)\|_{B(C^{\ell,\beta},P^{\ell,\beta})} \leq C$.

We proceed by induction: Suppose the Lemma holds for $k = 1,\ldots,j$, so that $|D^i\hat{\Theta}(s;\varphi_1,\ldots,\varphi_i)|_{P^{\ell,\beta}} \leq C\prod_{n=1}^{i}|\varphi_n|_{C^{\ell,\beta}}$ for $\varphi_1,\ldots,\varphi_i$ in $C^{\ell,\beta}$, and $i = 1,\ldots,j$. Then Equation (17) has the form

$$\frac{\partial}{\partial t}f = \alpha\psi\mathfrak{r}[\hat{\Theta}(s)]^{-(1+\alpha)}\mathfrak{r}[f] + g,$$

where $f = D^{(j+1)}\hat{\Theta}(s;\varphi_1,\ldots,\varphi_{j+1})$, and $|g|_{P^{\ell-2,\beta}} \leq C\prod_{n=1}^{j+1}|\varphi_n|_{C^{\ell,\beta}}$ by the inductive hypothesis. Proposition 16 therefore implies $|f|_{P^{\ell,\beta}} \leq C\prod_{n=1}^{j+1}|\varphi_n|_{C^{\ell,\beta}}$ and therefore $\|D^{j+1}\hat{\Theta}(s;.)\|_{B(\otimes^{j+1}C^{\ell,\beta},P^{\ell,\beta})} \leq C$. □

Lemma 19. *$\hat{\Theta}$ is a Lipschitz map from $\mathcal{K}^{\ell,\beta}_{M,T,r,R}$ to $P^{\ell,\beta}$.*

Proof. Suppose u and v are in $\mathcal{K}^{\ell,\beta}_{M,T,r,R}$. We compute

$$\frac{\partial}{\partial t}\left(\hat{\Theta}(u) - \hat{\Theta}(v)\right) = -\psi\mathfrak{r}[\hat{\Theta}(u)]^{-\alpha} + \psi\mathfrak{r}[\hat{\Theta}(v)]^{-\alpha} = a\mathfrak{r}[\hat{\Theta}(u) - \hat{\Theta}(v)],$$

where $a = \alpha\psi \int_0^1 \mathfrak{r}[\varepsilon\hat{\Theta}(u) + (1-\varepsilon)\hat{\Theta}(v)]^{-(1+\alpha)} d\varepsilon$ satisfies $|a|_{P^{\ell-2,\beta}} \leq C$ by Lemma 17. Therefore Proposition 16 applies to show $|\hat{\Theta}(u) - \hat{\Theta}(v)|_{P^{\ell,\beta}} \leq C|u-v|_{C^{\ell,\beta}}$. □

Lemma 20. *Suppose Θ is a $C^{j,1}$ map from $\mathcal{K}^{\ell,\beta}_{M,T,r,R}$ to $P^{\ell,\beta}$, and the following estimates hold for some constant C:*

$$\|D^i\hat{\Theta}(u;.) - D^i\hat{\Theta}(v;.)\|_{B(\otimes^i C^{\ell,\beta},P^{\ell,\beta})} \leq C|u-v|_{C^{\ell,\beta}} \tag{18}$$

for all u and v in $\mathcal{K}^{\ell,\beta}_{M,T,r,R}$ and $i = 0,\ldots,j$, and

$$\|D^i\hat{\Theta}(u;.) - D^i\hat{\Theta}(v;.) - D^{(i+1)}\hat{\Theta}(v;.,u-v)\|_{B(\otimes^i C^{\ell,\beta},P^{\ell,\beta})} \leq C|u-v|^2_{C^{\ell,\beta}} \tag{19}$$

for all u and v in $\mathcal{K}^{\ell,\beta}_{M,T,r,R}$ and $i = 0,\ldots,j-1$. Then Θ is a $C^{j+1,1}$ map from $\mathcal{K}^{\ell,\beta}_{M,T,r,R}$ to $P^{\ell,\beta}$; estimate (18) holds with $i = j+1$; and (19) holds with $i = j$.

Proof. We begin by establishing (19) for $i = j$. Writing

$$F = D^j\hat{\Theta}(u;\varphi_1,\ldots,\varphi_j) - D^j\hat{\Theta}(v;\varphi_1,\ldots,\varphi_j) - D^{(j+1)}\hat{\Theta}(v;\varphi_1,\ldots,\varphi_j,u-v)$$

for some $\varphi_1,\ldots,\varphi_j$ in $C^{\ell,\beta}$, we have by Equation (17)

$$\begin{aligned}
&\frac{\partial F}{\partial t}\\
&= \alpha\psi\mathfrak{r}[\hat{\Theta}(v)]^{-(1+\alpha)}\mathfrak{r}[F]\\
&+ \frac{\alpha(1+\alpha)\psi}{\mathfrak{r}[\hat{\Theta}v]^{(2+\alpha)}}\mathfrak{r}[D\hat{\Theta}(v;u-v)]\mathfrak{r}[D^j\hat{\Theta}(u;.) - D^j\hat{\Theta}(v;.)]\\
&+ \alpha\psi\mathfrak{r}[D^j\hat{\Theta}(u;.)]\left(\mathfrak{r}[\hat{\Theta}u]^{-(1+\alpha)} - \mathfrak{r}[\hat{\Theta}v]^{-(1+\alpha)} + (1+\alpha)\frac{\mathfrak{r}[D\hat{\Theta}(v;u-v)]}{\mathfrak{r}[\hat{\Theta}v]^{2+\alpha}}\right)\\
&- \psi\sum_{n=2}^{j}\sum_{\{I_1,\ldots,I_n\}\in Z_{j,n}}(-1)^n\left(\prod_{i=0}^{n-1}(\alpha+i)\right)\left(\prod_{i=1}^{n}\mathfrak{r}[D^{|I_i|}\hat{\Theta}(v;\varphi_{I_i})]\right)\\
&\times\left(\mathfrak{r}[\hat{\Theta}u]^{-(\alpha+n)} - \mathfrak{r}[\hat{\Theta}v]^{-(\alpha+n)} + (\alpha+n)\mathfrak{r}[\hat{\Theta}v]^{-(\alpha+n+1)}\mathfrak{r}[D\hat{\Theta}(v;u-v)]\right)\\
&- \psi\sum_{n=2}^{j}\sum_{\{I_1,\ldots,I_n\}\in Z_{j,n}}(-1)^n\prod_{i=0}^{n}(\alpha+i)\mathfrak{r}[\hat{\Theta}v]^{-(\alpha+n+1)}\mathfrak{r}[D\hat{\Theta}(v;u-v)]
\end{aligned}$$

$$\times\left(\prod_{i=1}^{n}\mathfrak{r}[D^{|I_i|}\hat{\Theta}(v;\varphi_{I_i})]-\prod_{i=1}^{n}\mathfrak{r}[D^{|I_i|}\hat{\Theta}(u;\varphi_{I_i})]\right)$$
$$-\psi\sum_{n=2}^{j}\sum_{\{I_1,\dots,I_n\}\in Z_{j,n}}(-1)^n\left(\prod_{i=0}^{n-1}(\alpha+i)\right)\mathfrak{r}[\hat{\Theta}v]^{-(\alpha+n)}$$
$$\times\left(\prod_{i=1}^{n}\mathfrak{r}[D^{|I_i|}\hat{\Theta}(u;\varphi_{I_i})]-\prod_{i=1}^{n}\mathfrak{r}[D^{|I_i|}\hat{\Theta}(v;\varphi_{I_i})]\right.$$
$$\left.-\sum_{i=1}^{n}\prod_{i'\neq i}\mathfrak{r}[D^{|I_i|}\hat{\Theta}(v;\varphi_{I_{i'}})]\mathfrak{r}[D^{1+|I_i|}\hat{\Theta}(v;\varphi_{I_{i'}},u-v)]\right).$$

By (18) and (19) and Lemma 18, the $P^{\ell-2,\beta}$ norm of all terms but the first on the right-hand side is bounded by $C|u-v|^2_{C^{\ell,\beta}}\prod_{i=1}^{j}|\varphi_i|_{C^{\ell,\beta}}$. Proposition 16 implies that the estimate (19) holds for $i=j$.

The estimate (18) for $i=j+1$ follows somewhat more straightforwardly: We have for $G=D^{j+1}\hat{\Theta}(u;\varphi_1,\dots,\varphi_{j+1})-D^{j+1}\hat{\Theta}(v;\varphi_1,\dots,\varphi_{j+1})$,

$$\frac{\partial G}{\partial t}=\frac{\alpha\psi}{\mathfrak{r}[\hat{\Theta}v]^{1+\alpha}}\mathfrak{r}[G]+\alpha\psi\mathfrak{r}[D^{j+1}\hat{\Theta}(u;\varphi_1,\dots,\varphi_{j+1})]$$
$$\times\left(\frac{1}{\mathfrak{r}[\hat{\Theta}u]^{1+\alpha}}-\frac{1}{\mathfrak{r}[\hat{\Theta}v]^{1+\alpha}}\right)$$
$$-\psi\sum_{n=2}^{j+1}\sum_{\{I_1,\dots,I_n\}\in Z_{j,n}}(-1)^n\prod_{i=0}^{n-1}(\alpha+n)$$
$$\times\left(\frac{\prod_{i=1}^{n}\mathfrak{r}[D^{|I_i|}\hat{\Theta}(u;\varphi_{I_i})]}{\mathfrak{r}[\hat{\Theta}u]^{\alpha+n}}-\frac{\prod_{i=1}^{n}\mathfrak{r}[D^{|I_i|}\hat{\Theta}(v;\varphi_{I_i})]}{\mathfrak{r}[\hat{\Theta}v]^{\alpha+n}}\right).$$

The terms other than the first on the right-hand side have $P^{\ell-2,\beta}$ norm bounded by $C|u-v|_{C^{\ell,\beta}}\prod_{n=1}^{j+1}|\varphi_n|_{C^{\ell,\beta}}$, by the boundedness estimates in Lemma 18 and the Lipschitz estimates of Equation (18) for $i=0,\dots,j$. Proposition 16 implies the result. □

By induction on j, $\hat{\Theta}$ is a smooth map from $\mathcal{K}^{\ell,\beta}_{M,T,r,R}$ to $\mathcal{P}^{\ell,\beta}$, since Lemma 19 provides the result for $j=0$, and Lemma 20 provides the iteration step. The result of Proposition 15 follows, since Θ is obtained from $\hat{\Theta}$ by smooth rescaling and smooth reparametrisation of time. □

Theorem 1. *There exists a symmetric solution $s:(-\infty,T)\times S^1\to\mathbb{R}$ of* *equation* (6) *which converges in* C^∞ *to* σ *as* $\tau \to -\infty$ *and such that the* C^0 *distance of* s_τ *from* σ *is no less than* $\min\{C(\omega), e^{\omega\tau}\}$ *for some constant* $C(\omega) > 0$ *and any* $\omega \in (0, 1 - \alpha\lambda_{\text{sym}})$.

Proof. Let φ be the symmetric eigenfunction of $\mathcal{L}_\sigma$ corresponding to the eigenvalue λ_{sym}, and define for $\varepsilon > 0$ sufficiently small and $\tau \geq \log\varepsilon$

$$s_{\tau,\varepsilon} = \Theta_{\tau-\log\varepsilon}\left(\sqrt{\frac{\pi}{A[\sigma + \varepsilon^{1-\alpha\lambda_{\text{sym}}}\varphi]}}\left(\sigma + \varepsilon^{1-\alpha\lambda_{\text{sym}}}\varphi\right)\right).$$

Taking $\omega = 1 - \alpha\lambda_{\text{sym}}$, we have

$$\begin{aligned}
&\frac{\partial}{\partial\tau}\left(s_{\tau,\varepsilon} - \sigma - e^{\omega\tau}\varphi\right) \\
&= -\psi\mathfrak{r}[s_{\tau,\varepsilon}]^{-\alpha} + \frac{s_{\tau,\varepsilon}}{2\pi}\int_{S^1}\psi\mathfrak{r}[s_{\tau,\varepsilon}]^{1-\alpha}d\theta - \omega e^{\omega\tau}\varphi \\
&= (\alpha\mathcal{L} + 1)\left(s_{\tau,\varepsilon} - \sigma - e^{\omega\tau}\varphi\right) \\
&\quad - \frac{\alpha(1+\alpha)}{2}\left(\frac{\mathfrak{r}[\sigma]}{\mathfrak{r}_*}\right)^{1+\alpha}\left(\frac{\mathfrak{r}[s_{\tau,\varepsilon} - \sigma]}{\mathfrak{r}[\sigma]}\right)^2 \\
&\quad + \frac{(1-\alpha)s_{\tau,\varepsilon}}{4\pi}\int_{S^1}(s_{\tau,\varepsilon} - \sigma)\mathfrak{r}[s_{\tau,\varepsilon} - \sigma]d\theta \\
&\quad - \frac{s_{\tau,\varepsilon}\alpha(1-\alpha)}{4\pi}\int_{S^1}\sigma\mathfrak{r}[\sigma]\left(\frac{\mathfrak{r}[\sigma]}{\mathfrak{r}_*}\right)^{1+\alpha}\left(\frac{\mathfrak{r}[s_{\tau,\varepsilon} - \sigma]}{\mathfrak{r}[\sigma]}\right)^2 d\theta
\end{aligned}$$

Therefore

$$\left|\left(\frac{\partial}{\partial\tau} - \alpha\mathcal{L} - 1\right)\left(s_{\tau,\varepsilon} - \sigma - e^{\omega\tau}\varphi\right)\right| \leq C\|s_{\tau,\varepsilon} - \sigma - e^{\omega\tau}\varphi\|^2_{C^2(S^1)} + Ce^{2\omega\tau}.$$

We write $x_{\tau,\varepsilon} = \sup_{S^1}|s_{\tau,\varepsilon} - \sigma - e^{\omega\tau}\varphi|$. The Gagliardo-Nirenberg interpolation inequalities give for any $\ell > 2$

$$\begin{aligned}
\|s_{\tau,\varepsilon} - \sigma - e^{\omega\tau}\varphi\|_{C^2(S^1)} &\leq C(x + x^{1-2/\ell}\|s_{\tau,\varepsilon} - \sigma - e^{\omega\tau}\varphi\|_{C^\ell(S^1)}) \\
&\leq C_\ell x^{1-2/\ell},
\end{aligned}$$

since Theorem II2.5 of [5] gives bounds on all higher derivatives of $s_{\tau,\varepsilon}$. For our purposes it suffices to take $\ell > 4$, say $\ell = 8$. By the maximum principle,

$$\frac{\partial}{\partial\tau}x_{\tau,\varepsilon} \leq x_{\tau,\varepsilon} + Cx^{3/2}_{\tau,\varepsilon} + Ce^{2\omega\tau}, \tag{20}$$

and we have for each ε that

$$s_{\tau,\varepsilon} - \sigma - e^{\omega\tau}\varphi|_{\tau=\log\varepsilon} = \left(\sqrt{\frac{\pi}{A[\sigma + \varepsilon^{\omega}\varphi]}} - 1\right)(\sigma + \varepsilon^{\omega}\varphi).$$

Now $A[\sigma+\varepsilon^{\omega}\varphi] = \pi + \varepsilon^{\omega}\int_{S^1}\varphi\mathfrak{r}[\sigma]d\theta + \frac{1}{2}\varepsilon^{2\omega}\int_{S^1}\varphi\mathfrak{r}[\varphi]d\theta$, and $\int_{S^1}\varphi\mathfrak{r}[\sigma]d\theta = 0$, so we have

$$x_{\log\varepsilon,\varepsilon} \leq C\varepsilon^{2\omega}. \tag{21}$$

It follows from (20) and (21) that

$$x_{\tau,\varepsilon} \leq Ce^{2\omega\tau}$$

for $\log\varepsilon \leq \tau \leq \tau_0$ where C and τ_0 are independent of ε for ε sufficiently small. It follows in particular that $s_{\tau,\varepsilon}$ is bounded away from σ, independent of ε, and that the functions $s_{\tau,\varepsilon}$ converge in C^{∞} to a limit s_{τ} as $\varepsilon \to 0$, which is defined for all $\tau \in (-\infty, \tau_0]$, and satisfies (6). By estimate (21), the Hausdorff distance of s_{τ} from σ approaches zero as $\tau \to -\infty$. C^{∞} convergence follows from the regularity bounds of Theorem II2.5 in [5], together with interpolation inequalities. The monotonicity of the entropy $\mathcal{Z}$ implies that the Hausdorff distance from s_{τ} to σ is bounded away from zero for large times. □

Now we proceed to the proof of the global instability result (Theorem 2).

Define $U = \left\{s \in \mathcal{K}^0 : \sup_{0\leq\tau\leq T(s)} I[\Theta_{\tau}s] = \infty\right\}$. We aim to show that $U \cap \mathcal{K}^{\ell,\beta}$ is a generic set in $\mathcal{K}^{\ell,\beta}$ for every integer $\ell \geq 0$ and $\beta \in (0,1]$. We define for any $I_0 > 0$ and $Z_0 > 0$

$$U_{I_0,Z_0} = \left\{s \in \mathcal{K}^0 : \sup_{0\leq\tau<T(s)} I(\Theta_{\tau}s) > I_0 \quad \text{or} \quad \inf_{0<\tau<T(s)} \mathcal{Z}(\Theta_{\tau}s) < Z_0\right\}.$$

Then $\cap_{i=1}^{\infty}U_{i,1/i} = U$: The intersection certainly contains U; conversely, if s is not in U, then $I[\Theta_{\tau}s]$ remains bounded, and by Lemma II2.4 of [5], the radius of curvature remains bounded, so $\mathcal{Z}$ remains bounded below by a positive constant, and s_0 is not in $\cap_{i=1}^{\infty}U_{i,1/i}$. We will prove U_{I_0,Z_0} is open and dense in $\mathcal{K}^{\ell,\beta}$, by showing openness in $\mathcal{K}^0$ and density in $\mathcal{K}^{\infty}$. This implies U is generic in $\mathcal{K}^{\ell,\beta}$.

The openness of U_{I_0,Z_0} follows immediately from Proposition 2, which shows that both $I[\Theta_{\tau}s_0]$ and $Z[\Theta_{\tau}s_0]$ are continuous on $\{(s_0,\tau) : s_0 \in \mathcal{K}^0,\ 0 < \tau < T(s_0)\}$.

It remains to show density in $\mathcal{K}^\infty$. The first step is to show that a small perturbation of the initial conditions can be made to avoid a given stationary solution:

Proposition 21. *Assume $0 < \alpha \le 1/3$, with ψ non-constant if $\alpha = 1/3$. If σ is a symmetric stationary solution, and s_τ is a solution of Eq. (6) which converges to σ as $\tau \to \infty$, then in any neighbourhood of s_0 in C^∞ there exists $\tilde{s}_0$ such that the solution $\tilde{s}_\tau$ of (6) with initial condition $\tilde{s}_0$ has $\inf_\tau \mathcal{Z}[\tilde{s}_\tau] < \mathcal{Z}[\sigma] - \varepsilon$, where ε depends only on ψ, α, and $\mathcal{I}[\sigma]$.*

Proof. In the proof we will make use of the Hilbert spaces H^1_σ and H^2_σ which are the completions of $C^\infty(S^1)$ with respect to the inner products

$$\langle f, g\rangle_{H^1_\sigma} = 2\langle f, g\rangle_{L^2_\sigma} - \langle f, \mathcal{L}g\rangle_{L^2_\sigma}$$

and

$$\langle f, g\rangle_{H^2_\sigma} = 2\langle f, g\rangle_{L^2_\sigma} + \langle \mathcal{L}f, \mathcal{L}g\rangle_{L^2_\sigma}.$$

These inner products define norms equivalent to the usual H^1 and H^2 norms.

The plan of the proof is as follows: First, we show that for sufficiently large times τ, the solution of Eq. (6) given by $\{\Theta_{\tau'}(\Theta_\tau(s) + \delta v)\}_{\tau' \ge 0}$ eventually has $\mathcal{Z} < \mathcal{Z}[\sigma] - \varepsilon$, whenever δ is sufficiently small and v is sufficiently close in angle in H^2_σ to an eigenfuction of $\mathcal{L}_\sigma$ with eigenfunction λ satisfying $1 - \alpha\lambda > 0$. Then we show that for any given $v \in H^2_\sigma$ we can find $\tilde{s}_0$ arbitrarily close to s_0 such that $\Theta_\tau \tilde{s}_0 = \Theta_\tau(s) + \delta\tilde{v}$, with δ as small as desired, and $\tilde{v}$ as close as desired to v in H^2_σ.

Proposition 22. *Let σ be a stationary solution of (6), and suppose φ satisfies $\mathcal{L}_\sigma\varphi + \lambda\varphi = 0$, with $1 - \alpha\lambda > 0$. Let $s \in \mathcal{K}^0$ be such that $\Theta_t s \to \sigma$ in C^∞ as $t \to \infty$. Then there exist $\delta > 0$ and $\varepsilon > 0$ depending only on λ, α, ψ and $\mathcal{I}[\sigma]$, and a time $t_0 \ge 0$ such that for any $t \ge t_0$ and any $u \in \Theta_1(\mathcal{K}^0)$ satisfying*

$$\|u - \Theta_t s\|_{L^2_\sigma} \le \delta \quad \text{and} \quad \langle u - \Theta_t s, \varphi\rangle_{H^2_\sigma} \ge (1-\delta)\|u - \Theta_t s\|_{H^2_\sigma}\|\varphi\|_{H^2_\sigma}$$

the following holds: If T is the maximal time of existence for Eq. (6) with initial condition u, then

$$\lim_{\tau \to T} \mathcal{Z}[\theta_\tau u] \le \mathcal{Z}[\sigma] - \varepsilon.$$

Proof. For simplicity the argument will be given only in the symmetric case, so that σ, s, u, φ and ψ are invariant under the involution $\theta \to \theta + \pi$ on S^1. The general case is similar but somewhat more involved.

The assumptions of the Proposition imply in particular the following estimate, where we write $s_t = \Theta_t s$:

$$\frac{\|\mathcal{L}(u - s_t)\|^2}{\|u - s_t\|^2} \le \frac{\langle u - s_t, \varphi\rangle^2_{H^2_\sigma}}{\|u - s_t\|^2_{L^2_\sigma}} - 2 \le \lambda^2 + \delta\frac{(2+\lambda^2)(2-\delta)}{(1-\delta)^2}. \tag{22}$$

Thus the quotient on the left can be made less than $\frac{1}{2}(\lambda^2 + \frac{1}{\alpha^2})$ by choosing $\delta > 0$ sufficiently small (depending only on λ). The key step of the proof is to show that this quotient remains controlled (in particular, strictly less than $1/\alpha^2$) as long as $\Theta_\tau u$ remains close to $\Theta_{\tau+t} s$. This will follow by estimating the evolution equation for the quantity

$$\mathcal{Q}(\tau) = \frac{\|\mathcal{L}(\Theta_\tau u - \Theta_{t+\tau} s)\|^2}{\|\Theta_\tau u - \Theta_{t+\tau} s\|^2}.$$

We first compute an evolution equation for $\Theta_\tau u - \Theta_{t+\tau} s$:

$$\begin{aligned}
&\frac{d}{d\tau}(\Theta_\tau u - \Theta_{t+\tau} s)\\
&= -\psi\mathfrak{r}[\Theta_\tau u]^{-\alpha} + \psi\mathfrak{r}[\Theta_{t+\tau} s]^{-\alpha} + \Theta_\tau u\tilde{\mathcal{Z}}[\Theta_\tau u] - \Theta_{t+\tau} s\tilde{\mathcal{Z}}[\Theta_{t+\tau} s]\\
&= \alpha\psi\mathfrak{r}_*^{-(1+\alpha)}\mathfrak{r}[\Theta_\tau u - \Theta_{t+\tau} s] + (\Theta_\tau u - \Theta_{t+\tau} s)\,\tilde{\mathcal{Z}}[\Theta_\tau u]\\
&\quad + \Theta_{t+\tau} s\left(\tilde{\mathcal{Z}}[\Theta_\tau u] - \tilde{\mathcal{Z}}[\Theta_{t+\tau} s]\right)\\
&= \alpha\mathcal{Z}[\sigma]\mathcal{L}[\Theta_\tau u - \Theta_{t+\tau} s] + (\Theta_\tau u - \Theta_{t+\tau} s)\,\tilde{\mathcal{Z}}[\Theta_\tau u]\\
&\quad - \alpha(1+\alpha)\psi\tilde{\mathfrak{r}}_*^{-(2+\alpha)}\mathfrak{r}[\Theta_\tau u - \Theta_{t+\tau} s](\mathfrak{r}_* - \mathfrak{r}[\sigma])\\
&\quad + \Theta_{t+\tau} s\left(\tilde{\mathcal{Z}}[\Theta_\tau u] - \tilde{\mathcal{Z}}[\Theta_{t+\tau} s]\right)
\end{aligned}$$

where $\mathfrak{r}_*$ and $\tilde{\mathfrak{r}}_*$ are defined by $\mathfrak{r}_*^{-(1+\alpha)} = \int_0^1 \mathfrak{r}[\xi\Theta_\tau u + (1-\xi)\Theta_{t+\tau} s]^{-(1+\alpha)} d\xi$ and $\tilde{\mathfrak{r}}_*^{-(2+\alpha)} = \int_0^1 (\xi\mathfrak{r}_* + (1-\xi)\mathfrak{r}[\sigma])^{-(2+\alpha)}\, d\xi$. This implies the following evolution equation for $\mathcal{Q}$:

(23)

$$\begin{aligned}
&\frac{d}{d\tau}\mathcal{Q}\\
&= \frac{2\langle \mathcal{L}^2(\Theta_\tau u - \Theta_{t+\tau} s) - \mathcal{Q}(\Theta_\tau u - \Theta_{t+\tau} s), \frac{d}{d\tau}(\Theta_\tau u - \Theta_{t+\tau} s)\rangle}{\|\Theta_\tau u - \Theta_{t+\tau} s\|^2}
\end{aligned}$$

$$
\begin{aligned}
&= 2\alpha\tilde{\mathcal{Z}}[\sigma]\frac{\langle\left(\mathcal{L}^2-\mathcal{Q}\right)\left(\Theta_\tau u-\Theta_{t+\tau}s\right),\mathcal{L}\left(\Theta_\tau u-\Theta_{t+\tau}s\right)\rangle}{\|\Theta_\tau u-\Theta_{t+\tau}s\|^2}\\
&+2\left(\tilde{\mathcal{Z}}[\Theta_\tau u]-\tilde{\mathcal{Z}}[\Theta_{t+\tau}s]\right)\frac{\langle\left(\mathcal{L}^2-\mathcal{Q}\right)\left(\Theta_\tau u-\Theta_{t+\tau}s\right),\Theta_{t+\tau}s\rangle}{\|\Theta_\tau u-\Theta_{t+\tau}s\|^2}\\
&-\alpha(1+\alpha)\frac{\langle\left(\mathcal{L}^2-\mathcal{Q}\right)\left(\Theta_\tau u-\Theta_{t+\tau}s\right),\psi\tilde{\mathfrak{r}}_*^{-(2+\alpha)}\mathfrak{r}[\Theta_\tau u-\Theta_{t+\tau}s](\mathfrak{r}_*-\mathfrak{r}[\sigma])\rangle}{\|\Theta_\tau u-\Theta_{t+\tau}s\|^2}.
\end{aligned}
$$

We consider the three terms on the right separately. The first term is essentially negative:

Lemma 23. *For any $f \in C^\infty(S^1)$ with $\langle f,\sigma\rangle = 0$ we have*

$$\|f\|^2\langle\mathcal{L}^2 f,\mathcal{L}f\rangle \le \|\mathcal{L}f\|^2\langle f,\mathcal{L}f\rangle.$$

Proof of Lemma. We can write $f=\sum_i f_i\varphi_i$ where $\{\varphi_i\}$ is an orthonormal basis for L^2_σ with $\mathcal{L}\varphi_i+\lambda_i\varphi_i=0$ and $\lambda_i\ge 0$. Then

$$\|f\|^2\langle\mathcal{L}^2 f,\mathcal{L}f\rangle-\|\mathcal{L}f\|^2\langle f,\mathcal{L}f\rangle=-\frac{1}{2}\sum_{i,j}(\lambda_i+\lambda_j)(\lambda_i-\lambda_j)^2 f_i^2 f_j^2\le 0,$$

and equality holds precisely when f is an eigenfunction of $\mathcal{L}$. □

In our case, the function $f=\Theta_\tau u-\Theta_{t+\tau}s$ need not be orthogonal to σ, but it is approximately so: We have

$$
\begin{aligned}
f_\sigma=\langle f,\sigma\rangle &= \langle\Theta_\tau u-\Theta_{t+\tau}s,\mathcal{L}[\sigma]\rangle\\
&= \langle\Theta_\tau u-\Theta_{t+\tau}s,\mathcal{L}[\sigma-\Theta_{t+\tau}s]\rangle\\
&\quad+\langle\Theta_\tau u-\Theta_{t+\tau}s,\mathcal{L}[\Theta_{t+\tau}s]\rangle\\
&= \langle\Theta_\tau u-\Theta_{t+\tau}s,\mathcal{L}[\sigma-\Theta_{t+\tau}s]\rangle\\
&\quad-\frac{1}{2}\langle\Theta_\tau u-\Theta_{t+\tau}s,\mathcal{L}[\Theta_\tau u-\Theta_{t+\tau}s]\rangle,
\end{aligned}
$$

since $\langle\Theta_\tau u,\mathcal{L}[\Theta_\tau u]\rangle=\langle\Theta_{t+\tau}s,\mathcal{L}[\Theta_{t+\tau}s]\rangle=2\pi$. It follows that

$$|f_\sigma|\le C\left(|\Theta_{t+\tau}s-\sigma|_{C^2}\|f\|+\mathcal{Q}^{1/2}\|f\|^2\right). \tag{24}$$

Writing $f=f_\sigma\sigma+f_\perp$ with $\langle f_\perp,\sigma\rangle=0$, we find

$$\|f\|^2\langle\mathcal{L}^2 f,\mathcal{L}f\rangle-\|\mathcal{L}f\|^2\langle f,\mathcal{L}f\rangle$$

$$
\begin{aligned}
&= \frac{\|f\|^2\left(\|f_\perp\|^2\langle\mathcal{L}^2 f_\perp, \mathcal{L}f_\perp\rangle - \|\mathcal{L}f_\perp\|^2\langle f_\perp, \mathcal{L}f_\perp\rangle\right)}{\|f_\perp\|^2} \\
&\quad + 2\pi f_\sigma^2\left(1 + \frac{\|\mathcal{L}f_\perp\|^2}{\|f_\perp\|^2}\right)\langle f_\perp, \mathcal{L}f_\perp\rangle \\
&\quad + 2\pi f_\sigma^2\left(\|f_\perp\|^2 - \|\mathcal{L}f_\perp\|^2\right) \\
&\le 2\pi f_\sigma^2\|f\|^2.
\end{aligned}
$$

In the second term on the right in (23), we can estimate as follows:

$$
\begin{aligned}
\left|\tilde{\mathcal{Z}}[\Theta_\tau u] - \tilde{\mathcal{Z}}[\Theta_{t+\tau}s]\right| &= \left|\int \psi\left(\mathfrak{r}[\Theta_\tau u]^{1-\alpha} - \mathfrak{r}[\Theta_{t+\tau}s]^{1-\alpha}\right)d\theta\right| \\
&= \left|(1-\alpha)\int \psi\hat{\mathfrak{r}}_*^{-\alpha}\mathfrak{r}[f]d\theta\right| \\
&\le C\mathcal{Q}^{1/2}\|f\|,
\end{aligned}
$$

where $\hat{\mathfrak{r}}_*^{-\alpha} = \int_0^1 \mathfrak{r}[\xi\Theta_\tau u + (1-\xi)\Theta_{t+\tau}s]^{-\alpha}d\theta$. We also have

$$
\frac{\langle(\mathcal{L}^2 - \mathcal{Q})f, \Theta_{t+\tau}s\rangle}{\|f\|^2} = \frac{\langle f, (\mathcal{L}^2 - \mathcal{Q})(\Theta_{t+\tau}s - \sigma)\rangle + (1-\mathcal{Q})f_\sigma}{\|f\|^2}.
$$

Combining these, we find

$$
\begin{aligned}
&\left|\left(\tilde{\mathcal{Z}}[\Theta_\tau u] - \tilde{\mathcal{Z}}[\Theta_{t+\tau}s]\right)\frac{\langle(\mathcal{L}^2 - \mathcal{Q})(\Theta_\tau u - \Theta_{t+\tau}s), \Theta_{t+\tau}s\rangle}{\|\Theta_\tau u - \Theta_{t+\tau}s\|^2}\right| \\
&\qquad\le C(1 + \mathcal{Q}^{3/2})\left(|\Theta_{t+\tau}s - \sigma|_{C^4} + \|f\|\right).
\end{aligned}
$$

Finally, the third term in (23) can be estimated as follows:

$$
\begin{aligned}
&\left|\left\langle(\mathcal{L}^2 - \mathcal{Q})(\Theta_\tau u - \Theta_{t+\tau}s), \psi\tilde{\mathfrak{r}}_*^{-(2+\alpha)}\mathfrak{r}[\Theta_\tau u - \Theta_{t+\tau}s](\mathfrak{r}_* - \mathfrak{r}[\sigma])\right\rangle\right| \\
&\qquad\le C\mathcal{Q}\|f\|^2\sup|\mathfrak{r}_* - \mathfrak{r}[\sigma]| \\
&\qquad\quad - C|\mathfrak{r}_* - \mathfrak{r}[\sigma]|_{C^1}\left(\langle\mathcal{L}^2 f, \mathcal{L}f\rangle - \mathcal{Q}\langle\mathcal{L}f, f\rangle\right),
\end{aligned}
$$

and we note $|\mathfrak{r}_* - \mathfrak{r}[\sigma]|_{C^1} \le C(|f|_{C^3} + |\Theta_{t+\tau}s - \sigma|_{C^3}) \le C(\|f\|^{1/2} + |\Theta_{t+\tau}s - \sigma|_{C^3})$ by the Gagliardo-Nirenberg inequality and the bounds on all derivatives of σ, $\Theta_\tau u$ and $\Theta_{t+\tau}s$ supplied by Theorem II2.5 in [5]. This gives (assuming that $\|f\|$ and $\|s - \sigma\|$ are bounded by some constant depending on α, ψ and $\mathcal{I}[\sigma]$)

(25)

$$
\frac{d}{d\tau}\mathcal{Q} \le \frac{1}{2\|f\|^2}\left(\langle\mathcal{L}^2 f, \mathcal{L}f\rangle - \mathcal{Q}\langle\mathcal{L}f, f\rangle\right) + C\left(\|f\|^{1/2} + |\Theta_{t+\tau}s - \sigma|_{C^3}\right).
$$

To make use of this we need to make better use of the non-positive leading term. We noted in the previous Lemma that this term is strictly negative unless f is an eigenfunction. The following result makes this more quantitative:

Lemma 24. *Suppose that there exist constants $a < b$ such that there are no eigenvalues of $\mathcal{L}$ in the interval (a,b). If $f \perp \sigma$ and $\|\mathcal{L}f\|^2 = q\|f\|^2$ with $q \in (a^2, b^2)$, then*

$$\langle \mathcal{L}^2 f, \mathcal{L}f\rangle - q\langle \mathcal{L}f, f\rangle \leq -\frac{1}{2}(b-a)(q-a^2)(1-q/b^2)^2\|f\|^2.$$

Proof of Lemma. Choose an orthonormal basis $\{\varphi_i\}_{i\geq 0}$ for L^2_σ consisting of eigenfunctions of $\mathcal{L}$, where $\varphi_0 = \sigma$ and λ_i is nondecreasing in i. Fix N such that $\lambda_i \leq a$ for $i \leq N$, and $\lambda_i \geq b$ for $i \geq N+1$. From the proof of Lemma 23, if $f = \sum_i f_i\varphi_i$,

$$\begin{aligned}(26)\qquad -\|f\|^2\langle \mathcal{L}^2 f, \mathcal{L}f\rangle + \|\mathcal{L}f\|^2\langle \mathcal{L}f, f\rangle &= \frac{1}{2}\sum_{i,j} f_i^2 f_j^2(\lambda_i+\lambda_j)(\lambda_i-\lambda_j)^2 \\ &\geq \sum_{\substack{1\leq i\leq N\\ N+1\leq j<\infty}} f_i^2 f_j^2 \lambda_j(\lambda_j-a)^2.\end{aligned}$$

The assumptions of the Lemma give us two useful inequalities:

$$a^2\sum_{i=1}^N f_i^2 + \sum_{i=N+1}^\infty \lambda_i^2 f_i^2 \geq \|\mathcal{L}f\|^2 = q\|f\|^2 = q\sum_{i=1}^N f_i^2 + q\sum_{i=N+1}^\infty f_i^2$$

so that

$$(27)\qquad (q-a^2)\sum_{i=1}^N f_i^2 \leq \sum_{i=N+1}^\infty (\lambda_i^2 - q) f_i^2,$$

and

$$b^2\sum_{i=N+1}^\infty f_i^2 \leq \|\mathcal{L}f\|^2 = q\|f\|^2 = q\sum_{i=1}^N f_i^2 + q\sum_{i=N+1}^\infty f_i^2$$

so that

$$(28)\qquad \sum_{i=N+1}^\infty f_i^2 \leq \frac{q}{b^2-q}\sum_{i=1}^N f_i^2.$$

We apply (27) on the right-hand side of (26) after noting that $\lambda_j(\lambda_j - a)^2 \geq (b-a)\lambda_j(\lambda_j - a) \geq \frac{1}{2}(b-a)(\lambda_j^2 - a^2) \geq \frac{1}{2}(b-a)(\lambda_j^2 - q)$, yielding

$$-\|f\|^2\langle \mathcal{L}^2 f, \mathcal{L}f\rangle + \|\mathcal{L}f\|^2\langle \mathcal{L}f, f\rangle \geq \frac{1}{2}(b-a)(q-a^2)\left(\sum_{i=1}^N f_i^2\right)^2.$$

Now (28) implies $\sum_{i=1}^N f_i^2 \geq (1 - q/b^2)\|f\|^2$, so that

$$-\|f\|^2\langle \mathcal{L}^2 f, \mathcal{L}f\rangle + \|\mathcal{L}f\|^2\langle \mathcal{L}f, f\rangle \geq \frac{1}{2}(b-a)(q-a^2)(1-q/b^2)^2\|f\|^4.$$

The Lemma follows after dividing through by $\|f\|^2$. □

To make use of this, we must show that there is a suitable gap between the eigenvalues of $\mathcal{L}$.

Lemma 25. *There exists a constant N depending only on λ, $\mathcal{I}[\sigma]$, α and ψ such that at most N eigenfunctions of $\mathcal{L}$ have eigenvalues less than $\sqrt{\frac{1}{2}\lambda(\lambda + 1/\alpha)}$.*

Proof. The eigenfunction equation can be written as follows:

$$\varphi_{\theta\theta} + \left(1 + \lambda_i \frac{\mathfrak{r}[\sigma]}{\sigma}\right)\varphi = 0.$$

The Sturm comparison theorem gives a bound on the number of zeroes of φ in terms of λ_i: If $A_- \leq \mathfrak{r}[\sigma]/\sigma \leq A_+$, then the distance between zeroes is bounded between $\pi/\sqrt{1 + \lambda_i A_+}$ and $\pi/\sqrt{1 + \lambda_i A_-}$, so the number of zeroes is no greater than $2\sqrt{1 + \lambda_i A_+}$. However the number of zeroes of φ_i is precisely $2\lceil i/2 \rceil$ (see for example Theorem 3.1 in Chapter 8 of [10]). Therefore the number of eigenfunctions with $\lambda_i \leq c$ is no greater than $1 + 2\sqrt{1 + cA_+}$, for any $c > 0$. □

Corollary 26. *There exists $\delta_1 > 0$ depending only on ψ, α, λ and $\mathcal{I}[\sigma]$, and positive constants a and b with*

$$\lambda^2 \leq a^2 < b^2 \leq \frac{1}{2}\lambda\left(\lambda + \frac{1}{\alpha}\right)$$

such that $b \geq a + \delta_1$, and there are no eigenvalues of $\mathcal{L}$ in the interval (a, b).

Now we can complete the proof of Proposition 22: Let a and b be as in Corollary 26. We will prove that while $\Theta_\tau u$ and $\Theta_{t+\tau}s$ are close enough to σ, $\mathcal{Q}$ will never increase above $(a^2+b^2)/2$. Indeed, $q = \|\mathcal{L}f_\perp\|^2/\|f_\perp\|^2$ lies within $C(|\theta_{t+\tau}s-\sigma|^2_{C^2}+\|f\|^2)$ of $\mathcal{Q}$ (by Eq. (23)), so we can find $\delta'>0$ depending only on λ, α, ψ and $\mathcal{I}[\sigma]$ such that if $\mathcal{Q}=(a^2+b^2)/2$ and $\|\Theta_{t+\tau}s-\sigma\|+\|f\|\le\delta'$ then $(3a^2+b^2)/4\le q\le(a^2+3b^2)/4$, and Lemma 24 and Equation (23) imply that $d\mathcal{Q}/d\tau\le 0$, possibly after again reducing δ'.

For any $\delta''>0$ we can choose t sufficiently large to ensure $|\Theta_{t+\tau}s-\sigma|_{C^4}\le\delta''$ for every $\tau\ge 0$. By the estimate (22), we can ensure that $\mathcal{Q}(0)<(a^2+b^2)/2$ by decreasing δ if necessary. The estimate just established shows that this remains true as long as $\|\Theta_\tau u-\Theta_{t+\tau}s\|\le\delta'$.

Given these estimates, we can show that $\|\Theta_\tau u-\Theta_{t+\tau}s\|$ increases exponentially, and so eventually reaches δ': We compute

$$
\begin{aligned}
\frac{d}{d\tau}\|f\|^2 &= 2\langle f, -\psi\mathfrak{r}[\Theta u]^{-\alpha}+\psi\mathfrak{r}[\Theta s]^{-\alpha}+\Theta u\tilde{\mathcal{Z}}[\Theta u]-\Theta s\tilde{\mathcal{Z}}[\Theta s]\rangle\\
&\ge 2\alpha\tilde{\mathcal{Z}}[\sigma]\langle\mathcal{L}f,f\rangle+2\tilde{\mathcal{Z}}[\sigma]\|f\|^2\\
&\quad -C\left(\|f\|\,\|\mathcal{L}f\|\,|\Theta_{t+\tau}s-\sigma|_{C^4}+\|f\|\,\|\mathcal{L}f\|^2\right)\\
&\ge 2\tilde{\mathcal{Z}}[\sigma]\left(\alpha\langle\mathcal{L}f_\perp,f_\perp\rangle+\|f_\perp\|^2\right)\\
&\quad -C\left(\delta''\mathcal{Q}^{1/2}\|f_\perp\|^2+\mathcal{Q}\|f_\perp\|^3\right)\\
&\ge 2\tilde{\mathcal{Z}}[\sigma]\left(-\frac{\alpha}{\lambda}\|\mathcal{L}f_\perp\|^2+\|f_\perp\|^2\right)-C(\delta'+\delta'')\|f_\perp\|^2\\
&\ge \frac{1-\alpha\lambda}{2}\tilde{\mathcal{Z}}[\sigma]\|f\|^2
\end{aligned}
$$

after decreasing δ' and δ'' if necessary. It follows that $\|f\|$ increases until it reaches δ' for some $\tau'>0$. Then we compute

$$
\begin{aligned}
&\tilde{\mathcal{Z}}[\Theta_{\tau'}u]-\tilde{\mathcal{Z}}[\sigma]\\
&=\tilde{\mathcal{Z}}[\Theta_{t+\tau'}s]-\tilde{\mathcal{Z}}[\sigma]+\int\psi\left(\mathfrak{r}[\Theta_{\tau'}u]^{1-\alpha}-\mathfrak{r}[\Theta_{t+\tau'}s]^{1-\alpha}\right)\,d\theta\\
&\ge -\frac{1-\alpha}{2}\tilde{\mathcal{Z}}[\sigma]\left(\alpha\|\mathcal{L}f_\perp\|^2+\langle\mathcal{L}f_\perp,f_\perp\rangle\right)-C\delta'\|f_\perp\|^2-C(\delta'')^2\\
&\ge \frac{1-\alpha}{2}\tilde{\mathcal{Z}}[\sigma](\lambda-\alpha\mathcal{Q})\|f\|^2-C\delta'\|f\|^2-C(\delta'')^2\\
&\ge \frac{1-\alpha}{8}\tilde{\mathcal{Z}}[\sigma]\lambda(1-\alpha\lambda)(\delta')^2-C(\delta'')^2,
\end{aligned}
$$

again after decreasing δ' if necessary. Now choosing δ'' sufficiently small (that is, t sufficiently large), we obtain

$$\tilde{\mathcal{Z}}[\Theta_{T'}u] - \tilde{\mathcal{Z}}[\sigma] \geq \frac{1-\alpha}{16}\tilde{\mathcal{Z}}[\sigma]\lambda(1-\alpha\lambda)(\delta')^2.$$

This completes the proof of Proposition 22. $\square$

Remark. The assumption that $u \in \Theta_1(\mathcal{K}^0)$ was used only to guarantee that C^k estimates hold for u for each k, depending only on $\mathcal{I}[\sigma]$, α and ψ. The assumption is well suited to our intended applications, but can easily be weakened.

The result of Proposition 22 will be combined with the following result on the density of the range of the solution operator of a linear parabolic equation:

Lemma 27. *Fix $k \geq 2$. Let $\Theta_T : H^k(S^1) \to H^k(S^1)$ be the operator which takes $u(.,0)$ to $u(.,T)$, where $u(\theta,t)$ is the solution of*

$$\frac{\partial u}{\partial t} = au_{\theta\theta} + bu_\theta + cu$$

where a, b, and c are smooth on $S^1 \times [0,T]$, and $C_0^{-1} \leq a(\theta,t) \leq C_0$ for some positive constant C_0. Then Θ_T has image which is dense in H^k.

Proof. We use the H^k inner product

$$\langle f,g\rangle_{H^k} = \int_{S^1} D^k f D^k g + fg\,d\theta.$$

If the image of Θ_T is not dense, then there is some $f \in H^k(S^1)$ such that

$$\int_{S^1} f\Theta_T u\,d\theta = 0$$

for every $u \in H^k(S^1)$. Now let $\varphi : S^1 \times [0,T]$ be the unique solution of the backwards parabolic final-value problem

$$\frac{\partial\varphi}{\partial t} = -a\varphi_{\theta\theta} + (b-2a_\theta)\varphi_\theta + (b_\theta - c - a_{\theta\theta})\varphi$$
$$\varphi(\theta,T) = f(\theta) + (-1)^k\Delta^k f(\theta),$$

and let $\phi_t(\theta)$ be the unique solution of

$$(-1)^k\Delta^k\phi_t + \phi_t = \varphi(.,t)$$

for each $t \in [0,T]$. Note that $\varphi(.,t) \in H^{-k}$ for each t, and hence $\phi_t \in H^k$, with $|\phi_t|_{H^k} \le C|f|_{H^k}$. In particular, $\phi_T = f$. Furthermore for any $u \in H^k(S^1)$,

$$\begin{aligned}\frac{d}{dt}\langle \phi_t, \Theta_T u\rangle_{H^k} &= \langle au_{\theta\theta} + bu_\theta + cu, \varphi\rangle \\ &\qquad + u(-a\varphi_{\theta\theta} + (b - 2a_\theta)\varphi_\theta + (b_\theta - c - a_{\theta\theta})\varphi)\,d\theta \\ &= 0.\end{aligned}$$

Therefore

$$0 = \int_{S^1} f\Theta_T u\,d\theta = \int_{S^1} \phi_0 u\,d\theta,$$

for all $u \in H^k(S^1)$, so that ϕ_0 vanishes, and φ_0 vanishes. But then f must also vanish, because of the following statement on backwards uniqueness of solutions of a parabolic equations, which we apply to φ with time reversed:

Lemma 28. *If $v : S^1 \times [0,T]$ satisfies*

$$\frac{\partial v}{\partial t} = av_{\theta\theta} + bv_\theta + cv$$

and $\varphi(\theta,T) = 0$ for all θ, then $\varphi(\theta,0) = 0$ for all θ.

Proof. Suppose that $v(T) \equiv 0$, but $v(0)$ is not identically zero. Without loss of generality assume that T is the first time at which v vanishes. Consider the energy functional $E(t) = \int_{S^1} v^2$. This is smooth in t for $t > 0$. We have

$$\begin{aligned}\frac{d}{dt}E(t) &= \int_{S^1} 2v(av_{\theta\theta} + bv_\theta + cv)\,d\theta \\ &\le -2\int_{S^1} av_\theta^2\,d\theta + C_0E(t) \\ \frac{d^2}{dt^2}E(t) &\ge 4\int_{S^1} (av_{\theta\theta} + bv_\theta + cv)^2\,d\theta \\ &\qquad - C_1\int_{S^1} v_\theta^2\,d\theta - C_2\int_{S^1} v^2\,d\theta\end{aligned}$$

for some constants C_0, C_1, and C_2 depending on a, b, c and their derivatives. Thus on the interval $(0,T)$ we have the inequality

$$\frac{d^2}{dt^2}\ln E + A\frac{d}{dt}\ln E + B \ge 0$$

for some non-negative constants A and B. By the maximum principle this implies that for any t in $[0,T)$, $\ln E(t) \leq f(t)$, where $f(t)$ satisfies

$$\begin{aligned} &\frac{d^2 f}{dt^2} + A\frac{df}{dt} + B = 0 \\ &f(0) = \ln E(0), \quad f(T) = \ln E_1, \end{aligned}$$

for any $E_1 > 0$. Thus

$$\begin{aligned} f(t) &= (1 - t/T)\log E_0 + (t/T)\log E_1 + \frac{1}{2}Bt(t-T) \qquad \text{if } A = 0 \\ &= \left(\frac{e^{-At} - e^{-AT}}{1 - e^{-AT}}\right)\log E_0 + \left(\frac{1 - e^{-At}}{1 - e^{-AT}}\right)\log E_1 \\ &\quad + \frac{B}{A}\frac{\left(T - t - Te^{-At} + te^{-AT}\right)}{1 - e^{-AT}} \qquad \text{if } A \neq 0, \end{aligned}$$

where $E_0 = E(0)$. In either case we have

$$f(t) = \alpha(t)\log E_0 + (1 - \alpha(t))\log E_1 + Q(t)$$

where α is a smooth positive function on $(0,T)$ with values between 0 and 1, and Q is a smooth, bounded function of t. Therefore

$$E(t) \leq E_0^{\alpha(t)} E_1^{1-\alpha(t)} e^{Q(t)}.$$

Taking $E_1 \to 0$ gives $E(t) = 0$ for all $t \in (0,T)$, a contradiction. □

This completes the proof of Lemma 27. □

Now we complete the proof of Proposition 21: For any $\mu > 0$ and $k \geq 2$, we will find $\tilde{s}_0$ such that $|\tilde{s}_0 - s_0|_{H^k} < \mu$ such that the conclusion of the Proposition holds.

By Proposition 22, we can find $\delta > 0$ and $t_0 > 0$ such that the Proposition holds provided

$$\|\Theta_{t_0}\tilde{s}_0 - \Theta_{t_0}s_0\|_{L^2_\sigma} < \delta$$

and

$$\langle \Theta_{t_0}\tilde{s}_0 - \Theta_{t_0}s_0, \varphi \rangle_{H^2_\sigma} \geq (1-\delta)\|\Theta_{t_0}\tilde{s}_0 - \Theta_{t_0}s_0\|_{H^2_\sigma}\|\varphi\|_{H^2_\sigma},$$

where φ is the eigenfunction of $\mathcal{L}_\sigma$ with eigenvalue $\lambda < 3 \leq 1/\alpha$ guaranteed by Theorem 7.

Choose $t_1 > 0$ sufficiently small to ensure that $|\Theta_{t_0}s_0 - s_0|_{H^k} < \delta/2$. Then for every ℓ and β, $\Theta_{t_1}s_0 \in \mathcal{K}^{\ell,\beta}_{t_0,M,r,R}$ for some M, r and R. By Proposition 15, $\Theta_{t_0-t_1}$ is a C^∞ map from a $C^{\ell,\beta}$ neighbourhood of $\Theta_{t_1}s_0$ into $C^{\ell,\beta}$.

Lemma 27 implies that $D\hat{\Theta}$ has H^k-dense image for any k, and hence we can find v such that $D\Theta_{t_0-t_1}(\Theta_{t_1}s_0; v)$ is as close as desired in H^k to φ. It follows that $u = \Theta_{t_0-t_1}(\Theta_{t_1}s_0 + \delta_1 v)$ satisfies the requirements of Proposition 22 for δ_1 sufficiently small. We take $\tilde{s}_0 = \Theta_{t_1}s_0 + \delta_1 v$. Proposition 21 is proved. □

Now we will complete the proof of Theorem 2. It remains to show that for any initial $s_0 \in \mathcal{K}^{\ell,\beta}$ we can choose $\bar{s}$ as close as desired to s_0 in $\mathcal{K}^{\ell,\beta}$ with $\bar{s} \in U_{I_0,Z_0}$.

Fix $\delta_0 > 0$, and suppose that $s_0 \notin U_{I_0,Z_0}$ (otherwise we have nothing to prove).

Then $\mathcal{I}[\Theta_t s_0] \le I_0$ for all $t > 0$, so $\Theta_t s_0$ converges in C^∞ to a stationary solution σ_0 with $\mathcal{I}[\sigma_0] \le I_0$. By Proposition 21, we can choose $s_1 \in \mathcal{K}^{\ell,\beta}$ such that $|s_1 - s_0|_{C^{\ell,\beta}} < \delta_0/4$ and $\lim_{t\to T(s_1)} \mathcal{Z}[\Theta_t s_1] \le \lim_{t\to T(s_0)} \mathcal{Z}[\Theta_t s_0] - \varepsilon = \mathcal{Z}[\sigma_0] - \varepsilon$.

There are two possibilities: Either $s_1 \in U_{I_0,Z_0}$, or $\Theta_t s_1$ converges in C^∞ to a stationary solution σ_1 with $\mathcal{I}[\sigma_1] \le I_0$. In the first case we are done; in the second, we apply Proposition 21 again to find s_2 with $|s_2 - s_1|_{C^{\ell,\beta}} < \delta_0/8$ and $\lim_{t\to T(s_2)} \mathcal{Z}[\Theta_t s_2] \le \mathcal{Z}[\sigma_0] - 2\varepsilon$.

Proceeding by induction, we either eventually have $s_k \in U_{I_0,Z_0}$, or we have a sequence $\{s_k\}$ with $|s_{k+1} - s_k|_{C^{\ell,\beta}} < \delta_0/2^{k+2}$ and $\lim_{t\to T(s_k)} \mathcal{Z}[\Theta_t s_k] \le \mathcal{Z}[\sigma_0] - k\varepsilon$. But then for large k we have $\lim_{t\to T(s_k)} \mathcal{Z}[\Theta_t s_k] < Z_0$, and so $s_k \in U_{I_0,Z_0}$. Therefore $U_{I_0,Z_0} \cap \mathcal{K}^{\ell,\beta}$ is dense and open in $\mathcal{K}^{\ell,\beta}$ for every ℓ and β.

This completes the proof of Theorem 2.

6. Counterexamples in the non-symmetric case.

The condition of symmetry in the main result cannot be removed: Consider the case where the support function has the form $\sigma = 1 + \varepsilon\cos 3\theta$, with $\varepsilon < \frac{1}{8}$.

Proposition 29. *If $\sigma = 1 + \varepsilon\cos 3\theta$, then $\lambda = 3 + \frac{270}{7}\varepsilon^2 + O(\varepsilon^3)$ as $\varepsilon \to 0$.*

Proof. For $\varepsilon = 0$ the eigenfunctions are $\varphi_{2j-1} = \cos(j\theta)$ and $\varphi_{2j} = \sin(j\theta)$, with eigenvalues $\lambda_{2j-1} = \lambda_{2j} = j^2 - 1$ for $j > 0$. Note that for ε small the eigenvalues change only by a small amount, and in particular do not become too small: The eigenvalue $\lambda_j(\varepsilon)$ is given by

$$\sup\left\{\inf\left\{\frac{\int\varphi_\theta^2-\varphi^2d\theta}{\int\frac{\mathfrak{r}[\sigma_\varepsilon]}{\sigma_\varepsilon}\varphi^2d\theta}:\varphi\in V\right\}:\mathrm{codim}V=j\right\}$$

On the subspace generated by $\{\varphi_k : k\geq j\}$,

$$\begin{aligned}\frac{\int\varphi_\theta^2-\varphi^2d\theta}{\int\frac{\mathfrak{r}[\sigma_\varepsilon]}{\sigma_\varepsilon}\varphi^2d\theta}&\geq\frac{\int\varphi_\theta^2-\varphi^2d\theta}{\frac{1+8\varepsilon}{1-\varepsilon}\int\varphi^2d\theta}\\&\geq\frac{1-\varepsilon}{1+8\varepsilon}\lambda_j,\end{aligned}$$

and therefore $\lambda_{2j}(\varepsilon)\geq\lambda_{2j-1}(\varepsilon)\geq\frac{1-\varepsilon}{1+8\varepsilon}(j^2-1)$.

Let

$$f_1=\cos 2\theta+\frac{9}{2}\varepsilon\cos\theta-\frac{9}{14}\varepsilon\cos 5\theta-\frac{31}{3}\varepsilon^2\cos 4\theta-\frac{18}{35}\varepsilon^2\cos 8\theta$$

and

$$f_2=\sin 2\theta-\frac{9}{2}\varepsilon\sin\theta-\frac{9}{14}\sin 5\theta+\frac{31}{3}\varepsilon^2\sin 4\theta-\frac{18}{35}\varepsilon^2\sin 8\theta.$$

Then

$$\left(\mathcal{L}_\varepsilon+\left(3+\frac{270}{7}\varepsilon^2\right)\right)f_i=O(\varepsilon^3)\tag{29}$$

for ε sufficiently small, $i=1,2$. Write f_i in terms of the eigenfunctions $\varphi_k(\varepsilon)$ of $\mathcal{L}_\varepsilon$:

$$f_i=\sum_k a_{i,k}\varphi_k(\varepsilon).$$

Then equation (29) gives:

$$\sum_k a_{i,k}^2\left(3+\frac{270}{7}\varepsilon^2-\lambda_k(\varepsilon)\right)^2\leq C\varepsilon^6.$$

Therefore for each k,

$$a_{i,k}^2\left(3+\frac{270}{7}\varepsilon^2-\lambda_k\right)^2\leq C\varepsilon^6.\tag{30}$$

If $k=0,1,2$ then $\lambda_k=-1$ or 0, so that $\left(3+\frac{270}{7}\varepsilon^2-\lambda_k\right)^2\geq 4$ and therefore $a_{i,k}^2\leq\frac{1}{4}C\varepsilon^6$. If $k\geq 5$ then $\lambda_k\geq\frac{1-\varepsilon}{1+8\varepsilon}(\left[\frac{j+1}{2}\right]^2-1)$, and so

$\left(3+\frac{270}{7}\varepsilon^2-\lambda_k\right)^2 \geq C(j-4)^2$ for some C. Again $a_{i,k}^2 \leq \frac{C\varepsilon^6}{(j-4)^4}$. But $\sum a_{i,k}^2 = \langle f_i, f_i\rangle \geq C$, so it follows that $a_{i,3}^2 + a_{i,4}^2 \geq C - C\varepsilon^6 - \varepsilon^6 \sum_{k\geq 5} \frac{1}{(j-4)^4} \geq C - C\varepsilon^6$. Therefore the cases $k=3$ and $k=4$ of equation (30) in the two cases $i=1,2$ imply $(3+\frac{270}{7}\varepsilon^2-\lambda_3) \leq C\varepsilon^6$ and $(3+\frac{270}{7}\varepsilon^2-\lambda_4) \leq C\varepsilon^6$. □

Other kinds of asymmetry do not lead to large λ. In particular, translating a symmetric curve to place it off-centre yields a smaller value of λ:

Proposition 30. *Let γ be any smooth, strictly convex curve with support function s. Let Ω be the region enclosed by γ, and define a function $f:\Omega\to\mathbb{R}$ by taking $f(\xi)$ to be the reciprocal of the first positive eigenvalue of the operator $\mathcal{L}_\xi = \frac{s-\xi\cdot z}{\mathfrak{r}[s-\xi\cdot z]}\mathfrak{r}$. Then f is a strictly convex function on Ω. If s is symmetric, then f is symmetric and $f \geq \frac{1}{3}$, with equality if and only if γ is an ellipse.*

Proof. From the Rayleigh quotient characterisation of λ,

$$f(\xi) = \sup\left\{\frac{\int_{S^1}\frac{\mathfrak{r}[s]}{s-\xi\cdot z}\tilde{\varphi}^2\,d\theta}{\int_{S^1}(\tilde{\varphi}_\theta)^2-\tilde{\varphi}^2\,d\theta} : \varphi\in C^\infty(S^1), \varphi\notin E_{-1}\oplus E_0\right\},$$

since $\mathfrak{r}[s-\xi\cdot z]=\mathfrak{r}[s]$, and $E_{-1}\oplus E_0$ is independent of $\xi\in\Omega$. Here $\tilde{\varphi}$ is the component of φ orthogonal in $L^2_{s-\xi\cdot z}$ to $E_{-1}\oplus E_0$:

$$\tilde{\varphi} = \varphi - \frac{\langle\varphi, s-\xi\cdot z\rangle_\xi}{\|s-\xi\cdot z\|_\xi^2}(s-\xi\cdot z) - (M_\xi^{-1})^{ij}\langle\varphi, z_i\rangle_\xi z_j \tag{31}$$

where $\langle\phi_1,\phi_2\rangle_\xi = \int_{S^1}\frac{\mathfrak{r}[s]}{s-\xi\cdot z}\phi_1\phi_2\,d\theta$, and $\|.\|_\xi$ is the corresponding norm; $z=(z_1,z_2)=(\cos\theta,\sin\theta)$, and $(M_\xi)_{ij}=\langle z_i,z_j\rangle_\xi$ for $1\leq i,j\leq 2$. Thus f is a supremum of functions, each of which will be shown to be convex.

Take φ fixed in $C^\infty(S^1)\backslash E_{-1}\oplus E_0$, and define $f_\varphi:\Omega\to\mathbb{R}$ by

$$f_\varphi(\xi) = \frac{\int_{S^1}\frac{\mathfrak{r}[s]}{s-\xi\cdot z}\tilde{\varphi}^2\,d\theta}{\int_{S^1}(\tilde{\varphi}_\theta)^2-\tilde{\varphi}^2\,d\theta}$$

Note that the denominator is independent of ξ: In expression (31), $\langle\varphi, s-\xi\cdot z\rangle_\xi = \int_{S^1}\frac{\mathfrak{r}[s]}{s-\xi\cdot z}(s-\xi\cdot z)\varphi\,d\theta = \int_{S^1}\mathfrak{r}[s]\varphi\,d\theta$ is independent of ξ. Also $\|s-\xi\cdot z\|_\xi^2 = \int_{S^1}\mathfrak{r}[s](s-\xi\cdot z)\,d\theta = \int_{S^1}s\mathfrak{r}[s]\,d\theta$ is independent of C. Therefore

$\tilde{\varphi}(\xi) = \varphi - Cs + v\cdot z$ where $C = \frac{\int \varphi\mathfrak{r}[s]d\theta}{\int s\mathfrak{r}[s]d\theta}$ is independent of ξ, and $v \in \mathbb{R}^2$ depends on ξ. Then

$$\begin{aligned}
\int_{S^1} (\tilde{\varphi}_\theta)^2 - \tilde{\varphi}^2\, d\theta &= -\int_{S^1} \tilde{\varphi}\mathfrak{r}[\tilde{\varphi}]\, d\theta \\
&= -\int_{S^1} (\varphi - Cs + v\cdot z)\mathfrak{r}[\varphi - Cs + v\cdot z]\, d\theta \\
&= -\int_{S^1} (\varphi - Cs + v\cdot z)\mathfrak{r}[\varphi - Cs]\, d\theta \\
&= -\int_{S^1} \mathfrak{r}[\varphi - Cs + v\cdot z](\varphi - Cs)\, d\theta \\
&= -\int_{S^1} \mathfrak{r}[\varphi - Cs](\varphi - Cs)\, d\theta
\end{aligned}$$

which is independent of ξ. Note also that f_φ is unchanged by adding multiples of s and z_i, $i = 1, 2$ to φ, since $\tilde{\varphi}$ is unchanged by such an operation.

The change in the numerator of f_φ as ξ is varied can be computed as follows:

$$\begin{aligned}
(32)\qquad D_i\|\tilde{\varphi}\|_\xi^2 = D_i \Bigg(&\langle\varphi,\varphi\rangle_\xi - \frac{\langle\varphi, s - \xi\cdot z\rangle_\xi^2}{\|s-\xi\cdot z\|^2} \\
&- \left(M_\xi^{-1}\right)^{k\ell}\langle\varphi, z_k\rangle_\xi\langle\varphi, z_\ell\rangle_\xi\Bigg).
\end{aligned}$$

The second term in the bracket is independent of ξ, as noted above. The change in the inner product is given by

$$\begin{aligned}
D_i\langle\phi_1,\phi_2\rangle_\xi &= D_i\int_{S^1} \frac{\mathfrak{r}[s]}{s-\xi\cdot z}\phi_1\phi_2 d\theta \\
&= \int_{S^1} \frac{\mathfrak{r}[s]}{(s-\xi\cdot z)^2} z_i\phi_1\phi_2 d\theta.
\end{aligned}$$

Substituting this in equation (32) gives

(33)

$$\begin{aligned}
D_i\|\tilde{\varphi}\|_\xi^2 = &\int_{S^1}\frac{\mathfrak{r}[s]}{(s-\xi\cdot z)^2} z_i\varphi^2\, d\theta - 2\left(M_\xi\right)^{k\ell}\int_{S^1}\frac{\mathfrak{r}[s]}{(s-\xi\cdot z)^2}\varphi z_k z_i d\theta\langle\varphi, z_\ell\rangle_\xi \\
&- \left(D_i\left(M_\xi^{-1}\right)^{k\ell}\right)\langle\varphi, z_k\rangle_\xi\langle\varphi, z_\ell\rangle_\xi.
\end{aligned}$$

To calculate the second derivative, use the invariance of f_φ under addition to φ of multiples of s and z_i to ensure that $\varphi = \tilde{\varphi}$ at the point ξ. The integrals $\langle\varphi, z_j\rangle$ are zero for each j, and only contribute to the final result when they are differentiated. In particular, the derivative of the last term in equation (33) is zero. This gives

$$\begin{aligned}
&D_iD_j\|\tilde{\varphi}\|_\xi^2 \\
&= 2\int_{S^1}\frac{\mathfrak{r}[s]}{(s-\xi\cdot z)^3}z_iz_j\tilde{\varphi}^2d\theta \\
&\quad - 2\left(M_\xi\right)^{k\ell}\int_{S^1}\frac{\mathfrak{r}[s]}{(s-\xi\cdot z)^2}\tilde{\varphi}z_iz_kd\theta\int_{S^1}\frac{\mathfrak{r}[s]}{(s-\xi\cdot z)^2}\tilde{\varphi}z_jz_\ell d\theta \\
&= 2\langle\frac{z_i\tilde{\varphi}}{s-\xi\cdot z},\frac{z_j\tilde{\varphi}}{s-\xi\cdot z}\rangle_\xi - 2\left(M_\xi^{-1}\right)^{k\ell}\langle\frac{z_i\tilde{\varphi}}{s-\xi\cdot z},z_k\rangle_\xi\langle\frac{z_j\tilde{\varphi}}{s-\xi\cdot z},z_\ell\rangle_\xi.
\end{aligned}$$

This leads to the following expression for the Hessian of f_φ:

$$\mathrm{Hess}f_\varphi(\eta,\eta) = \frac{2}{\langle\tilde{\varphi},-\mathcal{L}_\xi\tilde{\varphi}\rangle_\xi}\left\|\pi_{E_0^\perp}\left(\frac{\eta\cdot z\tilde{\varphi}}{s-\xi\cdot z}\right)\right\|_\xi^2 \geq 0,$$

where $\mathcal{L}_\xi = \frac{s-\xi\cdot z}{\mathfrak{r}[s]}\mathfrak{r}$, $E_0^\perp$ is the subspace orthogonal to E_0 with respect to the inner product $\langle\ ,\ \rangle_\xi$, and π is the orthogonal projection. Thus f_φ is convex for each φ.

Therefore f is a supremum of convex functions, hence convex. The symmetry properties claimed in the proposition are immediate, as is the lower bound in the symmetric case, by Theorem 3. □

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