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Harmonic and Quasi-Harmonic Spheres, Part II

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1. Introduction.

This is in the sequel of our previous work [LW] on the study of the approximated harmonic maps in high dimensions. The main purpose of the present article is to understand the bubbling phenomena as well as the energy quantization beyond the natural conformal dimension two for the Dirichelet integral. This will be important toward our understandings of the defect measures and the energy concentration sets introduced and studied already for approximated harmonic maps in [LW]. We shall examine here the static situation, that is, the studies of harmonic spheres. In our forthcoming work [LW2], we will study the rectifiablity of defect measures in the parabolic case as well as the quasi-harmonic sphere bubblings and the so-called generalized varifold flow.

As bi-products of our study are improvements of the "energy identity" as well as the "no necks formations" thorems for approximated harmonic maps from Riemannian surfaces. In all previous works one needs to assume the tension fields to be bounded in L^2, that is not a conformally invariant condition. We find an essential optimal condition on tension fields, which is also scaling(up) invariant, and which is always satisfied whenever the tension fields are bounded in L^p, for any $p > 1$.

To describe the main results more precisely, we let M be a m dimensional compact Riemannian manifold (with possibly non-empty boundary ∂M), $N \subset R^k$ be a compact Riemannian manifold without boundary. For $\epsilon > 0$, let $u_\epsilon \in C^2(M, R^k)$ be a critical point of the generalized Ginzburg-Landau functional

$$I_\epsilon(u) = \int_M \left(\frac{1}{2}|Du|^2 + \frac{1}{\epsilon^2}F(u)\right) dx,$$

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where $F \in C^\infty(R^k, R)$ satisfies:

$$F(p) = d^2(p, N), \quad \text{if d(p,N)} \le \delta,$$
$$= 4\delta^2, \qquad \text{if d(p,N)} \ge 2\delta.$$

Here d denotes the Euclidean distance in R^k and $d(\cdot, N) = \inf\{d(\cdot, p) : p \in N\}$. Note that $\delta > 0$ is chosen to be so small that $d^2(p, N)$ is smooth for $p \in N_{2\delta} \equiv \{p : d(p, N) \le 2\delta\}$. It is easy to check that u_ϵ satisfies:

$$\Delta u_\epsilon + \frac{1}{\epsilon^2} f(u_\epsilon) = 0. \tag{1.1}$$

Here $f(u_\epsilon) = -(DF)(u_\epsilon)$. We assume henceforth that if $N = S^{k-1}$, then $F(p) = \frac{1}{4}(1 - |p|^2)^2$ so that $f(p) = p(1 - |p|^2)$ and (1.1) becomes

$$\Delta u_\epsilon + \frac{1}{\epsilon^2} u_\epsilon (1 - |u_\epsilon|^2) = 0. \tag{1.1'}$$

For $\epsilon > 0$, let u_ϵ be solutions to (1.1) with

$$\sup_{\epsilon > 0} I_\epsilon(u_\epsilon) < +\infty. \tag{1.2}$$

Our interest is to study the limit behavior of u_ϵ's, as ϵ tends to 0.

It is well-known, via Chen-Struwe [CS] and Chen-Lin [CL]), that one can always find a subsequence of u_ϵ, still denoted by u_ϵ, such that $u_\epsilon \to u$ weakly in $H^1(M, R^k)$ and $u \in H^1(M, N)$ is a weakly harmonic map. Moreover, u is smooth away from a closed subset $\Sigma \subset M$ with locally finite $(m-2)$ dimensional Hausdorff measure. Very recently, we showed in [LW] that if N doesn't support harmonic S^2 (i.e., nontrivial harmonic maps from S^2) then $u_\epsilon \to u$ strongly in $H^1(M, R^k)$. In particular, u is a stationary harmonic map whose singular set has Hausdorff dimension at most $m - 4$ (see, Lin [L]).

The aim of this paper is to extend the blow-up techniques developed in [L] and [LW] to the case that N does support harmonic S^2. We obtain the bubbling result in the two dimension case, $m = 2$. For $m \ge 3$, we prove a quantization result for the density function of the defect measure on the concentration set associated with the process of convergence, provided that $N = S^{k-1}$. These ideas for the generalized Ginzburg-Landau functionals, which are motivated by an earlier work of Helein [H] and some recent works by [LR] and [LR1] in higher dimensions, can also be used to extend the known results on the energy identity and the bubble tree convergence for approximated harmonic maps from surfaces with bounded L^2 tension field to the case that the tension field of the approximated harmonic maps from surfaces with bounded L^p tension field for any $p > 1$, provided that $N = S^{k-1}$.

Now, let us state our main results.

Theorem A. *Assume that $m = 2$. For $\epsilon > 0$, let $u_\epsilon \subset H^1(M, R^k)$ be solutions to* (1.1) *and satisfy* (1.2) *(for $\partial M \neq \emptyset$, $u_\epsilon = g$ for some fixed $g \in C^1(\partial M, N)$). Then for any $\epsilon_n \to 0$ there exist a subsequence of u_{ϵ_n}, denoted as itself, and a harmonic map $u_0 \in C^\infty(M, N)$ and a finite number of harmonic S^2's, $\{\omega_i\}_{i=1}^l$, along with points $\{a_i^n\}_{i=1}^l \subset M$, and $\{\lambda_i^n\}_{i=1}^l \subset R_+$ such that*

$$\lim_{\epsilon_n \to 0} I_{\epsilon_n}(u_{\epsilon_n}) = E(u_0) + \sum_{i=1}^l E(\omega_i). \tag{1.3}$$

$$\lim_{\epsilon_n \to 0} \left\| u_{\epsilon_n} - u_0 - \sum_{i=1}^l \omega_i^n \right\|_{L^\infty(M)} = 0. \tag{1.4}$$

Here $E(u_0) = \int_M \frac{1}{2}|Du_0|^2$, $E(\omega_i) = \int_{S^2} \frac{1}{2}|D\omega_i|^2$, and $\omega_i^n(x) = \omega_i(\frac{x-a_i^n}{\lambda_i^n}) - \omega(\infty)$.

Throughout this paper, it is assumed that all sequential convergences is taken after passing to possible subsequences if not mentioned explicitly. (1.3) is called as *energy identity* and (1.4) is called as *bubble tree convergence*. It will be clear from the proof in the below that (1.3) implies

$$\lim_{\epsilon_n \to 0} \|u_{\epsilon_n} - u_0 - \sum_{i=1}^l \omega_i^n\|_{H^1(M)} = 0.$$

However, since $H^1(R^2) \not\subset L^\infty(R^2)$, the convergence (1.4) asserting that there is no neck formation during the process of convergence is one of the most difficult issues in the study of bubbling phenomena for approximated harmonic maps with bounded L^2 tension field in the two dimension case. For previous works on two dimensional harmonic map bubblings, see [J], [P], [Q], [DT], [QT], [W], [LW1].

For $m \geq 3$, assume that $M = \Omega \subset R^m$ is a bounded domain , Note that if u_{ϵ_n} satisfies (1.1) and (1.2), then, for $\epsilon_n \to 0$, there exists a nonnegative Radon measure ν on Ω such that

$$e_{\epsilon_n}(u_{\epsilon_n})(x)\,dx \equiv \left(\frac{1}{2}|Du_{\epsilon_n}|^2 + \frac{1}{\epsilon_n^2}F(u_{\epsilon_n})\right)(x)\,dx \to \frac{1}{2}|Du|^2(x)\,dx + \nu,$$

as convergence of Radon measures. Moreover, we showed in [LW] that there exists a closed subset $\Sigma \subset \Omega$, with locally finite $(m-2)$ dimensional Hausdorff measure, such that

(i) $\Sigma = \mathrm{spt}(\nu) \cup \mathrm{sing}(u)$, here $\mathrm{spt}(\nu)$ denotes the support of ν and $\mathrm{sing}(u)$ denotes the singular set of u;

(ii) there exists a H^{m-2} measurable function $0 < \epsilon_0^2 \leq \Theta < \infty$ such that

$$\nu(x) = \Theta(x) H^{m-2} L\Sigma, \quad \text{for} \quad H^{m-2} \quad \text{a.e.} \quad x \in \Sigma.$$

(iii) If u_{ϵ_n} does not converge to u strongly in $H^1(\Omega, R^k)$, then $H^{m-2}(\Sigma) > 0$ and there exists at least one harmonic S^2 in N.

Claim (iii) suggests that if N does support harmonic S^2, then the strong convergence may fail. Hence, in order to understand the blowing up behaviors of the convergence, it is important to understand the nontrivial defect measure ν and describe its density function Θ. We employ the ideas introduced in the recent works [LR] and [LR1] to show the following:

Theorem B. *If, in addition, $N = S^{k-1}$. Then, for H^{m-2} a.e. $x \in \Sigma$, there exist $1 \leq l_x < \infty$ and harmonic S^2's, $\{\phi_j\}_{j=1}^{l_x}$, such that*

$$\Theta(x) = \sum_{l=1}^{l_x} E(\phi_j). \tag{1.5}$$

One shall view (1.5) as a higher dimensional version of *energy identity* for weakly convergent sequences of critical points of the Ginzburg-Landau functionals. We also believe that theorem B remains to be true for any Riemannian manifold N.

When $m = 3$, for any fixed $\epsilon > 0$, suitable scalings of a solution u_ϵ to (1.1) yield either a harmonic map $u : R^3 \to N$:

$$\Delta u + A(u)(Du, Du) = 0, \tag{1.6}$$

with bounded normalized energy:

$$\sup_{R>0} R^{-1} \int_{B_R} |Du|^2(x)dx < \infty \tag{1.7}$$

or a map $v : R^3 \to R^k$ which solves:

$$\Delta v + f(v) = 0, \tag{1.8}$$

with

$$\sup_{R>0} R^{-1} \int_{B_R} \left(\frac{1}{2}|Du|^2 + F(v)\right)(x)\,dx < \infty. \tag{1.9}$$

In order to understand these maps, we look at its *tangent maps* at the infinity. This has been done by Lin-Riviere [LR] for maps satisfying (1.6) and (1.7). Here we carry out the analysis for (1.8) and (1.9).

First, recall a tangent map for $v : R^3 \to R^k$ satisfying (1.8) and (1.9) is a map $\phi : R^3 \to R^k$ obtained as a weak limit of $v_{R_n}(x) \equiv v(R_n x)$ in $H^1_{\text{loc}}(R^3, R^k)$ for some $R_n \to +\infty$. Let T_∞ denote the set consisting of all possible tangent maps of v at infinity. Then we can prove

Theorem C. *Let $v : R^3 \to R^k$ be a solution to (1.8) and (1.9). Then for any $\phi \in T_\infty$*

(a) *$\phi(x) = \phi(\frac{x}{|x|})$ for $x \neq 0$, and $\phi|_{S^2}$ is a harmonic map into N. Moreover, there exist $R_n \to \infty$ and a nonnegative Radon measure ν on R^3 such that*

$$\mu_n \equiv \frac{1}{2}|Dv_{R_n}|^2 + R_n^2 F(v_n))\,dx \to \frac{1}{2}|D\phi|^2\,dx + \nu$$

as convergence of Radon measures.

(b) *ν is a cone-measure, i.e. $\nu_\lambda = \nu$ for any $\lambda > 0$. Here $\nu_\lambda(A) = \lambda^{-1}\nu(\lambda A)$ for any Borel set $A \subset R^3$. Moreover, there exist $1 \le l < \infty$, $\{P_j\}_{j=1}^l \subset S^2$, and $\{\theta_j\}_{j=1}^l \subset R_+$ such that*

$$spt(\nu) = \cup_{j=1}^l \overline{OP_j},$$

where $\overline{OP_j}$ denotes the ray emitting from the origin to P_j. For $1 \le j \le l$,

$$\nu L\overline{OP_j} = \theta_j H^1 L\overline{OP_j}.$$

(c) *The following balancing condition holds:*

$$\int_{S^2} x|D\phi|^2(x)dH^2(x) + \sum_{j=1}^{l} \theta_j P_j = 0.$$

(d) *If, in addition, $N = S^{k-1}$. Then, for $1 \le j \le l$, there exist $1 \le p_j < \infty$ and harmonic S^2's, $\{\phi_q\}_1^{p_j}$, such that*

$$\theta_j = \sum_{q=1}^{p_j} E(\phi_q).$$

Note that if $\phi : S^2 \to S^2$ is a harmonic map, then one has

$$E(\phi) = 4\pi|\deg(\phi)|,$$

where $\deg(\phi)$ denotes the topological degree of ϕ. Hence, as a consequence of (d) in theorem C, we have the quantization result for a entire solution in of (1.8)-(1.9) in R^3.

Corollary D. *Let $v : R^3 \to S^2$ be a solution to* (1.8) *and* (1.9). *Then*

$$\lim_{R\to\infty} R^{-1} \int_{B_R} \left(\frac{1}{2}|Dv|^2 + \frac{1}{4}(1-|v|^2)^2\right)(x)\,dx = 4\pi k, \tag{1.10}$$

for some nonnegative integer k.

One shall compare Corollary D with the two dimensional quantization effect result by Brezis-Merle-Riviere [BMR]. The similar result for three dimensional entire solutions to (1.6) and (1.7) was obtained by [LR], [LR1]. The basic idea is the estimate for the gradient in $L^{2,1}$ and $L^{2,\infty}$ (both are the Lorentz spaces). As an application of such analysis , we obtain the bubbling result for sequences of maps into the sphere with bounded L^p tension field, for any $p > 1$. This extends previous known results, where the tension field is assumed to be bounded in L^2.

Theorem E. *For $m = 2$ and M without boundary. Assume that $u_n \subset H^1(M, S^{k-1})$ converges to u weakly in $H^1(M, S^{k-1})$. For any $p > 1$, if the tension field:*

$$h_n \equiv \Delta u_n + |Du_n|^2 u_n, \tag{1.11}$$

is bounded in $L^p(M)$. Then there exist a finite many harmonic S^2's, $\{\omega_j\}_{j=1}^L$, $\{a_n^j\}_{j=1}^L \subset M$, $\{\lambda_n^j\}_{j=1}^L \subset R_+$ such that

$$\lim_{n\to\infty} \|u_n - u - \sum_{j=1}^L \omega_n^j\|_{L^\infty(M)} = 0. \tag{1.12}$$

In particular, (1.12) holds, with $L^\infty(M)$ replaced by $H^1(M)$. Here $\omega_n^j(\cdot) = \omega_j(\frac{\cdot - a_n^j}{\lambda_n^j}) - \omega_j(\infty)$.

Here we would like to remark that the condition on the tension fields h_n can be further weakened to a local scaling invariant one (see Proposition 6.2 in the last section for the precise statement of these conditions).

2. Basic Estimates.

This section is devoted to establishing a priori estimates needed in later sections. We assume that $M = \Omega \subset R^m$ is a bounded smooth domain. Denote $B_R(x) \subset R^m$ as the ball centered at x with radius $R > 0$.

Lemma 2.1. *Assume that $u_\epsilon : \Omega \to R^k$ solves (1.1). Then we have*

(2.1)
$$R^{2-m}\int_{B_R(x)} e_\epsilon(u_\epsilon)(x)\,dx - r^{2-m}\int_{B_r(x)} e_\epsilon(u_\epsilon)(x)\,dx$$
$$= \int_{B_R(x)\setminus B_r(x)} |y-x|^{2-m}\left|\frac{\partial u_\epsilon}{\partial|y-x|}\right|^2 dy + 2\int_r^R t^{1-m}\int_{B_t(x)} \frac{F(u_\epsilon)}{\epsilon^2}\,dx$$

for any $x \in \Omega$ and $0 < r \le R < d(x, \partial\Omega)$.

One can find the proof in [CS] or [CL].

Lemma 2.2. *There exist $\epsilon_0 > 0$ and $C_0 > 0$ such that if $u_\epsilon : B_{2R} \to R^k$ solves (1.1) and $(2R)^{2-m}\int_{B_{2R}} e_\epsilon(u_\epsilon)\,dx \le \epsilon_0^2$, then*

$$R^2 \sup_{x\in B_{\frac{3R}{2}}} e_\epsilon(u_\epsilon)(x) \le C_0 R^{2-m}\int_{B_{2R}} e_\epsilon(u_\epsilon)\,dx. \tag{2.2}$$

Moreover, for $\epsilon << 1$,

$$R^2 \sup_{x\in B_R} \frac{1}{\epsilon^2}|f(u_\epsilon)|(x) \le C_0\left(\epsilon_0^2 + \frac{R^2}{\epsilon^2}e^{-C_0\frac{R}{\epsilon}}\right). \tag{2.3}$$

Proof. One can refer to [CS] for the proof of (2.2). One can also find the proof of (2.3) in the last section of [CL]. However, we would like to outline a proof of (2.3) in the case that that $N = S^{k-1}$. Note that the maximum principle implies $|u_\epsilon|(x) \le 1$. By scaling argument, it suffices to prove (2.3) for $R = 1$. Let $\Phi_\epsilon = 1 - |u_\epsilon|^2$. Then it follows from $(1.1')$ that

$$\begin{aligned} -\epsilon^2\Delta\Phi_\epsilon + 2\Phi_\epsilon \le 4\epsilon^2 e_\epsilon(u_\epsilon) \le C\epsilon_0^2\epsilon^2, &\quad \text{in } B_1, \\ 0 \le \Phi_\epsilon \le 1, &\quad \text{on } \partial B_1. \end{aligned} \tag{2.4}$$

Here we used (2.2), which implies that $e_\epsilon(u_\epsilon) \le C\epsilon_0^2$ in B_1. Let $w_\epsilon(x) = w_\epsilon(|x|) : B_1 \to R$ be a solution to

$$\begin{aligned} -\epsilon^2\Delta w_\epsilon + 2w_\epsilon = 0, &\quad \text{in } B_1, \\ w_\epsilon = 1, &\quad \text{on } \partial B_1. \end{aligned} \tag{2.5}$$

Then one can check that for $\epsilon \le 1$ $f_\epsilon(x) = e^{\frac{1}{2\epsilon}(|x|^2-1)}$ is a super-solution to (2.5). Hence the maximum principle implies that

$$w_\epsilon(x) \le e^{\frac{1}{2\epsilon}(|x|^2-1)},\ x \in B_1,$$

and

$$\Phi_\epsilon(x) \le C\epsilon_0^2\epsilon^2 + e^{\frac{1}{2\epsilon}(|x|^2-1)},\ x \in B_1.$$

This yields (2.3).

Lemma 2.3. *For $m = 2$. Let $u_\epsilon : B_2(0) \to R^k$ solve* (1.1). *Then, for any $0 < R < 2$,*

$$\int_{\partial B_R} \left|\frac{\partial u_\epsilon}{\partial r}\right|^2 + \frac{1}{R}\int_{B_R} \frac{F(u_\epsilon)}{\epsilon^2} \le \int_{\partial B_R} |D_T u_\epsilon|^2 + 2\int_{\partial B_R} \frac{F(u_\epsilon)}{\epsilon^2}. \tag{2.6}$$

Here D_T denotes the tangential derivative on ∂B_R.

Proof. Multiplying (1.1) by $x \cdot Du_\epsilon$ and integrating it over B_R, integrations by parts yield (2.6).

3. Proof of Theorem A.

In this section $m = 2$. The idea is based on that developed by Lin-Wang [LW1]. The first step is to show the convexity of tangential energy of u_ϵ on S^1; the second step is to use the Pohozaev inequality of Lemma 2.3 to control the radial energy of u_ϵ by its tangential energy. To make the proof clear and self-contained, we first recall the process for the first bubble.

For $\epsilon_n \to 0$, we assume that u_{ϵ_n} does not converge to u strongly in $H^1(M, R^k)$. Then there exist $\{x_j\}_{j=1}^L \subset M$ and $\{m_j\}_{j=1}^L \in R_+$ such that

$$e_{\epsilon_n}(u_{\epsilon_n})(x)\,dx \to \frac{1}{2}|Du|^2(x)dx + \sum_{j=1}^L m_j\delta_{x_j}.$$

Here δ_{x_j} denotes the Dirac mass at x_j. For simplicity, we assume $L = 1$. Consider the maximum concentration function:

$$Q_{\epsilon_n}(t) = \max_{B_t(x)\subset B_\delta(x)} \int_{B_t(x)} e_{\epsilon_n}(u_{\epsilon_n})(x)\,dx.$$

Then there are $x_n \to x_1$ and $\lambda_n \to 0$ such that

$$Q_{\epsilon_n}(\lambda_n) = \frac{\epsilon_0^2}{2} = \int_{B_{\lambda_n}(x_n)} e_{\epsilon_n}(u_{\epsilon_n})(x)dx.$$

Define $v_n(x) = u_{\epsilon_n}(x_n + \lambda_n x) : \Omega_n \to R^k$, here $\Omega_n = \lambda_n^{-1}(B_\delta(x_1) \setminus \{x_1\})$. Then

$$\Delta v_n + \frac{1}{\tilde{\epsilon}_n^{\,2}} f(v_n) = 0, \text{ in } \Omega_n,$$

and

$$\int_{B_1(z)} e_{\tilde{\epsilon}_n}(v_n) \le \frac{\epsilon_0^2}{2},$$

for any $z \in \Omega_n$, with equality for $z = 0$. Therefore Lemma 2.2 implies that $v_n \to \omega_1 \neq$ constant in $C^1 \cap H^1(R^2)$ locally. Moreover, we claim that

$$\tilde{\epsilon}_n = \frac{\epsilon_n}{\lambda_n} \to 0. \tag{3.3}$$

Otherwise, for a subsequence, either $\frac{\epsilon_n}{\lambda_n} \to c > 0$ and ω_1 satisfies

$$\Delta\omega_1 + c^{-2} f(\omega_1) = 0,$$

$$\frac{\epsilon_0^2}{2} \le \int_{R^2} \frac{1}{2}|D\omega_1|^2 + \frac{1}{c^2}F(\omega_1) < \infty.$$

It follows from [BMR] or [LW] that ω_1 is constant, which is impossible. Or $\tilde{\epsilon}_n \to \infty$, which implies that ω_1 is a harmonic function in R^2 with positive and finite energy and hence constant. This is impossible again. It follows that $\omega_1 : R^2 \to N$ is a nontrivial harmonic map with finite energy. Hence the removable result of [SU] implies that ω_1 can then be lifted to a harmonic S^2 to N. One can repeat the same process to find all possible harmonic S^2's, $\{\omega_j\}_{j=1}^l$. Moreover, $l \le C(M, N) < \infty$, since the energy of harmonic S^2's has a uniform positive low-bound. It is clear that

$$\lim_{n\to\infty} I_{\epsilon_n}(u_{\epsilon_n}) \ge E(u) + \sum_{j=1}^{l} E(\omega_j). \tag{3.4}$$

To prove (3.4) is an equality. We use the induction procedure illustrated in [DT] and assume that there is only one harmonic S^2 obtained as above. Then theorem follows from the following Lemma.

Lemma 3.1. *Assume that* $l = 1$. *Then we have*

$$\lim_{R\to+\infty,\delta\to 0}\lim_{\epsilon_n\to 0}\int_{B_\delta\setminus B_{R\lambda_n}} e_{\epsilon_n}(u_{\epsilon_n})(x)\,dx = 0, \tag{3.5}$$

$$\lim_{R\to+\infty,\delta\to 0}\lim_{\epsilon_n\to 0} osc_{x\in B_\delta\setminus B_{R\lambda_n}} u_{\epsilon_n}(x) = 0. \tag{3.6}$$

Proof. For simplicity, we assume that $x_n = x_1 = 0 \in R^2$. To make it clear, we further assume that $N = S^{k-1}$. Since $l = 1$, it follows from the argument of [DT] that

$$\lim_{\epsilon_n\to 0}\int_{B_{2r}\setminus B_r} e_{\epsilon_n}(u_{\epsilon_n})(x)\,dx \le \frac{\epsilon_0^2}{2},\ \forall R\lambda_n \le r \le \delta. \tag{3.7}$$

Therefore Lemma 2.2 implies that for n sufficiently large

$$\sup_{r\in[R\lambda_n,\delta]} r^2 \max_{B_{\frac{15r}{8}}\setminus B_{\frac{5r}{4}}}\left(|Du_{\epsilon_n}|^2 + \frac{1}{\epsilon_n^2}(1-|u_{\epsilon_n}|^2)\right) \le C\epsilon_0^2. \tag{3.8}$$

Let (r,θ) be the polar coordinate in R^2. Define $v_n : \Sigma_n \equiv [|\log\delta|, |\log R\lambda_n|] \times S^1 \to R^k$ by $v_n(r,\theta) = u_{\epsilon_n}(e^{-r},\theta)$. Then we have

$$\Delta v_n + \frac{e^{-2r}}{\epsilon_n^2}(1-|v_n|^2)v_n = 0, \text{ in } \Sigma_n. \tag{3.9}$$

Here $\Delta v_n = \frac{\partial^2 v_n}{\partial r^2} + \frac{\partial^2 v_n}{\partial \theta^2}$. Moreover, (3.8) gives, for $n \gg 1$,

$$\sup_{(r,\theta)\in\Sigma_n}\left(|Dv_n|^2 + \frac{e^{-2r}}{\epsilon_n^2}(1-|v_n|^2)\right)(r,\theta) \le C\epsilon_0^2. \tag{3.10}$$

By adapting the calculation of [LW1], we now claim that, for $n >> 1$,

$$\frac{d^2}{dr^2}\int_{S^1}|v_{n,\theta}|^2 \ge \int_{S^1}|v_{n,\theta}|^2,\ \forall r \in [|\log\delta|, |\log R\lambda_n|]. \tag{3.11}$$

In fact, by direct calculations and integrations by parts, we have

$$\begin{aligned}
\frac{d^2}{dr^2}\int_{S^1}|v_{n,\theta}|^2 &= 2\int_{S^1}|v_{n,\theta r}|^2 + 2\int_{S^1} v_{n,\theta}v_{n,\theta rr}\\
&= 2\int_{S^1}|v_{n,\theta r}|^2 - 2\int_{S^1} v_{n,\theta\theta}v_{n,rr}\\
&= 2\int_{S^1}|v_{n,\theta r}|^2 + 2\int_{S^1}|v_{n,\theta\theta}|^2\\
&+ 2\int_{S^1} e^{-2r}\epsilon_n^{-2}(1-|v_n|^2)v_n v_{n,\theta\theta}\\
&= 2\int_{S^1}|v_{n,\theta r}|^2 + 2\int_{S^1}|v_{n,\theta\theta}|^2\\
&- 2\int_{S^1} e^{-2r}\epsilon_n^{-2}((1-|v_n|^2)v_n)_\theta v_{n,\theta}\\
&= 2\int_{S^1}|v_{n,\theta r}|^2 + 2\int_{S^1}|v_{n,\theta\theta}|^2\\
&+ 4\int_{S^1} e^{-2r}\epsilon_n^{-2}|v_n v_{n,\theta}|^2 - 2\int_{S^1} e^{-2r}\epsilon_n^{-2}(1-|v_n|^2)|v_{n,\theta}|^2\\
&\geq 2\int_{S^1}|v_{n,\theta\theta}|^2 - C\epsilon_0^2\int_{S^1}|v_{n,\theta}|^2\\
&\geq \int_{S^1}|v_{n,\theta}|^2.
\end{aligned}$$

Here we have used the Poincaré inequality on S^1:

$$\int_{S^1}|v_{n,\theta\theta}|^2 \geq \int_{S^1}|v_{n,\theta}|^2.$$

Let $0(\delta, R^{-1})$ denote the quantities such that $\lim_{R\to\infty,\delta\to 0} 0(\delta,R^{-1}) = 0$. Since $u_{\epsilon_n}\to u$ in $C^1(M\setminus B_\delta)$ and $u_{\epsilon_n}(\lambda_n\cdot)\to\omega_1$ in $C^1(B_R)$ for any $R>0$, we can choose δ sufficiently small and R sufficiently large such that

$$a_n = \int_{\{|\log\delta|\}\times S^1}|Dv_n|^2 = 0(\delta, R^{-1})$$

$$b_n = \int_{\{|\log R\lambda_n|\}\times S^1}|Dv_n|^2 = 0(\delta, R^{-1}). \tag{3.12}$$

Denote $T_0 = |\log\delta|$ and $T_n = |\log R\lambda_n|$. Applying the maximum principle to (3.11), we have

$$\int_{S^1}|v_{n,\theta}|^2(r,\cdot) \leq A_n e^r + B_n e^{-r},\ \forall r\in[T_0,T_n], \tag{3.13}$$

where

$$A_n = \frac{e^{T_n} b_n - e^{T_0} a_n}{e^{2T_n} - e^{2T_0}}, \ B_n = \frac{e^{T_0+2T_n} a_n - e^{2T_0+T_n} b_n}{e^{2T_n} - e^{2T_0}}. \tag{3.14}$$

Now we can easily see

$$\lim_{n\to\infty} \int_{\Sigma_n} |v_{n,\theta}|^2 = 0(\delta, R^{-1}), \tag{3.15}$$

and

$$\lim_{n\to\infty} \int_{T_0}^{T_n} (\int_{S^1} |v_{n,\theta}|^2)^{\frac{1}{2}} = 0(\delta, R^{-1}). \tag{3.16}$$

Therefore Lemma 2.3 to conclude that

$$\begin{aligned}\int_{\Sigma_n} |\frac{\partial v_n}{\partial r}|^2 &\le \int_{\Sigma_n} |v_{n,\theta}|^2 + 2\int_{\Sigma_n} e^{-2r}\frac{(1-|v_n|^2)^2}{\epsilon_n^2} \\ &\le \int_{\Sigma_n} |v_{n,\theta}|^2 + \epsilon_n^2 \sup_{\Sigma_n}(e^{-2r}\frac{(1-|v_n|^2)^2}{\epsilon_n^2})^2 \int_{T_0}^{T_n} e^{2r} \\ &\le \int_{\Sigma_n} |v_{n,\theta}|^2 + CR^{-2}(\frac{\epsilon_n}{\lambda_n})^2.\end{aligned}$$

It tends to zero as $n \to \infty$, $\delta \to 0$, and $R \to \infty$. Putting these two estimates together, we obtain (3.5). Moreover, (3.16) and Lemma 2.3 imply

$$\begin{aligned}\int_{T_0}^{T_n} (\int_{S^1} |\frac{\partial v_n}{\partial r}|^2)^{\frac{1}{2}} &\le \int_{T_0}^{T_n} (\int_{S^1} |v_{n,\theta}|^2)^{\frac{1}{2}} \\ &+ \int_{T_0}^{T_n} e^{-\frac{r}{2}} (\int_{\partial B_{e^{-r}}} \frac{(1-|u_{\epsilon_n}|^2)^2}{\epsilon_n^2})^{\frac{1}{2}} \\ &\le \int_{T_0}^{T_n} (\int_{S^1} |v_{n,\theta}|^2)^{\frac{1}{2}} + \epsilon_n \sup_{r\in[T_0,T_n]} e^{-2r}\frac{1-|u_{\epsilon_n}|^2}{\epsilon_n^2} \int_{T_0}^{T_n} e^r \, dr \\ &\to 0,\end{aligned}$$

as $n \to \infty$, $\delta \to 0$, and $R \to \infty$. In particular, we have

$$\lim_{n\to\infty} \int_{\Sigma} |Dv_n| = 0(\delta, R^{-1}). \tag{3.17}$$

Note that the L^1 norm of gradient of v_n essentially controls the oscillation of v_n. Hence we conclude that the oscillation over the neck region goes to zero. Note that the inequality (2.6)also implies

$$\int_{B_\delta \setminus R\lambda_n} \frac{1}{\epsilon_n^2} F(u_{\epsilon_n})(x)dx$$

goes to zero as $n \to \infty$, $\delta \to 0$, and $R \to \infty$. Hence (3.5) is proven.

4. Proof of Theorem C.

Note claim (4) of theorem C is a consequence of theorem B, we will prove the first three statements of theorem C.

It follows from the definition of T_∞ that for any $\phi \in T_\infty$ there exist $R_n \to \infty$ such that $u_{R_n}(x) \equiv u(R_n x) \to \phi$ weakly in $H^1_{\text{loc}}(R^3, R^k)$. Moreover, it follows from (1.9) that for any $R > 0$,

$$R^{-1} \int_{B_R} e_{R_n^{-1}}(u_{R_n})(x)\, dx \le \lim_{R_n \to \infty} \frac{1}{RR_n} \int_{B_{RR_n}} e_1(u)(x)\, dx \equiv \Lambda_1 < \infty,$$

we assert that there exists a nonnegative Radon measure ν on R^3 such that

$$e_{R_n^{-1}}(u_{R_n})\, dx \to \frac{1}{2}|D\phi|^2(x) dx + \nu,$$

as convergence of Radon measures. Moreover, (2.1) and (1.9) imply

$$\lim_{R\to\infty} \int_{R^3 \backslash B_R} \rho^{-1} |\frac{\partial u}{\partial \rho}|^2 + 2 \int_R^\infty \rho^{-2} \int_{B_\rho} F(u) = 0. \tag{4.1}$$

By the lower semi-continuity, we then have, for any $r < R < \infty$,

$$\begin{aligned} \int_{B_R \backslash B_r} \rho^{-1} |\frac{\partial \phi}{\partial \rho}|^2 &\le \lim_{n\to\infty} \int_{B_R \backslash B_r} \rho^{-1} |\frac{\partial u_{R_n}}{\partial \rho}|^2 \\ &= \lim_{n\to\infty} \int_{B_{RR_n} \backslash B_{rR_n}} \rho^{-1} |\frac{\partial u}{\partial \rho}|^2 = 0. \end{aligned}$$

This implies that $\phi(x) = \phi(\frac{x}{|x|})$ for $x \neq 0$. It also follows that

$$\int_{B_R \backslash B_r} \rho^{-2} \int_{B_\rho} F(\phi) = 0,$$

this implies that $\phi : R^3 \to N$ is a harmonic map. Furthermore, we claim that $\mu \equiv \frac{1}{2}|D\phi|^2(x)\, dx + \nu$ is a cone-measure, i.e., for any $\lambda > 0$, $\mu_\lambda = \mu$. Suppose that we have achieved this. Then we see that ν is also a cone measure, since $\frac{1}{2}|D\phi|^2(x)\, dx$ is a cone measure. In particular, $\Sigma = \text{spt}(\nu)$ is a 1-dimensional cone in R^3 with locally finite H^1 measure so that there exist $1 \le l < \infty$ $\{P_j\}_{j=1}^l \subset S^2$, and $\{\theta_j\}_{j=1}^l$ such that

$$\Sigma = \cup_{j=1}^l \overline{OP_j},$$

and

$$\nu L \overline{OP_j} = \theta_j H^1 L \overline{OP_j},$$

for $1 \le j \le l$. To prove that μ is a cone measure, it suffices to show

$$d\mu(r,\theta) = dr\, d\alpha(\theta), \forall (r,\theta) \in R_+ \times S^2. \tag{4.2}$$

Here α is a Radon measure on S^2. In fact, if (4.2) is true then for any Borel set $A \subset R^3$ and $\lambda > 0$,

$$\begin{aligned}\mu_\lambda(A) &= \lambda^{-1}\mu(\lambda A) = \lambda^{-1}\int_{\lambda A} d\mu(r,\theta)\\ &= \lambda^{-1}\int_{\lambda A} dr d\alpha(\theta) = \int_A dr d\alpha(\theta) = \mu(A).\end{aligned}$$

Note that (4.2) is equivalent to that μ is invariant under radial directional translations, namely

$$d\mu(r+a,\theta) = d\mu(r,\theta), \forall r, a > 0, \theta \in S^2. \tag{4.3}$$

Let $\psi_\epsilon \in C^\infty(R_+, R)$ be a family of mollifies and $\eta \in C^\infty(S^2)$. For $a > 0$, denote $u_n = u_{R_n}$ and

$$E(u_n,\eta,a,\epsilon) = \int_0^\infty \int_{S^2} e_{R_n^{-1}}(u_n)(r+a,\theta)\eta(\theta)\psi_\epsilon(r)(r+a)^2 dr d\theta.$$

Note that

$$e_{R_n^{-1}}(u_n)(r+a,\theta)(r+a)^2 dr d\theta \to d\mu(r+a,\theta),$$

so that

$$\lim_{n\to\infty,\epsilon\to 0} E(u_n,\eta,a,\epsilon) = \int_{S^2} \eta(\theta) d\mu(r+a,\theta).$$

Therefore, we need to show

$$\frac{d}{da}|_{a=0} E(u_n,\eta,a,\epsilon) \to 0,$$

as $n \to \infty$ and $\epsilon \to 0$ in the sense of distribution. In fact,

(4.4)

$$
\begin{aligned}
&\frac{d}{da}|_{a=0}E(u_n, \eta, a, \epsilon) \\
&= 2\int_0^{\infty}\int_{S^2}(r+a)(|u_{n,r}|^2 + R_n^2F(u_n))(r+a,\theta)\eta(\theta)\psi_\epsilon(r)d\theta dr \\
&+ \int_0^{+\infty}\int_{S^2}((r+a)^2u_{n,r}u_{n,rr} + u_{n,\theta}u_{n,\theta r} \\
&- (r+a)^2R_n^2f(u_n)u_{n,r})(r+a,\theta)\eta(\theta)\psi_\epsilon(r)d\theta dr \\
&= 2\int_0^{+\infty}\int_{S^2}((r+a)|u_{n,r}|^2 + (r+a)R_n^2F(u_n))(r+a,\theta)\eta(\theta)\psi_\epsilon(r)d\theta dr \\
&+ \int_0^{+\infty}\int_{S^2}[2(r+a)^2(u_{n,r}u_{n,rr} + u_{n,r}u_{n,\theta\theta}) + u_{n,\theta}u_{n,\theta}]\eta(\theta)\psi_\epsilon(r)d\theta dr \\
&= 2\frac{d}{da}\int_0^{+\infty}\int_{S^2}(r+a)[|u_{n,r}|^2 + R_n^2F(u_n)](r+a,\theta)\eta(\theta)\psi_\epsilon(r)d\theta dr \\
&+ 2\int_0^{+\infty}\int_{S^2}(r+a)^2u_{n,r}u_{n,rr}(r+a,\theta)\phi(\theta)\psi_\epsilon(r)d\theta dr \\
&- \int_0^{+\infty}\int_{S^2}u_{n,r}u_{n,\theta}\phi_\theta(\theta)\psi_\epsilon(r)d\theta dr.
\end{aligned}
$$

Here we have used that

$$u_{n,\theta\theta} + (r+a)^2u_{n,rr} + R_n^2f(u_n) = 0.$$

Integrating (4.4) with respect to $a \in (0, R-\rho)$ and taking ϵ into zero, we have

$$
\begin{aligned}
(4.5)\qquad &\int_{S^2}e_{R_n^{-1}}(u_n)(R,\theta)\eta(\theta)d\theta - \int_{S^2}e_{R_n^{-1}}(u_n)(\rho,\theta)\eta(\theta)d\theta \\
&= 2\int_{S^2}R^2|u_{n,r}|^2(R,\theta)\eta(\theta)d\theta - 2\int_{S^2}\rho^2|u_{n,r}|^2(\rho,\theta)\eta(\theta)d\theta \\
&+ 2\int_\rho^R\int_{S^2}r^2R_n^2F(u_n)(r,\theta)\eta(\theta)d\theta dr \\
&- \int_\rho^R\int_{S^2}u_{n,r}u_{n,\theta}\eta_\theta(\theta)d\theta dr.
\end{aligned}
$$

Passing n into infinity and using (2.1), we obtain

$$\int_{S^2}\eta(\theta)d\mu(R,\theta) = \int_{S^2}\eta(\theta)d\mu(\rho,\theta).$$

This gives (4.3). To show (c), we observe that the same argument as in Lemma 2.3 implies that, for any $X \in C_0^\infty(R^3, R^3)$,

$$\int_{R^3} e_{R_n^{-1}}(u_n)\mathrm{div}X - u_{n,k}u_{n,l}X_l^k = 0. \tag{4.6}$$

Hence, by choosing $X(x) = x_j$ for $1 \le j \le 3$ and letting n into infinity, we have

$$\int_{S^2} \frac{1}{2}x|D\phi|^2(x)dH^2(x) + \sum_{j=1}^{l} \theta_j P_j = 0.$$

This yields (c).

5. Proof of Theorem B.

In this section, As in [LR] and [LR1] we use the estimate on suitable Lorentz spaces norm of Du_ϵ to give a proof of the energy identity of theorem B. We assume that $M = \Omega \subset R^m$ is a bounded smooth domain.

First note that (2.1) implies that

$$R^{2-m}\mu(B_R(x)) \ge r^{2-m}\mu(B_r(x)),\ \forall x \in \Omega, 0 < r \le R < d(x, \partial\Omega). \tag{5.1}$$

Here $\mu = \frac{1}{2}|Du|^2(x)\,dx + \nu$ is the limiting of $e_{\epsilon_n}(u_{\epsilon_n})(x)dx$. Hence

$$\Theta^{m-2}(\mu, x) = \lim_{R\to 0} R^{2-m}\mu(B_R(x))$$

exists for all $x \in \Omega$ and is upper semi-continuous. Moreover, it follows from the definition of Σ that

$$x \in \Sigma \text{ if and only if } \epsilon_0^2 \le \Theta^{m-2}(\mu, x) < \infty. \tag{5.2}$$

In fact, the rectifiablity theorem of Preiss [P] and Lin [L] yields that Σ is $(m-2)$-rectifiable. Note that $\Theta^{m-2}(u, x) \equiv \lim_{r\to 0} r^{2-m}\int_{B_r(x)} |Du|^2 = 0$ for H^{m-2} a.e. $x \in \Sigma$. Therefore, for H^{m-2} a.e. $x \in \Sigma$, $\epsilon_0^2 \le \Theta^{m-2}(\nu, x) \equiv \lim_{r\to 0} r^{2-m}\nu(B_r(x)) = \Theta^{m-2}(\mu, x) < \infty$. This verifies the condition of the rectifiablity theorem of [P1] and [L]. Based on this, we know that there exists a H^{m-2}-measurable function $\epsilon_0^2 \le \Theta < \infty$ such that

$$\nu(x) = \Theta(x)H^{m-2}L\Sigma, \quad \text{for } H^{m-2} \text{ a.e. x} \in \Sigma.$$

Since Θ is H^{m-2}-measurable, it is approximately continuous for H^{m-2} a.e. $x \in \Sigma$. This, combines with the $(m-2)$-rectifiablity of Σ, implies that for

H^{m-2} a.e $x_0 \in \Sigma$, Σ has a $(m-2)$-dimensional tangent plane $T_{x_0}\Sigma$, Θ is approximately continuous at x_0, and $\Theta^{m-2}(u, x_0) = 0$. We may assume that $x_0 = 0$ and $T_{x_0}\Sigma = \{(0,0,y) : y \in R^{m-2}\}$. Let $r_n \to 0$ and $v_n(x) = u_{\epsilon_n}(r_n x) : B_2^m \to R^k$. Then

$$\Delta v_n + \tilde{\epsilon}_n^{-2} v_n(1 - |v_n|^2) = 0, \text{ in } B_2^m.$$

Here $\tilde{\epsilon}_n = \frac{\epsilon_n}{r_n}$. Now we follow the blow-up scheme of [LW] to conclude that there exist a tangent measure μ_* of μ at 0 such that

$$e_{\tilde{\epsilon}_n}(v_n)(x)dx \to \mu_*$$

and

$$v_n \to \text{constant},$$

weakly in $H^1(R^m)$ locally. Moreover, $\text{spt}(\mu_*) = \{(0,0)\} \times R^{m-2} \subset R^m$, and

$$\mu_* = \Theta(0)H^{m-2}L(\{(0,0)\} \times R^{m-2}).$$

Furthermore, applying (2.1) with various centers on $T_0\Sigma$, we can assume that

$$\lim_{n\to\infty} \int_{Q_1} \sum_{j=3}^{m} \left|\frac{\partial v_n}{\partial y_j}\right|^2 + \tilde{\epsilon}_n^{-2} F(v_n) = 0. \tag{5.3}$$

Here $Q_1 = B_1^2 \times B_1^{m-2} \subset R^2 \times R^{m-2}$.

Now we want to show that $\Theta(0)$ is a finite sum of energies of nontrivial harmonic maps from S^2 into N.

Let us first recall how the first bubble is obtained from the blow-up analysis from [LW]. Let $X = (x_1, x_2) \in R^2$ and $Y = (y_3, \cdots, y_m) \in R^{m-2}$. Define $f_n : B_1^{m-2} \to R$ by

$$f_n(Y) = \int_{B_1^2} \left(\sum_{j=3}^{m} \left|\frac{\partial v_n}{\partial y_j}\right|^2 + \tilde{\epsilon}_n^{-2} F(v_n) \right) (X, Y) dX,$$

and $g_n : B_1^{m-2} \to R$ by

$$g_n(Y) = \int_{B_1^2} e_{\tilde{\epsilon}_n}(v_n)(X, Y) dX.$$

Then the Fubini's theorem and (5.3) imply

$$\lim_{n\to\infty} \|f_n\|_{L^1(B_1^{m-2})} = 0.$$

Therefore the weak L^1-estimate of the Hardy-Littlewood maximal function implies that for any $0 < \delta < 1$ there exists $E_\delta^n \subset B_1^{m-2}$, with $|E_\delta^n| \geq 1 - \delta$, such that

$$(5.4) \qquad \lim_{n\to\infty} \sup_{0<r\leq\frac{1}{2}} r^{2-m} \int_{B_r(Y)} f_n(Y)dY = 0, \forall Y \in E_\delta^n.$$

Note also that for H^{m-2} a.e. $Y \in B_1^{m-2}$, g_n is bounded. It follows [LW] that there exists $F_\delta^n \subset E_\delta^n$, with $|F_\delta^n| \geq 1 - 2\delta$, such that

$$(5.5) \qquad \lim_{n\to\infty} g_n(Y) = \Theta(0), \forall Y \in F_\delta^n.$$

As in the section 3, we may assume that $0 \in F_\delta^n$. For $\epsilon_0 > 0$ given by Lemma 2.2, there exist $\{X_n\}(\subset B_1^2) \to 0$ and $\lambda_n \to 0$ such that

$$(5.6) \quad \int_{B^2_{\lambda_n}(X_n)} e_{\tilde{\epsilon}_n}(v_n)(X,0)dX = \frac{\epsilon_0^2}{C(m)} = \max_{Z\in B^2_{\frac{1}{2}}} \int_{B_{\lambda_n}(Z)} e_{\tilde{\epsilon}_n}(v_n)(X,0)dX,$$

here $C(m) > 0$ is to be chosen later. Define rescaling maps $w_n(X,Y) = v_n((X_n,0) + \lambda_n(X,Y))$. Then w_n satisfies (1.1′) in $\lambda_n^{-1}(B_1^2(X_n) \setminus \{X_n\}) \times B^{m-2}_{\lambda_n^{-1}}$, with $\tilde{\epsilon}_n$ replaced by $\delta_n = \frac{\tilde{\epsilon}_n}{\lambda_n}$. It also follows from (5.4) and (5.6) that
(5.7)

$$\lim_{n\to\infty} \sup_{0<r\leq\lambda_n^{-1}} r^{2-m} \int_{B_r^{m-2}(0)} \int_{B^2_{\lambda_n^{-1}}(0)} \left(\sum_{j=3}^m \left|\frac{\partial w_n}{\partial y_j}\right|^2 + \delta_n{}^{-2} F(w_n) \right) = 0,$$

$$(5.8) \qquad \int_{B_1^2(0)} e_{\delta_n}(w_n)(X,0)dX = \frac{\epsilon_0^2}{C(m)}$$

$$= \max_{Z\in B^2_{\lambda_n^{-1}}} \int_{B_1^2(Z)} e_{\delta_n}(w_n)(X,0)dX.$$

Moreover, if we let $\phi \in C_0^\infty(B_2^2(0))$, then direct calculations show that for $3 \leq j \leq m$,

$$\frac{\partial}{\partial y_j} \int_{B_2^2(0)} \phi^2(X) e_{\delta_n}(w_n)(X,Y)\, dX$$

$$= -2 \sum_{k=1}^2 \int_{B_2^2(0)} \phi_0(X)\phi(X) w_{n,x_k}(X,Y) w_{n,y_j}(X,Y)\, dX$$

$$+ \sum_{l=3}^m \frac{\partial}{\partial y_l} \int_{B_2^2(0)} \phi^2(X) w_{n,y_l} w_{n,y_j}(X,Y) dX.$$

This, combines with (5.7) and (5.8), implies

$$(2\lambda_n)^{2-m}\int_{B^{m-2}_{2\lambda_n}(0)\times B^2_1(0)} e_{\delta_n}(w_n)(X,Y) \le \frac{\epsilon_0^2}{2}. \tag{5.9}$$

Hence Lemma 2.2 implies that

$$w_n(X,Y) \to w \text{ in } C^1_{\text{loc}}(B^{m-2}_2 \times R^2).$$

Moreover, $\delta_n \to 0$ and $w(X,Y) = w(Y) : R^2 \to S^{k-1}$ is a nontrivial harmonic map with finite energy, which can be lifted to be a harmonic map from S^2.

Let $\overline{w}_n(X) = w_n(X,0) : B^2_1 \to S^{k-1}$. Since (5.5) gives

$$\lim_{n\to\infty}\int_{B^2_{\frac{1}{2}}} e_{\delta_n}(\overline{w}_n)(X)dX = \Theta(0). \tag{5.10}$$

In order to prove theorem B, we need to show that

$$\lim_{n\to\infty} e_{\delta_n}(\overline{w}_n)(X)dX = \sum_{j=1}^{l} E(\phi_j), \tag{5.11}$$

where $\phi_j : S^2 \to S^{k-1}$ are nontrivial harmonic maps.

Denote $A(R,n) = \{X \in R^2 : R\lambda_n \le |X| \le \frac{1}{2}\}$. Then it follows from the first bubbling process shown as above that (5.11) is equivalent to

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{A(R,n)} e_{\tilde{\epsilon}_n}(v_n)(X,0)dX = \sum_{j=2}^{l} E(\phi_j). \tag{5.12}$$

As in §3, it follows from the induction argument of [DT] that we only need to show that (5.12) is true for $l = 1$, i.e.,

$$\lim_{n\to\infty}\int_{A(R,n)} e_{\tilde{\epsilon}_n}(v_n)(X,0)dX = 0(R^{-1}). \tag{5.13}$$

Here $\lim_{R\to\infty} 0(R^{-1}) = 0$. We first claim that for any $\epsilon_1 > 0$, there are sufficiently large R and n_0 such that for $n \ge n_0$, one has

$$r^{2-m}\int_{(B^2_{2r}\setminus B^2_r)\times B^{m-2}_r} e_{\tilde{\epsilon}_n}(v_n)(X,Y)dXdY \le \epsilon_1^2, \tag{5.14}$$

for all $r \in [R\lambda_n, \frac{1}{2}]$. Indeed, if (5.14) fails, then we may assume that there exists $r_n \in [R\lambda_n, \frac{1}{2}]$ such that

$$r_n^{2-m} \int_{(B^2_{2r_n} \setminus B^2_{r_n}) \times B^{m-2}_{r_n}} e_{\tilde{\epsilon}_n}(v_n)(X,Y)dXdY$$

is equal to the maximum of the left hand side of (5.14) over all $r \in [R\lambda_n, \frac{1}{2}]$, and is larger than ϵ_1^2. Moreover $\frac{r_n}{\lambda_n} \to \infty$. Scale v_n and define $\overline{v}_n(X,Y) : (B^2_{r_n^{-1}} \setminus B^2_{\frac{R\lambda_n}{r_n}}) \times B^{m-2}_2 \to R^k$ by $\overline{v}_n(X,Y) = v_n(r_n X, r_n Y)$. Then we have

$$\int_{(B^2_2 \times B^2_1) \times B^{m-2}_1} e_{\eta_n}(\overline{v}_n)(X,Y)dXdY \geq \epsilon_1^2. \tag{5.15}$$

Here $\eta_n = \frac{\tilde{\epsilon}_n}{r_n} \to 0$. By the energy bound of $\overline{v}_n$, we may assume that $\overline{v}_n \to v_\infty$ weakly in $H^1_{\text{loc}}(R^2 \times B^{m-2}_2, R^k)$. Moreover, using (5.7), we can conclude that $v_\infty(X,Y) = v_\infty(X) : R^2 \to S^{k-1}$ is a harmonic map with finite energy. Hence, if the convergence is strong in $H^1_{\text{loc}}(R^2 \times B^{m-2}_2, R^k)$, then (5.15) implies that v_∞ is nontrivial, which contradicts with $l = 1$. Hence the convergence fails to be strong and the blow-up argument of [LW] yields that there exist $\{x_d\}_{d=1}^p \subset R^2$ for some $1 \leq p < \infty$ and $C_d > 0$ for $1 \leq d \leq p$ such that

$$|D\overline{v}_n|^2(X,Y)dXdY \to |Dv_\infty|^2(X)dXdY + \sum_{d=1}^p C_d H^{m-2} L(\{x_d\} \times R^{m-2}),$$

as convergence of Radon measures. Moreover, there exists at least one nontrivial harmonic S^2 along with the convergence process, this again contradicts with $l = 1$. It follows from (5.14) and Lemma 2.2 that we have

$$r^2 \sup_{(B^2_{2r} \setminus B^2_r) \times B^{m-2}_r} e_{\tilde{\epsilon}_n}(v_n)(X,Y) \leq C\epsilon_1^2, \tag{5.16}$$

for all $r \in [R\lambda_n, \frac{1}{2}]$. If we define $w_n(X,Y) : B^2_{\lambda_n^{-1}} \times B^{m-2}_{\lambda_n^{-1}} \to R^k$ by $w_n(X,Y) = v_n(\lambda_n X, \lambda_n Y)$. Then (5.16) implies, in particular,

$$|X||Dw_n|(X,Y) \leq C\epsilon_1, \forall (X,Y) \in (B^2_{(2\lambda_n)^{-1}} \setminus B^2_R) \times B^{m-2}_R. \tag{5.17}$$

Now let us recall the definition of the two Lorentz spaces we need, the reader can refer to Ziemer [Z] for details.

Definition 5.1. Let $\Omega \subset R^2$ be a given domain. A $f : \Omega \to R$ is in $L^{2,1}(\Omega)$ (or $L^{2,\infty}(\Omega)$) if

$$\|f\|_{L^{2,1}(\Omega)} = \int_0^\infty t^{\frac{1}{2}} f^*(t) \frac{dt}{t} < \infty$$

($\|f\|_{L^{2,\infty}(\Omega)} = \sup_{t>0} t^{\frac{1}{2}} f^*(t) < \infty$ respectively) where $f^*(t)$ is the rearrangement function of f.

Lemma 5.2. *Let w_n be given as above. Then we have, for any $Y \in B_R^{m-2}$,*

$$\|Dw_n(\cdot, Y)\|_{L^{2,\infty}(B^2_{(2\lambda_n)^{-1}} \setminus B^2_R)} \le C\epsilon_1. \tag{5.18}$$

Proof. It is easy to see that $\frac{1}{|X|} \in L^{2,\infty}(R^2)$. Hence (5.17) implies, for any $Y \in B_R^{m-2}$,

$$\begin{aligned} \|Dw_n(\cdot, Y)\|_{L^{2,\infty}(B^2_{(2\lambda_n)^{-1}} \setminus B^2_R)} &\le C\epsilon_n \||X|^{-1}\|_{L^{2,\infty}(B^2_{(2\lambda_n)^{-1}} \setminus B^2_R)} \\ &\le C\epsilon_1 \||X|^{-1}\|_{L^{2,\infty}(R^2)} \le C\epsilon_1. \end{aligned} \tag{5.19}$$

Let $\mathcal{H}^1(R^m)$ denote the Hardy space in R^m. The following lemma is well-known and can be found in Stein [S].

Lemma 5.3. *Assume that $g \in \mathcal{H}^1(R^m)$, the Hardy space in R^m. Let $\psi \in H^1(R^m)$ be a solution to*

$$\Delta \psi = g.$$

Then $\psi \in W^{2,1}(R^m)$ and

$$\|D^2\psi\|_{L^1(R^m)} \le C\|f\|_{\mathcal{H}^1(R^m)}. \tag{5.20}$$

Now we need

Proposition 5.4. *For $1 \le j, l \le k$, we have, in $B^2_{\lambda_n^{-1}} \times B^{m-2}_{\lambda_n^{-1}}$,*

(1)

$$\sum_{\alpha=1}^m (w_n^j w_{n,\alpha}^l - w_n^l w_{n,\alpha}^j)_\alpha = 0. \tag{5.21}$$

(2)

$$\Delta(w_n^j dw_n^l - w_n^l dw_n^j) = 2d^*(dw_n^j \wedge dw_n^l). \tag{5.22}$$

Here d denotes the exterior derivative and d^* denotes its adjoint in R^m.

Proof. (1) Note that

$$\begin{aligned}&\sum_{\alpha=1}^{m}(w_n^j w_{n,\alpha}^l - w_n^l w_{n,\alpha}^j)_\alpha \\ &= w_n^j \Delta w_n^l - w_n^l \Delta w_n^j \\ &= w_n^j \tilde{\epsilon}_n^{-2}(|w_n|^2-1)w_n^l - w_n^l \tilde{\epsilon}_n^{-2}(|w_n|^2-1)w_n^j = 0.\end{aligned}$$

(2) Note that (1) is equivalent to that $d^*(w_n^j dw_n^l - w_n^l dw_n^j) = 0$. Hence

$$\begin{aligned}\Delta(w_n^j dw_n^l - w_n^l dw_n^j) &= (dd^* + d^*d)(w_n^j dw_n^l - w_n^l dw_n^j) \\ &= d^*d(w_n^j dw_n^l - w_n^l dw_n^j) \\ &= 2d^*(dw_n^j \wedge dw_n^l).\end{aligned}$$

Lemma 5.5.

$$\begin{aligned}&\int_{B^{m-2}_{\frac{R}{2}}} \|dw_n^i w_n^j - dw_n^j w_n^i(\cdot, Y)\|_{L^{2,1}(B^2_{(2\lambda_n)^{-1}})} dY \\ &\le C \int_{B^2_{\lambda_n^{-1}} \times B^{m-2}_R} |Dw_n|^2(X,Y)\,dX\,dY.\end{aligned} \tag{5.23}$$

Proof. Let $\tilde{w}_n(X,Y) : R^m \to R^k$ be an extension of w_n such that

$$\|D\tilde{w}_n\|_{L^2(R^m)} \le C\|Dw_n\|_{L^2(B^2_{\lambda_n^{-1}} \times B^{m-2}_R)}. \tag{5.24}$$

Let $F_n^{ij} \in H^1(R^m, \wedge^2(R^m))$ be a solution to

$$\Delta F_n^{ij} = 2(d\tilde{w}_n^i \wedge d\tilde{w}_n^j). \tag{5.25}$$

Then Lemma 5.3 implies that $F_n^{ij} \in W^{2,1}(R^m, \wedge^2(R^m))$ and

$$\begin{aligned}\|D^2 F_n^{ij}\|_{L^1(R^m)} &\le C\|d\tilde{w}_n^i \wedge d\tilde{w}_n^j\|_{\mathcal{H}^1(R^m)} \\ &\le C\|D\tilde{w}_n\|^2_{L^2(R^m)} \\ &\le C\|Dw_n\|^2_{L^2(B^2_{\lambda_n^{-1}} \times B^{m-2}_R)}.\end{aligned}$$

By the Fubini's theorem, we know, for H^{m-2} a.e. $Y \in B_R^{m-2}$, $DF_n^{ij}(\cdot, Y) \in W^{1,1}(R^2)$. By using the equation (5.22), we know that if G_n^{ij} is a harmonic 1-form in $B^2_{\lambda_n^{-1}} \times B_R^{m-2}$, with

$$j^*(G_n^{ij} + d^*(F_n^{ij}) - (dw_n^i w_n^j - w_n^i dw_n^j)) = 0,$$

here $j : \partial(B^2_{\lambda_n^{-1}} \times B_R^{m-2}) \to R^m$ denotes the inclusion map, then

$$dw_n^i w_n^j - w_n^i dw_n^j = d^*(F_n^{ij}) + G_n^{ij}, \tag{5.26}$$

in $B^2_{\lambda_n^{-1}} \times B_R^{m-2}$. By choosing R suitably, we may assume that

$$\begin{aligned} &\|dw_n^i w_n^j - dw_n^j w_n^i\|_{L^2\left(\partial\left(B^2_{(\frac{4\lambda_n}{3})^{-1}} \times B^{m-2}_{\frac{3R}{4}}\right)\right)} \\ &\leq CR^{-\frac{1}{2}} \|dw_n^i w_n^j - dw_n^j w_n^i\|_{L^2(B^2_{\lambda_n^{-1}} \times B_R^{m-2})}, \end{aligned} \tag{5.27}$$

and

$$\|D^2 F_n^{ij}\|_{L^1\left(\partial\left(B^2_{(\frac{4\lambda_n}{3})^{-1}} \times B^{m-2}_{\frac{3R}{4}}\right)\right)} \leq C \|D^2 F_n^{ij}\|_{L^1(B^2_{\lambda_n^{-1}} \times B_R^{m-2})}. \tag{5.28}$$

Hence, applying the estimate on harmonic functions, we have $D^2 G_n^{ij} \in L^1(B^2_{(\frac{4\lambda_n}{3})^{-1}} \times B^{m-2}_{\frac{3R}{4}})$ and

$$\begin{aligned} &\|D^2 G_n^{ij}\|_{L^1(B^2_{(2\lambda_n)^{-1}} \times B^{m-2}_{\frac{R}{2}})} \\ &\leq C \left(\|D^2 F_n^{ij}\|_{L^1(B^2_{\lambda_n^{-1}} \times B_R^{m-2})} + \|dw_n^i w_n^j - dw_n^j w_n^i\|_{L^2(B^2_{\lambda_n^{-1}} \times B_R^{m-2})} \right) \\ &\leq C \int_{B^2_{\lambda_n^{-1}} \times B_R^{m-2}} |Dw_n|^2(X, Y) dX dY. \end{aligned} \tag{5.29}$$

Combining these estimates together, we can conclude that

$$dw_n^i w_n^j - dw_n^j w_n^i \in W^{1,1}\left(B^2_{(2\lambda_n)^{-1}} \times B^{m-2}_{\frac{R}{2}}\right)$$

and

$$\begin{aligned} &\|dw_n^i w_n^j - dw_n^j w_n^i\|_{W^{1,1}\left(B^2_{(2\lambda_n)^{-1}} \times B^{m-2}_{\frac{R}{2}}\right)} \\ &\leq C \int_{B^2_{\lambda_n^{-1}} \times B_R^{m-2}} |Dw_n|^2(X, Y) dX dY. \end{aligned} \tag{5.30}$$

Thus, for H^{m-2} a.e. $Y \in B^{m-2}_{\frac{R}{2}}$, $(dw^i_n w^j_n - dw^j_n w^i_n)(\cdot, Y) \in W^{1,1}(B^2_{(2\lambda_n)^{-1}})$. Hence, using the embedding $W^{1,1}(R^2) \subset L^{2,1}(R^2)$ (cf. Hélein [H]), we know that, for H^{m-2} a.e. $Y \in B^{m-2}_{\frac{R}{2}}$, $dw^i_n w^j_n - dw^j_n w^i_n(\cdot, Y) \in L^{2,1}(B^2_{(2\lambda_n)^{-1}})$ and

$$\begin{aligned}
&\int_{B^{m-2}_{\frac{R}{2}}} \|dw^i_n w^j_n - dw^j_n w^i_n(\cdot, Y)\|_{L^{2,1}(B^2_{(2\lambda_n)^{-1}})} dY \\
&\le C \int_{B^{m-2}_{\frac{R}{2}}} \|dw^i_n w^j_n - dw^j_n w^i_n\|_{W^{1,1}(B^2_{(2\lambda_n)^{-1}})} \\
&\le C \int_{B^2_{\lambda_n^{-1}} \times B^{m-2}_R} |Dw_n|^2(X, Y) dX dY.
\end{aligned}$$

This proves Lemma 5.5.

Using the duality between $L^{2,1}$ and $L^{2,\infty}$ and putting Lemma 5.3 and Lemma 5.4 together, we then have

$$\begin{aligned}
&\int_{(B^2_{(2\lambda_n)^{-1}} \setminus B^2_R) \times B^{m-2}_{\frac{R}{2}}} |dw^i_n w^j_n - dw^j_n w^i_n|^2(X, Y) dX dY \\
&= \int_{B^{m-2}_{\frac{R}{2}}} \|(dw^i_n w^j_n - dw^j_n w^i_n)(\cdot, Y)\|^2_{L^2(B^2_{(2\lambda_n)^{-1}} \setminus B^2_R)} \\
&\le \int_{B^{m-2}_{\frac{R}{2}}} \|dw^i_n w^j_n - dw^j_n w^i_n\|_{L^{2,1}(B^2_{(2\lambda_n)^{-1}} \setminus B^2_R)} \\
&\qquad \cdot \|dw^i_n w^j_n - dw^j_n w^i_n\|_{L^{2,\infty}(B^2_{(2\lambda_n)^{-1}} \setminus B^2_R)} \\
&\le C\epsilon_1 \int_{B^2_{\lambda_n^{-1}} \times B^{m-2}_R} |Dw_n|^2(X, Y) dX dY. \qquad (5.31)
\end{aligned}$$

On the other hand, it follows from (5.16) that

$$|w_n|(X, Y) \ge \frac{1}{2}, \forall (X, Y) \in (B^2_{(2\lambda_n)^{-1}} \setminus B_R) \times B^{m-2}_R.$$

Hence, we can write (1.1′) into the polar coordinate form as follows. Since $w_n = |w_n| \frac{w_n}{|w_n|} = \rho_n \omega_n$, we have $\rho_n \ge \frac{1}{2}$ and $|\omega_n| = 1$. Moreover, (ρ_n, ω_n) satisfies:

$$\Delta \rho_n + \tilde{\epsilon}_n^{-2} \rho_n (1 - |\rho_n|^2) - \rho_n |D\omega_n|^2 = 0, \tag{5.32}$$

$$\operatorname{div}(\rho_n^2 D\omega_n) + \rho_n^2 |D\omega_n|^2 \omega_n = 0. \tag{5.33}$$

Observe that

$$\sum_{ij=1}^{k} |dw_n^i w_n^j - dw_n^j w_n^i|^2(X,Y) = \rho_n^2 |D\omega_n|^2(X,Y).$$

Hence, (5.31) implies

$$\begin{aligned}
&\int_{(B^2_{(2\lambda_n)^{-1}} \backslash B^2_R)\times B^{m-2}_R} \rho_n^2 |D\omega_n|^2(X,Y)dXdY \\
(5.34) \qquad &\le C\epsilon_1 \int_{B^2_{(\lambda_n)^{-1}} \times B^{m-2}_R} |Dw_n|^2(X,Y)dXdY.
\end{aligned}$$

Now we need to estimate the L^2 norm of $D\rho_n$. In order to do so, we multiply (5.32) by $(1-\rho_n)$ and integrate it over the domain, we then have

$$\begin{aligned}
&\int_{(B^2_{(2\lambda_n)^{-1}} \backslash B^2_R)\times B^{m-2}_R} \Delta\rho_n(1-\rho_n) + \tilde{\epsilon}_n^{-2}(1-\rho_n)^2\rho_n(1+\rho_n)(X,Y)dXdY \\
&= \int_{(B^2_{(2\lambda_n)^{-1}} \backslash B^2_R)\times B^{m-2}_R} \rho_n(1-\rho_n)|D\omega_n|^2(X,Y)dXdY \\
&\le C\epsilon_1 \int_{B^2_{(\lambda_n)^{-1}} \times B^{m-2}_R} |Dw_n|^2(X,Y)dXdY.
\end{aligned}$$

It follows from (5.16) that

$$\lim_{n\to\infty} R^{2-m} \int_{(B^2_{(2\lambda_n)^{-1}} \backslash B^2_R)\times B^{m-2}_R} \tilde{\epsilon}_n^{-2}(1-\rho_n)^2\rho_n(1+\rho_n)(X,Y)dXdY = 0.$$

Using the integration by parts, we also have

$$\begin{aligned}
&\int_{(B^2_{(2\lambda_n)^{-1}} \backslash B^2_R)\times B^{m-2}_R} \Delta\rho_n(1-\rho_n)(X,Y)dXdY \\
&= \int_{(B^2_{(2\lambda_n)^{-1}} \backslash B^2_R)\times B^{m-2}_R} |D\rho_n|^2(X,Y)dXdY \\
&\quad + \int_{\partial((B^2_{(2\lambda_n)^{-1}} \backslash B^2_R)\times B^{m-2}_R)} (1-\rho_n)\rho_{n,\nu}.
\end{aligned}$$

Here ν denotes the unit outward normal of the boundary. It is not difficult to see that the second term goes to zero as long as $n \to \infty$, $R \to \infty$. In fact, we have

$$R^{2-m}\int_{(B^2_{(2\lambda_n)^{-1}}\backslash B^2_R)\times B^{m-2}_R}|D\rho_n|^2(X,Y)dXdY$$

(5.35) $$\le C\epsilon_1 R^{2-m}\int_{B^2_{\lambda_n^{-1}}\times B^{m-2}_R}|Dw_n|^2(X,Y)+0(n^{-1},R^{-1}).$$

Here $\lim_{R\to\infty}\lim_{n\to\infty}0(n^{-1},R^{-1})=0$. Putting (5.34) and (5.35) together, we then obtain

$$\begin{aligned}&R^{2-m}\int_{(B^2_{(2\lambda_n)^{-1}}\backslash B^2_R)\times B^{m-2}_R}|Dw_n|^2+\tilde{\epsilon}_n^{-2}(1-|w_n|^2)^2(X,Y)dXdY\\&\le C\epsilon_1 R^{2-m}\int_{B^2_{\lambda_n^{-1}}\times B^{m-2}_R}|Dw_n|^2(X,Y)+0(n^{-1},R^{-1})\\&\le C\epsilon_1+0(n^{-1},R^{-1}).\end{aligned}$$

This, combines with the strong constancy argument as that used in the first bubble process, implies

(5.36) $$\int_{B^2_{(2\lambda_n)^{-1}}\backslash B^2_R}|Dw_n|^2(X,0)dX\le C\epsilon_1+0(n^{-1},R^{-1}).$$

This finishes the proof of (5.23) and the proof of Theorem B is complete.

6. Proof of Theorem E.

In this section, we modify the ideas developed in the previous section to show both the energy identity and the oscillation convergence results stated in theorem E. Note that all the known results on this aspect require that the tension fields are bounded in L^2, and various methods developed in [J], [P], [Q], [DT], [W], [QT], [LW1] seem to be difficult to generalize the case that the tension field is bounded in L^p for any $p>1$. By an example of [P], one knows that the energy identity fails to hold for tension fields belonging to L^1 only. We believe the idea here may be useful for other problems as well.

By following the bubbling scheme developed by Brezis-Coron [BC] (see, Qing [Q] or Wang [W]), one only needs to consider the situation where two bubbles by different scales generated at the same point and prove that there is no energy concentration and oscillation at the neck region between these two bubbles. More precisely, we assume that there exist $\lambda_n \to 0$, $\mu_n \to 0$, with $\frac{\mu_n}{\lambda_n} \to \infty$, such that $u_n(x_1+\lambda_n\cdot)$ converges to a nontrivial harmonic map ω_1 strongly in $H^1(B_R, N)$ for any $0 < R < \infty$, and $u_n(x_1 + \mu_n\cdot)$ converges to another nontrivial harmonic map ω_2 strongly in $H^1_{\text{loc}}(R^2 \setminus B_r, N)$ for any small $r > 0$. Moreover, for some universal small constant ϵ_0 to be chosen later, we can assume that

$$\int_{B_{r\mu_n} \setminus B_{R\lambda_n}} |Du_n|^2 \leq \epsilon_0^2. \tag{6.1}$$

We need to show:

Lemma 6.1. *Under the same notions as above. We have*

$$\lim_{n\to\infty} \int_{B_{\frac{r\mu_n}{2}} \setminus B_{2R\lambda_n}} |Du_n|^2 = 0(r, R^{-1}). \tag{6.2}$$

Here $\lim_{r\to 0, R\to\infty} 0(r, R^{-1}) = 0$.

Proof. Denote $r_n = \frac{r\mu_n}{\lambda_n}$. Note that $r_n \to \infty$. Define $v_n(x) = u_n(x_1 + r_n x)$. Then v_n satisfies:

$$\Delta v_n + |Dv_n|^2 v_n = \overline{h}_n, \text{ in } B_{r_n}, \tag{6.3}$$

where $\overline{h}_n(x) = \lambda_n^2 h_n(\lambda_n x)$. It follows from the conformal invariance of the Dirichlet energy in two dimension that

$$\int_{B_{r_n} \setminus B_R} |Dv_n|^2 \leq \epsilon_0^2. \tag{6.4}$$

For $1 \leq i, j \leq k$. Consider the 1- forms $dv_n^i v_n^j - v_n^i dv_n^j$. Then the equation (6.3) gives

$$d^*(dv_n^i v_n^j - v_n^i dv_n^j) = \overline{h}_n^i v_n^j - \overline{h}_n^j v_n^i \equiv H_n^{ij}, \tag{6.5}$$

and

$$\Delta(dv_n^i v_n^j - v_n^i dv_n^j) = dH_n^{ij} + 2d^*(dv_n^i \wedge dv_n^j). \tag{6.6}$$

Now, let $\overline{v}_n : R^2 \to R^k$ be an extension of v_n from $B_{r_n} \setminus B_R$ such that

$$\int_{R^2} |D\overline{v}_n|^2 \leq C \int_{B_{r_n} \setminus B_R} |Dv_n|^2 \leq C\epsilon_0^2. \tag{6.7}$$

Let $\overline{H}_n^{ij} : R^2 \to R^{k\times k}$ be an extension of H_n from B_{r_n} by letting $\overline{H}_n^{ij} = 0$ outside B_{r_n}. Hence

$$\int_{R^2} |\overline{H}_n|^p \le C \int_{B_{r_n}} |H_n|^p. \tag{6.8}$$

Define two functions $\Psi_n \in H^1(R^2, \wedge^2(R^{k\times k}))$, $F_n \in W^{2,p}(R^2, R^{k\times k})$ by letting:

$$\Delta \Psi_n^{ij} = d\overline{v}_n^i \wedge d\overline{v}_n^j, \tag{6.9}$$

and

$$\Delta F_n^{ij} = \overline{H}_n^{ij}. \tag{6.10}$$

Then, we have

$$\Delta(dv_n^i v_n^j - v_n^i dv_n^j - dF_n^{ij} - 2d^*\Psi_n^{ij}) = 0, \text{ in } B_{r_n} \setminus B_R. \tag{6.11}$$

Therefore, if we define the 1-forms $G_n \in H^1(B_{r_n} \setminus B_R, \wedge(R^{k\times k}))$ by

$$\Delta G_n^{ij} = 0, \text{ in } B_{r_n} \setminus B_R, \tag{6.12}$$
$$i^*(G_n^{ij} - (dv_n^i v_n^j - v_n^i dv_n^j - dF_n^{ij} - 2d^*\Psi_n^{ij})) = 0. \tag{6.13}$$

Here $i : \partial(B_{r_n} \setminus B_R) \to R^2$ denotes the inclusion map and i^* denotes the pull-back map over one forms. Then we know, for $1 \le ij \le k$,

$$dv_n^i v_n^j - v_n^i dv_n^j - dF_n^{ij} - 2d^*\Psi_n^{ij} = G_n^{ij}, \text{ in } B_{r_n} \setminus B_R. \tag{6.14}$$

For Ψ_n, we observe that the right hand side of (6.9) is in $\mathcal{H}^1(R^2)$ (the Hardy space in R^2)(see [CLMS] for the details), and in $H^{-1}(R^2)$ by Brezis-Coron [BC]. Hence, we know that $\Psi_n^{ij} \in W^{2,1}(R^2)$ and satisfies:

$$\begin{aligned} \|D^2\Psi_n^{ij}\|_{L^1(R^2)} &\le \|d\overline{v}_n^i \wedge d\overline{v}_n^j\|_{\mathcal{H}^1(R^2)} \\ &\le C \int_{R^2} |D\overline{v}_n|^2 \\ &\le C \int_{B_{r_n}\setminus B_R} |Dv_n|^2 \le C\epsilon_0^2, \end{aligned} \tag{6.15}$$

and

$$\int_{R^2} |D\Psi_n^{ij}|^2 \le C \int_{R^2} |D\overline{v}_n|^2 \le C \int_{B_{r_n}\setminus B_R} |Dv_n|^2 \le C\epsilon_0^2. \tag{6.16}$$

For F_n, the $W^{2,p}$-estimate implies

$$\begin{aligned}\|D^2F_n^{ij}\|_{L^p(R^2)} &\le C\|\overline{H}_n^{ij}\|_{L^p(R^2)}\\ &\le C\|H_n^{ij}\|_{L^p(B_{r_n})}\\ &\le C\lambda_n^{2-\frac{2}{p}}\|h_n\|_{L^p(B_{\alpha\mu_n})}.\end{aligned} \tag{6.17}$$

Hence, by Hölder inequality, we have

$$\begin{aligned}\|D^2F_n^{ij}\|_{L^1(B_{r_n})} &\le \|D^2F_n^{ij}\|_{L^p(B_{r_n})}r_n^{2(1-\frac{1}{p})}\\ &\le \|D^2F_n^{ij}\|_{L^p(R^2)}r_n^{2(1-\frac{1}{p})}\\ &\le C\|h_n\|_{L^p(B_{\alpha\mu_n})}\mu_n^{2(1-\frac{1}{p})}.\end{aligned} \tag{6.18}$$

This, combines with the embedding (see, Hélein [H]) that $W^{1,1}(R^2) \subset L^{2,1}(R^2)$ yields

$$\|DF_n^{ij}\|_{L^{2,1}(B_{r_n})} \le C\|h_n\|_{L^p(B_{\alpha\mu_n})}\mu_n^{2(1-\frac{1}{p})}. \tag{6.19}$$

Moreover, using the $L^{2,\infty}$ estimate for DF_n and (6.8) (see, also [H]), we have

$$\begin{aligned}\|DF_n^{ij}\|_{L^{2,\infty}(R^2)} &\le C\|\overline{H}_n^{ij}\|_{L^1(R^2)}\\ &\le C\|H_n\|_{L^1(B_{r_n})}\\ &\le C\|h_n\|_{L^1(B_{\alpha\mu_n})}\\ &\le C\|h_n\|_{L^p(B_{\alpha\mu_n})}\mu_n^{2(1-\frac{1}{p})}.\end{aligned} \tag{6.20}$$

Using the duality between $L^{2,1}$ and $L^{2,\infty}$, we have

$$\begin{aligned}\|DF_n^{ij}\|_{L^2(B_{r_n})} &\le \|DF_n^{ij}\|_{L^{2,1}}^{\frac{1}{2}}\|DF_n^{ij}\|_{L^{2,\infty}}^{\frac{1}{2}}\\ &\le C\|h_n\|_{L^p(B_{\alpha\mu_n})}\mu_n^{2(1-\frac{1}{p})}.\end{aligned} \tag{6.21}$$

For G_n, we can choose suitable $\alpha > 0$ and $R > 0$ so that

$$R^{\frac{1}{2}}\|G_n^{ij}\|_{L^2(\partial B_R)} \le C, r_n^{\frac{1}{2}}\|G_n^{ij}\|_{L^2(\partial B_{r_n})} \le C. \tag{6.22}$$

Since G_n is a harmonic 1-form, it is well-known that

$$\|G_n\|_{L^{2,1}(B_{\frac{r_n}{2}}\setminus B_{2R})} \le CR^{-\frac{1}{2}}, \|G_n\|_{L^{2,\infty}(B_{\frac{r_n}{2}}\setminus B_{2R})} \le CR^{-\frac{1}{2}}. \tag{6.23}$$

Substituting these estimates into (6.14), we conclude that $(dv_n^i v_n^j - v_n^i dv_n^j - 2d^*\Psi_n^{ij}) \in L^{2,1} \cap L^{2,\infty}(B_{\frac{r_n}{2}} \setminus B_{2R})$ and

$$\|(dv_n^i v_n^j - v_n^i dv_n^j - 2d^*\Psi_n^{ij})\|_{L^{2,1}(B_{\frac{r_n}{2}} \setminus B_{2R})} \leq C\left(R^{-\frac{1}{2}} + \mu_n^{2(1-\frac{1}{p})}\right), \tag{6.24}$$

$$\|(dv_n^i v_n^j - v_n^i dv_n^j - 2d^*\Psi_n^{ij})\|_{L^{2,\infty}(B_{\frac{r_n}{2}} \setminus B_{2R})} \leq C\left(R^{-\frac{1}{2}} + \mu_n^{2(1-\frac{1}{p})}\right). \tag{6.25}$$

In particular,

$$\|(dv_n^i v_n^j - v_n^i dv_n^j - 2d^*\Psi_n^{ij})\|_{L^{2}(B_{\frac{r_n}{2}} \setminus B_{2R})} \leq C\left(R^{-\frac{1}{2}} + \mu_n^{2(1-\frac{1}{p})}\right). \tag{6.26}$$

Therefore, by (6.15) and (6.16), we have

$$\begin{aligned}
&\|dv_n^i v_n^j - v_n^i dv_n^j\|_{L^2(B_{\frac{r_n}{2}} \setminus B_{2R})} \\
&\leq 2\|D\psi_n^{ij}\|_{L^2(R^2)} + CR^{-\frac{1}{2}} + C\mu_n^{2(1-\frac{1}{p})} \\
&\leq C\int_{B_{\frac{r_n}{2}} \setminus B_R} |Dv_n|^2 + CR^{-\frac{1}{2}} + C\mu_n^{2(1-\frac{1}{p})} \\
&\leq C\epsilon_0\|Dv_n\|_{L^2(B_{\frac{r_n}{2}} \setminus B_{2R})} + C\int_{B_{r_n} \setminus B_{\frac{r_n}{2}}} |Dv_n|^2 \\
&\quad + C\int_{B_{2R} \setminus B_R} |Dv_n|^2 + CR^{-\frac{1}{2}} + C\mu_n^{2(1-\frac{1}{p})}.
\end{aligned}$$

Notice that

$$\sum_{ij=1}^{k} |dv_n^i v_n^j - v_n^i dv_n^j|^2 = 2|Dv_n|^2.$$

Hence, by choosing ϵ_0 sufficiently small and summing the left hand side of the above inequality over $1 \leq i, j \leq k$, we obtain

$$\begin{aligned}
\|Dv_n\|_{L^2(B_{\frac{r_n}{2}} \setminus B_{2R})} \leq\ & C\left(\int_{B_{r_n} \setminus B_{\frac{r_n}{2}}} |Dv_n|^2 + \int_{B_{2R} \setminus B_R} |Dv_n|^2\right) \\
& + CR^{-\frac{1}{2}} + C\mu_n^{2(1-\frac{1}{p})}.
\end{aligned} \tag{6.27}$$

Note that

$$\int_{B_{2R} \setminus B_R} |Dv_n|^2 = \int_{B_{2R} \setminus B_R} |D\omega_1|^2 + 0(n^{-1}),$$

and

$$\int_{B_{r_n}\setminus B_{\frac{r_n}{2}}} |Dv_n|^2 \le \int_{B_\alpha \setminus B_{\frac{\alpha}{2}}} |D\omega_2|^2 + 0(n^{-1}).$$

Here $\lim_{n\to\infty} 0(n^{-1}) = 0$. It is clear that if we choose R sufficiently large and α sufficiently small, then both terms in the right hand sides of the above two inequality can be as arbitrarily small. Hence, the proof of Lemma 6.1 is complete.

The oscillation convergence in theorem E also follows from Lemma 6.1. In fact, it follows from the proof of Lemma 6.1 that

$$\|D^2 v_n\|_{L^1(B_{\frac{r_n}{2}}\setminus B_{2R})} \le C \int_{B_{r_n}\setminus B_R} |Dv_n|^2 + C\|h_n\|_{L^p(M)} \mu_n^{2(1-\frac{1}{p})} \to 0,$$

as $n \to 0$ and $R \to \infty$. Let $\tilde{v}_n$ be an extension of v_n to R^2 such that it is compactly supported and

$$\|D^2 \tilde{v}_n\|_{L^1(R^2)} \le C\|D^2 v_n\|_{L^1(B_{\frac{r_n}{2}}\setminus B_{2R})}.$$

Note that

$$\begin{aligned}
|\tilde{v}_n|(x) &= \left| \int_{R^2} \log|x-y| \Delta \tilde{v}_n(y) dy \right| \\
&= \left| \int_{R^2} \frac{x-y}{|x-y|^2} D\tilde{v}(y) dy \right| \\
&\le \left\| \frac{y}{|y|^2} \right\|_{L^{2,\infty}(R^2)} \|D\tilde{v}_n\|_{L^{2,1}(R^2)} \\
&\le C\|D^2 \tilde{v}_n\|_{L^1(R^2)} \\
&\le C\|D^2 v_n\|_{L^1(B_{\frac{r_n}{2}}\setminus B_{2R})}.
\end{aligned}$$

In particular,

$$\lim_{R\to\infty} \lim_{n\to\infty} \max_{x\in B_{\frac{r_n}{2}}\setminus B_{2R}} |v_n|(x) = 0.$$

This implies that there is no neck formation between the two bubbles.

At the end of this paper, we provide a weaker condition of the tension field h_n, which seems to be optimal in certain sense, such that theorem E remains to be true.

For an bounded domain $\Omega \subset R^2$. Let $\mathcal{H}^1(\Omega)$ be the local Hardy space on Ω defined in the usual way (see, Semmes [Se] for the detail). For a Riemannian surface M, we can define $\mathcal{H}^1(M)$ by using the coordinate charts.

Proposition 6.2. *The energy identity part of theorem E remains to be true if*

(a) *the tension fields h_n is bounded in $\mathcal{H}^1(M)$;*

(b) *h_n is equi-integrable,*

i.e., for any $\epsilon > 0$ there is a $\delta > 0$ such that for any $E \subset M$, with $|E| \leq \delta$, $\int_E |h_n|(x)dx \leq \epsilon$ for any $n \geq 1$. The oscillation convergence part of theorem E remains to be true if h_n is equi-integrable in $\mathcal{H}^1(M)$ in the sense that for any $\epsilon > 0$ there is a $\delta > 0$ such that for any open set $E \subset M$, with $|E| \leq \delta$, $\|h_n\|_{\mathcal{H}^1(E)} \leq \epsilon$ for any $n \geq 1$.

It is easy to check that for $p > 1$ if h_n is bounded in $L^p(M)$ then it satisfies the above conditions.

Proof. It follows the same line of the proof of Lemma 6.1, except that we need to estimate the $L^{2,1}$ norm of DF_n^{ij} in a different way. To do it, let $\eta \in C_0^1(R^2, R_+)$ be such that $\eta = 1$ in B_{r_n} and let

$$c_n^{ij} = \frac{\int_{R^2} \eta H_n^{ij}}{\int_{R^2} \eta}$$

Then it follows from a Lemma of Semmes [Se] that $\eta(H_n^{ij} - c_n^{ij}) \in \mathcal{H}^1(R^2)$ and

$$\begin{aligned} \|\eta(H_n^{ij} - c_n^{ij})\|_{\mathcal{H}^1(R^2)} &\leq C\|\overline{h}_n\|_{\mathcal{H}^1(B_{rn})} \\ &\leq C\|h_n\|_{\mathcal{H}^1(M)} \leq C < \infty. \end{aligned} \tag{6.28}$$

Now Let $F_{n,1}^{ij}$ and $F_{n,2}^{ij}$ be 1-forms on R^2 and solve:

$$\Delta F_{n,1}^{ij} = \eta(H_n^{ij} - c_n^{ij}), \tag{6.29}$$

$$\Delta F_{n,2}^{ij} = c_n^{ij}\eta. \tag{6.30}$$

This, combines with (6.10), implies

$$F_n^{ij} = F_{n,1}^{ij} + F_{n,2}^{ij} + L_n^{ij}, \text{ in } B_{r_n}. \tag{6.31}$$

Here L_n^{ij} is a harmonic 1-form so that we can estimate $L^{2,1}$ norm of DL_n^{ij} in the same way as that of G_n^{ij}. For $F_{n,1}^{ij}$ and $F_{n,2}^{ij}$, we have

$$\|D^2 F_{n,1}^{ij}\|_{L^1(R^2)} \leq C\|\eta(H_n^{ij} - c_n^{ij})\|_{\mathcal{H}^1(R^2)} \leq C,$$

$$\begin{aligned}\|D^2 F^{ij}_{n,2}\|_{L^1(B_{r_n})} &\le Cr_n \|D^2 F^{ij}_{n,2}\|_{L^2(B_{r_n})} \\ &\le Cr_n \|c^{ij}_n \eta\|_{L^2(R^2)} = Cr_n^2 |c^{ij}_n| \\ &\le C\|\overline{h}_n\|_{L^1(B_{r_n})} \\ &\le C\|h_n\|_{L^1(B_\delta)} \le C.\end{aligned}$$

From these, we obtain the bound of $\|D^2 F^{ij}_n\|_{L^1(B_{r_n})}$. The smallness of $\|DF^{ij}_n\|_{L^{2,\infty}(B_{r_n})}$ follows from the equi-integrable condition of h_n. In fact, for any $\epsilon > 0$, we can choose $\delta > 0$ sufficiently small so that

$$\|\eta H^{ij}_n\|_{L^1(R^2)} \le \|\overline{h}_n\|_{L^1(B_{r_n})} \le \|h_n\|_{L^1(B_\delta)} \le \epsilon.$$

If, in addition, h_n is equi-integrable in $\mathcal{H}^1(M)$, then the above argument implies that $\|DF^{ij}_n\|_{L^{2,1}(B_{r_n})} = 0(\delta)$ so that the oscillation convergence follows as well.

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