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Curvature Estimates and the Positive Mass Theorem

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The Positive Mass Theorem implies that any smooth, complete, asymptotically flat 3-manifold with non-negative scalar curvature which has zero total mass is isometric to $(\mathbb{R}^3, \delta_{ij})$. In this paper, we quantify this statement using spinors and prove that if a complete, asymptotically flat manifold with non-negative scalar curvature has small mass and bounded isoperimetric constant, then the manifold must be close to $(\mathbb{R}^3, \delta_{ij})$, in the sense that there is an upper bound for the L^2 norm of the Riemannian curvature tensor over the manifold except for a set of small measure. This curvature estimate allows us to extend the case of equality of the Positive Mass Theorem to include non-smooth manifolds with generalized non-negative scalar curvature, which we define.

1. Introduction.

We introduce our problem in the context of General Relativity. Consider a $3+1$ dimensional Lorentzian manifold N with metric $g_{\alpha\beta}$ of signature $(-+++)$. We denote the induced Levi-Civita connection by $\bar{\nabla}$. Then the corresponding Ricci tensor $\bar{R}_{\alpha\beta}$ satisfies Einstein's equations

$$\bar{R}_{\alpha\beta} - \frac{1}{2}\,\bar{R}\,g_{\alpha\beta} = 8\pi\, T_{\alpha\beta}, \tag{1}$$

where $T_{\alpha\beta}$ is the energy-momentum tensor (which describes the distribution of matter in space-time). Furthermore, we are given a complete, oriented, space-like hypersurface M. The Lorentzian metric $g_{\alpha\beta}$ induces on M a Riemannian metric g_{ij} (we always use Latin indices on the hypersurface and Greek indices in the embedding manifold). Choosing on M a normal vector field ν, the exterior curvature of M is given by the second fundamental form $h_{jk} = (\bar{\nabla}_j \nu)_k$. We make the following assumptions:

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(I) M is asymptotically flat. Thus we assume that there is a compact set K such that $M \setminus K$ is diffeomorphic to the region $\mathbb{R}^3 \setminus B_r(0)$ outside a ball of radius r. Under this diffeomorphism, the metric should be of the form

$$(2) \qquad g_{jk}(x) = \delta_{jk} + a_{jk}(x), \qquad x \in \mathbb{R}^3 \setminus B_r(0),$$

where a_{ij} decays at infinity as

$$(3) \quad a_{ij} = O(1/r), \quad \partial_k a_{ij} = O(1/r^2), \quad \text{and} \quad \partial_{kl} a_{ij} = O(1/r^3).$$

The second fundamental form should decay as

$$(4) \qquad h_{ij} = O(1/r^2) \qquad \text{and} \qquad \partial_k h_{ij} = O(1/r^3).$$

(II) The energy-momentum tensor satisfies on M the dominant energy condition [4], i.e., for each point $p \in M$ and for each time-like vector $u \in T_pN$, the vector $T^\alpha_\beta u^\beta$ is non-spacelike and $T_{\alpha\beta}\, u^\alpha u^\beta \leq 0$.

For the hypersurface M, one can define the total energy and momentum, as first introduced by Arnowitt, Deser, and Misner [1]. For this, one considers in the asymptotic end the coordinate spheres S_R, $R > r$, around the origin and takes limits of integrals over these spheres,

$$(5) \qquad E = \frac{1}{16\pi} \lim_{R\to\infty} \int_{S_R} (\partial_j g_{ij} - \partial_i g_{jj})\, d\Omega^i$$

$$(6) \qquad P_k = \frac{1}{8\pi} \lim_{R\to\infty} \int_{S_R} (h_{ki} - \delta_{ki}\, h_{jj})\, d\Omega^i,$$

where $d\Omega^i = \nu^i\, du$, du is the area form, and ν is the normal vector to S_R in the coordinate chart. The Positive Energy Theorem [6, 7] states that, under the considered assumptions, $E \geq |P|$. In the case that the second fundamental form is identically zero, the total momentum vanishes. Then the total energy is also called the total mass m, and the Positive Mass Theorem states that $m \geq 0$.

In this paper, our aim is to study how total energy and momentum control the Riemannian curvature tensor. Following Witten's proof of the Positive Energy Theorem, we consider the massless Dirac equation on the hypersurface M. We derive an integral estimate for the Riemannian curvature tensor $\bar{R}_{\alpha\beta\gamma\delta}$ involving total energy/momentum and the Dirac wave function Ψ. We then restrict our attention to the case of zero second fundamental form. By substituting in a-priori estimates for the Dirac wave function, we get an L^2 estimate for the Riemannian curvature tensor of M in terms of its total mass. More precisely, our main theorem is the following:

Theorem 1.1. *There exist positive constants c_1, c_2, and c_3 such that for any smooth, complete, asymptotically flat manifold (M^3, g) with non-negative scalar curvature and total mass m and any smooth, bounded function η with bounded gradient on M,*

$$\begin{aligned}(7)\qquad &\int_{M\setminus D} \eta\, R_{ijkl}R^{ijkl}\, d\mu \\ &\leq m\, c_1 \sup_M \left(|\eta||R_{ijkl}| + |\Delta\eta|\right) + \sqrt{m}\, c_2\, \|\eta\, |\nabla_k R_{ij\alpha\beta}|\|_{L^2}\,,\end{aligned}$$

where the set D depends on M with

$$(8)\qquad \mathrm{Vol}\,(D)^{1/3} \;\leq\; 64\pi\, c_3\, \frac{m}{k^2}\quad ,$$

$k = \inf \frac{A}{V^{2/3}}$ is the isoperimetric constant of M, R_{ijkl} is the Riemannian curvature tensor of M, and $\|.\|_{L^2}$ is the L^2-norm on (M^3, g).

As an application of this theorem, we finally extend the case of equality of the Positive Mass Theorem to non-smooth manifolds.

2. Spinors, the Hypersurface Dirac Operator.

We begin with a brief introduction to Dirac spinors in curved space-time. Following [3], we work in a coordinate chart (for a coordinate-free formulation see e.g., [5]). The *Dirac operator* G is a differential operator of first order,

$$(9)\qquad G = iG^\alpha(x)\frac{\partial}{\partial x^\alpha} + B(x),$$

where B and the Dirac matrices G^α are (4×4)-matrices. The Dirac matrices and the Lorentzian metric are related by the anti-commutation relations

$$(10)\qquad -g^{\alpha\beta}(x) = \frac{1}{2}\left\{G^\alpha(x),\, G^\beta(x)\right\} \;\equiv\; \frac{1}{2}\left(G^\alpha G^\beta + G^\beta G^\alpha\right)(x).$$

The four-component, complex wave function Ψ of a Dirac particle satisfies the Dirac equation

$$(G - m_0)\,\Psi = 0,$$

where m_0 is the rest mass of the Dirac particle. At every space-time point x, the wave functions are endowed with an indefinite scalar product, which we call *spin scalar product.* For two wave functions Ψ and Φ, it takes the form

$$\prec\!\Psi \,|\, \Phi\!\succ\!(x) = \Psi^*(x) \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix} \Phi(x),$$

where '*' denotes complex conjugation and where $1\!\!1$, 0 are (2×2)-submatrices (in physics, this scalar product is currently written in the form $\prec\!\Psi \,|\, \Phi\!\succ = \overline{\Psi}\Phi$ with the "adjoint spinor" $\overline{\Psi}$). The Dirac matrices $G^\alpha(x)$ are Hermitian with respect to the spin scalar product. By integrating the spin scalar product over the hypersurface M, we form the scalar product

$$(11) \qquad \langle \Psi \,|\, \Phi \rangle = \int_M \prec\!\Psi \,|\, G^\alpha\, \Phi\!\succ \nu_\alpha \, d\mu,$$

where $d\mu = \sqrt{\det\, g_{ij}}\, d^3x$ is the invariant measure on M. This scalar product is definite; we can assume it to be positive. The integrand of (11) has the physical interpretation as the probability density of the particle. Since it will appear in our calculations very often, we introduce the short notation

$$(\Psi \,|\, \Phi) \;\equiv\; \prec\!\Psi \,|\, G^\alpha \nu_\alpha\, \Phi\!\succ.$$

For a given Lorentzian metric, the Dirac matrices $G^\alpha(x)$ are not uniquely determined by the anti-commutation relations (10). One way to handle this problem is to work with spin bundles and orthonormal frames [5]. More generally, it is shown in [3] that all possible choices of Dirac matrices lead to unitarily equivalent Dirac operators. One must keep in mind, however, that the matrix $B(x)$ in (9) depends on the choice of the G^α; it can be given explicitly in terms of the Dirac matrices G^α and their first partial derivatives.

The Dirac matrices induce a connection D on the spinors, which we call *spin derivative.* In a chart, it has the representation

$$(12) \qquad D_\alpha = \frac{\partial}{\partial x^\alpha} - iE_\alpha(x),$$

where the (4×4)-matrices $E_\alpha(x)$ are functions of $G^\alpha(x)$ and $\partial_\alpha G^\beta(x)$ (see [3] for an explicit formula). The spin derivative is compatible with the spin scalar product, i.e.,

$$(13) \qquad \partial_j \prec\!\Psi \,|\, \Phi\!\succ = \prec\! D_j\Psi \,|\, \Phi\!\succ + \prec\!\Psi \,|\, D_j\Phi\!\succ.$$

Furthermore, the combined spin and covariant derivative of the Dirac matrices vanishes,

$$(14) \qquad [D_j, G^k] + \bar{\Gamma}^k_{jl}\, G^l = 0$$

($\bar{\Gamma}^i_{jk}$ denote the Christoffel symbols of the Levi-Civita connection). The curvature of the spin connection is given by the commutator $[D_j, D_k] \equiv D_j D_k - D_k D_j$. It is related with the Riemannian curvature tensor by

$$[D_\alpha, D_\beta] \;=\; \frac{1}{8}\,\bar{R}_{\alpha\beta\gamma\delta}\,[G^\gamma, G^\delta]. \tag{15}$$

Using the spin derivative, one can write the Dirac operator (9) in the alternative form

$$G = i\sum_{\alpha=1}^{4} G^\alpha(x)\,D_\alpha.$$

Similar to the procedure in [7, 5], we next define the so-called *hypersurface Dirac operator* $\mathcal{D}$. For this, we consider the Dirac matrices G^α and the spin derivative (12) of the Lorentzian manifold, but take only the derivatives tangential to M,

$$\mathcal{D} := i\sum_{j=1}^{3} G^j(x)\,D_j.$$

The hypersurface Dirac operator can be considered as a differential operator on the four-component wave functions on the hypersurface M. According to [7, 5], the square of the hypersurface Dirac operator satisfies the Weitzenböck formula

$$\begin{aligned} \mathcal{D}^2 &= D_j^* D^j + \mathcal{R} \qquad \text{with} \\ \mathcal{R} &= \frac{1}{4}\,(\bar{R} + 2\bar{R}_{\alpha\beta}\,\nu^\alpha\nu^\beta + 2\bar{R}_{\alpha i}\,\nu^\alpha\,\nu_\beta\,G^\beta G^i), \end{aligned} \tag{16}$$

where D_j^* denotes the formal adjoint of the operator D_j. As a consequence of the dominant energy condition **(II)** and Einstein's equations (1), the (4×4)-matrix $\mathcal{R}$ is positive definite.

Let us introduce a convenient notation for the covariant and spin derivatives. The Levi-Civita connection $\bar{\nabla}_\alpha$ and the spin connection D_α give a parallel transport of tensor and spinor fields, respectively. Furthermore, the induced metric g_{jk} yields a Levi-Civita connection on M. For clarity, we denote this last connection and all its derived "intrinsic" curvature objects without a bar; i.e., we have the covariant derivative ∇_j with Christoffel symbols Γ^j_{kl}, the curvature tensor R^j_{klm}, etc.. For a derivative tangential to M, the connections $\bar{\nabla}$ and ∇ are related to each other by

$$\bar{\nabla}_j u = \nabla_j u + h_{jk}\,u^k\,\nu, \tag{17}$$

where u denotes a vector field tangential to the hypersurface. Combining the spin and Levi-Civita connections, we can differentiate all objects with spin and tensor indices. With a slight abuse of notation, we write this derivative with a nabla. A bar indicates that we treat the tensor indices with the Christoffel symbols $\bar{\Gamma}$; otherwise, the connection Γ is used. For example, we have

$$\begin{aligned} \bar{\nabla}_i \Psi &= \nabla_i \Psi = D_i \Psi \\ \nabla_i \nabla_j \Psi &= D_i D_j \Psi - \Gamma^k_{ij} \, D_k \Psi \\ \bar{\nabla}_i \bar{\nabla}_j \Psi &= D_i D_j \Psi - \bar{\Gamma}^k_{ij} \, D_k \Psi, \text{ etc.} \end{aligned}$$

For the proof of our curvature estimates, we shall choose a constant spinor Ψ_0 in the asymptotic end and consider the solution of the massless hypersurface Dirac equation with asymptotic boundary values Ψ_0,

$$\mathcal{D}\Psi = 0 \qquad \text{with} \qquad \lim_{|x|\to\infty} \Psi(x) = \Psi_0. \tag{18}$$

The existence of such a solution is proved in [5]. The wave function behaves at infinity like

$$\partial_j \Psi = O(1/r^2), \qquad \partial_{jk} \Psi = O(1/r^3). \tag{19}$$

We remark that the massless Dirac equation (18) decouples into two two-spinor equations, the so-called Weyl equations (which separately describe the left and right handed components of the Dirac spinor). But this is not very useful for us; we prefer working with four-component Dirac spinors.

In order to illustrate our notation, we finally outline Witten's proof of the Positive Energy Theorem in our setting. We take the solution Ψ of the Dirac equation (18) and compute the following divergence:

(20)

$$\begin{aligned} &\nabla_j (D^j \Psi \,|\, \Psi) = \nabla_j \prec\! D^j \Psi \,|\, G^\alpha \nu_\alpha \, \Psi \!\succ \\ &\overset{(13)}{=} \prec\! \nabla_j D^j \Psi \,|\, G^\alpha \nu_\alpha \, \Psi \!\succ + \prec\! D^j \Psi \,|\, \partial_j (G^\alpha \nu_\alpha) \, \Psi \!\succ + \prec\! D^j \Psi \,|\, G^\alpha \nu_\alpha \, D_j \Psi \!\succ \\ &\overset{(14)}{=} (\nabla_j D^j \Psi \,|\, \Psi) + \prec\! D^j \Psi \,|\, h_{jk} G^k \, \Psi \!\succ + (D^j \Psi \,|\, D_j \Psi) \\ &= ((\nabla_j + G^\alpha \nu_\alpha \, h_{jk} G^k) \, D^j \Psi \,|\, \Psi) + (D^j \Psi \,|\, D_j \Psi). \end{aligned}$$

Using that the formal adjoints of the operators D_j are

$$D_j^* = -\nabla_j - G^\alpha \nu_\alpha \, h_{jk} G^k, \tag{21}$$

we can write (20) in the shorter form

$$(D^j\Psi \,|\, D_j\Psi) = \nabla_j(D^j\Psi \,|\, \Psi) + (D_j^* D^j\Psi \,|\, \Psi).$$

We now integrate both sides and substitute the Weitzenböck formula (16). This gives the identity

$$\langle D\Psi \,|\, D\Psi\rangle + \langle \Psi \,|\, \mathcal{R}\,\Psi\rangle = \int_M \nabla_j(D^j\Psi \,|\, \Psi)\, d\mu. \tag{22}$$

Since $\mathcal{R}$ is a positive matrix, the left side of (22) is positive. The right side of this equation is an integral over a divergence. If this integral is approximated by integrals over the balls B_R, $R > r$, we can apply Gauss' theorem to rewrite them in terms of boundary integrals over the spheres S_R. As explained in detail in [5], these boundary integrals can be identified with the integrals in (5) and (6). More precisely,

$$\begin{aligned}\int_M \nabla_j(D^j\Psi \,|\, \Psi)\, d\mu &= \lim_{R\to\infty} \int_{S_R} (D_j\Psi \,|\, \Psi)\; d\Omega^j \\ &= 4\pi\,(E\,|\Psi_0|^2 + \prec\!\Psi_0 \,|\, P_k G^k\, \Psi_0\!\succ).\end{aligned}$$

The Positive Energy Theorem follows by choosing Ψ_0 such that $\prec\!\Psi_0 \,|\, P_k G^k\, \Psi_0\!\succ \;=\; -|P|$ and $|\Psi_0|^2 = 1$. Namely, in this case, one gets in combination with (22) the inequalities

$$0 \leq \langle D\Psi \,|\, D\Psi\rangle \leq 4\pi\,(E - |P|). \tag{23}$$

3. Estimates of the Riemann Tensor.

We begin with a pointwise estimate of the Riemann tensor of the Lorentzian manifold in terms of the second derivative of the Dirac wave function.

Lemma 3.1. *For any solution* Ψ *of the hypersurface Dirac operator* (18),

$$\left(\sqrt{\bar{R}_{ijkl}\bar{R}^{ijkl}} - \sqrt{2\,\bar{R}_{ijk\alpha}\nu^\alpha\,\bar{R}^{ijk\beta}\nu_\beta}\right)^2 \;(\Psi \,|\, \Psi) \leq 32\,(\nabla_j\nabla_k\Psi \,|\, \nabla^j\nabla^k\Psi). \tag{24}$$

Proof. Relation (15) and the Schwarz inequality yield the following estimate:

$$\begin{aligned}(\bar{R}_{jk\alpha\beta}\, G^\alpha G^\beta\Psi \,|\, \bar{R}^{jk\gamma\delta}\, G_\gamma G_\delta\Psi) &= 16\,([D_j, D_k]\Psi \,|\, [D^j, D^k]\Psi) \qquad (25)\\ &= 32\,\Big((\nabla_j\nabla_k\Psi \,|\, \nabla^j\nabla^k\Psi) - (\nabla_j\nabla_k\Psi \,|\, \nabla^k\nabla^j\Psi)\Big) \\ &\leq 64\,(\nabla_j\nabla_k\Psi \,|\, \nabla^j\nabla^k\Psi) \qquad (26)\end{aligned}$$

Let us analyze the curvature term on the left side of (25) more explicitly. For simplicity in notation, we choose a chart with $\nu = \frac{\partial}{\partial x^0}$. We decompose the Riemann tensor into the tangential and normal components,

$$\bar{R}_{jk\alpha\beta}\, G^\alpha G^\beta = \bar{R}_{jklm}\, G^l G^m + 2\bar{R}_{jkl0}\, G^l G^0.$$

Since the Dirac matrices are Hermitian with respect to the spin scalar product and the matrix G^0 anti-commutes with the G^j, we obtain

$$\begin{aligned}
(27)\qquad & (\bar{R}_{ij\alpha\beta}\, G^\alpha G^\beta \Psi \,|\, \bar{R}^{ij\gamma\delta}\, G_\gamma G_\delta \Psi) \\
&= \prec \bar{R}_{ij\alpha\beta}\, G^\alpha G^\beta \Psi \,|\, (-G^0)\, \bar{R}^{ij\gamma\delta}\, G_\gamma G_\delta \Psi \succ \\
&= (\Psi \,|\, \bar{R}_{ijkl}\, G^l G^k\, \bar{R}^{ijmn}\, G_m G_n \Psi) \\
&\quad - 4\, (\Psi \,|\, \bar{R}_{ijk0}\, G^0 G^k\, \bar{R}^{ijm0}\, G_m G_0 \Psi) \\
&\quad - 2\, (\Psi \,|\, \bar{R}_{ijk0}\, G^0 G^k\, \bar{R}^{ijmn}\, G_m G_n \Psi) \\
(28)\qquad &\quad + 2\, (\Psi \,|\, \bar{R}_{ijkl}\, G^l G^k\, \bar{R}^{ijm0}\, G_m G_0 \Psi).
\end{aligned}$$

The products of Dirac matrices can be simplified with the anti-commutation rules (10). The important point is that, in both summands in (27), the Dirac matrices combine to a positive multiple of the identity,

$$\begin{aligned}
\bar{R}_{ijkl}\, G^l G^k\, \bar{R}^{ijmn}\, G_m G_n &= 2\, \bar{R}_{ijkl} \bar{R}^{ijkl}\, \mathbb{1} \\
-4\, \bar{R}_{ijk0}\, G^0 G^k\, \bar{R}^{ijm0}\, G_m G_0 &= 4\, \bar{R}_{ijk\alpha} \nu^\alpha\, \bar{R}^{ijk\beta} \nu_\beta \mathbb{1}
\end{aligned}$$

In the two summands in (28), the products of Dirac matrices is more complicated, and the sign of the terms is undetermined. But we can bound both summands from below with the Schwarz inequality,

$$\begin{aligned}
& - 2\, (\Psi \,|\, \bar{R}_{ijk0}\, G^0 G^k\, \bar{R}^{ijmn}\, G_m G_n \Psi) \\
&\geq -\sqrt{2 \bar{R}_{ijkl} \bar{R}^{ijkl}}\, \sqrt{4\, \bar{R}_{ijk\alpha} \nu^\alpha\, \bar{R}^{ijk\beta} \nu_\beta}\, (\Psi \,|\, \Psi), \\
& 2\, (\Psi \,|\, \bar{R}_{ijkl}\, G^l G^k\, \bar{R}^{ijm0}\, G_m G_0 \Psi) \\
&\geq -\sqrt{2 \bar{R}_{ijkl} \bar{R}^{ijkl}}\, \sqrt{4\, \bar{R}_{ijk\alpha} \nu^\alpha\, \bar{R}^{ijk\beta} \nu_\beta}\, (\Psi \,|\, \Psi).
\end{aligned}$$

By substituting into (27) and (28), we obtain the estimate

$$\begin{aligned}
& (\bar{R}_{ij\alpha\beta}\, G^\alpha G^\beta \Psi \,|\, \bar{R}^{ij\gamma\delta}\, G_\gamma G_\delta \Psi) \\
&\geq \left(\sqrt{2 \bar{R}_{ijkl} \bar{R}^{ijkl}} - \sqrt{4\, \bar{R}_{ijk\alpha} \nu^\alpha\, \bar{R}^{ijk\beta} \nu_\beta} \right)^2 (\Psi \,|\, \Psi).
\end{aligned}$$

□

In the following lemma, we estimate the integral over the right side of (24) from above. Similar as in Witten's proof of the Positive Energy Theorem, this is done by integrating one spin derivative by parts. The higher order of the derivatives makes the calculation more complicated; on the other hand, we do not get boundary terms.

Lemma 3.2. *There are constants c_1 and c_2 independent of the geometry of M and N such that for any smooth, bounded function η with bounded gradient on M and the Dirac wave function of the Positive Mass Theorem* (23),

$$\begin{aligned}
&\int_M \eta\, (\nabla^j \nabla^k \Psi \,|\, \nabla_j \nabla_k \Psi) \\
&\leq c_1\, (E - |P|) \sup_M \left(|\partial_j \eta|\, |h_{kl}| + |\eta|\, (|\nabla_j h_{kl}| + |R_{ijkl}| + |h_{ij}|^2) + |\Delta \eta| \right) \\
&\quad + c_2 \sqrt{E - |P|}\, \left\| \eta\, (|\nabla_k \bar{R}_{ij\alpha\beta}| + |h_{ij}\, \nabla_k h_{lm}| + |h_{ij}\, \bar{R}_{klmn}|) \right\|_{L^2} \sup_M |\Psi|.
\end{aligned}$$

In this formula, Δ denotes the Laplace-Beltrami operator on M, and $|\Psi| \equiv (\Psi|\Psi)^{\frac{1}{2}}$.

Proof. Exactly as in (20), we compute the following divergence:

$$\begin{aligned}
\nabla_j (\nabla^j \nabla^k \Psi \,|\, \nabla_k \Psi) = {}& ((\nabla_j + G^\alpha \nu_\alpha\, h_{jk} G^k)\, \nabla^j \nabla^k \Psi \,|\, \nabla_k \Psi) \\
&+ (\nabla^j \nabla^k \Psi \,|\, \nabla_j \nabla_k \Psi).
\end{aligned}$$

Using the short notation with the formal adjoint

$$\nabla_j^* \equiv -\nabla_j - G^\alpha \nu_\alpha\, h_{jk} G^k, \tag{29}$$

we can also write

$$(\nabla^j \nabla^k \Psi \,|\, \nabla_j \nabla_k \Psi) = \nabla_j (\nabla^j \nabla^k \Psi \,|\, \nabla_k \Psi) + (\nabla_j^*\, \nabla^j \nabla^k \Psi \,|\, \nabla_k \Psi).$$

We multiply this equation by η and integrate over M. According to the decay properties (3), (4), and (19) we can integrate by parts without boundary terms and obtain

$$\begin{aligned}
&\int_M \eta\, (\nabla^j \nabla^k \Psi \,|\, \nabla_j \nabla_k \Psi)\, d\mu \\
&= -\int_M (\partial_j \eta)\, (\nabla^j \nabla^k \Psi \,|\, \nabla_k \Psi)\, d\mu + \int_M \eta\, (\nabla_j^* \nabla^j \nabla^k \Psi \,|\, \nabla_k \Psi)\, d\mu.
\end{aligned} \tag{30}$$

We estimate the resulting integrals. Since the left side of (30) is real, we must only consider the real parts of all terms. In the first integral on the right side of (30), we can integrate by parts once again. Using again the decay properties (3), (4), and (19), and the fact that $\partial\eta$ is bounded, we obtain, as in (20),

$$\begin{aligned}
&\mathrm{Re} \int_M (\partial_j \eta)\, (\nabla^j \nabla^k \Psi \,|\, \nabla_k \Psi)\, d\mu \\
&= \frac{1}{2} \int_M (\partial_j \eta) \left(\partial^j (D^k \Psi \,|\, D_k \Psi) - \prec\! D^k \Psi \,|\, h^{jl} G_l D_k \Psi \!\succ \right) d\mu \\
&= \frac{1}{2} \int_M (D^k \Psi \,|\, \left(-\Delta\eta - G^\alpha \nu_\alpha\, G_l\, h^{lj} (\partial_j \eta) \right) D_k \Psi)\, d\mu .
\end{aligned}$$

We bound the obtained integral with the sup-norm and substitute in the Positive Energy Theorem (23),

$$\begin{aligned}
&\left| \mathrm{Re} \int_M (\partial_j \eta)\, (\nabla^j \nabla^k \Psi \,|\, \nabla_k \Psi)\, d\mu \right| \\
&\leq \frac{1}{2} \sup_M \left(|\Delta\eta| + |\partial_j \eta|\, |h_{kl}| \right) \int_M |D\Psi|^2\, d\mu \\
&\leq 2\pi \sup_M \left(|\Delta\eta| + |\partial_j \eta|\, |h_{kl}| \right) (E - |P|).
\end{aligned}$$

The second summand on the right side of (30) is more difficult because it involves third derivatives of the wave function. Our method is to commute the ∇^k-derivative to the very left using the transformation

$$\nabla_j^* \nabla^j \nabla^k \Psi = \nabla_j^* \left[\nabla^j, \nabla^k \right] \Psi + \left[\nabla_j^*, \nabla^k \right] \nabla^j \Psi + \nabla^k\, \nabla_j^* \nabla^j \Psi . \tag{31}$$

In the resulting third order term, we can apply the Weitzenböck formula,

$$\nabla^k\, \nabla_j^* \nabla^j \Psi = \nabla^k (D_j^* D^j) \Psi = -\nabla^k (\mathcal{R}\, \Psi) = -(D^k \mathcal{R})\, \Psi - \mathcal{R}\, (D^k \Psi). \tag{32}$$

The two commutators in (31) yield terms involving curvature and the second fundamental form, more precisely

$$\nabla_j^* \left[\nabla^j, \nabla^k \right] \Psi = \frac{1}{4} \nabla_j^* \left(\bar{R}^{jk\alpha\beta}\, G_\alpha G_\beta\, \Psi \right) \tag{33}$$

$$\left[\nabla_j^*, \nabla^k \right] \nabla^j \Psi \stackrel{(29)}{=} - \left[\nabla_j, \nabla^k \right] \nabla^j \Psi - \left[G^\alpha \nu_\alpha h_{jl} G^l, \nabla^k \right] \nabla^j \Psi \tag{34}$$

$$\begin{aligned}
&= -\frac{1}{4} \bar{R}^{jk\alpha\beta}\, G_\alpha G_\beta\, D_j \Psi \;-\; R^{kl}\, D_l \Psi \\
&\quad + G^\alpha \nu_\alpha\, (\nabla^k h_{jl})\, G^l D^j \Psi .
\end{aligned} \tag{35}$$

We mention for clarity that the first summand in (35) comes about as the curvature of the spin connection, whereas the second summand arises as the Riemannian curvature of M; this can be seen more explicitly by writing out ∇ with the spin derivative D and the Christoffel symbols of the Levi-Civita connection on M. The third summand in (35) is obtained by combining (14) with (17).

With the transformations (32), (33), and (35), we can reduce the third order derivative of the wave function $\nabla^*_j \nabla^j \nabla^k \Psi$ to expressions which contain derivatives of Ψ of at most first order. More precisely, using the Gauss equation, this allows us to estimate the scalar product $(\nabla^*_j \nabla^j \nabla^k \Psi \,|\, \nabla_l \Psi)$ in the form

$$(\nabla^*_j \nabla^j \nabla^k \Psi \,|\, \nabla_l \Psi) \;\leq\; C_1 \left(|R_{ijkl}| + |h_{jk}|^2 + |\nabla_j h_{kl}|\right) (D\Psi \,|\, D\Psi) \tag{36}$$

$$+\, C_2 \left(|\nabla_i \bar{R}_{jklm}| + |h_{ij}|\, |\bar{R}_{klmn}|\right) (\Psi \,|\, D\Psi) \tag{37}$$

with suitable constants C_1 and C_2 which are independent of the geometry of M and N. We multiply both sides of this inequality by η and integrate over M. In the integral over (36), we estimate with the sup-norm, whereas the integral over (37) can be bounded using the Schwarz inequality,

$$\begin{aligned}
&\int_M \eta \, (\nabla^*_j \nabla^j \nabla^k \Psi \,|\, \nabla_l \Psi) \, d\mu \\
&\leq C_1 \sup_M (\eta \, (|R_{ijkl}| + |h_{jk}|^2 + |\nabla_j h_{kl}|)) \int_M (D\Psi \,|\, D\Psi) \, d\mu \\
&\quad + C_2 \sup_M |\Psi| \; \left\| \eta \, (|\nabla_i \bar{R}_{jklm}| + |h_{ij}|\, |\bar{R}_{klmn}|) \right\|_{L^2} \||D\Psi|\|_{L^2}.
\end{aligned}$$

Finally, we substitute the Positive Energy Theorem (23). □

We now combine the results of Lemma 3.1 and 3.2.

Corollary 3.3. *There are constants c_1 and c_2 independent of the geometry of M and N such that for any smooth, bounded function η with bounded gradient on M and the Dirac wave function of the Positive Mass Theorem (23),*

$$\begin{aligned}
&\int_M \eta \left(\sqrt{\bar{R}_{ijkl} \bar{R}^{ijkl}} - \sqrt{2\, \bar{R}_{ijk\alpha} \nu^\alpha \, \bar{R}^{ijk\beta} \nu_\beta} \right)^2 (\Psi \,|\, \Psi) \, d\mu \\
&\leq c_1 \, (E - |P|) \sup_M \left(|\partial_j \eta| \, |h_{kl}| + |\eta| \, (|\nabla_j h_{kl}| + |R_{ijkl}| + |h_{ij}|^2) + |\Delta \eta| \right) \\
&\quad + c_2 \sqrt{E - |P|} \; \left\| \eta \, (|\nabla_k \bar{R}_{ij\alpha\beta}| + |h_{ij} \, \nabla_k h_{lm}| + |h_{ij} \, \bar{R}_{klmn}|) \right\|_{L^2} \sup_M |\Psi|.
\end{aligned}$$

It seems likely to the authors that this inequality is not optimal in the case of non-zero fundamental form, in the sense that $E - |P| = 0$ does not imply that M^3 is a submanifold of Minkowski space. Improvements of the estimate are still under investigation. However, in the case $h_{ij} \equiv 0$, the above inequality is very useful, as we shall see in what follows.

For the rest of this paper we will assume that $h_{ij} \equiv 0$. Hence, all of the remaining theorems will concern Riemannian 3-manifolds (M^3, g) which, by the dominant energy condition (II) and the Gauss equation, must have non-negative scalar curvature. Then in this zero second fundamental form setting, it follows from the Gauss and Codazzi equations that $\bar{R}_{ijkl} = R_{ijkl}$ and $\bar{R}_{ijk\alpha}\nu^\alpha = 0$, where R is the Riemannian curvature tensor of (M^3, g). It also follows in this setting that the total momentum is zero, so that the total energy E equals the total mass m.

Corollary 3.4. *There exist positive constants c_1 and c_2 such that for any smooth, complete, asymptotically flat manifold (M^3, g) with non-negative scalar curvature and total mass m and any smooth, bounded function η with bounded gradient on M^3,*

$$
\begin{aligned}
(38) \qquad & \int_M \eta\, R_{ijkl} R^{ijkl}\, (\Psi \mid \Psi)\, d\mu \\
& \leq c_1\, m \sup_M \left(|\eta| |R_{ijkl}| + |\Delta \eta| \right) + c_2 \sqrt{m}\, \left\| \eta\, |\nabla_k R_{ij\alpha\beta}| \right\|_{L^2} \sup_M |\Psi|,
\end{aligned}
$$

where Ψ is the Dirac wave function of the Positive Mass Theorem (23).

The interesting point of this estimate is that the terms on the right side of our inequality all contain factors m or $\sqrt{m}$, which vanish when the total mass goes to zero. The disadvantage of our estimate is that it involves the Dirac wave function. In order to get a more explicit estimate, we shall in the next section derive a-priori bounds for Ψ.

4. Upper and Lower Bounds for the Norm of the Spinor.

First, we observe that we can use the maximum principle to prove that $|\Psi(x)| \leq 1$. To do this, we must derive a second order scalar inequality for $|\Psi(x)|^2$. Recall that we are still assuming that M^3 has zero second fundamental form, as we will do for the remainder of the paper. Then we have

$$\begin{aligned}
(39)\quad \partial_j(\Psi \mid \Psi) &= (D_j\Psi \mid \Psi) + (\Psi \mid D_j\Psi) \\
\Delta(\Psi \mid \Psi) &= \nabla_j\left((D^j\Psi \mid \Psi) + (\Psi \mid D^j\Psi)\right) \\
&= (\nabla_j D^j\Psi \mid \Psi) + 2\,(D_j\Psi \mid D^j\Psi) + (\Psi \mid \nabla_j D^j\Psi) \\
&\overset{(21),(17)}{=} -(D_j^* D^j\Psi \mid \Psi) + 2\,|D\Psi|^2 - (\Psi \mid D_j^* D^j\Psi).
\end{aligned}$$

Substituting in the Weitzenböck formula (23), we find that, by (16),

$$\Delta|\Psi|^2 \geq \frac{1}{2}\,(\Psi \mid R\,\Psi) + 2\,|D\Psi|^2,$$

where R is the scalar curvature of (M^3, g). Since $R \geq 0$ in the zero second fundamental form context, it follows that $|\Psi(x)|^2$ is subharmonic. Using that $|\Psi(x)|^2$ goes to one at infinity by construction, the maximum principle yields that

$$(40)\qquad |\Psi(x)| \leq 1$$

for all x.

To get a lower bound for $|\Psi(x)|$, we let $f(x) = |\Psi(x)|^2$ and observe that equations (39), (40), and (23) imply that

$$\int_M |\nabla f|^2 \leq 4 \int_M |D\Psi|^2 |\Psi|^2 \leq 16\pi\, m.$$

Then the Sobolev inequality applied to $1 - f(x)$ yields for some positive constant c_3

$$(41)\qquad k^2 \left(\int_M (1 - f(x))^6\right)^{1/3} \leq c_3 \int_M |\nabla f|^2 \leq 16\pi\, c_3\, m,$$

where k is the isoperimetric constant of M defined to be

$$k = \inf \frac{A}{V^{2/3}},$$

and the infimum is taken over all smooth regions with volume V and boundary area A (the Sobolev constant of M^3 can be bounded by $\sqrt{c_3}/k$, where c_3 is a constant independent of the geometry of M). The inequality (41) immediately implies the following lemma, which gives a lower bound for $|\Psi|^2$ except on a set of small measure.

Lemma 4.1. *Let k be the isoperimetric constant of M. Then for any $c < 1$,*

$$|\Psi|^2 \geq c$$

except on a set $D(c)$ with

$$\mathrm{Vol}\,(D(c))^{1/3} \leq \frac{16\pi\, c_3}{(1-c)^2}\, \frac{m}{k^2}. \tag{42}$$

5. Proof of the Main Theorem and Applications.

Our main Theorem 1.1 immediately follows by combining the bound (40) and Lemma 4.1 (where we set $c = 1/2$) with Corollary 3.4. We note that the constants c_1 ,c_2, and c_3 could be computed in a straightforward manner, although we do not carry this out since their actual value is not important for our applications of the theorem. Also, we see that by choosing η to be zero everywhere except in a neighborhood of a given point, Theorem 1.1 yields the following corollary.

Corollary 5.1. *Suppose $\{g_i\}$ is a sequence of smooth, complete, asymptotically flat metrics on M^3 with non-negative scalar curvature and total masses $\{m_i\}$ which converge to a possibly non-smooth limit metric g in the C^0 sense. Let U be the interior of the set of points where this convergence of metrics is locally C^3 and nondegenerate.*

Then if the metrics $\{g_i\}$ have uniformly positive isoperimetric constants and their masses $\{m_i\}$ converge to zero, then g is flat in U.

Equivalently, we can restate the above corollary in a manner which extends the case of equality of the Positive Mass Theorem to manifolds which are not necessarily smooth.

Definition 5.2. Given a metric g on a manifold M^3 which is not necessarily smooth, we say that it has **generalized non-negative scalar curvature** if it is the limit in the C^0 sense of a sequence of smooth, complete, asymptotically flat metrics $\{g_i\}$ on M^3 which have non-negative scalar curvature. We will also require that g is smooth outside a bounded set, and that its total mass equals the limit of the total masses of the smooth metrics. Furthermore, given such a manifold, let $\mathbf{U(M^3)}$ denote the interior of the set of points in M^3 where the convergence of metrics is locally C^3 and nondegenerate.

Theorem 5.3. *Suppose that (M^3, g) is not necessarily smooth but is complete, asymptotically flat, and has generalized non-negative scalar curvature, total mass m, and isoperimetric constant k. Then $m = 0$ and $k > 0$ implies that g is flat in $U(M^3)$.*

We note that the above theorem is not true without the requirement that the isoperimetric constant $k > 0$. For example, the induced metric on a 3-plane which is tangent to a round 3-sphere in Euclidean 4-space has generalized non-negative scalar curvature and is the limit of portions of space-like Schwarzschild metrics of small mass joined to the 3-sphere minus a small neighborhood of the north pole. This singular manifold has zero mass, but it is not flat everywhere in $U(M^3)$ since $U(M^3)$ equals the whole manifold minus the point where the 3-plane and 3-sphere are tangent. However, Theorem 5.3 is not contradicted by this example since this singular manifold has zero isoperimetric constant.

Among other possible applications, Theorem 5.3 is used in [2] to handle the cases of equality of Theorems 1, 9, and 10 of that paper. These three theorems are generalizations of the Positive Mass Theorem and give lower bounds on the total mass of an asymptotically flat manifold in terms of the geometry of the manifold. In particular, Theorem 1, which is the main theorem of [2], is a slight generalization of the Riemannian Penrose Inequality, which states that the total mass of a 3-manifold with non-negative scalar curvature is greater than or equal to the mass contributed by any black holes it may contain. The above Theorem 5.3 is then needed to prove that, if the total mass of the 3-manifold exactly equals the mass contributed by the black holes it contains, then the 3-manifold is a Schwarzschild 3-manifold (defined to be a time-symmetric, space-like slice of the usual 3+1 dimensional Schwarzschild metric) which corresponds to a single non-rotating black hole in vacuum.

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References.

[1] R. Arnowitt, S. Deser and C. Misner, *Energy and the Criteria for Radiation in General Relativity*, Phys. Rev., **118** (1960), 1100.

[2] H. Bray, *Proof of the Riemannian Penrose Conjecture in General Relativity Using the Positive Mass Theorem*, in preparation.

[3] F. Finster, *Local $U(2,2)$ Symmetry in Relativistic Quantum Mechanics*, hep-th/9703083, J. Math. Phys., **39** (1998), 6276–6290.

[4] S. Hawking and G. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press, 1973.

[5] T. Parker and C.H. Taubes, *On Witten's Proof of the Positive Energy Theorem*, Commun. Math. Phys., **84** (1982), 223–238.

[6] R. Schoen and S.-T. Yau, *On the Proof of the Positive Mass Conjecture in General Relativity*, Commun. Math. Phys., **65** (1976), 45–76.

[7] E. Witten, *A New Proof of the Positive Energy Theorem*, Commun. Math. Phys., **80** (1981), 381–402.

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