A Gap Theorem For Complete Noncompact Manifolds With Nonnegative Ricci Curvature

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In this paper we obtain a gap theorem for the Ricci curvature over complete noncompact locally conformally flat manifolds. We prove that if the Ricci curvature is nonnegative and decays faster than quadratic at infinity, then the manifold is flat.

1. Introduction.

Let M^n be an n-dimensional complete noncompact Riemannian manifold with nonnegative (sectional or Ricci) curvature. We are interested in the question that how much the curvature could have near the infinities. In views of the classical Bonnet-Meyers theorem there cannot too much positive curvature there otherwise the manifold would close up, ceasing to be noncompact. At first sight, there seems to be no restriction in the other direction, no limit on how less positive curvature can have. For example, it is easy to construct complete metrics on $\mathbf{R}^2$ from surface of revolution such that their curvatures are zero outside some compact set, nonnegative everywhere, and positive somewhere. But it is rather surprising that the corresponding situation for a complete manifold M^n with dimension $n > 2$, can not occur. In [MSY] Mok, Siu and Yau proved that if a complete noncompact Kähler-Stein manifold of nonnegative holomorphic bisectional curvature of complex dimension $m \geq 2$ has maximal volume growth and the scalar curvature decays faster than quadratic, in the sense that, for some $C > 0$ and $\varepsilon > 0$,

$$(1.1) \qquad \sup R(x) \leq Cr^{-(2+\varepsilon)},$$

where the supremum of the scalar curvature is taken over all points x at distance r from a fixed point, then **M** is isometrically biholomorphic to $\mathbf{C}^m$.

This result shows that there is a gap between the flat metric and the other metrics of nonnegative curvature on $\mathbf{C}^m$. Subsequently, Greene and Wu [GW] studied the corresponding, more general Riemannian situation and proved that a manifold with a pole, with faster than quadratic curvature decay (1.1), and with nonnegative sectional curvature is necessarily flat, if the dimension of the manifold is ≥ 3 except when the dimension is 4 or 8. Recently the pole assumption was removed by Eschenburg, Schroeder and Strake [ESS] in the odd-dimensional case and by Dress [Dr] in the even-dimensional case.

Naturally one would ask if there are similar gap phenomena for the Ricci curvature, instead of the sectional curvature, over complete noncompact Riemannian manifolds. In general, this seems no longer possible because there are many nonflat Ricci-flat metrics, for examples, the Eguchi-Hanson metric described in [P] and the Ricci-flat complete Kähler manifolds constructed in [TY]. The Eguchi-Hanson metric has Ricci $\equiv 0$, $|R_m| \leq \frac{C}{r^6}$, and $\mathrm{vol} B(x, r) \geq \frac{1}{2}\omega_n r^n$ for all x, where ω_n is the volume of the unit sphere in the Euclidean space. However, Bando,Kasue and Nakajima [BKN] could still get such a gap result for the Ricci curvature on a complete noncompact Riemannian manifold with maximal volume growth. More precisely, they proved in [BKN] that:

Suppose a complete noncompact Riemannian manifold M^n of dimension $n \geq 3$ satisfies

$$
\begin{cases}
\mathrm{Ricci} \ \geq 0, \\
\mathrm{vol}\,(B(x_0, r)) \geq \gamma \omega_n r^n, & \text{for some } \gamma > \frac{1}{2}, \ x_0 \in M^n, \text{ and all } r > 0, \\
\sup |k(\sigma)| \leq Cr^{-2}, & \text{if } r \text{ is sufficiently large,}
\end{cases}
$$

and either

$$
(\sup |k(\sigma)|) \cdot r^{-2} \to 0, \quad \text{as } r \to +\infty,
$$

or

$$
\int_{M^n} |R_m|^{\frac{n}{2}}\, dv < +\infty,
$$

where the supremum of the sectional curvature is taken over all two-planes σ at distance r from x_0 and R_m is the curvature operator. Then M^n is isometric to Euclidean space $\mathbf{R}^n$.

The argument in [BKN] is via the Gromov convergence theorem. The maximal volume growth assumption which guarantees non-collapsing limit is crucial for their method. Moreover in view of the Eguchi-Hanson metric the assumption that the volume growth factor $\gamma > \frac{1}{2}$ is necessary for the

above theorem. Thus it is interesting to investigate the gap phenomena on non-maximal volume growth manifolds. In the recent work [N], Ni gave some of such gap results on complete noncompact Kähler manifolds. In particular, Mok-Siu-Yau's gap theorem on maximal volume growth manifolds was extended to non-parabolic manifolds in [N]. The purpose of this present paper is also to provide such a gap phenomena for the Ricci curvature over non-maximal volume growth manifolds. The main result which we prove is the following:

Main Theorem. *Let M^n be an n-dimensional $(n \geq 3)$ complete non-compact locally conformally flat manifold with nonnegative Ricci curvature. Suppose the scalar curvature is bounded and there exists a positive function $\varepsilon : \mathbf{R} \to \mathbf{R}$ with $\lim_{r \to \infty} \varepsilon(r) = 0$ such that*

$$(1.2) \qquad \frac{1}{\text{vol}(B(x_0, r))} \int_{B(x_0, r)} R(x) dv \leq \frac{\varepsilon(r)}{r^2}, \quad \text{for } x_0 \in M^n, \ r > 0,$$

where $B(x_0, r)$ denotes the geodesic ball of radius r and centered at x_0, vol $(B(x_0, r))$ denotes the volume of $B(x_0, r)$. Then M^n is flat.

We briefly describe the proof of the theorem. Our method is via the study of the Yamabe flow on complete noncompact manifolds with nonnegative Ricci curvature. Suppose there exists such a manifold satisfying the quadratic decay condition (1.2). We evolve the metric by Yamabe flow. In section 2, we establish a short time existence result which concludes the solution either exists for all time or extracts to point in a finite time. Then in section 3, by using the decay assumption we can derive a dedicate zero order estimate for the evolving metric. Particularly, this will imply that the solution of the Yamabe flow exists for all time. Finally in section 4, by combining the zero order estimate with Chow's Li-Yau-Hamilton inequality [C] on locally conformally flat manifolds, we are able to show that the initial metric is actually Ricci flat. However any locally conformally flat manifold with vanishing Ricci curvature must be flat.

2. Short Time Existence.

Let M^n be an n-dimensional complete noncompact smooth Riemannian manifold with dimension $n \geq 3$. Suppose M^n has bounded scalar curvature. Denote by g_{ij}, R_{ij} and R the metric tensor, the Ricci tensor and the scalar curvature. The Yamabe flow is the following evolution equation for

the metric

$$(2.1) \qquad \begin{cases} \frac{\partial g_{ij}}{\partial t} = -R(x,t)g_{ij}(x,t), & x \in M^n, \ t > 0, \\ g_{ij}(x,0) = g_{ij}(x), & x \in M^n. \end{cases}$$

Write $g_{ij}(x,t) = [u(x,t)]^{\frac{4}{n-2}} g_{ij}(x)$ for some positive function $u(x,t)$. Then (2.1) can be written in the equivalent form

$$(2.2)$$
$$\begin{cases} \frac{\partial u^N(x,t)}{\partial t} = (n-1)N \left[\triangle u(x,t) - \frac{n-2}{4(n-1)} R(x)u(x,t) \right], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in M^n, \ t > 0, \\ u(x,t) > 0, \qquad\qquad\qquad\qquad\qquad\qquad x \in M^n, \ t > 0, \\ u(x,0) = 1, \qquad\qquad\qquad\qquad\qquad\qquad x \in M^n, \end{cases}$$

where $N = \frac{n+2}{n-2}$, $\triangle$ is the Laplace operator of the initial metric $g_{ij}(x)$, and $R(x)$ is the scalar curvature of the initial metric $g_{ij}(x)$.

To solve the problem (2.2), we first consider the corresponding Dirichlet problem

$$(2.3)$$
$$\begin{cases} \frac{\partial u^N(x,t)}{\partial t} = (n-1)N \left[\triangle u(x,t) - \frac{n-2}{4(n-1)} R(x)u(x,t) \right], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in \Omega, \ t > 0, \\ u(x,t) > 0, \qquad\qquad\qquad\qquad\qquad\qquad x \in \Omega, \ t > 0, \\ u(x,t) = 1, \qquad\qquad\qquad\qquad\qquad\qquad x \in \partial\Omega, \ t > 0, \\ u(x,0) = 1, \qquad\qquad\qquad\qquad\qquad\qquad x \in \Omega, \end{cases}$$

where Ω is a bounded smooth domain of M^n. In the following we will show that the solution of (2.3) exists for all times. Let us start with the local existence.

Lemma 2.1. *For each $\delta \in (0,1)$ and $C > 1$, there exists a constant $T > 0$, depending only on δ, C and the geometry of Ω such that if the initial data $u_0(x) \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies*

$$\delta \leq u_0(x) \leq C, \quad x \in \Omega$$

and

$$u_0|_{\partial\Omega} = 1,$$

then the Dirichlet problem

$$(2.4) \quad \begin{cases} \frac{\partial u^N(x,t)}{\partial t} = (n-1)N \left[\triangle u(x,t) - \frac{n-2}{4(n-1)} R(x)u(x,t) \right], \\ \qquad\qquad\qquad\qquad\qquad (x,t) \in Q \hat{=} \Omega \times (0,T), \\ u(x,t) > 0, \qquad\qquad\qquad (x,t) \in Q, \\ u(x,t) = 1, \qquad\qquad\qquad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \qquad\qquad x \in \Omega, \end{cases}$$

has a unique classical (i.e., $C(\bar{\Omega}) \cap C^2(Q)$) solution.

Proof. By applying the Leray-Schauder principle and the barrier argument in a standard way as in [LSU, §6, Chapter V], we only need to get a prior interior $C^{2,\alpha}$ estimate for the solution u. Assume that $u(x,t)$ achieves its maximum $u_{\max}(t)$ in the interior of Ω. Then by a result of Hamilton (see Lemma 3.5 in [H1]) we have

$$\frac{du_{\max}^N(t)}{dt} \le \frac{n+2}{4} \max_{x \in \Omega} |R(x)| \cdot u_{\max}(t),$$

which implies that

$$(2.5) \qquad u_{\max}(t) \le \left(C^{\frac{4}{n-2}} + \max_{x \in \Omega} |R(x)| \cdot t \right)^{\frac{n-2}{4}}, \qquad \text{for } t > 0.$$

Similarly the minimum $u_{\min}(t)$ of $u(\cdot,t)$ on Ω satisfies

$$(2.6) \qquad u_{\min}(t) \ge \left(\delta^{\frac{4}{n-2}} - \min_{x \in \Omega} |R(x)| \cdot t \right)^{\frac{n-2}{4}}, \qquad \text{for } t > 0.$$

Then there exists a positive constant T depending only on δ, C and $\max_{x \in \Omega} |R(x)|$ such that

$$(2.7) \qquad \frac{1}{2}\delta \le u(x,t) \le 2C, \qquad \text{for } x \in \bar{\Omega}, \, t \in [0,T].$$

Thus we write the equation in the following form

$$(2.8) \quad \frac{\partial u(x,t)}{\partial t} = \frac{n-1}{u^{N-1}(x,t)} \left[\triangle u(x,t) - \frac{n-2}{4(n-1)} R(x)u(x,t) \right], (x,t) \in Q.$$

The uniform estimate (2.7) implies that the equation (2.8) is uniformly parabolic. So it follows from the Krylov-Safonov estimate [KS] that there is a prior interior C^α estimate for the solution $u(x,t)$ for some $0 < \alpha < 1$. And then we get a prior interior $C^{2,\alpha}$ estimate from Schauder theory. □

Proposition 2.2. *The Dirichlet problem (2.3) has a unique classical solution for all time $t \in [0, \infty)$.*

Proof. By lemma 2.1, we know that the problem (2.3) has a unique classical solution $u(x,t)$ on a maximal time interval $[0, t_{\max})$ such that when $t_{\max} < \infty$, then as $t \to t_{\max}$ either the minimum $u_{\min}(t)$ (of $u(x,t)$ on Ω) tends to zero or the maximum $u_{\max}(t)$ (of $u(x,t)$ on Ω) tends to infinity. Whenever the maximum $u_{\max}(t)$ is achieved in the interior of Ω, it follows that

$$\frac{du_{\max}^N(t)}{dt} \leq \frac{n+2}{4} \max_{x \in \Omega} |R(x)| \cdot u_{\max}(t).$$

Thus

$$(2.9) \qquad u_{\max}(t) \leq \left[1 + \max_{x \in \Omega} |R(x)| \cdot t \right]^{\frac{n-2}{4}}, \quad \forall \ t \in [0, t_{\max}).$$

Suppose $t_{\max} < +\infty$, we then deduce that the minimum $u_{\min}(t)$ must tend to zero as $t \to t_{\max}$. Let us drop the restriction on positivity of the solution in (2.3). That is, we consider the problem

$$(\widetilde{2.3}) \qquad \begin{cases} \frac{\partial u^N(x,t)}{\partial t} = (n-1)N \left[\triangle u(x,t) - \frac{n-2}{4(n-1)} R(x)u(x,t) \right], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in \Omega, \ t > 0, \\ u(x,t) = 1, \qquad\qquad\qquad\qquad\quad x \in \partial\Omega, \ t > 0, \\ u(x,0) = 1, \qquad\qquad\qquad\qquad\quad x \in \Omega. \end{cases}$$

By using DiBennedetto's estimate on the modulus of continuity of general porous medium equation [Di] (see also [Z]), we know from (2.9) that the solution $u(x,t)$ of $(\widetilde{2.3})$ can be extended continuously to $t_{\max}$. In the following we adapt an argument of Ye in [Y] to show that the extension is positive everywhere which will give a contradiction.

Since $u(x,t) = 1$ on $\partial\Omega \times [0, t_{\max}]$, we can find a positive constant $r_0 > 0$ such that $u(x,t)$ is positive on the r_0−neighborhood of $\partial\Omega \times [0, t_{\max}]$. Without loss of generality, we may assume r_0 is less than the infimum of the injectivity radius of the points on Ω. By the compactness of $\bar{\Omega}$, we only need to prove that $u(x, t_{\max})$ is positive on the geodesic ball $B(x_0, \frac{r_0}{4})$ whenever $u(x_0, t_{\max}) > 0$. By continuity, there exist two small positive constants ε, δ such that

$$(2.10) \qquad u(x,t) \geq \varepsilon, \quad \text{for } x \in B(x_0, \delta), \ t \in [0, t_{\max}].$$

Let $\vec{e}$ be an arbitrary unit tangent vector at x_0. Denote by $P_0 = \exp_{x_0}\left(\frac{r_0}{2}\vec{e}\right)$, $D = B(x_0, \delta) \cap \partial B(P_0, \frac{r_0}{2})$, and C to be the geodesic cone with tip P_0 and cap D. Let r denote the distance to P_0 and set $S_r = \partial B(P_0, r)$ and $D_r = C \cap S_r$. Then the Laplacian operator of $g_{ij}(x)$ can be written

$$\triangle = \frac{\partial^2}{\partial r^2} + H(r)\frac{\partial}{\partial r} + \triangle_{S_r}$$

where $H(r)$ is the mean curvature function of $S_r = \partial B(P_0, r)$ and $\triangle_{S_r}$ is the Laplacian operator on S_r for the induced metric. Let $\varphi(r, \cdot)$ denote the positive first eigenfunction of $r^2 \triangle_{S_r}$ on D_r with zero boundary values and the maximum one. Put $v = r^b \varphi$, where b is to be determined. If we choose b sufficiently large (depending only on $n, \varepsilon, \delta, \Omega, \max_{x \in \Omega} |R(X)|, t_{\max}$), then by the same calculation as in [Y],

$$\triangle v - \frac{n-2}{4(n-1)}Rv \geq 0, \qquad for \quad r \leq \frac{r_0}{2}.$$

Now choose a small $\varepsilon' > 0$ such that $\varepsilon'v \leq u$ on $D \times [0, t_{\max}]$ and $C \times \{0\}$. The inequality $\varepsilon'v \leq u$ also holds on the remaining part of the parabolic boundary of $C \times [0, t_{\max}]$, because v vanishes there. Since we have

$$\frac{\partial}{\partial t}\left[u^N - (\varepsilon'v)^N\right] \geq (n-1)N\left[\triangle(u - \varepsilon'v) - \frac{(n-2)}{4(n-1)}R(u - \varepsilon'v)\right],$$

on $C \times [0, t_{\max}]$, which implies $u \geq \varepsilon'v$ on $C \times [0, t_{\max}]$ by the maximum principle. Consequently u is positive along the geodesic $\exp_{x_0}(r\,\vec{e}), 0 \leq r \leq \frac{r_0}{2}$. since $\vec{e}$ is arbitrary, we conclude that u is positive in $B(x_0, \frac{r_0}{4})$. This completes the proof of proposition. $\qquad\square$

The main result of this section is the following

Theorem 2.3. *The Yamabe flow (2.1) has a smooth solution on a maximal time interval* $[0, t_{\max})$ *with* $t_{\max} > 0$ *such that either* $t_{\max} = +\infty$, *or the evolving metric contracts to a point at a finite time* $t_{\max}$ *in the sense that for any curve* γ *on* M^n, *the length of* γ *with respect to the evolving metric* $g_{ij}(\cdot, t)$ *tends to zero as* $t \to t_{\max}$.

Proof. Let $\Omega_1 \subset \Omega_2 \subset \cdots$ be a sequence of exhausting bounded smooth domains of M^n. We solve the problem (2.3) for $\Omega = \Omega_j, j = 1, 2, \cdots$. to get a sequence of solutions $u_j(x, t)$ on $\Omega_j \times [0, +\infty), j = 1, 2, \cdots$. Let x_0 be

any fixed point in Ω_1. For simplicity, we may assume that the geodesic ball (w.r.t. the initial metric $g_{ij}(x)$) $B(x_0, 1)$ is contained in Ω_1.

Define

$$\mu_j(t) = \inf_{(x,t) \in B(x_0,1) \times [0,t]} u_j(x,t),$$

and

$$\mu(t) = \liminf_{j \to \infty} \mu_j(t).$$

By assumption, the initial metric $g_{ij}(x)$ has bounded scalar curvature . We know from (2.6) that $\mu(t)$ is positive for t small enough. It is also clear that $\mu(t)$ is non-increasing in t. Thus we can distinguish two cases:

(i) $\mu(t) > 0$ for all $t \in [0, +\infty)$;

(ii) there exists a finite time $t_{\max} > 0$ such that $\mu(t) > 0$ for $t \in [0, t_{\max})$ and $\mu(t_{\max}) = 0$.

First, we consider the case (i). For any fixed $T < +\infty$, we know from (2.9) that $u_j(x,t)$ has a uniform upper bound on $M^n \times [0, T]$. Since $\mu(T) > 0$, we still known from the proof of Proposition 2.2 that for any fixed compact subset K of M^n, $u_j(x,t)$ has a positive lower bound on $K \times [0, T]$ for j large enough. That says, the equation is uniformly parabolic on $K \times [0, T]$ Thus we can get the uniform $C^\alpha (0 < \alpha < 1)$ estimate from Krylov-Safonov estimate [KS] and then the uniform $C^{2,\alpha}(0 < \alpha < 1)$ estimate on $K \times [0, T]$. from Schauder theory for j. Hence by the arbitrariness of K and T, it follows from a standard diagonal argument that (2.2) has a positive solution $u(x,t)$ on $M^n \times [0, +\infty)$.

Next, we consider the case (ii). That is, $0 < t_{\max} < +\infty$, $\mu(t) > 0$ on $[0, t_{\max})$ and $\mu(t_{\max}) = 0$. Denote by $t_k = (1 - \frac{1}{k+1})t_{\max}$, $k = 1, 2, \cdots$. Since $\mu(t_1) > 0$, by repeating the above argument we can get a subsequence of u_j so that it converges on every compact subset of $M^n \times [0, t_1)$. Similarly by $\mu(t_2) > 0$, we can further get a subseqence from the previous subsequence so that it converges on every compact subset of $M^n \times [0, t_2)$. Then by repeating this procedure and the standard diagonal argument, we can get a smooth positive solution $u(x,t)$ defined on $M^n \times [0, t_{\max})$, moreover by applying the result of DiBennedetto [Di] we can extend the solution $u(x,t)$ to $t_{\max}$ continuously. Again, the same argument in the proof of Proposition2.2 shows that $u(x, t_{\max})$ must be identically zero on M^n. This implies that the length of any arc γ with respect to the metric $g_{ij}(x,t) = (u(x,t))^{\frac{4}{n-2}} g_{ij}(x)$ tends to zero as $t \to t_{\max}$. $\square$

3. Zero Order Estimate.

In this section we use the decay assumption on the curvature to drive a lower bound estimate for the solution of the Yamabe flow (2.2). Our argument is inspired by the work of Shi in [S].

Let (M^n, g_{ij}) be a smooth complete noncompact Riemannian manifold with nonnegative Ricci curvature. Suppose the scalar curvature is bounded and satisfies (1.2).

From Theorem 2.3 we have a smooth solution $g_{ij}(x,t)$ of (2.1) on a maximal time interval $[0, t_{\max})$. Write the solution as $g_{ij}(x,t) = (u(x,t))^{\frac{4}{n-2}} g_{ij}(x)$, where $u(x,t)$ is a positive solution of (2.2) on $[0, t_{\max})$.

We first present the following lemma.

Lemma 3.1. *For any $1 \leq r < +\infty$, we have*

$$(3.1) \quad -\frac{1}{\mathrm{vol}\,(B(x_0,r))} \int_{B(x_0,r)} \log u(x,t) dv \leq C\left[1 - (\log u)_{\min}(t)\right] \cdot \left(\frac{t}{r^2}\right),$$

$$t > 0,$$

where $(\log u)_{\min}(t)$ is the infimum of $\log u(x,t)$ for $x \in M^n$, and C is some positive constant depending only on n.

Proof. Since the Ricci curvature of the initial metric $g_{ij}(x)$ is nonnegative, there exists a constant $C(n) > 0$ depending only on the dimension such that for any fixed point $x_0 \in M^n$ and any number $1 \leq r < +\infty$, there exists a smooth function $\varphi(x) \in C^\infty(M^n)$ such that

$$(3.2) \quad \begin{cases} \exp\left[-C(n)\left(1 + \frac{d(x,x_0)}{r}\right)\right] \leq \varphi(x) \leq \exp\left[-\left(1 + \frac{d(x,x_0)}{r}\right)\right], \\ |\nabla \varphi| \leq \frac{C(n)}{r}\varphi, \\ |\triangle \varphi| \leq \frac{C(n)}{r^2}\varphi, \end{cases}$$

where $d(x, x_0)$ is the distance between x and x_0 in the metric g_{ij}. The construction of this kind of exponential decay function can be found in Shi [S] or Schoen-Yau [SY].

Recall the solution $u(x,t)$ is obtained by a sequence of approximating solutions $u_j(x,t)$, $j = 1, 2, \cdots$, which solves the corresponding Dirichlet

problem (2.3) for a sequence of exhausting domains Ω_j, $j = 1, 2, \cdots$ By a direct computation,

$$(3.3) \qquad \frac{\partial}{\partial t} \int_{\Omega_j} \varphi u_j^N dv = \int_{\Omega_j} \varphi \left[(n-1)N \triangle u_j - \frac{(n+2)}{4} R u_j \right] dv$$

$$= (n-1)N \int_{\Omega_j} \triangle \varphi \cdot u_j dv - \frac{n+2}{4} \int_{\Omega_j} R \varphi u_j dv$$

$$+ (n-1)N \left(\int_{\partial \Omega_j} \frac{\partial u_j}{\partial v} \varphi ds - \int_{\partial \Omega_j} u_j \frac{\partial \varphi}{\partial v} ds \right)$$

where v denotes the outer unit normal vector of $\partial \Omega_j$.

By noting that $R(x)$ is nonnegative, it follows from (2.3) and the maximal principle that $u_j(x,t) \leq 1$ on $\bar{\Omega}_j \times [0, +\infty)$. Since $u_j|_{\partial \Omega_j} = 1$, we deduce that $\frac{\partial u_j}{\partial v} \geq 0$ on $\partial \Omega_j$. To estimate the other boundary term, without loss of generality, we may assume $\Omega_j = B(x_0, j)$. Thus vol $(\partial \Omega_j) \leq C j^{n-1}$ for some constant C (depending only on n) by the volume comparison theorem. And by (3.2) we have

$$|\nabla \varphi| \leq \frac{C(n)}{r} e^{-\frac{j}{r}}, \qquad on \quad \partial \Omega_j.$$

Then we get

$$\left| \int_{\partial \Omega_j} u_j \frac{\partial \varphi}{\partial v} ds \right| \leq \frac{C \cdot C(n)}{r} j^{n-1} e^{-\frac{j}{r}} \to 0, \qquad as \quad j \to \infty.$$

So by integrating (3.3) from 0 to t and setting $j \to \infty$, we obtain

$$(3.4) \qquad \int_{M^n} \varphi(1 - u^N) dv \leq \frac{C}{r^2} \int_0^t \int_{M^n} u\varphi dv + C \int_0^t \int_{M^n} u\varphi R dv$$

$$\leq Ct \left(\frac{1}{r^2} \int_{M^n} \varphi dv + \int_{M^n} \varphi R dv \right),$$

for some positive constant C depending only on n.

From (1.2) and (3.2), we have (see also [S])

$$\int_{M^n} R\varphi dv \le \int_{M^n} R(x)e^{-\left(1+\frac{d(x,x_0)}{r}\right)} dv$$

$$= \int_{B(x_0,r)} R(x)e^{-\left(1+\frac{d(x,x_0)}{r}\right)} dv$$

$$+ \sum_{k=0}^{\infty} \int_{B(x_0,2^{k+1}r)\setminus B(x_0,2^k r)} R(x)e^{-\left(1+\frac{d(x,x_0)}{r}\right)} dv$$

$$\le \frac{\varepsilon(r)}{r^2}\text{vol}\left(B(x_0,r)\right) + \sum_{k=0}^{\infty} e^{-2^k}\frac{\varepsilon(2^{k+1}r)}{(2^{k+1}r)^2}\text{vol}\left(B\left(x_0,2^{k+1}r\right)\right)$$

$$(3.5) \qquad \le \frac{C}{r^2}\left[\text{vol}\left(B(x_0,r)\right) + \sum_{k=0}^{\infty} e^{-2^k}2^{-2(k+1)}\text{vol}\left(B\left(x_0,2^{k+1}r\right)\right)\right].$$

By the volume comparison theorem,

$$\sum_{k=0}^{\infty} e^{-2^k}2^{-2(k+1)}\text{vol}\left(B\left(x_0,2^{k+1}r\right)\right)$$

$$\le \sum_{k=0}^{\infty} e^{-2^k}2^{-2(k+1)}\cdot 2^{n(k+1)}\text{vol}\left(B(x_0,r)\right)$$

$$\le C\text{vol}\left(B(x_0,r)\right),$$

thus (3.5) becomes

$$(3.6) \qquad \int_{M^n} R\varphi dv \le \frac{C}{r^2}\text{vol}\left(B(x_0,r)\right),$$

for some positive constant C depending only on n.

Similarly, by (3.2) we have

$$(3.7) \qquad C^{-1}\text{vol}\left(B(x_0,r)\right) \le \int_{M^n} \varphi dv \le C\text{vol}\left(B(x_0,r)\right)$$

for some positive constant C depending only on n.

Substituting (3.6) and (3.7) into (3.4) we deduce

$$(3.8) \qquad \int_{M^n} \varphi(1-u^N)dv \le C\frac{t}{r^2}\text{vol}\left(B(x_0,r)\right)$$

for some positive constant C depending only on n.

On the other hand, since $u \leq 1$ and $1 - e^x \geq -\frac{x}{2}$ for $-1 \leq x \leq 0$, we get

$$
(3.9) \qquad \int_{M^n} \varphi(1 - u^N) dv \geq \int_{M^n} \varphi(1 - u) dv
$$

$$
= \int_{\{\log u \geq -1\}} \varphi(1 - e^{\log u}) dv
$$

$$
+ \int_{\{\log u \leq -1\}} \varphi(1 - e^{\log u}) dv
$$

$$
\geq -\frac{1}{2} \int_{\{\log u \geq -1\}} \varphi \log u \, dv + \frac{1}{2} \int_{\{\log u \leq -1\}} \varphi dv.
$$

Note that the last two terms are positive. We deduce

$$
- \int_{\{\log u \geq -1\}} \varphi \log u \, dv \leq 2 \int_{M^n} \varphi \left(1 - u^N\right) dv,
$$

and

$$
- \int_{\{\log u \leq -1\}} \varphi \log u \, dv \leq -(\log u)_{\min}(t) \int_{\{\log u \leq -1\}} \varphi dv
$$

$$
\leq -2(\log u)_{\min}(t) \int_{M^n} \varphi(1 - u^N) dv.
$$

That is,

$$
(3.10) \qquad - \int_{M^n} \varphi \log u \, dv \leq 2(1 - (\log u)_{\min}(t)) \int_{M^n} \varphi \left(1 - u^N\right) dv.
$$

By (3.2), we have

$$
(3.11) \qquad - \int_{M^n} \varphi \log u \, dv \geq - \int_{M^n} \log u \cdot e^{-C(n)\left(1 + \frac{d(x_0, x)}{r}\right)} dv
$$

$$
\geq -e^{-2C(n)} \int_{B(x_0, r)} \log u \, dv.
$$

The combination of (3.8), (3.10) and (3.11) implies the estimate (3.1). $\qquad \square$

We next want to estimate $-(\log u)_{\min}(t)$ in terms of $\frac{1}{\operatorname{vol}(B(x_0,r))} \cdot$ $\int_{B(x_0,r)} (-\log u) dv$ by means of the positive Green function. Since in general the manifold M^n is not necessary nonparabolic, we adapt a trick of Shi in [S] by considering a new manifold $\tilde{M} = M^n \times \mathbf{R}^3$, where $\mathbf{R}^3$ is equipped with the flat Euclidean metric and $\tilde{M}$ is equipped with the product metric. Let x, y denote the points in M^n and $\mathbf{R}^3$ respectively, $R(x)$ denote the scalar curvature of M^n at x, and $\tilde{R}(x,y)$ denote the scalar curvature of $M^n \times \mathbf{R}^3$ at (x,y). It is clear that $\tilde{R}(x,y) = R(x)$ for all $x \in M^n, y \in \mathbf{R}^3$. Thus the function $\tilde{u}$, defined by $\tilde{u}(x,y,t) = u(x,t)$ for $x \in M^n, y \in \mathbf{R}^3$ and $t > 0$, is a solution of the following evolution equation.

(3.12)
$$
\begin{cases}
\frac{\partial \tilde{u}^N(x,y,t)}{\partial t} = (n-1)N \left[\tilde{\triangle}\tilde{u}(x,y,t) - \frac{n-2}{4(n-1)}\tilde{R}(x,y)\tilde{u}(x,y,t) \right], \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,y) \in \tilde{M}, \ t > 0, \\
\tilde{u}(x,y,t) > 0, \qquad\qquad\qquad\qquad (x,y) \in \tilde{M}, t > 0, \\
\tilde{u}(x,y,0) = 1, \qquad\qquad\qquad\qquad (x,y) \in \tilde{M},
\end{cases}
$$

where $\tilde{\triangle}$ is the Laplacian operator of $\tilde{M} = M^n \times \mathbf{R}^3$.

We note that the manifold $\tilde{M}$ still has nonnegative Ricci curvature and

$$
B_{M^n}\left(x_0, \frac{r}{2}\right) \times B_{\mathbf{R}^3}\left(y_0, \frac{r}{2}\right) \subset \tilde{B}((x_0,y_0), r) \subset B_{M^n}(x_0, r) \times B_{\mathbf{R}^3}(y_0, r).
$$

Then by the volume comparison theorem, we have

$$
(3.13) \qquad C(n)\left(\frac{r_2}{r_1}\right)^3 \le \frac{\operatorname{vol} \tilde{B}((x_0,y_0), r_2)}{\operatorname{vol} \tilde{B}((x_0,y_0), r_1)} \le \left(\frac{r_2}{r_1}\right)^{n+3}, \quad \text{for } r_1 \le r_2,
$$

and

$$
\int_{\tilde{B}((x_0,y_0),r)} \tilde{R}(x,y) \, d\tilde{v} \le \int_{B_{M^n}(x_0,r) \times B_{\mathbf{R}^3}(y_0,r)} \tilde{R}(x,y) \, d\tilde{v}
$$

$$
\le Cr^3 \int_{B_{M^n}(x_0,r)} R(x) dv
$$

$$
\le \tilde{C}\frac{\varepsilon(r)}{r^2} \operatorname{vol} \tilde{B}((x_0,y_0), r),
$$

that is,

$$
(3.14) \qquad \frac{r^2}{\operatorname{vol} \tilde{B}((x_0,y_0), r)} \int_{\tilde{B}((x_0,y_0),r)} \tilde{R}(x,y) d\tilde{v} \le \tilde{C}\varepsilon(r),
$$

where $C, C(n)$ and $\tilde{C}$ are positive constants depending only on n. It is also clear that to estimate $-(\log u)_{\min}(t)$ in terms of $\frac{1}{\operatorname{vol}(B(x_0,r))} \cdot \int\limits_{B(x_0,r)} (-\log u)dv$

is equivalent to estimate $-(\log \tilde{u})_{\min}(t)$ in terms of $\frac{1}{\operatorname{vol}\tilde{B}((x_0,y_0),r)}$ $\int\limits_{\tilde{B}((x_0,y_0),r)} [-\log \tilde{u}(x,t)]\, d\tilde{v}$. So in the next lemma we study the solution $\tilde{u}(x,y,t)$ of (3.12) with the manifold $\tilde{M}$ satisfying (3.13) and (3.14). For simplicity, we drop the symbol "$\sim$" and the variable "y" off.

Lemma 3.2. *There exist a positive constant $C(n)$ and a positive function $\tilde{\varepsilon}(r)$ defined on $[1,+\infty)$ satisfying $\tilde{\varepsilon}(r) \to 0$ as $r \to +\infty$ such that*

(3.15)

$$
\log u(x_0,t)
$$

$$
\geq C \left[-\tilde{\varepsilon}(r) \log r - \sup_{x \in M^n} R(x) + \frac{1}{\operatorname{vol}(B(x_0,r))} \int\limits_{B(x_0,r)} \log u(x,t)dv \right],
$$

for all $t > 0$ and $r \geq 1$.

Proof. Since the Ricci curvature of M is nonnegative and (3.13) holds, we know from [LY] (see also Lemma 6.6 of [S]) that there exists a positive Green function $G(x_0,x)$ satisfying

(3.16) $\quad \dfrac{d^2(x_0,x)}{C\operatorname{vol}(B(x_0,d(x_0,x)))} \leq G(x_0,x) \leq \dfrac{Cd^2(x_0,x)}{\operatorname{vol}(B(x_0,d(x_0,x)))}, \quad x \in M,$

and

(3.17) $\qquad |\nabla G(x_0,x)| \leq \dfrac{Cd(x_0,x)}{\operatorname{vol}(B(x_0,d(x_0,x)))}, \qquad x \in M.$

Here and in the following we denote C to be variant positive constant depending only on n. For any $\alpha > 0$, we define

$$
\Omega_\alpha = \{ x \in M \,|\, G(x_0,x) > \alpha \}.
$$

It is easy to see from (3.13) and (3.16) that

$$
\frac{1}{C} \cdot \frac{1}{d(x_0,x)^{n+1}\operatorname{vol}(B(x_0,1))} \leq G(x_0,x) \leq \frac{C}{d(x_0,x)\operatorname{vol}(B(x_0,1))}
$$

for $d(x_0, x) \geq 1$. Thus $\overline{\Omega}_\alpha$ is compact subset of M and $\partial \Omega_\alpha = \{x \in M \mid G(x_0, x) = \alpha\}$. Moreover for each $\alpha > 0$ there is a $d(\alpha)$ such that

$$(3.18) \qquad \frac{d^2(\alpha)}{\text{vol}\,(B(x_0, d(\alpha))} = \alpha \quad \text{and} \quad d(\alpha) \to 0 \quad \text{as} \quad \alpha \to \infty.$$

This implies that for any $x \in \partial \Omega_\alpha$,

$$\frac{d^2(x_0, x)}{C\text{vol}\,(B(x_0, d(x_0, x))} \leq \frac{d^2(\alpha)}{\text{vol}\,(B(x_0, d(\alpha))} \leq \frac{Cd^2(x_0, x)}{\text{vol}\,(B(x_0, d(x_0, x))},$$

and then

$$(3.19) \qquad \frac{\text{vol}\,(B(x_0, d(\alpha))}{C\text{vol}\,(B(x_0, d(x_0, x)))} \leq \frac{d^2(\alpha)}{d^2(x_0, x)} \leq \frac{C\text{vol}\,(B(x_0, d(\alpha)))}{\text{vol}\,(B(x_0, d(x_0, x)))}.$$

Combining (3.13) and (3.19) we get

$$(3.20) \qquad C^{-1} d(\alpha) \leq d(x_0, x) \leq C d(\alpha), \qquad \text{for} \;\; x \in \partial \Omega_\alpha,$$

and

$$(3.21) \qquad B(x_0, C^{-1} d(\alpha)) \subset \Omega_\alpha \subset B(x_0, C d(\alpha)).$$

Note by the Green formula, it follows

$$\log u(x_0, t) = \int_{\Omega_\alpha} (\alpha - G(x_0, x)) \triangle \log u(x, t) dv$$

$$(3.22) \qquad \qquad - \int_{\partial \Omega_\alpha} \log u(x, t) \cdot \frac{\partial G(x_0, x)}{\partial \nu} ds,$$

where ν denotes the outer unit normal vector of $\partial \Omega_\alpha$.

By a direct computation (see also Lemma 2.2 in [C]) we known that the scalar curvature $R(x, t)$ of the solution evolves by

$$\frac{\partial R(x, t)}{\partial t} = (n-1) \triangle_t R(x, t) + (R(x, t))^2,$$

where $\triangle_t$ is the Laplacian operator with respect to the metric $g_{ij}(x, t)$. Since the scalar curvature of the initial metric is nonnegative, we know from the maximum principle that $R(x, t) \geq 0$, for $x \in M$ and $t > 0$. Thus by the fact that $g_{ij}(x, t) = (u(x, t))^{\frac{4}{n-2}} g_{ij}(x)$ for $x \in M$ and $t > 0$, we obtain

$$\triangle u(x, t) - \frac{(n-2)}{4(n-1)} R(x) u(x, t) \leq 0, \qquad \text{for} \;\; x \in M, \; t > 0.$$

This concludes that

$$(3.23) \qquad \triangle \log u(x,t) \leq \frac{n-2}{4(n-1)} R(x), \qquad \text{for } x \in M, \ t > 0.$$

Substituting (3.23) into (3.22), we get

$$
\begin{aligned}
\log u(x_0, t) \quad &\geq \quad \frac{n-2}{4(n-1)} \int_{\Omega_\alpha} (\alpha - G(x_0, x)) \, R(x) dv \\
&\qquad - \int_{\partial\Omega_\alpha} \log u(x,t) \cdot \frac{\partial G(x_0, x)}{\partial \nu} ds \\
&\geq \quad -\frac{n-2}{4(n-1)} \int_{\Omega_\alpha} G(x_0, x) R(x) dv \\
&\qquad + \int_{\partial\Omega_\alpha} \log u(x,t) \cdot |\nabla G(x_0, x)| \, ds.
\end{aligned}
$$

(3.24)

We estimate the last two terms as follows:

For any $\alpha > 0$, by (3.16), (3.20) and (3.21), we obtain

$$
\begin{aligned}
\int_{\Omega_\alpha} G(x_0, x) R(x) dv \quad &\leq \quad \int_{B(x_0, Cd(\alpha))} G(x_0, x) \, R(x) dv \\
&\leq \quad \int_{B(x_0, 1)} G(x_0, x) R(x) dv \\
&\qquad + \sum_{k=1}^{s} \int_{B(x_0, 2^k) \setminus B(x_0, 2^{k-1})} G(x_0, x) R(x) dv,
\end{aligned}
$$

where s is the integer satisfying $2^{s-1} \leq Cd(\alpha) \leq 2^s$, and

$$\int_{B(x_0,1)} G(x_0,x)R(x)dv$$

$$\leq \left(\sup_{x \in M} R(x)\right) \int_{B(x_0,1)} G(x_0,x)dv$$

$$\leq \left(\sup_{x \in M} R(x)\right) \sum_{k=0}^{\infty} \int_{B(x_0,2^{-k})\setminus B(x_0,2^{-k-1})} \frac{Cd^2(x_0,x)}{\mathrm{vol}\,(B(x_0,d(x_0,x)))} dv$$

$$\leq C\left(\sup_{x \in M} R(x)\right) \sum_{k=0}^{\infty} \frac{2^{-2k}}{\mathrm{vol}\,(B(x_0,2^{-k-1}))} \mathrm{vol}\,(B(x_0,2^{-k}))$$

$$\leq C\left(\sup_{x \in M} R(x)\right),$$

$$\sum_{k=1}^{s} \int_{B(x_0,2^k)\setminus B(x_0,2^{k-1})} G(x_0,x)\ R(x)dv$$

$$\leq \sum_{k=1}^{s} \int_{B(x_0,2^k)\setminus B(x_0,2^{k-1})} \frac{Cd^2(x_0,x)}{\mathrm{vol}\,(B(x_0,d(x_0,x)))} R(x)dv$$

$$\leq C\sum_{k=1}^{s} \frac{2^{2k}}{\mathrm{vol}\,(B(x_0,2^{k-1}))} \int_{B(x_0,2^k)} R(x)dv$$

$$\leq C\sum_{k=1}^{s} \frac{2^{2k}}{\mathrm{vol}\,(B(x_0,2^{k-1}))} \cdot \frac{\varepsilon(2^k)}{2^{2k}} \mathrm{vol}\,(B(x_0,2^k))$$

$$\leq C\sum_{k=1}^{s} \varepsilon(2^k)$$

$$\leq C\left(\frac{1}{s}\sum_{k=1}^{s} \varepsilon\left(2^k\right)\right) \cdot (1 + \log d(\alpha)),$$

thus

$$(3.25) \quad \int_{\Omega_\alpha} G(x_0,x)R(x)dv \leq C\left[\sup_{x \in M} R(x) + \frac{1}{s}\sum_{k=1}^{s} \varepsilon\left(2^k\right) \cdot (1 + \log d(\alpha))\right] \cdot$$

By (3.17), (3.13) and (3.20), we have

$$(3.26) \quad \int_{\partial\Omega_\alpha} \log u(x,t) \cdot |\nabla G(x_0,x)|\, ds \geq \frac{Cd(\alpha)}{\mathrm{vol}\,(B(x_0,d(\alpha)))} \int_{\partial\Omega_\alpha} \log u(x,t)ds.$$

We know from the definition of $d(\alpha)$ that for $\beta \geq \alpha$,

$$\frac{\beta}{\alpha} = \left(\frac{d^2(\beta)}{d^2(\alpha)}\right) \bigg/ \left(\frac{\text{vol}\,(B(x_0, d(\beta)))}{\text{vol}\,(B(x_0, d(\alpha)))}\right),$$

and by (3.13)

$$(3.27) \qquad \left(\frac{d(\alpha)}{d(\beta)}\right)^{n+1} \leq \frac{\beta}{\alpha} \leq C \cdot \frac{d(\alpha)}{d(\beta)}.$$

Thus by integrating (3.26) from α to 2α, we get

$$(3.28) \qquad \int_\alpha^{2\alpha} \int_{\partial\Omega_\beta} \log u(x,t) \cdot |\nabla G(x_0, x)| \, ds d\beta$$

$$\geq \frac{Cd(\alpha)}{\text{vol}\,(B(x_0, d(\alpha)))} \int_\alpha^{2\alpha} \int_{\partial\Omega_\beta} \log u(x,t) ds d\beta.$$

Since $d\beta = \frac{\partial G}{\partial \nu} d\nu$, we have

$$ds d\beta = \left|\frac{\partial G}{\partial \nu}\right| ds \, |d\nu| \leq |\nabla G| \, dv \leq C \frac{d(\beta)}{\text{vol}\,(B(x_0, d(\beta)))} dv, \quad \text{on } \partial\Omega_\beta.$$

Then (3.28) becomes

$$(3.29) \qquad \int_\alpha^{2\alpha} \int_{\partial\Omega_\alpha} \log u(x,t) \cdot |\nabla G(x_0, x)| \, ds d\beta$$

$$\geq C \left(\frac{d(\alpha)}{\text{vol}\,(B(x_0, d(\alpha)))}\right)^2 \int_{\Omega_\alpha \backslash \Omega_{2\alpha}} \log u(x,t) dv$$

$$\geq C \frac{\alpha}{\text{vol}\,(B(x_0, Cd(\alpha)))} \int_{B(x_0, Cd(\alpha))} \log u(x,t) dv.$$

By integrating (3.24) from α to 2α and substituting the estimates (3.25) and (3.29) into the expression, we obtain

$$(3.30) \qquad \log u(x_0, t) \geq C \frac{1}{\text{vol}\,(B(x_0, Cd(\alpha)))} \int_{B(x_0, Cd(\alpha))} \log u(x,t) dv$$

$$- C \left[\sup_{x \in M} R(x) + \frac{1}{s} \sum_{k=1}^s \varepsilon\left(2^k\right) \cdot (1 + \log d(\alpha))\right].$$

Finally for any $r > 0$, set $\alpha = \frac{r^2}{\mathrm{vol}\,(B(x_0, r))}$. From the definition of $d(\alpha)$, we have

$$\frac{r^2}{\mathrm{vol}\,(B(x_0, r))} = \frac{d^2(\alpha)}{\mathrm{vol}\,(B(x_0, d(\alpha)))},$$

which together with (3.13) implies

(3.31) $C^{-1}r \le d(\alpha) \le Cr.$

It is easy to know that

(3.32) $\frac{1}{s}\sum_{k=1}^{s}\varepsilon\left(2^k\right) \to 0 \qquad as \quad d(\alpha) \to +\infty,$

because $\varepsilon(r) \to 0$ as $r \to +\infty$. Therefore by combining (3.30), (3.31) and (3.32) we get the desired estimate (3.15) with $\tilde{\varepsilon}(r)$ defined by $\frac{1}{s}\sum_{k=1}^{s}\varepsilon(2^k)$. $\square$

The combination of Lemma 3.1 and Lemma 3.2 is the following lower bound estimate for the solution of (2.2).

Theorem 3.3. *Suppose (M^n, g_{ij}) is a smooth n-dimensional $(n \ge 3)$ complete, noncompact Riemannian manifold with nonnegative Ricci curvature. Suppose also the scalar curvature is bounded and satisfies (1.2). Then the solution $u(x, t)$ of (2.2) obtained in section 2 exists for all times and satisfies*

(3.33) $(\log u)_{\min}(t) \ge -\tilde{\varepsilon}(r)\log r - C\left[1 + \sup_{x \in M} R(x) + \frac{t}{r^2}(\log u)_{\min}(t)\right],$

for all $t > 0$ and $1 \le r < +\infty$, where C is some positive constant depending only on n and $\tilde{\varepsilon}(r)$ is some positive function defined on $[1, +\infty)$ satisfying $\tilde{\varepsilon}(r) \to 0$ as $r \to +\infty$.

4. The Proof of the Main Theorem.

In this section we will give a proof for the main theorem based on the lower bound estimate of the previous section and the Li-Yau-Hamilton inequality of Chow [C] on locally conformally flat manifolds. We first recall the Li-Yau-Hamilton inequality as follows.

Theorem 4.1 (Chow). *Suppose (M^n, g_{ij}) is an n-dimensional $(n \ge 3)$ complete locally conformally flat Riemannian manifold with bounded and*

nonnegative Ricci curvature. Let $R(x,t)$ be the scalar curvature of the solution of the Yamabe flow with g_{ij} as initial metric. Then we have

$$(4.1) \qquad \frac{\partial R}{\partial t} + \langle \nabla R, X \rangle_t + \frac{1}{2(n-1)} R_{ij} X^i X^j + \frac{R}{t} \geq 0$$

for any vector X. Particularly if let $X = -\frac{\nabla R}{R}$ in (4.1), we have

$$(4.2) \qquad \frac{\partial R}{\partial t} + \frac{R}{t} \geq \frac{1}{2} \frac{|\nabla R|_t^2}{R},$$

where $\langle \cdot, \cdot \rangle_t$ and $|\cdot|_t$ are the inner product and norm with respect to the evolving metric $g_{ij}(x,t)$.

In his paper [C], Chow proved the above theorem for compact locally comformally flat manifolds with positive Ricci curvature. However, by a perturbation argument as in [H2], it is pretty clear that the Li-Yau-Hamilton inequality actually holds for complete locally conformally flat manifolds with nonnegative Ricci curvature.

Proof of the Main Theorem.

Suppose (M^n, g_{ij}) is an $n-$dimensional $(n \geq 3)$ complete noncompact locally conformally flat manifold with nonnegative Ricci curvature. Suppose also the scalar curvature is bounded and satisfies (1.2). We evolve the metric by the Yamabe flow (2.1). From Theorem 3.3 we know that the solution exists for all time $t \geq 0$ and satisfies the lower bound estimate (3.33). By setting $t = r$ in (3.33), it follows

$$(4.3) \qquad \lim_{t \to +\infty} \frac{-(\log u)_{\min}(t)}{\log t} = 0.$$

Thus by the evolution equation (2.1) and $g_{ij}(x,t) = (u(x,t))^{\frac{4}{n-2}} g_{ij}(x)$, we have

$$(4.4) \qquad \int_0^t R(x,\tau)d\tau = -\frac{4}{n-2} \log u(x,t) \leq -\frac{4}{n-2} (\log u)_{\min}(t).$$

Let $t > 1$. It follows from (4.2) that

$$R(x,\tau) \geq R(x,\sqrt{t}) \frac{\sqrt{t}}{\tau}, \qquad \text{for} \ \ \tau \geq \sqrt{t}.$$

Then we obtain

$$(4.5) \qquad \int_0^t R(x, \tau)d\tau \geq \int_{\sqrt{t}}^t R(x, \tau)d\tau$$

$$\geq R(x, \sqrt{t})\sqrt{t} \int_{\sqrt{t}}^t \frac{d\tau}{\tau}$$

$$\geq \frac{1}{2}R(x, \sqrt{t})\sqrt{t}\log t.$$

By combining (4.4) and (4.5), we have

$$R(x, \sqrt{t})\sqrt{t} \leq -\frac{8}{n-2}\frac{(\log u)_{\min}(t)}{\log t}$$

which together with (4.3) implies

$$\lim_{t \to +\infty} R(x, \sqrt{t})\sqrt{t} = 0.$$

But (4.2) tells us that $R(x, \sqrt{t})\sqrt{t}$ is nondecreasing in time. This shows that $R(x, t) \equiv 0$ for all $x \in M$, $t > 0$. Therefore the manifold M with the initial metric must be flat. $\qquad\qquad \square$

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