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Explicit construction of extremal Hermitian metrics with finite conical singularities on S^2

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In this paper, we will use the ODE method and geometry of Gaussian curvature of HCMU metrics to construct a class of nonradial extremal Hermitian metrics with finite conical singular angles $2\pi \cdot integer$ on S^2.

0. Introduction.

It is well-known that there is a metric with constant Gaussian curvature in each conformal class of any compact Riemann surface, by the classical uniformization theorem. It is a natural question to generalize this classical uniformization theory to compact surfaces with conical singularities. However, there are surfaces with conical singularities which do not admit a metric with constant curvature. For example, a football with two different singular angles does not admit a metric with constant curvature (for the existence or non-existence results of constant curvature metric in a surface with conical singularities, to see [T], [M], [CY], [LT]). Recently, instead of using metrics of constant curvature, X. Chen in [Ch1], [Ch2] and [Ch3] started to use the extremal Hermitian metrics to generalize the classical uniformization theory to Riemann surfaces with finite conical singularities. Besides a class of radial extremal Hermitian metrics on footballs (cf. [Ch2]), some nonradial examples of these metrics on S^2 with three conical angles 4π and nonconstant Gaussian curvature were found by E. Calabi and X. Chen (cf. [Ch3]). In [WZ], Wang and the second author also studied the extremal Hermitain metrics on Riemann surfaces with finite conical singularities and generalized Chen's results in [Ch2]. Let M be a compact Riemann surface and $\{p_i\}_{i=1,\dots,n} \subset M$. For any Hermitian metric g_0 on $M \setminus \{p_i\}_{i=1,\dots,n}$, consider the set $\mathcal{G}(M)$ of metrics with same area which are pointwise conformal to g_0 and agree with g_0 in a small neighborhood of $\{p_i\}_{i=1,\dots,n}$. In the closure of this set $\mathcal{G}(M)$ under some suitable $H^{2,2}$-norm, we define the energy functional

$$E(g) = \int_M K^2 dg,$$

where K denotes the Gaussian curvature of g. A critical point of this functional is called an extremal Hermitian metric. It is easy to see that the Euler-Lagrange equation is

$$\Delta_g K + K^2 = c, \quad \text{on} \quad M \setminus \{p_i\}_{i=1,\ldots,n}, \tag{0.1}$$

for some constant c, where Δ_g denotes the Lapalace operator associated to metric g. Let $g = e^{2\phi}|dz|^2$ be written in local coordinates. Then equation (0.1) is equivalent to a system,

$$\begin{cases} \Delta\phi = -Ke^{2\phi} \\ \Delta K = (-K^2 + c)e^{2\phi}, \end{cases} \tag{0.2}$$

where Δ is the standard Lapalace operator on $\mathbb{R}^2$.

Let

$$K_{,zz} = \frac{\partial^2 K}{\partial z \partial z} - 2\frac{\partial \phi}{\partial z}\frac{\partial K}{\partial z}$$

be the second derivative of K with respect to g. Then, by (0.2), one can check

$$(K_{,zz})_{\bar{z}} = 0. \tag{0.3}$$

A special case of (0.3) is

$$K_{,zz} = 0, \tag{0.4}$$

which means that

$$e^{-2\phi} K_{\bar{z}} \frac{\partial}{\partial z}$$

is a holomorphic vector field on $M \setminus \{p_i\}_{i=1,\ldots,n}$. In particular, in case $M = S^2$,

$$F(z) = e^{-2\phi} K_{\bar{z}} \tag{0.5}$$

is a rationally holomorphic function on $\mathbb{C}$ (cf. Proposition 1.2). Usually, a Hermitian metric satisfying (0.4) is called HCMU ([Ch3]), which including the case of $K \equiv$ const. A HCMU metric on a compact Riemann surface with finite singularities is the simplest case of Calabi's extremal metrics in the singular spaces ([Ca1], [Ca2]).

In this paper, we will use the ODE method and geometry of Gaussian curvature of HCMU metrics to construct a class of nonradial extremal Hermitian metrics with finite conical singular angles $2\pi \cdot integer$ on S^2. In fact, we shall classify all these HCMU metrics on S^2 and give an explicit formula via rationally holomorphic functions on $\mathbb{C}$.

Definition 0.1. A HCMU metric on a compact Riemann surface M with finite singularities $\{p_i\}_{i=1,\dots,n}$ is called exceptional if all singular points have weak, integer conical singular angles $2\pi\alpha_i$ and $F(z)$ defined by (0.5) has following expansion near those singular points p_i,

$$F(z) = (z - z_i)^{-(\alpha_i - 1)}(c_i + zg_i(z)),$$

for some complex-valued numbers $c_i \neq 0$ and holomorphic functions $g_i(z)$ near p_i.

Our main theorem can be stated as follows (cf. Theorem 2.1).

Theorem 0.1. *Let $p_1 = (\infty)$ and $p_i = (z_i), i = 2, \dots, n$, be n points on $S^2 = \mathbb{C} \cup \{\infty\}$, and $\alpha_1 \geq 2, \dots, \alpha_n \geq 2$, n positive integer numbers. Let*

$$\alpha = \alpha_1 + \sum_{i=2}^{n}(\alpha_i - 1).$$

Then $g = e^{2\phi}|dz|^2$ is an exceptional HCMU metric with finite conical singular angles $2\pi\alpha_i$ at p_i on S^2 if and only if there are a positive integer k, a complex-valued $B \neq 0$ and a holomorphic polynomial function $f(z)$ on $\mathbb{C}$ with degree $(\alpha + 1)$ and different $(\alpha + 1)$ roots γ_l of $f(z) = 0$ such that $3k \leq \alpha + 1$, and $(\alpha + 1)$ roots γ_l satisfy

$$\sum_{l=1}^{k} \frac{-2a}{z - \gamma_l} + \sum_{l'=k+1}^{\alpha+1} \frac{2}{z - \gamma_{l'}} = \frac{B\prod_{i=2}^{n}(z - z_i)^{(\alpha_i - 1)}}{\prod_{l=1}^{\alpha+1}(z - \gamma_l)},$$

where $a = \frac{\alpha+1-k}{k}$. Furthermore, K and ϕ are given by

$$\frac{1}{\left(K + \frac{(a-2)\beta}{2a-1}\right)^{a}}(\beta - K)\left(K + \frac{\beta(a+1)}{2a-1}\right)^{a-1} = A\prod_{l=1}^{k}|z - \gamma_l|^{-2a}\prod_{l'=k+1}^{\alpha+1}|z - \gamma_{l'}|^2,$$

and

$$\phi = \frac{1}{2}\ln\left(\frac{1}{|F(z)|^2}\left(-\frac{1}{3}K^3 + cK + c'\right)\right),$$

where $A > 0$ is some constant and

$$\beta = \sqrt{c}(2a - 1)/\sqrt{a^2 - a + 1},$$

$$c' = \frac{(a+1)(a-2)\beta^3}{3(2a-1)^2},$$

$$F(z) = (z-z_2)^{-(\alpha_2-1)} \cdots (z-z_n)^{-(\alpha_n-1)} f(z),$$

and

$$f(z) = B^{-1}\frac{-3a(a-1)\beta^2}{(2a-1)^2}\prod_{l=1}^{\alpha+1}(z-\gamma_l).$$

From Theorem 0.1, we see that an explicit HUMC metric on S^2 is determined by a pair (a, k).

Definition 0.2. An exceptional HCMU metrics on S^2 is called minimal if $(a, k) = (\alpha, 1)$ in Theorem 0.1.

In case of minimal exceptional HCMU metrics on S^2, we can give an explicit formula (cf. Theorem 3.1). In particular, for any n singular points with conical angles $2\pi \cdot \mathit{integer}$ and a positive number c_0, there is a family of extremal Hermitian metrics on S^2 with same area c_0, which vary energy $E(g)$ with three parameters (cf. Remark 3.1). This shows that a HCMU metric with finite integer conical singular angles on S^2 could not be in general a local minimizer for energy functional E with respect to a general deformation, which preserves the area and conical angle structure at each singular point, although any HCMU metrics is a local minimizer in the class $\mathcal{G}(M)$ (cf. [Ca2], [Ch3]).

The organization of paper is as follows. In Section 1, by using the asymptotic expansion of Gaussian curvature, we study the local behavior of Gaussian curvature of HCMU metric with finite energy and area near the singular point. In Section 2, we will use the ODE method and geometry of Gaussian curvature of HCMU metrics to classify all exceptional metrics with finite singularities on S^2, and then prove Theorem 0.1 (Theorem 2.1). In Section 3, we can give an explicit formula for any minimal exceptional, HCMU metrics (cf. Theorem 3.1). In Section 4, we shall classify all HCMU metrics with finite energy and area on R^2 as well as radial, extremal Hermitian metrics with two different weakly conical singular angles at the origin and infinity. In Section 5, as two examples, we discuss all exceptional, HCMU metrics on S^2 with two and three weakly conical singular angles 4π respectively.

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1. Local behavior of Gaussian curvature K.

In this section, we discuss the local behavior of Gaussian curvature of a HCMU metric on $D \setminus \{0\}$, where $D \subset R^2$ is a disk.

Let $g = e^{2\phi}|dz|^2$ be an extremal Hermitian metric with finite energy and area on $D \setminus \{0\}$. Then we have the following system,

$$\text{(1.1)} \qquad \begin{cases} \triangle\phi = -Ke^{2\phi} \\ \triangle K = (-K^2 + c)e^{2\phi}, \quad \text{on } D \setminus \{0\}, \end{cases}$$

with

$$\text{(1.2)} \qquad \int_{D\setminus\{0\}} e^{2\phi}dx < \infty \ \text{ and } \int_{D\setminus\{0\}} K^2e^{2\phi}dx < \infty,$$

where $\triangle$ is the standard Lapalace operator on $\mathbb{R}^2$.

Let $x = (z) = (r\cos\theta, r\sin\theta)$. Then following result was proved in [WZ].

Lemma 1.1 ([WZ]). *Let $g = e^{2\phi}|dz|^2$ be an extremal Hermitian metric on $D \setminus \{0\}$. Suppose that* (1.2) *is satisfied. Then there is a number $\alpha \geq 0$ such that*

$$\lim_{r\to 0}\int_0^{2\pi} r\frac{\partial\phi}{\partial r}d\theta = 2\pi(\alpha - 1).$$

Furthermore, if $\alpha > 0$, then

i) $\lim_{|x|\to 0}|\phi(x) - (\alpha - 1)\ln|x|| < C$, *for some* C;

ii) $\lim_{|x|\to 0}|K(x)||x|^\alpha = 0$.

Remark 1.1. We call the singular point $\{0\}$ weakly conical with singular angle $2\pi\alpha$ if $\alpha > 0$ ([WZ]). Note that the singular point $\{0\}$ is called conical with singular angle $2\pi\alpha$ if $\phi(x) = (\alpha - 1)\ln|x| + \rho(x)$ for some smooth function $\rho(x)$ on D. In this paper, we always assume $\alpha > 0$. For the case $\alpha = 0$, we refer reader to [Ch2].

Next we shall use the technique of Kelvin transformation to refine Lemma 1.1. Let $w = \frac{1}{z}$, and

$$\psi(w) = -2\ln|w| + \phi\left(\frac{1}{w}\right) \ \text{ and } \ \overline{K}(w) = K\left(\frac{1}{w}\right).$$

Then $(\psi, \overline{K})$ is a solution of system (1.1) on $\mathbb{R}^2 \setminus B(\frac{R}{2})$ for some sufficiently large number R. Furthermore, by Lemma 1.1, if $\alpha > 0$, then we have

$$\lim_{|x|\to\infty} |\psi(x) - (\alpha+1)\ln|x|| < C, \quad \text{for some } \; C, \tag{1.3}$$

and

$$\lim_{|x|\to\infty} |\overline{K}(x)||x|^{-\alpha} = 0. \tag{1.4}$$

Let $\eta(x) \geq 0$ be a smooth cut-off function on $\mathbb{R}^2$ so that $\eta(x) = 1$ on $\mathbb{R}^2 \setminus B(R)$, $\eta(x) = 0$, on $B(\frac{R}{2})$ and $0 \leq \eta \leq 1$, on $B(R) \setminus B(\frac{R}{2})$. Let $\psi^* = \eta\psi$ and $K^* = \eta\overline{K}$. Then (ψ^*, K^*) is a solution of system (1.1) on $R^2 \setminus B(R)$, and satisfies on $\mathbb{R}^2$,

$$\begin{cases} \triangle\psi^* = -\eta K e^{2\psi} + 2\nabla\eta\nabla\psi^* + \psi^*\triangle\eta = h \\ \triangle K^* = \eta(-K^2 + c)e^{2\psi} + 2\nabla\eta\nabla K^* + K^*\triangle\eta = g. \end{cases}$$

Clearly, $h, g \in L^1(\mathbb{R}^2)$.

Let

$$\tilde{K}(x) = \frac{1}{2\pi}\int_{R^2} (\ln(|x-y|) - \ln(|y|+1))g(y)dy.$$

Then we see that there are a constant c_0 and a holomorphic polynomial function with degree k,

$$f(z) = \sum_{l=1}^{k} e_l w^l$$

such that

$$K(x)^* = \hat{K}(x) + \tilde{K}(x) + c_0, \quad \text{and} \quad \hat{K}(x) = \operatorname{Re} f(w).$$

Without loss of generality, we may assume $e_k = 1$. By (1.4), we see $k < \alpha$. Hence by the standard regularity theorem, it follows

$$\phi(z) - (\alpha - 1)\ln|z| \in C^{2\alpha-k}(D),$$

and

$$\tilde{K}\left(\frac{1}{z}\right) + \beta\ln|z| \in C^{2\alpha-2k}(D),$$

where $\beta = \frac{1}{2\pi}\int_{R^2} g(x)dx$. So we may assume that

$$r\psi_r = -(\alpha+1) + \sum_{l=1}^{[2\alpha-k]'} c_l^1 \frac{1}{r^l} + a_1(\theta)\frac{1}{r^{2\alpha-k}} + o\left(\frac{1}{r^{2\alpha-k}}\right), \tag{1.5}$$

$$\psi_\theta = \sum_{l=1}^{[2\alpha-k]'} c_l^2 \frac{1}{r^l} + b_1(\theta)\frac{1}{r^{2\alpha-k}} + o\left(\frac{1}{r^{2\alpha-k}}\right), \tag{1.6}$$

$$r\tilde{K}_r = \beta + \sum_{l=1}^{[2\alpha-2k]'} d_l^1 \frac{1}{r^l} + a_2(\theta)\frac{1}{r^{2\alpha-2k}} + o\left(\frac{1}{r^{2\alpha-2k}}\right), \tag{1.7}$$

and

$$\tilde{K}_\theta = \sum_{l=1}^{[2\alpha-2k]'} d_l^2 \frac{1}{r^l} + b_2(\theta)\frac{1}{r^{2\alpha-2k}} + o\left(\frac{1}{r^{2\alpha-2k}}\right), \tag{1.8}$$

where $c_l^1, c_l^2, d_l^1, d_l^2, a_1, a_2, b_1, b_2$ are C^∞-functions about θ, and

$$[2\alpha - k]' = [2\alpha - k], \quad \text{if } \ 2\alpha \neq \text{ integer}$$

and

$$[2\alpha - k]' = 2\alpha - k - 1, \quad \text{if } \ 2\alpha = \text{ integer}.$$

Lemma 1.2. *Let $g = e^{2\phi}|dz|^2$ be a HCMU metric on $D\setminus\{0\}$. Suppose that (1.2) is satisfied. Then $\hat{K}(x) = 0$, i.e., the holomorphic polynomial function $f(w)$ vanishes.*

Proof. Let

$$\frac{\partial}{\partial w} = \frac{\partial}{\partial x_1} - \sqrt{-1}\frac{\partial}{\partial x_2} \quad \text{and} \quad \frac{\partial}{\partial \bar{w}} = \frac{\partial}{\partial x_1} + \sqrt{-1}\frac{\partial}{\partial x_2}$$

be two complex partial differential operators. Then the above two operators are equivalent to

$$\frac{\partial}{\partial w} = \frac{1}{w}\left(r\frac{\partial}{\partial r} - \sqrt{-1}\frac{\partial}{\partial \theta}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{w}} = \frac{1}{w}\left(r\frac{\partial}{\partial r} + \sqrt{-1}\frac{\partial}{\partial \theta}\right).$$

Thus, for any smooth function Φ, one obtains

$$\Phi_{ww} = \frac{-1}{w^2}(2r\Phi_r - (r(r\Phi_r)_r - \Phi_{\theta\theta}) - 2\sqrt{-1}(\Phi_\theta - r\Phi_{r\theta}))$$

and

$$\begin{aligned}\Phi_{,ww} &= \Phi_{ww} - 2\Phi_w\phi_w \\ &= -\frac{1}{w^2}((2r\Phi_r - (r(r\Phi_r)_r - \Phi_{\theta\theta}) + 2(r\Phi_r.r\phi_r - \Phi_\theta\phi_\theta) \\ &\quad - 2\sqrt{-1}((\Phi_\theta - 2r\Phi_{r\theta}) + (\Phi_\theta \cdot r\phi_r - \phi_\theta \cdot r\Phi_\theta))).\end{aligned}$$

By using (1.5)-(1.8), we have

$$\tilde{K}_w \phi_w = \frac{O(1)}{w^2} \quad \text{and} \quad \tilde{K}_{ww} = \frac{O(1)}{w^2}. \tag{1.9}$$

It follows

$$\tilde{K}_{,ww} = \frac{O(1)}{w^2}. \tag{1.10}$$

On the other hand, we see

$$\hat{K}_{ww} = \frac{1}{w^2} \sum_{l=1}^{k} 2l(l-1)e_l w^l, \tag{1.11}$$

and

$$\hat{K}_w \psi_w = -(\alpha+1)\frac{1}{w^2} \sum_{l=1}^{k} l e_l w^l + \frac{O(1)}{w^2}. \tag{1.12}$$

Hence, combining (1.10), (1.11) and (1.12), we get

$$\begin{aligned} K^*_{,ww} &= \tilde{K}_{,ww} + \hat{K}_{,ww} - 2\hat{K}_w \psi_w \\ &= 2k(\alpha+k)w^{k-2} + o(|w|^{k-2}). \end{aligned} \tag{1.13}$$

Since g is a HCMU metric, then by (0.4), we have

$$K^*_{,ww} = \overline{K}_{,ww} = z^4 K_{,zz} = 0, \quad \text{on} \quad \mathbb{R}^2 \setminus B(R).$$

Then by (1.13), it follows $k = 0$, and consequently, $\hat{K}(x) = 0$. □

Lemma 1.3. *Let $g = e^{2\phi}|dz|^2$ be a HCMU metric on $D \setminus \{0\}$. Suppose that* (1.2) *is satisfied. Then the number $\beta = 0$ in* (1.7).

Proof. By Lemma 1.2, we see that there is a constant C such that

$$|\overline{K} - \beta \ln r| < C, \quad \text{on } \mathbb{R}^2 \setminus B(R).$$

So we may assume that

$$r\psi_r = -(\alpha+1) + \sum_{l=1}^{[2\alpha]'} c_l^1 \frac{1}{r^l} + a_1 \ln r \frac{1}{r^{2\alpha}} + o\left(\ln r \frac{1}{r^{2\alpha}}\right),$$

$$\psi_\theta = \sum_{l=1}^{[2\alpha]'} c_l^2 \frac{1}{r^l} + b_1 \ln r \frac{1}{r^{2\alpha}} + o\left(\ln r \frac{1}{r^{2\alpha}}\right),$$

$$r\overline{K}_r = \beta + \sum_{l=1}^{[2\alpha]'} d_l^1 \frac{1}{r^l} + a_2 (\ln r)^2 \frac{1}{r^{2\alpha}} + o\left((\ln r)^2 \frac{1}{r^{2\alpha}}\right), \tag{1.14}$$

and

$$\overline{K}_\theta = \sum_{l=1}^{[2\alpha]'} d_l^2 \frac{1}{r^l} + b_2 (\ln r)^2 \frac{1}{r^{2\alpha}} + o\left((\ln r)^2 \frac{1}{r^{2\alpha}}\right),$$

where

1) $[2\alpha]' = [2\alpha]$, if $2\alpha \neq$ integer and $[2\alpha] = 2\alpha - 1$, if $2\alpha =$ integer;
2) a_1, b_1, a_2, b_2 are some constants;
3) $c_l^1 = \mathrm{Re}(c_l e^{\sqrt{-1}l\theta})$, $c_l^2 = \mathrm{Im}(c_l e^{\sqrt{-1}l\theta})$, $d_l^1 = \mathrm{Re}(d_l e^{\sqrt{-1}l\theta})$, $d_l^2 = \mathrm{Im}(d_l e^{\sqrt{-1}l\theta})$.

Then similar to (1.9), we obtain

$$\overline{K}_{ww} = \frac{-1}{w^2}\left(2\beta + O\left(\frac{1}{r}\right) + O\left((\ln r)^2 \frac{1}{r^{2\alpha}}\right)\right),$$

and

$$\overline{K}_w \psi_w = \frac{1}{w^2}\left(-(\alpha+1)\beta + O\left(\frac{1}{r}\right) + O\left((\ln r)^2 \frac{1}{r^{2\alpha}}\right)\right).$$

It follows

$$\begin{aligned}\overline{K}_{,ww} &= \overline{K}_{ww} - 2\overline{K}_w \psi_w \\ &= \frac{-1}{w^2}\left(2\alpha\beta + O(r^{-1}) + O\left((\ln r)^2 \frac{1}{r^{2\alpha}}\right)\right).\end{aligned}$$

This shows $\beta = 0$, since $\overline{K}_{,ww} = z^4 K_{,zz} = 0$ on $\mathbb{R}^2 \setminus B(R)$. □

By (1.14) and Lemma 1.3, we see

$$rK_r = O\left(\frac{1}{r}\right) + O\left((\ln r)^2 \frac{1}{r^{2\alpha}}\right).$$

This shows that the Guassian curvature K is continueous across the singular point $\{0\}$. In general, we prove

Proposition 1.1. *Let $g = e^{2\phi}|dz|^2$ be a HCMU metric on a compact Riemann surface M with finite weakly conical singular angles at $\{p_i\}_{i=1,\dots,n}$. Suppose that*

$$\int_{M\setminus\{p_i\}_{i=1,\dots,n}} e^{2\phi}dx < \infty \quad \text{and} \quad \int_{M\setminus\{p_i\}_{i=1,\dots,n}} K^2 e^{2\phi}dx < \infty.$$

Then the Guassian curvature of g is continueous across the singularities $\{p_i\}_{i=1,\dots,n}$.

Since

$$K_{,zz} = \frac{\partial^2 K}{\partial z \partial z} - 2\frac{\partial \phi}{\partial z}\frac{\partial K}{\partial z} = 0,$$

we see that there is a holomorphic function $F(x)$ on $D \setminus \{0\}$ such that

$$e^{-2\phi}K_{\overline{z}} = F(x).$$

Proposition 1.2. *Let $g = e^{2\phi}|dz|^2$ be a HCMU metric on $D \setminus \{0\}$ with a weakly conical singular angle $2\pi\alpha$ at the origin. Then $F(z)$ has the following expansion near the origin,*

$$F(z) = \begin{cases} zg(z), & \text{if } \alpha \neq \text{ integer;} \\ z^{-(\alpha-1)}(c+g(z)), & \text{for some } c \neq 0, \\ \text{or } = zg(z), & \text{if } \alpha = \text{ integer,} \end{cases}$$

where $g(z)$ is some holomorphic function on D.

Proof. By Lemma 1.3, we may assume that

$$r\psi_r = -(\alpha+1) + \sum_{l=1}^{[2\alpha]'} c_l^1 \frac{1}{r^l} + a_1 \frac{1}{r^{2\alpha}} + o\left(\frac{1}{r^{2\alpha}}\right),$$

$$\psi_\theta = \sum_{l=1}^{[2\alpha]'} c_l^2 \frac{1}{r^l} + b_1 \frac{1}{r^{2\alpha}} + o\left(\frac{1}{r^{2\alpha}}\right),$$

$$(1.15) \qquad r\overline{K}_r = \sum_{l=1}^{[2\alpha]'} d_l^1 \frac{1}{r^l} + a_2 \frac{1}{r^{2\alpha}} + o\left(\frac{1}{r^{2\alpha}}\right),$$

and

$$\overline{K}_\theta = \sum_{l=1}^{[2\alpha]'} d_l^2 \frac{1}{r^l} + b_2 \frac{1}{r^{2\alpha}} + o\left(\frac{1}{r^{2\alpha}}\right), \tag{1.16}$$

where a_1, b_1, a_2, b_2 are some constants, and $c_l^1 = \mathrm{Re}(c_l e^{\sqrt{-1}l\theta})$, $c_l^2 = \mathrm{Im}(c_l e^{\sqrt{-1}l\theta})$, $d_l^1 = \mathrm{Re}(d_l e^{\sqrt{-1}l\theta})$, $d_l^2 = \mathrm{Im}(d_l e^{\sqrt{-1}l\theta})$.

It follows

$$\begin{aligned}
\overline{K}_{,ww} &= \overline{K}_{ww} - 2\overline{K}_w \psi_w \\
&= \frac{-1}{w^2}\left(\sum_{l=1}^{[2\alpha]'} 2(l+1) d_l w^{-l} - 2(\alpha+1) \sum_{l=1}^{[2\alpha]'} d_l w^{-l} \right. \\
&\quad \left. + 2 \sum_{l=2}^{[2\alpha]'} \sum_{l=p+q} c_p d_q w^{-l} + O(r^{-2\alpha}) \right) \\
&= \frac{-1}{w^2}\left(\sum_{l=1}^{[2\alpha]'} (2(l+1) - 2\alpha - 2) d_l w^{-l} + 2 \sum_{l=2}^{[2\alpha]'} \sum_{l=p+q} c_p d_q w^{-l} \right. \\
&\quad \left. + O(r^{-2\alpha}) \right).
\end{aligned}$$

Since $\overline{K}_{,ww} = 0$ on $\mathbb{R}^2 \setminus B(R)$, we get

$$((l+1) - \alpha - 1) d_l + \sum_{l=p+q} c_p d_q = 0, \ l = 1, \ldots, [2\alpha]. \tag{1.17}$$

By using the induction to (1.17), we have

$$d_l = 0, \ l = 1, \ldots, [\alpha] - 1,$$

and

$$d_l = 0, \ l = 1, \ldots, [2\alpha], \text{ if } d_\alpha = 0.$$

Futhermore, if $\alpha \neq$ integer, then $d_\alpha = 0$ and consequently,

$$d_l = 0, \ l = 1, \ldots, [2\alpha].$$

Hence by (1.15) and (1.16), we get

a) $|e^{-2\phi} K_{\overline{z}}| \leq C|z|$, if $\alpha \neq$ *integer*;

b) $|e^{-2\phi} K_{\overline{z}}| \leq C|z|$, or $= O(|z|^{-(\alpha-1)})$, if $\alpha =$ *integer*.

This shows that there is a holomorphic function $g(z)$ on D such that

$$F(z) = \begin{cases} zg(z), & \text{if } \alpha \neq \text{ integer;} \\ z^{-(\alpha-1)}(c+g(z)), & \text{for some } c \neq 0, \\ \text{or } = zg(z), & \text{if } \alpha = \text{ integer.} \end{cases}$$

□

Definition 1.1. A HCMU metric on a compact Riemann surface M with finite singularities $\{p_i\}_{i=1,\ldots,n}$ is called exceptional if all singular points have weak, integer conical angles $2\pi\alpha_i$ and $F(z)$ has following expansion near those singular points p_i,

$$F(z) = (z - z_i)^{-(\alpha_i - 1)}(c_i + g_i(z)),$$

for some complex-valued numbers $c_i \neq 0$ and holomorphic functions $g_i(z)$ near p_i.

Remark 1.2. If $g = e^{2\phi}|dz|^2$ is a HCMU metric on $D \setminus \{0\}$ with finite area and energy, and a weakly conical singular angle 2π at the origin, then by the system (1.1) together with Proposition 1.1, one can prove that g is in fact smooth on D (cf. an argument in the proof of Proposition 4.1). So in Definition 1.1, we may assume that all weakly conical singular angles are more than 2π.

2. Classification of exceptional HCMU metrics on S^2.

Let $g = e^{2\phi}|dz|^2$ be an extremal Hermitian metric on $M \setminus \{p_i\}_{i=1,\ldots,n}$, and K its Gaussian curvature. Then (ϕ, K) is a local solution of system,

$$\begin{cases} \Delta\phi = -Ke^{2\phi}, \\ \Delta K = (-K^2 + c)e^{2\phi}. \end{cases} \tag{2.1 – 2.2}$$

Lemma 2.1. *Let $g = e^{-2\phi}|dz|^2$ be an extremal Hermitian metric on $M \setminus \{p_i\}$, i=1,...,n. Then g is HCMU if and only if there are some constant c' and a holomorphic function $F(z)$ such that $F(z)\frac{\partial}{\partial z}$ is a holomorphic vector field on $M \setminus \{p_i\}_{i=1,\ldots,n}$, and the Guassian curvature K of g is a solution of ODE,*

$$K_z = F(z)^{-1}\left(-\frac{K^3}{3} + cK + c'\right), \tag{2.3}$$

and ϕ is given by

$$e^{2\phi} = |F(z)|^{-2}\left(-\frac{K^3}{3} + cK + c'\right) > 0. \tag{2.4}$$

Proof. Since g is HCMU, there is a holomorphic vector field $F(z)\frac{\partial}{\partial z}$ on $M \setminus \{p_i\}_{i=1,\dots,n}$, such that

$$K_{\overline{z}} = e^{2\phi}F(z). \tag{2.5}$$

Then by using (2.2), we get

$$K_{z\overline{z}} = F(z)^{-1}\left(-\frac{K^3}{3} + cK + c''\right)_{\overline{z}}$$

for some constant c''. It follows

$$K_z = F(z)^{-1}\left(-\frac{K^3}{3} + cK + c''\right) + g(z) \tag{2.6}$$

for some meromorphic function $g(z)$ on $M \setminus \{p_i\}_{i=1,\dots,n}$.

By (2.5) and (2.6), we see

$$e^{2\phi} = |F(z)|^{-2}\left(-\frac{K^3}{3} + cK + c''\right) + \overline{\frac{g(z)}{F(z)}}$$

is a real-valued function, and consequently $g(z) = CF(z)^{-1}$ for some constant real-valued number C. Thus there is a real-valued number c' such that

$$e^{2\phi} = |F(z)|^{-2}\left(-\frac{K^3}{3} + cK + c'\right) > 0,$$

and consequently, K satisfies

$$K_z = F(z)^{-1}\left(-\frac{K^3}{3} + cK + c'\right).$$

Conversly, we assume that K is a solution of integrable equation (2.3) and ϕ is given by (2.4). Then it is clear

$$\phi = \frac{1}{2}\ln(K_{\overline{z}}F(z)^{-1}). \tag{2.7}$$

Differentiating (2.3) and using (2.7), we have

$$\begin{aligned} K_{z\bar{z}} &= (-K^2 + c)K_{\bar{z}}F(z)^{-1} \\ &= (-K^2 + c)e^{2\phi}. \end{aligned}$$

This shows (ϕ, K) satisfies equation (2.2). Now we need to check (K, ϕ) also satisfies (2.1).

Differentiating (2.7), we have

$$\begin{aligned} \phi_z &= F(z)\left(-\frac{F(z)'}{F(z)^2} + \frac{1}{2}\frac{K_{z\bar{z}}}{K_{\bar{z}}}F(z)^{-1}\right) \\ &= \left(-\frac{F(z)'}{F(z)^2} + \frac{1}{2}(-K^2 + c)F(z)^{-2}\right)F(z) \end{aligned}$$

Then again differentiating the above equation and using (2.3) and (2.7), it follows

$$\begin{aligned} \triangle\phi = \phi_{z\bar{z}} &= -KK_{\bar{z}}F(z)^{-1} \\ &= -Ke^{2\phi} \end{aligned}$$

□

Now we assume that $g = e^{2\phi}|dz|^2$ is an exceptional HCMU metric on $S^2 = \mathbb{C} \cup \{\infty\}$ with finite singularities $\{p_i\}_{i=1,\ldots,n}$. Without the loss of generality, we may assume $p_1 = (\infty)$. Let $2\pi\alpha_i$ $(\alpha_1 \geq 2)$ be the weak, integer conical singular angles at $p_i, i = 1, \ldots, n$. Then by Definition 1.1, it is easy to see that there is a meromorphic function $F(z)$ on $\mathbb{C}$ such that

$$F(z) = e^{-2\phi}K_{\bar{z}} = \prod_{i=2}^{n}(z - z_i)^{-(\alpha_i - 1)}f(z), \tag{2.8}$$

where $f(z)$ is a holomorphic function on $\mathbb{C}$ with degree $\alpha = 1 + \sum_{i=1}^{n}(\alpha_i - 1)$.

Proposition 2.1. *Let $g = e^{2\phi}|dz|^2$ be an exceptional HCMU metric on $S^2 = \mathbb{C} \cup \{\infty\}$ with finite singularities $\{p_i\}_{i=1,\ldots,n}$, and $f(z)$ and $F(z)$ are given in (2.8). Let $\gamma_1, \ldots, \gamma_{\alpha+1}$ be $(\alpha + 1)$ roots of $f(z) = 0$. Then*

$$\gamma_i \neq \gamma_j, \ \forall i \neq j,$$

and there are some real-valued numbers $c_l, l = 1, \ldots, \alpha + 1$, such that

$$\sum_{l=1}^{\alpha+1} c_l = 0,$$

and

$$\frac{1}{F(z)} = \sum_{l=1}^{\alpha+1} \frac{c_l}{z-\gamma_l}.$$

Before we prove Proposition 2.1, we need a series of lemmas.

Lemma 2.2. *Let c' be the constant determined in Lemma 2.1. Suppose that there are three roots $\beta_1, \beta_2, \beta_3$ of equation,*

$$\frac{-K^3}{3} + cK + c' = 0. \tag{2.9}$$

Then

$$\beta_i \neq \beta_j, \ \forall i \neq j.$$

Proof. On the contrary, we assume $\beta_1 \neq \beta_2 = \beta_3$. Then one can choose three real-valued numbers $a_1, a_2, a_3 \neq 0$ such that

$$\frac{1}{\frac{-K^3}{3} + cK + c'} = \frac{a_1}{(K-\beta_1)^2} + \frac{a_2}{K-\beta_1} + \frac{a_3}{K-\beta_2}. \tag{2.10}$$

Claim. *Let $\gamma_1, \dots, \gamma_{\alpha+1}$ be $(\alpha+1)$ roots of $f(z)=0$. Then there are some i and $j(\neq i)$ such that $\gamma_i = \gamma_j$.*

We also use an argument by contradiction, and assume

$$\gamma_i \neq \gamma_j, \ \forall i \neq j.$$

Then one can choose some $\alpha+1$ complex-valued numbers $c_l = c_l^1 + \sqrt{-1}c_l^2 \neq 0$ such that

$$\sum_{l=1}^{\alpha+1} c_l = 0, \tag{2.11}$$

and

$$\frac{1}{F(z)} = \sum_{l=1}^{\alpha+1} \frac{c_l}{z-\gamma_l}. \tag{2.12}$$

Thus by using (2.10) and (2.12), we can solve equation (2.3) as follow,

$$\begin{aligned} &\frac{-a_1}{K-\beta_1} + a_2 \ln|K-\beta_1| + a_3 \ln|K-\beta_2| \\ &= 2\sum_{l=1}^{\alpha+1} c_l^1 \ln|z-z_l|^2 + 4\sum_{l=1}^{\alpha+1} c_l^2 \theta_l(z-\gamma_l), \end{aligned} \tag{2.13}$$

where $\theta_l(z-\gamma_l)$ denote functions of complex angles of $z-\gamma_l$.

Since by Proposition 1.1, K is a continueous function on S^2, from (2.11), we see that there are at least two nonzero numbers c_i^1, c_j^1 such that

$$c_i^1 \cdot c_j^1 < 0.$$

This shows

$$K(\gamma_i) = \beta_1 \text{ and } K(\gamma_j) = \beta_2,$$

and consequently,

$$K - \beta_1 = O\left(\frac{-1}{\ln|z-\gamma_i|^2}\right), \quad \text{as } z \to \gamma_i. \tag{2.14}$$

On the other hand, since ϕ is smooth at γ_i, then by (2.4) in Lemma 2.1, one see that there is a constant $A' \neq 0$ such that

$$\lim_{z\to\gamma_i} \frac{K-\beta_1}{|z-\gamma_i|} = A',$$

which is contradict to (2.14). Claim is proved.

By Claim, we see that there are complex-valued number $c_i \neq 0$ and a root γ_i with multiple $k \geq 2$ of equation $f(z) = 0$ such that

$$\frac{1}{F(z)} = \frac{1}{(z-\gamma_i)^k} + o\left(\frac{c_i}{|z-\gamma_i|^k}\right).$$

Then by (2.10), one can solve equation (2.4) as follow,

$$\frac{-a_1}{K-\beta_1} = 4\,\text{real}\left(\frac{-c_i}{(k-1)(z-\gamma_i)^{k-1}}\right) + o\left(\frac{1}{|z-\gamma_i|^{k-1}}\right).$$

It follows

$$K - \beta_1 = O(|z-\gamma_i|^{k-1}), \quad \text{as } z \to \gamma_i. \tag{2.15}$$

On the other hand, by (2.4) in Lemma 2.1, we see that there is a constant $A' \neq 0$ such that

$$\lim_{z\to\gamma_i} \frac{K-\beta_1}{|z-\gamma_i|^k} = A',$$

which contradicts to (2.15). Lemma 2.2 is proved. □

Lemma 2.3. *Let $\gamma_1, \dots, \gamma_{\alpha+1}$ be $(\alpha+1)$ roots of $f(z)=0$. Then*

$$\gamma_i \neq \gamma_j, \ \forall i \neq j.$$

Proof. On the contrary, we assume that there is a root γ_i with multiple $k \geq 2$ of equation $f(z)=0$. Then there is a complex-valued number $c_i \neq 0$ such that

$$\frac{1}{F(z)} = \frac{c_i}{(z-\gamma_i)^k} + o\left(\frac{1}{|z-\gamma_i|^k}\right).$$

By Lemma 2.2, there is only one root β or are three different roots $\beta_1, \beta_2, \beta_3$ of equation (2.9). In the late case, one can choose three nonzero real numbers a_1, a_2, a_3, such that

$$\frac{1}{\frac{-K^3}{3} + cK + c'} = \frac{a_1}{K-\beta_1} + \frac{a_2}{K-\beta_2} + \frac{a_3}{K-\beta_3}.$$

Then by using the argument in Lemma 2.2, we solve equation (2.3) as follow,

$$\begin{aligned} &a_1 \ln|K-\beta_1| + a_2 \ln|K-\beta_2| + a_3 \ln|K-\beta_3| \\ &= 4\,\mathrm{real}\left(\frac{-c_i}{(k-1)(z-\gamma_i)^{k-1}}\right) + o\left(\frac{1}{|z-\gamma_i|^{k-1}}\right). \end{aligned}$$

This shows that there is one of roots β_1 (we may assume β_1) such that

$$\ln(K-\beta_1) = O(|z-\gamma_i|^{-(k-1)}), \text{ as } z \to \gamma_i. \tag{2.16}$$

On the other hand, by (2.4) in Lemma 2.1, we see there is constant $A' \neq 0$ such that

$$\lim_{z\to\gamma_i} \frac{K-\beta_1}{|z-\gamma_i|^{2k}} = A', \tag{2.17}$$

which is contradict to (2.16). Thus this case is impossible.

Now we assume that there is only one real-valued root β. Then there are some numbers a, b, c_1' such that

$$\left(\frac{-K^3}{3} + cK + c'\right)^{-1} = -3\left(\frac{K+c_1'}{K^2+aK+b} + \frac{-1}{K-\beta}\right)$$

and

$$0 < C_1 \leq K^2 + aK + b.$$

Thus by using the above argument, one can solve equation (2.3) as follow,

$$\begin{aligned}&\ln|K-\beta|+a_2'\ln(K^2+aK+b)+F(K)\\&=\frac{4}{3}\text{real}\left(\frac{-c_i}{(k-1)(z-\gamma_i)^{k-1}}\right)+o\left(\frac{1}{|z-\gamma_i|^{k-1}}\right),\end{aligned}$$

for some real number a_2', where function $|F(K)|\leq C$ for some constant C. By Proposition 1.1, it follows

$$\ln(K-\beta)=O(|z-\gamma_i|^{-(k-1)}),\quad \text{as}\quad z\to\gamma_i,$$

which contradicts to (2.17). Lemma 2.3 is proved. □

By Lemma 2.3, we see that there are some nonzero complex-valued numbers $c_l, l=1,\ldots,\alpha+1$, such that

$$\frac{1}{F(z)}=\sum_{i=1}^{\alpha+1}\frac{c_i}{z-\gamma_i}.$$

Lemma 2.4. *All numbers $c_l, l=1,\ldots,\alpha+1$, are real-valued.*

Proof. Similar to the proof Lemma 2.3, there are two cases: a) there is only one root β of equation (2.9); b) there are three different roots β_1,β_2,β_3 of equation (2.9).

In case a). From the proof of Lemma 2.3, one can solve equation (2.3) as follow,

$$\begin{aligned}&-\ln|K-\beta|+a_2'\ln(K^2+aK+b)+F(K)\\&=\frac{4}{3}\sum_{l=1}^{\alpha+1}c_l^2\theta_l(z-\gamma_l)+\frac{2}{3}\sum_{l=1}^{\alpha+1}c_l^1\ln|z-\gamma_l|^2.\end{aligned}\tag{2.18}$$

This shows

$$c_l^2=0,\ \forall\ l=1,\ldots,\alpha+1,\tag{2.19}$$

since functions $\theta_l(z-\gamma_l)$ are non-continueous and K is continueous by Proposition 1.1.

In case b). Also from the proof of Lemma 2.3, one can solve equation (2.3) as follow,

$$\begin{aligned}&a_1\ln|K-\beta_1|+a_2\ln|K-\beta_2|+a_3\ln|K-\beta_3|\\&=4\sum_{l=1}^{\alpha+1}c_i^2\theta_l(z-\gamma_l)+2\sum_{l=1}^{\alpha+1}c_i^1\ln|z-\gamma_l|^2.\end{aligned}$$

So we can also get (2.19). □

Proposition 2.1 follows from Lemma 2.3 and Lemma 2.4 immediately.

Proposition 2.2. *There are three different roots $\beta_1, \beta_2, \beta_3$ of equation (2.9). In particular, there are three nonzero real numbers a_1, a_2, a_3 such that*

$$a_1 + a_2 + a_3 = 0, \tag{2.20}$$

and

$$\frac{1}{\frac{-K^3}{3} + cK + c'} = \frac{a_1}{K - \beta_1} + \frac{a_2}{K - \beta_2} + \frac{a_3}{K - \beta_3}. \tag{2.21}$$

Proof. On the contrary, we assume that there is only one real-valued root β of equation (2.9). Then by Lemma 2.4 and (2.18) in Lemma 2.4, we solve (2.3) as follow,

$$\ln|K - \beta| + a_2' \ln(K^2 + aK + b) + F(K) = \frac{2}{3}\sum_{l=1}^{\alpha+1} c_l^1 \ln|z - \gamma_l|^2.$$

Since

$$\sum_{l=1}^{\alpha+1} c_l^1 = 0,$$

then there is some γ_i and another number $a' \neq 0$ such that

$$\lim_{z \to \gamma_i} |K(z) - \beta|(K^2 + aK + b)^{a'} = \infty,$$

which is impossible, since K is continueous by Proposition 1.1. This shows that there are three different roots $\beta_1, \beta_2, \beta_3$ of equation (2.9). Furthermore, by Lemma 2.2, $\beta_i \neq \beta_j$, $\forall i \neq j$. (2.20) and (2.21) follows directly. □

By Proposition 2.1 and Proposition 2.2, we see, if $g = e^{2\phi}|dz|^2$ is an exceptional HCMU metric on $S^2 = \mathbb{C} \cup \{\infty\}$ with finite singular angles $2\pi\alpha_i$ at $p_i, i = 1, \ldots, n$, and $F(z)$ and $f(z)$ is given in (2.8), then there are some nonzero numbers $a \geq 2$, $c_1, \ldots, c_{\alpha+1}$ and another three different numbers $\beta_1, \beta_2, \beta_3$ such that

$$\sum_{l=1}^{\alpha+1} c_l = 0, \tag{2.22}$$

and equation (2.3) is equivalent to

$$\left(\frac{1}{K-\beta_1}+\frac{a-1}{K-\beta_2}+\frac{-a}{K-\beta_3}\right)K_z=\sum_{l=1}^{\alpha+1}\frac{c_l}{z-\gamma_l}. \tag{2.23}$$

Moreover, we can determine

$$\begin{aligned}\beta=\beta_1&=\sqrt{c}(2a-1)/\sqrt{a^2-a+1},\\ \beta_2&=-\frac{\beta(a+1)}{2a-1},\\ \beta_3&=-\frac{(a-2)\beta}{2a-1},\end{aligned}$$

and

$$\begin{aligned}F(z)^{-1}&=\frac{(2a-1)^2}{-3a(a-1)\beta^2}\sum_{l=1}^{\alpha+1}\frac{c_l}{z-\gamma_l}\\ &=f(z)^{-1}\prod_{i=2}^{n}(z-z_i)^{\alpha_i-1}.\end{aligned} \tag{2.24}$$

Theorem 2.1. *Let $p_1=(\infty)$ and $p_i=(z_i), i=2,\dots,n$, be n points on $S^2=\mathbb{C}\cup\{\infty\}$, and $\alpha_1\geq 2,\dots,\alpha_n\geq 2$, n positive integer numbers. Let*

$$\alpha=\alpha_1+\sum_{i=2}^{n}(\alpha_i-1).$$

Then $g=e^{2\phi}|dz|^2$ is an exceptional HCMU metric with finite conical singular angles $2\pi\alpha_i$ at p_i on S^2 if and only if there are a positive integer k, a complex-valued $B\neq 0$ and a holomorphic polynomial function $f(z)$ on $\mathbb{C}$ with degree $(\alpha+1)$ and different $(\alpha+1)$ roots γ_l of $f(z)=0$ such that $3k\leq\alpha+1$, and $(\alpha+1)$ roots γ_l satisfy

$$\sum_{l=1}^{k}\frac{-2a}{z-\gamma_l}+\sum_{l'=k+1}^{\alpha+1}\frac{2}{z-\gamma_{l'}}=\frac{B\prod_{i=2}^{n}(z-z_i)^{(\alpha_i-1)}}{\prod_{l=1}^{\alpha+1}(z-\gamma_l)}, \tag{2.25}$$

where $a=\frac{\alpha+1-k}{k}$. Furthermore, K and ϕ are given by

$$\begin{aligned}&\frac{1}{(K+\frac{(a-2)\beta}{2a-1})^a}(\beta-K)\left(K+\frac{\beta(a+1)}{2a-1}\right)^{a-1}\\ &=A\prod_{l=1}^{k}|z-\gamma_l|^{-2a}\prod_{l'=k+1}^{\alpha+1}|z-\gamma_{l'}|^2,\end{aligned} \tag{2.26}$$

and

$$\phi = \frac{1}{2}\ln\left(\frac{1}{|F(z)|^2}\left(-\frac{1}{3}K^3 + cK + c'\right)\right), \tag{2.27}$$

where $A > 0$ *is some constant and*

$$\beta = \sqrt{c}(2a-1)/\sqrt{a^2 - a + 1},$$

$$c' = \frac{(a+1)(a-2)\beta^3}{3(2a-1)^2},$$

$$F(z) = (z - z_2)^{-(\alpha_2 - 1)} \cdots (z - z_n)^{-(\alpha_n - 1)} f(z),$$

and

$$f(z) = B^{-1}\frac{-3a(a-1)\beta^2}{(2a-1)^2}\prod_{l=1}^{\alpha+1}(z - \gamma_l).$$

Proof. Necessity. By (2.23), one solves the Gaussian curvature K as follow,

$$\begin{aligned}&\frac{1}{\left(K + \frac{(a-2)\beta}{2a-1}\right)^a}(\beta - K)\left(K + \frac{\beta(a+1)}{2a-1}\right)^{a-1} \\ &= A|z-\gamma_1|^{c_1}|z-\gamma_2|^{c_2}\cdots|z-\gamma_{\alpha+1}|^{c_{\alpha+1}},\end{aligned} \tag{2.28}$$

where A is some positive number. Then by (2.22) amd (2.4) in Lemma 2.1, we see that there is some $c_i < 0$ such that $c_i = -2a$ and

$$\lim_{z \to z_i} \frac{K + \frac{(a-2)\beta}{2\alpha-1}}{|z - \gamma_i|^2} = A'$$

for some constant A'. On the other hand, it is easy to see

$$\left(K + \frac{(a-2)\beta}{2a-1}\right)(\beta - K)\left(K + \frac{\beta(a+1)}{2a-1}\right) = 3\left(\frac{-1}{3}K^3 + cK + c'\right). \tag{2.29}$$

Thus the condition

$$\frac{-1}{3}K^3 + cK + c' > 0$$

implies

$$\frac{-(a-2)\beta}{2a-1} \le K \le \beta.$$

This shows, for any $l = 1, \dots, \alpha + 1$,

$$c_l = -2a, \text{ or } c_l = 2.$$

Let $k \geq 1$ be the integer such that

$$c_l = -2a, \; l = 1, \dots, k, \quad \text{and} \quad c_{l'} = 2, \; l' = k, \dots, \alpha + 1.$$

Then by (2.22), we have

$$2 \leq a = \frac{\alpha + 1 - k}{k}$$

and consequently,

$$3k \leq \alpha + 1.$$

(2.26) follows from (2.28) directly as well as (2.27) follows from (2.4).

Sufficiency. Let

$$c_0 = \frac{-3a(a-1)\beta^2}{2(2a-1)^2} B^{-1}.$$

Then by condition (2.25), we have,

$$\frac{(z - z_2)^{\alpha_2 - 1} \cdots (z - z_n)^{\alpha_n - 1}}{c_0 (z - \gamma_1) \cdots (z - \gamma_{\alpha+1})} = \frac{(2a-1)^2}{(-3a)(a-1)\beta^2} \left(\sum_{l=1}^{k} \frac{-2a}{z - \gamma_1} + \sum_{l'=k}^{\alpha+1} \frac{2}{z - \gamma_{l'}} \right)$$

Since

$$\frac{-3a(a-1)\beta^2}{(2a-1)^2} \frac{1}{-\frac{1}{3}K^3 + cK + c'} = \frac{-a}{K + \frac{(a-2)\beta}{2a-1}} + \frac{1}{K - \beta} + \frac{a-1}{K + \frac{\beta(a+1)}{2a-1}},$$

then function K defined by (2.26) satisfies equation

$$K_z \left(-\frac{K^3}{3} + cK + c' \right)^{-1} = \frac{(z - z_2)^{\alpha_2 - 1} \cdots (z - z_n)^{\alpha_n - 1}}{c_0 (z - \gamma_1) \cdots (z - \gamma_{\alpha+1})} = \frac{1}{F(z)}$$

From (2.26), it is clear

$$\frac{2-a}{2a-1}\beta \leq 0 \leq K \leq \beta$$

and

$$\min_{R^2} K(x) = K(\gamma_1) = \frac{2-a}{2a-1}\beta$$

and

$$\max_{R^2} K(x) = K(\gamma_i) = \beta,\ i = 2, \ldots, \alpha+1$$

This shows $-K^3 + cK + c' \geq 0$ and

$$\frac{1}{|F(z)|^2}\left(-\frac{1}{3}K^3 + cK + c'\right) > 0.$$

Thus ϕ defined by (2.27) is a smooth function on $\mathbb{R}^2$ away from points $\{p_i\}_{i=1,\ldots,n}$ and consequently , by Lemma 2.1, $g = e^{2\phi}|dz|^2$ is a HCMU metric.

Since

$$z_i \neq \gamma_l,\ \forall i = 2, \ldots, n, l = 1, \ldots \alpha+1,$$

we see that $\phi(x)$ have the following behaviors near these singular points $p_2, \ldots, p_n$,

$$\phi(x) = (\alpha_i - 1)\ln|z - z_i| + \rho_i(x),$$

where $\rho_i(x)$, $i = 2, \ldots, n$, are smooth functions near p_i. Furthermore, there is some constant c_0 such that

$$\phi(x) + (\alpha_1 + 1)\ln|x| \to c_0, \text{ as } |x| \to \infty.$$

Thus we also prove $g = e^{2\phi}|dz|^2$ is a conical metric on S^2 with finite conical singular angles $2\pi\alpha_i$ at each $p_i, i = 2, \ldots, n$. □

Theorem 0.1 follows from Theorem 2.1.

From the above theorem, any exceptianl HCMU metric on S^2 with n weak, integer conical singular angles $2\pi\alpha_i$ at $\{p_i\}_{i=1,\ldots,n}$, is determined by a pair $(a, k) = (\frac{\alpha+1-k}{k}, k)$.

Definition 2.1. An exceptionl HCMU metric with finite weak, integer conical angles on S^2 is called minimal if $(a, k) = (\alpha, 1)$, where $\alpha = \sum_{i=1}^n (\alpha_i - 1) + 1$.

3. An explicit formula for minimal exceptional HCMU metrics.

Theorem 3.1. *Let $p_1 = (\infty)$ and $p_i = (z_i), i = 2, \ldots, n$, be n points on $S^2 = \mathbb{C} \cup \{\infty\}$, and $\alpha_1 \geq 2, \ldots, \alpha_n \geq 2$, n positive integer numbers. Then for given $c > 0$, any minimal exceptional HCMU metric on S^2 with finite conical singular angles $2\pi\alpha_i$ at p_i are determined by a positive number A, and two complex-valued parameters γ_1 and $B \neq 0$ besides $\alpha_i, i = 1, \ldots, n$. In precise, K and ϕ are given by*

$$\frac{1}{\left(K + \frac{(\alpha-2)\beta}{2\alpha-1}\right)^{\alpha}}(\beta - K)\left(K + \frac{\beta(\alpha+1)}{2\alpha - 1}\right)^{\alpha-1}$$
$$= A|z - \gamma_1|^{-2\alpha}|z - \gamma_2|^2 \cdots |z - \gamma_{\alpha+1}|^2,$$

and

$$\phi = \frac{1}{2}\ln\left(\frac{1}{|F(z)|^2}\left(-\frac{1}{3}K^3 + cK + c'\right)\right),$$

where

i)

$$\alpha = \alpha_1 + \sum_{i=2}^{n}(\alpha_i - 1) \geq 2,$$
$$\beta = \sqrt{c}(2\alpha - 1)/\sqrt{\alpha^2 - \alpha + 1},$$
$$c' = \frac{(\alpha+1)(\alpha-2)\beta^3}{3(2\alpha-1)^2};$$

ii)

$$F(z) = (z - z_2)^{-(\alpha_2 - 1)} \cdots (z - z_n)^{-(\alpha_n - 1)} f(z),$$

and

$$f(z) = \frac{-3\alpha(\alpha-1)\beta^2}{2(2\alpha-1)^2} B^{-1}(z - \gamma_1) \cdots (z - \gamma_{\alpha+1}),$$

and $\gamma_2, \ldots, \gamma_{\alpha+1}$ are α roots of the polynomial function equation with degree α,

$$g(z) = z^{\alpha} - a_1 z^{\alpha-1} + \cdots + (-1)^{\alpha} a_{\alpha} = 0,$$

where

$$a_j = \begin{cases} C_\alpha^j \gamma_1^j, \ j = 1, \ldots, \alpha_1 - 1, \\ C_\alpha^j \gamma_1^j - B(-1)^j \frac{d_{\alpha-j}}{j} \\ -B\sum_{l=\alpha_1}^{j-1}(-1)^l d_{\alpha-l}\gamma_1^{j-l}\frac{(\alpha-j+1)(\alpha-j+2)\cdots(\alpha-l)}{l(l+1)\cdots j}, \ j = \alpha_1, \ldots \alpha, \end{cases}$$

and

$$d_l = (-1)^{\alpha-\alpha_1-l} \sum_{l=j_2+\cdots+j_n} (z_2^{\alpha_2-1-j_2} C_{\alpha_2-1}^{j_2} \cdots z_n^{\alpha_n-1-j_n} C_{\alpha_n-1}^{j_n}),$$

$$l = 0, \ldots, \alpha - \alpha_1.$$

Theorem 3.1 follows from Theorem 2.1 and the followimg lemma.

Lemma 3.1. *Let γ_1 and $B \neq 0$ be two complex-valued numbers and $z_2, \ldots, z_n$ are another $(n-1)$-different complex-valued numbers. Let*

$$d_l = (-1)^{\alpha-\alpha_1-l} \sum_{l=j_2+\cdots+j_n} \left(z_2^{\alpha_2-1-j_2} C_{\alpha_2-1}^{j_2} \cdots z_n^{\alpha_n-1-j_n} C_{\alpha_n-1}^{j_n}\right),$$

$$l = 0, \ldots, \alpha - \alpha_1,$$

and

$$a_j = \begin{cases} C_\alpha^j \gamma_1^j, \; j = 1, \ldots, \alpha_1 - 1, \\ C_\alpha^j \gamma_1^j - B(-1)^j \frac{d_{\alpha-j}}{j} \\ -B \sum_{l=\alpha_1}^{j-1} (-1)^l d_{\alpha-l} \gamma_1^{j-l} \frac{(\alpha-j+1)(\alpha-j+2)\cdots(\alpha-l)}{l(l+1)\cdots j}, \; j = \alpha_1, \ldots \alpha, \end{cases}$$

where $\alpha = \alpha_1 + \sum_{i=2}^n (\alpha_i - 1)$. Then $\gamma_2, \ldots, \gamma_{\alpha+1}$ are α roots of the polynomial function equation with degree α,

$$g(z) = z^\alpha - a_1 z^{\alpha-1} + \cdots + (-1)^\alpha a_\alpha = 0$$

if and only if $\gamma_2, \ldots, \gamma_{\alpha+1}$ satisfy

$$\frac{B(z-z_2)^{\alpha_2-1} \cdots (z-z_n)^{\alpha_n-1}}{(z-\gamma_1)\cdots(z-\gamma_{\alpha+1})} = \frac{-\alpha}{z-\gamma_1} + \sum_{i=2}^{\alpha+1} \frac{1}{z-\gamma_i}. \tag{3.1}$$

Proof. Let

$$\frac{-\alpha}{z-\gamma_1} + \sum_{i=2}^{\alpha+1} \frac{1}{z-\gamma_i} = \frac{\sum_{j=1}^{\alpha} b_j z^{\alpha-j}}{(z-\gamma_1)\cdots(z-\gamma_{\alpha+1})}.$$

Then a direct computation shows

$$(3.2)\qquad \begin{aligned} b_j &= (-1)^j(-\alpha \sum_{2\le i_1<\cdots<i_j\le\alpha+1} (\gamma_{i_1}\cdots\gamma_{i_j}) \\ &\quad + \sum_{l=2}^{\alpha+1} \sum_{\substack{2\le i_1<\cdots<i_j\le\alpha+1\\ i_1,\ldots,i_j\neq l}} (\gamma_{i_1}\cdots\gamma_{i_j}) \\ &\quad + \gamma_1 \sum_{l=2}^{\alpha+1} \sum_{\substack{2\le i_1<\cdots<i_{j-1}\le\alpha+1\\ i_1,\ldots,i_{j-1}\neq l}} (\gamma_{i_1}\cdots\gamma_{i_{j-1}})) \\ &= (-1)^j(-j \sum_{2\le i_1<\cdots<i_j\le\alpha} (\gamma_{i_1}\cdots\gamma_{i_j}) \\ &\quad + (\alpha-j+1)\gamma_1 \sum_{2\le i_1<\cdots<i_{j-1}\le\alpha+1} (\gamma_{i_1}\cdots\gamma_{i_{j-1}})), \end{aligned}$$

where $j = 1, \ldots, \alpha$. Let

$$\sum_{2\le i_1<\cdots<i_j\le\alpha+1} (\gamma_{i_1}\cdots\gamma_{i_j}) = a_j,\ j = 1, \ldots, \alpha.$$

Then (3.2) becomes

$$(3.3)\qquad b_j = (-1)^j(-ja_j + (\alpha-j+1)\gamma_1 a_{j-1}).$$

Since

$$\begin{aligned} &(z-z_2)^{\alpha_2-1}(z-z_3)^{\alpha_3-1}\cdots(z-z_n)^{\alpha_n-1} \\ &= \prod_{i=2}^{n} \left(\sum_{j=0}^{\alpha_i-1} (-1)^{\alpha_i-1-j} z_i^{\alpha_i-1-j} C_{\alpha_i-1}^{j} z^j \right) \\ &= \sum_{l=0}^{\alpha-\alpha_1} \left[(-1)^{\alpha-\alpha_1-l} \sum_{l=j_2+\cdots+j_n} (z_2^{\alpha_2-1-j_2} C_{\alpha_2-1}^{j_2} \cdots . z_n^{\alpha_n-1-j_n} C_{\alpha_n-1}^{j_n}) \right] z^l \\ &= \sum_{l=0}^{\alpha-\alpha_1} d_l z^l, \end{aligned}$$

where

$$d_l = (-1)^{\alpha-\alpha_1-l} \sum_{l=j_2+\cdots+j_n} (z_2^{\alpha_2-1-j_2} C_{\alpha_2-1}^{j_2} \cdots . z_n^{\alpha_n-1-j_n} C_{\alpha_n-1}^{j_n}),$$

then

$$B \sum_{l=0}^{\alpha-\alpha_1} d_l z^l = \sum_{j=1}^{\alpha} b_j z^{\alpha-j} \tag{3.4}$$

is equivalent to

$$b_j = \begin{cases} 0, \ j = 1, \dots, \alpha_1 - 1, \\ B d_{\alpha-j}, \ j = \alpha_1, \dots, \alpha. \end{cases} \tag{3.5}$$

Thus combining (3.3) and (3.5), we get

$$\begin{cases} a_1 = \alpha\gamma_1, \\ -ja_j + (\alpha - j + 1)\gamma_1 a_{j-1} = 0, \ j = 2, \dots, \alpha_1 - 1, \\ -ja_j + (\alpha - j + 1)\gamma_1 a_{j-1} = (-1)^j B d_{\alpha-j}, \ j = \alpha_1, \dots, \alpha. \end{cases} \tag{3.6}$$

By using iteration to (3.6), one obtains

$$a_j = \begin{cases} C_\alpha^j \gamma_1^j, \ j = 1, \dots, \alpha_1 - 1, \\ C_\alpha^j \gamma_1^j - B(-1)^j \frac{d_{\alpha-j}}{j} \\ -B \sum_{l=\alpha_1}^{j-1} (-1)^l d_{\alpha-l} \gamma_1^{j-l} \frac{(\alpha-j+1)(\alpha-j+2)\cdots(\alpha-l)}{l(l+1)\cdots j}, \ j = \alpha_1, \dots, \alpha. \end{cases} \tag{3.7}$$

This shows that (3.4) is true if and only if (3.7) is satisfied. Hence by the fundamental theorem, (3.1) is true if and only if $\gamma_2, \dots, \gamma_{\alpha+1}$ are α roots of the polynomial function equation,

$$g(z) = z^\alpha - a_1 z^{\alpha-1} + \cdots + (-1)^\alpha a_\alpha = 0.$$

The lemma is proved. □

Remark 3.1. By (2.29) and Theorem 3.1, one can prove that, for any n singular points with conical angles $2\pi \cdot integer$ on S^2, there are two uniform constnats C_1 and C_2 depending only on B, A, γ_1 such that the areas of the family of minimal exceptional HCMU metrics constructed in Theorem 3.1 satisfy,

$$\frac{C_1}{\sqrt{c}} \le \text{Area}\,(g) \le \frac{C_2}{\sqrt{c}}, \quad \text{as} \ \ c \to 0,$$

and

$$\frac{C_1}{\sqrt{c}} \le \text{Area}\,(g) \le \frac{C_2}{\sqrt{c}}, \quad \text{as} \ \ c \to \infty.$$

Hence for any n singular points with conical angles $2\pi \cdot integer$ on S^2 and a positive number c_0, there is a family of HCMU metrics on S^2 with same area c_0, which vary energy $E(g)$ with the parameters B, A, γ_1. This shows that a HCMU metric with finite integer conical singular angles on S^2 could not be in general a local minimizer for energy functional E with respect to a general deformation, which preserves the area and conical angle structure at each singular point, although any HCMU metrics is a local minimizer in the class $\mathcal{G}(M)$ defined in O. Introduction (cf. [Ca2], [Ch3]).

4. HCMU metrics on $\mathbb{R}^2$.

In this section, we shall classify all HCMU metrics with finite energy and area on $\mathbb{R}^2$. First by Proposition 1.1, we have

Proposition 4.1. *Let $g = e^{2\phi}|dz|^2$ be a HCMU metric with finite energy and area on $\mathbb{R}^2$. Let $2\pi\alpha > 0$ be the weakly conical singular angle at the infinity on $\mathbb{R}^2$. Then*

i) *If $\alpha \neq$ integer, g is a radial HCMU metric on $\mathbb{R}^2$.*

ii) *If $\alpha = 1$, then the Gaussian curvature $K = \sqrt{c}$ of g, and consequently the metric g can be extended to a smooth one with constant Gaussian curvature on S^2.*

iii) *If $\alpha =$ integer ≥ 2, then g is an either radial or exceptional metric on $\mathbb{R}^2$.*

Proof. Let $g = e^{2\phi}|dz|^2$ be a HCMU metric with finite energy and area on $\mathbb{R}^2$. Let K be the Gaussian curvature of g. Then (ϕ, K) is a global solution of system (1.1) on $\mathbb{R}^2$. If $K = const.$, then the metric g can be extended to a smooth one with constant Guassian curvature on S^2, which was proved in [CL]. In particular, the weakly conical singular angle is 2π. Hence, the case $K = const.$ is of case ii) in Proposition 4.1. Now we assume $K \neq const.$ Then there is a holomorphic function $F(z)$ on $\mathbb{R}^2$ such that $F(z) = e^{-2\phi}K_{\bar{z}}$. Therefore, by Proposition 1.2, we see that there is a holomorphic function $g(z)$ on $\mathbb{R}^2$ such that

$$F(z) = \begin{cases} zg(\frac{1}{z}), & \text{if } \alpha \neq \text{ integer;} \\ z^{(\alpha+1)}(c + g(\frac{1}{z})), & \text{for some } c \neq 0, \\ \text{or } zg(\frac{1}{z}), & \text{if } \alpha = \text{ integer.} \end{cases} \tag{4.1}$$

i) In case $\alpha \neq$ integer. By (4.1), we see that the holomorphic vector field on $\mathbb{R}^2$

$$F(z)\frac{\partial}{\partial z} = e^{-2\phi} K_{\overline{z}} \frac{\partial}{\partial z}$$

can be extended to a holomorphic vector field on S^2 with vanishing at the origin and infinity. Then by using a suitable rotation of coordinate, we may assume

$$e^{-2\phi} K_{\overline{z}} = az$$

for some real number $a \neq 0$. It follows the one-parameter actions generated by the imaginary part of holomorphic vector field $e^{-2\phi} K_{\overline{z}} \frac{\partial}{\partial z}$ are rotation transformations. By a well-known result, ϕ is invariant under these rotations. This shows that ϕ is radial and g is radial HCMU metric on $\mathbb{R}^2$.

ii) In case $\alpha = 1$. As in Section 1, it is convenient to use Kelvin transformation, $w = \frac{1}{z}$. Let

$$\psi(w) = -2\ln|w| + \phi\left(\frac{1}{w}\right) \quad \text{and} \quad \overline{K}(w) = K\left(\frac{1}{w}\right).$$

Then $(\psi, \overline{K})$ is a solution of system (1.1) on $D \setminus \{0\}$. By Lemma 1.1, it follows $|\psi| \leq C$ for some constant C as $\alpha = 1$. On the other hand, by Proposition 1.1, $K \in C^0(D)$. Thus $|Ke^{2\psi}| \leq C$ and $|(-K^2 + c)e^{2\psi}| \leq C$. By the standard regularity theorem, $\psi \in C^{1,\delta}(D)$ and $K \in C^{1,\delta}(D)$. Using the iteration method, we prove $\psi \in C^\infty(D)$. This shows the metric defined by $e^{2\phi}|dz|^2$ can be extended to a smooth one on S^2. By a well-known result of Kazdan-Warner ([KW]), we see that the vector field $e^{-2\phi} K_{\overline{z}} \frac{\partial}{\partial z}$ is holomorphic on S^2 implies $K =$ const, and consequently (ϕ, K) is radial by a result in [CL]. $K = \sqrt{c}$ follows from the relation

$$\int_{R^2} (-K^2 + c)e^{2\phi} dx = 0.$$

iii) In case $\alpha =$ integer ≥ 2, there are two cases: one is $F(z) = zg(\frac{1}{z})$, the other is $F(z) = z^{\alpha+1}(c + g(\frac{1}{z}))$. In the first case, g must be radial by the argument in case i). In the second case, g is an extremal metric according to Definition 1.1. □

Since $\mathbb{R}^2 \cong S^2 \setminus \{\infty\}$, then by Theorem 2.1, we can classify all exceptional HCMU metrics on $\mathbb{R}^2$. In the case of minimal, exceptional HCMU metrics, we have

Theorem 4.1. *Let $g = e^{2\phi}|dz|^2$ be a minimal, exceptional HCMU metric with a conical singular angle $2\pi\alpha \geq 4\pi$ at the infinity on $\mathbb{R}^2$. Then there are a positive number A and two complex-valued parameters γ_1 and Γ $(\neq \gamma_1^\alpha)$ such that K and ϕ are given by*

$$\begin{aligned}&\frac{1}{\left(K+\frac{(\alpha-2)\beta}{2\alpha-1}\right)^\alpha}(\beta-K)\left(K+\frac{\beta(\alpha+1)}{2\alpha-1}\right)^{\alpha-1}\\&=A|z-\gamma_1|^{-2\alpha}|z-\gamma_2|^2\cdots|z-\gamma_{\alpha+1}|^2,\end{aligned}$$

and

$$\phi=\frac{1}{2}\ln\left(\frac{1}{|f(z)|^2}\left(-\frac{1}{3}K^3+cK+c'\right)\right),$$

where

i)

$$\begin{aligned}\beta&=\sqrt{c}(2\alpha-1)/\sqrt{\alpha^2-\alpha+1},\\c'&=\frac{(\alpha+1)(\alpha-2)\beta^3}{3(2\alpha-1)^2};\end{aligned}$$

ii)

$$\begin{aligned}f(z)&=c_0(z-\gamma_1)\cdots(z-\gamma_{\alpha+1}),\\c_0&=\frac{-3\alpha(\alpha-1)\beta^2}{2(2\alpha-1)^2}\frac{(-1)^\alpha}{-\alpha\Gamma+\alpha\gamma_1^\alpha},\end{aligned}$$

and $\gamma_2, \dots, \gamma_{\alpha+1}$ are α roots of the polynomial function equation with degree α,

$$g(z)=z^\alpha-a_1z^{\alpha-1}+\cdots+(-1)^\alpha a_\alpha=0,$$

where

$$\begin{cases}a_j=C_\alpha^j\gamma_1^j=\frac{\alpha!}{j!(\alpha-j)!}\gamma_1^j,\ j=1,\dots,\alpha_1-1,\\a_\alpha=\Gamma.\end{cases}$$

Proof. Applying Theorem 3.1 to the case $\alpha = \alpha_1$, we have

$$\begin{cases}a_j=C_\alpha^j\gamma_1^j=\frac{\alpha!}{j!(\alpha-j)!}\gamma_1^j,\quad j=1,\dots,\alpha-1,\\a_\alpha=\gamma_1^\alpha-(-1)^\alpha\frac{B}{\alpha}.\end{cases}$$

Let $a_\alpha = \Gamma$. Then $B = (-1)^\alpha(\alpha\gamma_1^\alpha - \alpha\Gamma)$, and Theorem 4.1 follows immediately. □

Now we assume that $g = e^{2\phi}|dz|^2$ is a radial, extremal Hermitian metric on $\mathbb{R}^2$ with two different weakly conical singular angles $2\pi\alpha_1$ and $2\pi\alpha_2$ at the origin and infinity. In the other hand, ϕ satisfies

$$|\phi - (\alpha_1 - 1)\ln|x|| < \infty, \quad \text{as} \quad |x| \to 0,$$

and

$$|\phi - (\alpha_2 + 1)\ln|x|| < \infty\ , \quad \text{as} \quad |x| \to \infty.$$

Recently, X. Chen discussed these radial metrics by using ODE method ([Ch2]). In addition of HCMU, we can use a new ODE method to give an explicit formula for these metrics. In fact, we can prove that any radial, extremal Hermitian metric with finite energy and area on $\mathbb{R}^2 \setminus \{0\}$ must be HCMU (cf. Proposition A in Appendix).

Since the Gaussian curvature K of g is a function only on one variable r, then by the proof of Proposition 1.2, one see

$$|e^{-2\phi}K_{\bar{z}}| \le C|z|, \quad \text{near} \quad z = 0,$$

and

$$|e^{-2\phi}K_{\bar{z}}| \le C|z|, \quad \text{near} \quad z = +\infty.$$

This shows that the holomorphic vector field on R^2

$$e^{-2\phi}K_{\bar{z}}\frac{\partial}{\partial z}$$

can be extended to a holomorphic vector field on S^2 with vanishing at the origin and infinity. Hence there is some constant $c_1 \neq 0$ such that

$$e^{-2\phi}K' = c_1 r. \tag{4.2}$$

In particular, $a_2 \neq 0$ in (1.15).

From (2.1), we have

$$(r\phi')' = -Kre^{2\phi}. \tag{4.3}$$

Combining (4.2) and (4.3), it follows

$$r\phi' = \frac{-K^2}{2c_1} + c_2. \tag{4.4}$$

Similarly, by (2.2) and (4.2), we have

$$rK' = \frac{-K^3}{3c_1} + \frac{cK}{c_1} + c_3. \tag{4.5}$$

Lemma 4.1. *There are three real-valued roots of equation*

$$-K^3 + 3cK + 3c_1c_3 = 0. \tag{4.6}$$

Proof. On the contrary, we assume that there is only one real-valued roof β of eqaution (4.6). Then there are some a, b, c_1' such that

$$\left(-\frac{K^3}{3} + cK + c_1c_3\right)^{-1} = -3\left(\frac{K + c_1'}{K^2 + aK + b} + \frac{-1}{K - \beta}\right),$$

and

$$0 < C_1 \le K^2 + aK + b.$$

Thus one can solve equation (4.5) as follows,

$$e^{F(K)}(K^2 + aK + b)^{\alpha'}(K - \beta) = Ar^{\alpha}$$

for some $\alpha \neq 0$ and α', where function $|F(K)| \le C$. This shows

$$\lim_{|x|\to 0} (K^2 + aK + b)^{\alpha'}|K(x) - \beta| = \infty,$$

or

$$\lim_{|x|\to \infty} (K^2 + aK + b)^{\alpha'}|K(x) - \beta| = \infty.$$

But this is impossible since K is bounded by Proposition 1.1. The contradition implies Lemma 4.1. □

Proposition 4.2. *Let $g = e^{2\phi}|dz|^2$ be a radial, extremal Hermitian metric with finite energy and area on $\mathbb{R}^2 \setminus \{0\}$. Suppose that two weakly conical singular angles $2\pi\alpha_1$ and $2\pi\alpha_2$ at the origin and infinity are different. Then ϕ and K are given by*

$$(K - \beta_1)^{-\alpha_2/\alpha_1}(-K + \beta_2)(K - \beta_3)^{\alpha_2/\alpha_1 - 1} = Br^{-2\alpha_2},$$

and

$$\phi = \frac{1}{2}\ln\left(\frac{-K^3 + 3cK + C}{r^2}\right) - \frac{1}{2}\ln(3c_1^2),$$

where $B > 0$ is some constant and $\beta_1, \beta_2, \beta_3$, and C can be uniquely determined by c, α_1, α_2.

Proof. By Lemma 4.1, there are three roots $\beta_1, \beta_2, \beta_3$, of equation

$$-K^3 + 3cK + C = 0,$$

where $C = 3c_1c_3$ and

$$c = \int_{R^2} K^2 e^{2\phi} dx \Big/ \int_{R^2} e^{2\phi} dx > 0.$$

Moreover, by using the argument in Lemma 4.1, one can prove these three roots are different. So we may assume $\beta_2 > \beta_1 > \beta_3$. Since $a_2 \neq 0$, it is not hard to check $K(0), K(\infty) \neq \beta_3$. Hence we may further assume

$$\beta_1 = K(0) \quad \text{and} \quad \beta_2 = K(+\infty).$$

By (4.5), we see that there are numbers c_4, a_1, a_2 such that

$$a_1 + a_2 + 1 = 0,$$

and

$$\left(a_1 \frac{1}{K - \beta_1} + \frac{1}{K - \beta_2} + a_2 \frac{1}{K - \beta_3} \right) K' = \frac{c_4}{r}.$$

Then we solve the above equation as follow,

$$(K - \beta_1)^{a_1} (\beta_2 - K)(K - \beta_3)^{a_2} = B r^{c_4}, \tag{4.7}$$

for some positive number B. On the other hand, by (1.15) and the condition $a_2 \neq 0$, one can prove

$$\lim_{|x| \to 0} \frac{\beta_1 - K}{r^{-2\alpha_1}} = B' \text{ and } \lim_{|x| \to \infty} \frac{\beta_2 - K}{r^{-2\alpha_2}} = B'',$$

for some constant $B'' > 0$ and $B' < 0$. It follows

$$c_4 = -2\alpha_2, \; a_1 = \frac{-\alpha_2}{\alpha_1}, \; a_2 = \frac{\alpha_2}{\alpha_1} - 1,$$

and consequently,

$$(K - \beta_1)^{-\alpha_2/\alpha_1} (\beta_2 - K)(K - \beta_3)^{\alpha_2/\alpha_1 - 1} = B r^{-2\alpha_2}.$$

Moreover, there are three identies

$$\begin{cases} \beta_1 + \beta_2 + \beta_3 = 0, \\ \beta_2(\alpha_2 - 2\alpha_1) + \beta_1(2\alpha_2 - \alpha_1) = 0, \\ \beta_1^2 + \beta_2^2 + \beta_1\beta_2 = 3c. \end{cases}$$

This solves

$$\beta_1 = (2\alpha_1 - \alpha_2)\sqrt{\frac{c}{\alpha_1^2 - \alpha_1\alpha_2 + \alpha_2^2}},$$

and

$$\beta_2 = (2\alpha_2 - \alpha_1)\sqrt{\frac{c}{\alpha_1^2 - \alpha_1\alpha_2 + \alpha_2^2}}.$$

By (1.15), one can determine the constant c_1 in (4.2),

$$c_1 = \frac{\beta_2^2 - c}{2\alpha_2}.$$

Combining (4.2) and (4.5), we have

$$\phi = \frac{1}{2}\ln\left(\frac{-K^3 + 3cK + C}{r^2}\right) - \frac{1}{2}\ln(3c_1^2), \tag{4.8}$$

where $C = \beta_1\beta_2\beta_3$. □

Remark 4.1. By Proposition 4.2, the area of metric $g = e^{2\phi}|dz|^2$ on R^2 can be computed by

$$\text{Area} = \int_{R^2}\left(\frac{-K^3 + 3cK + C}{3c_1^2 r^2}\right)dx.$$

Then c is uniquely determined by Area, α_1 and α_2, if we normalize the area. This shows that any radial, extremal Hermitian metric on $\mathbb{R}^2 \setminus \{0\}$ with two different weakly conical singular angles at the origin and infinity is uniquely determined by angles under the nomalized area.

As a corollary of Proposition 4.2, we have

Corollary 4.1. *Let $g = e^{2\phi}|dz|^2$ be a radial, extremal Hermitian metric on R^2 with a weakly conical singular angle $2\pi\alpha \neq 2\pi$ at the infinity. Then ϕ and K are given by*

$$\frac{1}{\left(K + \frac{(\alpha-2)\beta}{2\alpha-1}\right)^{\alpha}}(\beta - K)\left(K + \frac{\beta(\alpha+1)}{2\alpha - 1}\right)^{\alpha-1} = Br^{-2\alpha},$$

and

$$\phi = \frac{1}{2}\ln\left(\frac{-K^3 + 3cK + C}{r^2}\right) - \frac{1}{2}\ln(3c_1^2),$$

where $B > 0$ is some constant and

$$\beta = \sqrt{c}(2\alpha - 1)/\sqrt{\alpha^2 - \alpha + 1},$$
$$C = \frac{(\alpha + 1)(\alpha - 2)\beta^3}{(2\alpha - 1)^2},$$
$$c_1 = \frac{3c(\alpha - 1)}{2(\alpha^2 - \alpha + 1)}.$$

Remark 4.2. It seems that any extremal Hermitain metrics on $\mathbb{R}^2$ should be HCMU. If it is true, then Proposition 4.1 classifies all extremal Hermitian metrics on S^2 with only one weakly conical singular point. In Appendix, we will show that any radial, extremal Hermitain metric with finite energy and area on $\mathbb{R}^2 \setminus \{0\}$ must be HCMU.

5. Examples.

Example 1. $S^2_{2,2}$.

In this case, there are two singular points with conical angles 4π on S^2. Without the loss of generality, we may assume two singular points are $p_1 = (\infty)$ and $p_2 = (0)$.

Proposition 5.1. *For fixed $c > 0$, any exceptional HCMU metrics on S^2 with two conical singular angles 4π at the origin and infinity respectivly are determined by a real number $A > 0$ and two complex-valued parameters γ_1 and Γ $(\neq \gamma_1^3)$. In precise, K and ϕ are given by*

$$\frac{1}{(K + \beta/5)^3}(\beta - K)(K + 4\beta/5)^2$$
$$= A|z - \gamma_1|^{-6}|z - \gamma_2|^2|z - \gamma_3|^2|z - \gamma_4|^2,$$

$$\phi = \frac{1}{2}\ln\left(\frac{1}{|F(z)|^2}\left(-\frac{1}{3}K^3 + cK + c'\right)\right),$$

where

i)
$$\beta = \frac{5}{7}\sqrt{7c} \quad \textit{and} \quad c' = \frac{20c}{147}\sqrt{7c};$$

ii)
$$F(z) = \frac{-3c}{14(-\Gamma/\gamma_1 + \gamma_1^2)}z^{-1}(z - \gamma_1)(z - \gamma_2)(z - \gamma_3)(z - \gamma_4),$$

where $\gamma_2, \gamma_3, \gamma_4$ *are three roots of the polynomial function equation with degree* 3*,*

$$g(z) = z^3 - a_1 z^2 + a_2 z^1 - a_3 = 0,$$

and $a_1 = 3\gamma_1, a_2 = \frac{3}{\gamma_1}\Gamma$ *and* $a_3 = \Gamma \neq \gamma_1^3$.

Proof. By the condition in Theorem 2.1, $3k \leq \alpha + 1 = 4$, we have $k = 1$. This shows any exceptional HCMU metrics with two singular conical angles 4π on S^2 are minimal. Then by using condition (2.27) in Theorem 2.1, we get

$$\begin{cases} a_1 = 3\gamma_1 \\ a_2 = \frac{3}{\gamma_1} a_3 \\ B = 2(-2a_2 + 2\gamma_1 a_1) = 12(\frac{-a_3}{\gamma_1} + \gamma_1^2). \end{cases}$$

Let $a_3 = \Gamma \neq \gamma_1^3$. Then

$$a_1 = 3\gamma_1 \text{ and } a_2 = \frac{3}{\gamma_1}\Gamma.$$

Moreover, by using Theorem 3.1, we can determine numbers β, c', and prove Proposition 5.1. □

Example 2. $S^2_{2,2,2}$.

In this case, there are three singular points with conical angles 4π on S^2. Without the loss of generality, we may assume three singular points are $p_1 = (\infty), p_2 = (0)$ and $p_3 = 1$.

Proposition 5.2. *Let* $p_1 = (\infty), p_2 = (0)$ *and* $p_3 = (1)$*. Then for fixed* $c > 0$*, any exceptional HCMU metrics on* S^2 *with three conical singular angles* 4π *at* p_1, p_2 *and* p_3 *respectivly are determined by a real number* $A > 0$ *and two complex-valued parameters* γ_1 *and* $B \neq 0$*. In precise,* K *and* ϕ *are given by*

$$\frac{1}{(K + 2\beta/7)^3}(\beta - K)(K + 5\beta/7)^2$$
$$= A|z - \gamma_1|^{-8}|z - \gamma_2|^2|z - \gamma_3|^2|z - \gamma_4|^2|z - \gamma_5|^2,$$

$$\phi = \frac{1}{2}\ln(\frac{1}{|F(z)|^2}(-\frac{1}{3}K^3 + cK + c')).$$

where

i)

$$\beta = \frac{91\sqrt{c}}{13} \text{ and } c' = \frac{70c\sqrt{13c}}{507};$$

ii)

$$F(z) = \frac{-18c}{13} B z^{-1}(z-1)^{-1}(z-\gamma_1)\cdots(z-\gamma_5),$$

where $\gamma_2, \ldots, \gamma_5$ *are three roots of the polynomial function equation with degree* 4*,*

$$g(z) = z^4 - a_1 z^3 + a_2 z^2 - a_3 z + a_4 = 0,$$

and $a_1, \ldots, a_4$ *determined by*

$$\begin{cases} a_1 = 4\gamma_1 \\ a_2 = 6\gamma_1^2 - \frac{B}{2} \\ a_3 = 4\gamma_1^3 + \frac{B}{3} - \frac{B\gamma_1}{3} \\ a_4 = \gamma_1^4 - B(\frac{\gamma_1^2}{12} + \frac{\gamma_1}{12}). \end{cases}$$

Proof. By Theorem 3.1, one can compute,

$$\begin{cases} a_1 = 4\gamma_1 \\ a_2 = 6\gamma_1^2 - B\frac{d_2}{2} = 6\gamma_1^2 - \frac{B}{2} \\ a_3 = 4\gamma_1^3 - C\frac{d_1}{3} - Cd_2\frac{\gamma_1}{3} = 4\gamma_1^3 + \frac{B}{3} - \frac{B\gamma_1}{3} \\ a_4 = \gamma_1^4 - C(\frac{d_2\gamma_1^2}{12} - \frac{d_1\gamma_1}{12}) = \gamma_1^4 - B(\frac{\gamma_1^2}{12} + \frac{\gamma_1}{12}). \end{cases}$$

Moreover, by Theorem 3.1, we can determine β, c', and prove Proposition 5.2. □

Remark 5.1. In [Ch3], X. Chen described the Calabi's example of HCMU metric on S^2 with three conical singular angles 4π. By Proposition 5.2, this metric is belonged to the family of exceptional HCMU metrics constructed in Proposition 5.2, since it is nonradial near each sigular point.

Appendix A.

The following proposition is needed in Section 4.

Proposition A. *Let* $g = e^{2\phi}|dz|^2$ *be a radial, extremal Hermitian metric with finite energy and area on* $R^2 \setminus \{0\}$*. Then* g *is HCMU.*

Proof. Let $2\pi\alpha_1$ and $2\pi\alpha_2$ be two weakly conical singular angles at the origin and infinity respectively. Since the Gaussian curvature K must be constant as $\alpha_1 = \alpha_2$ ([Ch2]), we may assume $\alpha_1 \neq \alpha_2$.

Let

$$\tilde{K}(x) = \frac{1}{2\pi}\int_{R^2\setminus\{0\}} (\ln(|x-y|) - \ln(|y|+1))(-K^2+c)e^{2\phi}dy.$$

Then it is clear

$$\triangle\tilde{K} = (-K^2+c)e^{2\phi}, \qquad \text{on } R^2\setminus\{0\} \tag{A.1}$$

Since K is also a solution of (A.1), we get

$$\triangle(K-\tilde{K}) = 0.$$

It follows

$$(K-\tilde{K}) = \mathrm{Re}\,(f(z)), \quad \text{on } \mathbb{R}^2\setminus\{0\}$$

for some meromorphic function $f(z)$ on R^2. On the other hand, one can prove

$$|\tilde{K} - \beta\ln r| < C, \qquad \text{as } r\to+\infty,$$

where

$$\beta = \frac{1}{2\pi}\int_{R^2\setminus\{0\}}(-K^2+c)e^{2\phi}dy.$$

Hence by using the condition that K is radial, we see that $f(z)$ must be constant, and consequently,

$$K = \tilde{K} + \ const..$$

Since ϕ and K are radial, as in the proof of Lemma 1.3, we may assume that

$$r\psi_r = -(\alpha_2+1) + a_1\ln r\frac{1}{r^{2\alpha_2}} + o\left(\ln r\frac{1}{r^{2\alpha_2}}\right),$$

$$\psi_\theta = b_1\ln r\frac{1}{r^{2\alpha_2}} + o\left(\ln r\frac{1}{r^{2\alpha_2}}\right),$$

$$rK_r = \beta + a_2(\ln r)^2\frac{1}{r^{2\alpha_2}} + o\left((\ln r)^2\frac{1}{r^{2\alpha_2}}\right),$$

and

$$K_\theta = b_2(\ln r)^2\frac{1}{r^{2\alpha_2}} + o\left((\ln r)^2\frac{1}{r^{2\alpha_2}}\right),$$

where a_1, b_1, a_2, b_2 are some constants. Then we can obtain

$$K_{,ww} = \frac{-1}{w^2}\left(2\alpha_2\beta + O\left((\ln r)^2 \frac{1}{r^{2\alpha_2}}\right)\right).$$

Since $K_{,ww}$ is holomorphic on $\mathbb{C}\setminus\{0\}$, we see that there is some holomorphic function $g(z)$ on $\mathbb{C}$ such that

(A.2) $$K_{,ww} = \frac{-1}{w^2}(2\alpha_2\beta + w^{-1}g(\frac{1}{w})).$$

On the other hand, by using Kelvin transfomation, $z = \frac{1}{w}$, and the above argument, we can also prove that there is another holomorphic function $g'(z)$ on $\mathbb{C}$ such that

(A.3) $$K_{,zz} = \frac{-1}{z^2}\left(2\alpha_1\beta + z^{-1}g'\left(\frac{1}{z}\right)\right).$$

Since

$$K_{,ww} = \frac{1}{w^4}K_{,zz} = -\frac{1}{w^2}(2\alpha_1\beta + wg'(w)),$$

then combining (A.2) and (A.3), we get

$$\beta = 0, \quad \text{and} \quad g(z) = g'(z) = 0,$$

and consequently, $K_{,ww} = 0$, i.e., g is HCMU. □

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