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Bohr-Sommerfeld tori and relative Poincaré series on a complex hyperbolic space

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Let Γ be a cocompact discrete subgroup of $SU(n, 1)$ which acts freely on $B^n = SU(n, 1)/U(n)$. We suggest a construction of relative Poincaré series associated to loxodromic elements of Γ. For $\Gamma \subset SU(2, 1)$ we describe Bohr-Sommerfeld tori in $\Gamma \backslash B^2$ associated to hyperbolic elements of Γ and compute the asymptotics of the relative Poincaré series associated to hyperbolic elements of Γ in semi-classical limit.

1. Introduction.

1.1. General definitions.

We shall start with a brief review of the general concept of an automorphic form. Let G be a connected non-compact real semi-simple Lie group with a finite center, which we also assume to be unimodular, K be a maximal compact subgroup of G, Γ be a discrete subgroup of G. Let V be a finite-dimensional vector space, $\rho : K \to GL(V)$ be an (anti)-representation of K. A smooth $Z(\mathfrak{g})$-finite function $f : G \to V$ is called an **automorphic form on G for Γ** if

$$f(\gamma g k) = f(g)\rho(k) \tag{1}$$

for any $\gamma \in \Gamma$, $g \in G$, $k \in K$, and there are a positive constant C and a non-negative integer m such that

$$|f(g)| \leq C||g||^m \tag{2}$$

for any $g \in G$, here $|.|$ is a norm in V, and $||g||^2 = tr(g^* g)$ is taken in the adjoint representation of G.

The automorphy law (1) means geometrically that f defines a Γ-invariant section of the vector bundle $G \times_K V \to G/K$ associated to the principal bundle $G \to G/K$, where $G \times_K V = G \times V/\sim$, and the equivalence relation is given by $(gk, v) \sim (g, v\rho(k))$.

The growth condition (2) is automatically satisfied with $m = 0$ in the case when $\Gamma\backslash G$ is compact.

Recall also that a function $f : G \to V$ is said to be **Z($\mathfrak{g}$)-finite** if it is annihilated by an ideal I of $Z(\mathfrak{g})$ of a finite codimension, here $Z(\mathfrak{g})$ is the center of the universal enveloping algebra $U(\mathfrak{g})$ (over $\mathbb{C}$). $U(\mathfrak{g})$ can be identified with the algebra $D(G)$ of all left-invariant differential operators on G (with complex coefficients): to $Y \in \mathfrak{g}$ is associated a differential operator $Yf(g) = \frac{d}{dt} f(ge^{tY})|_{t=0}$, this establishes a linear map $\mathfrak{g} \to D(G)$ which extends to an isomorphism from $U(\mathfrak{g})$ onto $D(G)$. $Z(\mathfrak{g})$ can be viewed as the subalgebra of all bi-invariant differential operators, it is isomorphic to a polynomial ring in l letters where l is the rank of G. A useful example to have in mind is $G = SL(2, R)$ and *codim* $I = 1$, then we have: $l = 1$, $Z(\mathfrak{g})$ is generated by the Casimir operator $\mathcal{C}$, and saying that a function f is $Z(\mathfrak{g})$-finite is equivalent to stating that f is an eigenfunction of $\mathcal{C}$.

A well-known construction of an automorphic form on G is **Poincaré series**

$$\sum_{\gamma \in \Gamma} q(\gamma g),$$

where the function $q : G \to V$ is $Z(\mathfrak{g})$-finite and K*-finite on the right* (i.e., the set of its right translates under elements of K is a finite-dimensional vector space), and $q \in L^1(G)$. One can also consider **relative Poincaré series**

$$\sum_{\gamma \in \Gamma_0 \backslash \Gamma} q(\gamma g),$$

where $q : G \to V$ is $Z(\mathfrak{g})$-finite, K-finite on the right, Γ_0-invariant, and $q \in L^1(\Gamma_0 \backslash G)$.

Let us explain now how to construct an automorphic form on G/K. An **automorphy factor** is a map $\mu : \Gamma \times G/K \to GL(V)$ such that $\mu(g_1 g_2, x) = \mu(g_1, g_2 x)\mu(g_2, x)$. It allows to define an **automorphic form on G/K** as a function $f : G/K \to V$ such that

$$f(\gamma x)\mu(\gamma, x) = f(x)$$

for any $\gamma \in \Gamma$, $x \in G/K$. Notice that then the function

$$F(g) = f(g(0))\mu(g, 0),$$

where $g \in G$, $x = g(0) \in G/K$, satisfies (1) with $\rho(k) = \mu(k, 0)$, where 0 is the fixed point of K in G/K. If f is holomorphic then F is $Z(\mathfrak{g})$-finite.

In particular, for a smooth function $q \in L^1(G/K)$ the Poincaré series on G/K is

$$\sum_{\gamma\in\Gamma} q(\gamma x)\mu(\gamma, x). \tag{3}$$

1.2. Automorphic forms on bounded symmetric domains and quantization.

Consider a classical system (M, ω), where M is a manifold, and ω is a symplectic form on M. The main problem of quantization is to associate a quantum system $(\mathcal{H}, \mathcal{O})$ to (M, ω), where $\mathcal{H}$ is a Hilbert space and $\mathcal{O}$ consists of symmetric operators with domain in $\mathcal{H}$. Elements of $\mathcal{H}$ are wave functions (quantum-mechanical states), and elements of $\mathcal{O}$ are quantum observables.

The map $f \mapsto \hat{f}$, where $f \in C^\infty(M)$ and $\hat{f} \in \mathcal{O}$, should satisfy the following requirements:

1) it is $\mathbb{R}$-linear,

2) if $f =$const then $\hat{f}$ is the corresponding multiplication operator,

3) if $\{f_1, f_2\} = f_3$ then $\hat{f}_1\hat{f}_2 - \hat{f}_2\hat{f}_1 = i\hbar\hat{f}_3$.

These are Dirac quantization conditions and they are famously impossible to satisfy in most cases, so one should consider a certain modification of them.

How do automorphic forms appear in the context of quantization ?

Suppose that M is a compact Kähler manifold of complex dimension n which is a quotient of a bounded symmetric domain $D = G/K$ by the action of a discrete subgroup Γ, i.e. $M = \Gamma\backslash D$. Then $\mathcal{H}$ consists of holomorphic automorphic forms on D for Γ. More precisely, let us consider the well-known quantization scheme for compact Kähler manifolds via Toeplitz operators (it is related to the standard scheme of geometric quantization with Kähler polarization). Automorphic forms are holomorphic sections of $L^{\otimes k}$, where the canonical line bundle $L = \Lambda^n T^*_{hol}M$ is the quantizing line bundle on M, here k is a positive integer which determines the weight of an automorphic form, and $\hbar = \frac{1}{k}$.

We also notice that the automorphic form (3) is a sum of coherent states associated to a holomorphic discrete series representation of G.

Let us describe all this in a bit more details. Let $D = G/K$ be a bounded symmetric domain, it is a Hermitian symmetric space of noncompact type. The irreducible Hermitian spaces of non-compact type are

I) $SU(p, q)/S(U(p) \times U(q))$,

II) $Sp(p, \mathbb{R})/U(p)$,

III) $SO^*(2p)/U(p)$,
IV) $SO_o(p,2)/SO(p)\times SO(2)$
(and also there is the case of an exceptional Lie group). We have a metric

$$ds^2 = g_{ij}dz^i d\bar{z}^j, \tag{4}$$

the corresponding Kähler form is $\omega = ig_{ij}dz^i \wedge d\bar{z}^j = i\partial\bar{\partial}\ln K(z,z)$, where $K(z,w)$ is the Bergman kernel of the domain D. Recall that $K(z,w) = \overline{K(w,z)}$ and

$$K(\gamma z, \gamma w) = [\det J(\gamma,z)]^{-1}[\det \bar{J}(\gamma,w)]^{-1}K(z,w).$$

A **quantizing line bundle** $L \to M = \Gamma\backslash D$ is defined as a line bundle such that the curvature of its natural connection is the Kähler form ω on M. Denoting the canonical line bundle by L we see that the potential 1-form corresponding to the natural connection on L is $\theta = i\partial\ln(s,s) = -i\partial\ln K(z,z)$, hence the curvature $d\theta = -i\bar{\partial}\partial\ln K(z,z) = \omega$ and this is indeed a quantizing line bundle for M.

A holomorphic function $f: D \to \mathbb{C}$ is called an **automorphic form of weight k** if

$$f(\gamma z)[\det J(\gamma,z)]^k = f(z) \tag{5}$$

for any $z \in D$, $\gamma \in \Gamma$; here $J(\gamma,z)$ is the Jacobi matrix of transformation γ at point z. In the context of 1.1 the automorphy factor $\mu(\gamma,z) = [\det J(\gamma,z)]^k$. The space of automorphic forms of weight k can be identified with the complex inner product space $H^0(M, L^{\otimes k})$ of holomorphic sections of $L^{\otimes k}$.

Now we consider a family of maps p_k, here k is a positive integer, such that $p_k(f) = T_f^{(k)}$, where f belongs to the Poisson algebra of smooth real-valued functions on M and $T_f^{(k)}$ is the Toeplitz operator on $H^0(M, L^{\otimes k})$ obtained from multiplication operator $M_f^{(k)}(g) = fg$ on $L^2(M, L^{\otimes k})$ by the orthogonal compression to the closed subspace $H^0(M, L^{\otimes k})$, i.e. $T_f^{(k)} = \Pi^{(k)} \circ M_f^{(k)} \circ \Pi^{(k)}$, where $\Pi^{(k)}$ is the orthogonal projection from $L^2(M, L^{\otimes k})$ to $H^0(M, L^{\otimes k})$.

In the Berezin scheme of quantization [1], [18] for each $\hbar = \frac{1}{k}$ we consider the space $\mathcal{F}_\hbar$ of functions holomorphic in D and satisfying (5) with the scalar product defined by

$$(f,g) = \text{const}\,(\hbar)\int_M f(z)\bar{g}(z)[K(z,z)]^{-\frac{1}{\hbar}}d\mu(z),$$

where $d\mu(z) = \omega^n$ is the G-invariant volume form on D corresponding to the metric (4). It is clear that $\mathcal{F}_\hbar$ is naturally identified with $H^0(M, L^{\otimes k})$. For the sake of completeness let us also explain briefly how the operator $\hat{A}$ corresponding a classical observable $A = A(z)$, is defined. First, we consider an analytic continuation $A(z,w)$ of the function $A(z)$ to $D \times D$. The covariant symbol $A(z,z)$ of $\hat{A}$ is defined as the diagonal value of the function

$$A(z,w) = \frac{\int_M \hat{A}[(K(u,w))^{\frac{1}{\hbar}}](K(z,u))^{\frac{1}{\hbar}} d\mu(u)}{\int_M (K(u,w))^{\frac{1}{\hbar}}(K(z,u))^{\frac{1}{\hbar}} d\mu(u)},$$

and

$$(\hat{A}f)(z) = \mathrm{const}\,(\hbar) \int_M A(z,w) f(w) [K(z,w)]^{\frac{1}{\hbar}} [K(w,w)]^{-\frac{1}{\hbar}} d\mu(w).$$

So we end up with the algebra $A_\hbar$ of covariant symbols of bounded operators acting in $\mathcal{F}_\hbar$. The $*$-product in $A_\hbar$ is given by

$$A_1 * A_2\ (z,z) = \mathrm{const}\,(\hbar) \int_M A_1(z,w) A_2(w,z) \left(\frac{K(z,w)K(w,z)}{K(z,z)K(w,w)} \right)^{\frac{1}{\hbar}} d\mu(w).$$

In conclusion let us discuss the Poincaré series (3). Consider a unitary representation of G in $L^2(G/K)$ given by the operators

$$[\pi^k(g)(q)](z) = [\det J(g,z)]^k q(gz).$$

It can be regarded as a subrepresentation of the right regular representation of G in $L^2(\Gamma\backslash G)$. Fix $q \in \mathcal{F}_\hbar$, then the set $\{\pi^k(g)(q) | g \in G\}$ is a system of (generalized) coherent states. Strictly speaking, we should regard two coherent states $\pi^k(g_1)(q)$ and $\pi^k(g_2)(q)$ as equivalent if $\pi^k(g_1)(q) = e^{i\alpha}\pi^k(g_2)(q)$. Now it is clear that (3) is a sum of coherent states which belong to the subsystem associated to Γ.

1.3. Comments on the subject of the present paper.

In [9] and in the present paper we consider holomorphic automorphic forms on $D = \mathbb{H}^n_{\mathbb{C}} = SU(n,1)/U(n)$. In [9] we construct sets of relative Poincaré series which span the spaces of $\mathbb{C}$-valued holomorphic cusp forms on a finite volume quotient of D. In the present paper we regard holomorphic $\mathbb{C}$-valued automorphic forms on $\mathbb{H}^n_{\mathbb{C}}$ as holomorphic sections of the line bundle $L^{\otimes k} \to \Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$, where L is a quantizing line bundle on $\Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$, k is an integer, and Γ is a discrete cocompact subgroup of $SU(n,1)$. We construct relative

Poincaré series associated to loxodromic elements of Γ and we address an interesting problem which is not resolved for Poincaré series in general: is it true that these series are not identically zero ? We restrict ourselves to the case of complex dimension 2 and answer "yes" to this question going through the following steps: 1) to each hyperbolic element of Γ we associate a sequence $\Lambda(l)$, $l \geq 1$, of Legendrian submanifolds of the unit circle bundle in L^* such that the corresponding Lagrangian tori in $\Gamma\backslash SU(2,1)/U(2)$ satisfy a Bohr-Sommerfeld condition, 2) following the method of [6] we compute the k-th component u_k of the delta-function associated to $\Lambda(l)$ and the leading order asymptotics of $||u_k||$, which allows us to conclude that the relative Poincaré series associated to hyperbolic elements are not zero for large values of k (i.e. in semi-classical limit $\hbar = \frac{1}{k} \to 0$).

2. Preliminaries.

2.1. Complex hyperbolic space.

Consider the complex hyperbolic space

$$\mathbb{H}^n_{\mathbb{C}} = SU(n,1)/S(U(n) \times U(1)) = \mathbb{P}(\{z \in \mathbb{C}^{n+1} \mid \langle z, z\rangle < 0\}) \simeq B^n,$$

here B^n is the open unit ball in $\mathbb{C}^n$, $\langle\ .\ ,\ .\ \rangle$ is the Hermitian product on $\mathbb{C}^{n+1}$ given by $\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n - z_{n+1}\bar{w}_{n+1}$.

A vector $z \in \mathbb{C}^{n+1} - \{0\}$ is called *negative* (resp. *null, positive*) if the value of $\langle z, z\rangle$ is negative (resp. null, positive).

For $z, w \in B^n$ the corresponding vectors in $\mathbb{C}^{n+1}$ are $\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} z_1 \\ \cdots \\ z_n \\ 1 \end{pmatrix}$ and $\begin{pmatrix} w \\ 1 \end{pmatrix} = \begin{pmatrix} w_1 \\ \cdots \\ w_n \\ 1 \end{pmatrix}$, and we denote $\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n - 1$.

The group of isometries of $\mathbb{H}^n_{\mathbb{C}}$ is $PU(n,1) = SU(n,1)/center$. The group $SU(n,1)$ acts on B^n and on the boundary sphere $\partial B^n = \mathbb{P}(\{z \in \mathbb{C}^{n+1} - \{0\} | \langle z, z\rangle = 0\})$ by fractional-linear transformations: for

$$\gamma = \begin{pmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} & b_n \\ c_1 & \cdots & c_n & d \end{pmatrix}$$

and $z \in B^n$ (or $z \in \partial B^n$) we have:

$$\gamma z = \left(\frac{a_{11}z_1 + \cdots + a_{1n}z_n + b_1}{c_1 z_1 + \cdots + c_n z_n + d}, \ldots, \frac{a_{n1}z_1 + \cdots + a_{nn}z_n + b_n}{c_1 z_1 + \cdots + c_n z_n + d} \right)^T,$$

and

$$\det J(\gamma, z) = (c_1 z_1 + \cdots + c_n z_n + d)^{-(n+1)},$$

where $J(\gamma, z)$ denotes the Jacobi matrix of transformation γ at point z.

An automorphism is called *loxodromic* if it has no fixed points in B^n and fixes two points in ∂B^n. Notice that the fixed points of the automorphisms correspond to the eigenvectors of the corresponding matrices in $U(n,1)$. A loxodromic automorphism is called *hyperbolic* if it has a lift to $U(n,1)$ all of whose eigenvalues are real.

A loxodromic element $\gamma_0 \in SU(n,1)$ has $n-1$ positive eigenvectors and two null eigenvectors.

Let $v_1, \ldots, v_{n-1}$ be the positive eigenvectors of γ_0 and $\tau_1, \ldots, \tau_{n-1}$ be the corresponding eigenvalues. Then $|\tau_j| = 1$, $1 \le j \le n-1$.

Let X, Y be the null eigenvectors of γ_0. Then the corresponding eigenvalues are λ and $\bar{\lambda}^{-1}$ for some $\lambda \in \mathbb{C}$, $|\lambda| > 1$.

A loxodromic transformation can always be represented by a matrix in $U(n,1)$ with eigenvalues $\tau_1, \ldots, \tau_{n-1}, \lambda, \lambda^{-1}$ where $\lambda \in \mathbb{R}$, $|\lambda| > 1$.

The geodesic connecting X and Y is the geodesic in the Poincaré metric on the complex line containing X and Y (so it is an arc of a circle orthogonal to ∂B^n or a diameter), it is γ_0-invariant and is called the *axis* of γ_0.

2.2. Automorphic forms and geometry of the quotient.

Consider a compact manifold $X := \Gamma \setminus B^n$, where Γ is a discrete cocompact subgroup of $SU(n,1)$ which acts freely on B^n.

The Bergman kernel for the domain B^n is $K(z,w) = \frac{1}{(-\langle z,w\rangle)^{n+1}}$ (up to a multiplicative constant) and an $SU(n,1)$-invariant Kähler form on B^n is

$$\Omega = i\partial\bar{\partial} \ln K(z,z) = -\frac{(n+1)i}{\langle z,z\rangle^2} \left(\langle z,z\rangle \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j - \langle dz, z\rangle \wedge \langle z, dz\rangle \right).$$

Remark 2.1. With this normalization the holomorphic sectional curvature is equal to $-\frac{4}{n+1}$ and the sectional curvature is pinched between $-\frac{4}{n+1}$ and $-\frac{1}{n+1}$.

A holomorphic function $f : B^n \to \mathbb{C}$ satisfying the automorphy law

$$f(\gamma z)(\det J(\gamma, z))^k = f(z) \tag{6}$$

for any $\gamma \in \Gamma$ is called an *automorphic form of weight* $(n+1)k$ *for* Γ. The corresponding automorphic form on $SU(n,1)$ is given by $F(g) = f(g(0))\zeta^k$, where $\zeta = \zeta(g) = \det J(g, 0)$ and the origin 0 of B^n is the fixed point of $K = S(U(n) \times U(1)) \simeq U(n)$. The automorphy law on the group is

$$F(\gamma g \kappa) = F(g)\rho(\kappa),$$

where $\rho(\kappa) = (\det J(\kappa, 0))^k$, for any $g \in G$, $\gamma \in \Gamma$, $\kappa \in K$. Notice that $\gamma : \zeta \to \zeta \det J(\gamma, z)$ for any $\gamma \in SU(n,1)$. The automorphic form $F(g)$ can be regarded as a section of L^k where $L = \Lambda^n T^*_{hol} X$ is the canonical line bundle, and $T^*_{hol} X$ denotes the holomorphic cotangent bundle on X.

We shall denote the space of automorphic forms of weight $(n+1)k$ for Γ on B^n by $S_{(n+1)k}(\Gamma)$ and the corresponding space of automorphic forms on $SU(n,1)$ by $\tilde{S}_{(n+1)k}(\Gamma)$. The inner product in each of these spaces is given by

$$(f, g) = (f(z)\zeta^k, g(z)\zeta^k) = \int_{\Gamma \backslash B^n} f\bar{g}(-\langle z, z\rangle)^{(n+1)k} dV,$$

where

$$dV = i^n \frac{dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n}{(-\langle z, z\rangle)^{n+1}}$$

is a constant multiple of the $SU(n,1)$-invariant volume form for the metric corresponding to Ω.

Given a subgroup Γ_0 of Γ and a holomorphic function $q(z)$ satisfying (6) for all $\gamma \in \Gamma_0$ and such that $\int_{\Gamma_0 \backslash B^n} |q(z)|(-\langle z, z\rangle)^{\frac{(n+1)k}{2}} dV < \infty$, the *relative Poincaré series* associated to Γ_0 is defined as

$$\Theta(z) = \sum_{\gamma \in \Gamma_0 \backslash \Gamma} q(\gamma z)(\det J(\gamma, z))^k.$$

By Theorem A.1 (Appendix) this series converges absolutely and uniformly on compact sets and belongs to the space $S_{(n+1)k}(\Gamma)$.

The potential 1-form θ on L^* is characterized by

$$\nabla s = -i\theta s,$$

where ∇ is a connection on L^* and s is the unit section. In local coordinates the potential 1-form corresponding to the natural connection on L^* is

$$\theta = i\partial \ln(s, s) = -i\partial \ln(-\langle z, z\rangle)^{n+1}.$$

The curvature on L^* is $d\theta = -\Omega$, hence L is the natural quantizing line bundle for X.

Let

$$E_k = H^0(X, L^{\otimes k})$$

be the complex inner product space of holomorphic sections of the k-th tensor power of L. Consider the unit circle bundle $P \subset L^*$. Denote also $\tilde{L} = \bigwedge^n T^*_{hol}B^n$, $\tilde{P}$ - the unit circle bundle in $\tilde{L}^*$.

The connection form $\alpha : TP \to \mathbb{R}$ on P is

$$\alpha = \theta + i\frac{d\zeta}{\zeta} = -(n+1)i\frac{\langle dz, z\rangle}{\langle z, z\rangle} + i\frac{d\zeta}{\zeta} = -d\psi + \frac{n+1}{2}i\frac{\langle z, dz\rangle - \langle dz, z\rangle}{\langle z, z\rangle}$$

in local coordinates $z \in B^n$, $\zeta = (-\langle z, z\rangle)^{\frac{n+1}{2}} e^{i\psi}$. It serves as a contact form on $\tilde{P}$ and P.

A Lagrangian submanifold $\Lambda_0 \subset X$ satisfies a *Bohr-Sommerfeld condition* if

$$\frac{k}{2\pi}\int_C \theta \in \mathbb{Z}$$

for any closed curve $C \subset \Lambda_0$. The constant $\frac{1}{k}$ plays role of the Planck constant.

The unit disk bundle in L^* is a compact, strictly pseudoconvex domain with smooth boundary P. Let us consider the Hardy space of P: $E \subset L^2(P)$ and the Szëgo projector

$$\Pi : L^2(P) \to E$$

given by the orthogonal projection of $L^2(P)$ onto E. We identify:

$$E = \oplus_{k=0}^{\infty} E_k.$$

We also denote

$$\tilde{E}_k = \left\{ f(z)\zeta^k \mid (z,\zeta) \in \tilde{P},\ f \text{ is holomorphic on } B^n, \right.$$
$$\left. \int_{B^n} |f(z)|^2 (-\langle z, z\rangle)^{(n+1)k} dV < \infty \right\}.$$

We shall denote the corresponding orthogonal projection by

$$\tilde{\Pi} : L^2(\tilde{P}) \to \oplus_{k=0}^{\infty} \tilde{E}_k.$$

Both projectors can be extended to a class of distributions including the delta function.

3. Construction of relative Poincaré series associated to certain loxodromic elements of Γ.

Consider a loxodromic automorphism of B^n, represent it by a matrix $\gamma_0 \in U(n,1)$ with eigenvalues $\tau_1, \ldots, \tau_{n-1}, \lambda, \lambda^{-1}$, $|\tau_j| = 1$, $j = 1, \ldots, n-1$, $\lambda \in \mathbb{R}$, $|\lambda| > 1$, denote the corresponding eigenvectors by $v_1, \ldots, v_{n-1}, X, Y$ ($v_1, \ldots, v_{n-1}$ are positive, X, Y are null). Notice that if each τ_j is a root of 1 then some power of γ_0 is a hyperbolic element.

Now consider an arbitrary loxodromic element $\gamma_0 \in \Gamma \subset SU(n,1)$ represented as described above and assume that γ_0 satisfies the following condition.

Assumption 3.1. Assume that 1 is among the eigenvalues of γ_0.

Remark 3.2. If $g \in U(n,1)$ is hyperbolic then g^2 is a hyperbolic element of $SU(n,1)$ which satisfies Assumption 3.1 and has the same eigenvectors as g.

Generalizing the construction suggested in [9], for any collection, w.l.o.g. $v_1, \ldots, v_m$, $m \leq n-1$, of positive eigenvectors corresponding to eigenvalue 1 we construct a relative Poincaré series

$$\Theta_{\gamma_0,l,k} = \sum_{\gamma \in \Gamma_0 \backslash \Gamma} q_l(\gamma z)(\det J(\gamma, z))^{2k} \in S_{2(n+1)k}(\Gamma),$$

where $\Gamma_0 = \langle \gamma_0 \rangle$,

$$q_l(z) = \frac{\langle z, v_1 \rangle^{l_1} \cdots \langle z, v_m \rangle^{l_m}}{(\langle z, X \rangle \langle z, Y \rangle)^{(n+1)k + \frac{l_1 + \cdots + l_m}{2}}},$$

$l_1, \ldots, l_m$ are positive integers such that $l_1 + \cdots + l_m$ is even, $l = (l_1, \ldots, l_m)$. The series converges absolutely and uniformly on the compact sets of B^n by the Theorem A.1 (Appendix) for $k \geq 1$.

If $n = 2$ then the loxodromic elements of Γ satisfying Assumption 3.1 are exactly the hyperbolic elements of Γ. The relative Poincaré series associated to a hyperbolic element $\gamma_0 \in \Gamma$ is

$$\Theta_{\gamma_0,l,k} = \sum_{\gamma \in \Gamma_0 \backslash \Gamma} q_l(\gamma z)(\det J(\gamma, z))^{2k} \in S_{6k}(\Gamma),$$

where $\Gamma_0 = \langle \gamma_0 \rangle$,

$$q_l(z) = \frac{\langle z, v \rangle^{2l}}{(\langle z, X \rangle \langle z, Y \rangle)^{3k+l}},$$

and l is a non-negative integer.

Remark 3.3. Let γ_1 and γ_2 be hyperbolic elements of Γ. If $\gamma_1 = \gamma_2^N$ for a positive integer N, then $\Theta_{\gamma_1,l,k} = N\Theta_{\gamma_2,l,k}$.

Remark 3.4. In further exposition we assume that $l > 0$. The relative Poincaré series with $l = 0$ studied in [9] are associated to closed geodesics (not to Lagrangian tori).

4. Bohr-Sommerfeld tori.

Consider a hyperbolic element $\gamma_0 \in \Gamma \subset SU(2,1)$, denote its null eigenvectors by $X = \begin{pmatrix} X_1 \\ X_2 \\ 1 \end{pmatrix}$, $Y = \begin{pmatrix} Y_1 \\ Y_2 \\ 1 \end{pmatrix}$, denote its positive eigenvector by v, then the corresponding eigenvalues are λ, λ^{-1}, 1, for $\lambda \in \mathbb{R}$, $|\lambda| > 1$. We have:

$$\langle v, X \rangle = \langle v, Y \rangle = \langle X, X \rangle = \langle Y, Y \rangle = 0.$$

We normalize v so that $\langle v, v \rangle = 1$, then the matrix

$$A := \left[v \quad \frac{X}{\langle X, Y \rangle} + \frac{Y}{2} \quad \frac{X}{\langle X, Y \rangle} - \frac{Y}{2} \right]$$

belongs to $SU(2,1)$.

The transformation $A^{-1} = s \begin{pmatrix} \bar{v}^T \\ \frac{\bar{X}^T}{\langle Y, X \rangle} + \frac{\bar{Y}^T}{2} \\ \frac{\bar{X}^T}{\langle Y, X \rangle} - \frac{\bar{Y}^T}{2} \end{pmatrix} s$, where $s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, maps the complex line containing X and Y to the complex line $\{z_1 = 0\}$ and maps the geodesic connecting X and Y to the geodesic $\tilde{C}$ connecting $(0, -1)$ and $(0, 1)$. More precisely

$$A^{-1} \cdot X = \begin{pmatrix} 0 \\ \frac{\langle X, Y \rangle}{2} \\ \frac{\langle X, Y \rangle}{2} \end{pmatrix}, \quad A^{-1} \cdot Y = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

where $\cdot$ stands for the standard linear action of $GL(3, \mathbb{C})$ on $\mathbb{C}^3$, so

$$A^{-1} X = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad A^{-1} Y = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},$$

also $A^{-1} \cdot v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

The following loxodromic element of $SU(2,1)$ preserves $\tilde{C}$ and the complex line $\{z_1 = 0\}$:

$$\gamma := A^{-1}\gamma_0 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix},$$

where

$$a = \frac{\lambda^2 + 1}{2\lambda}, \; b = \frac{\lambda^2 - 1}{2\lambda}.$$

Denote

$$w = (w_1, w_2)^T = A^{-1}z, \; w_1 = A^{-1}z_1, \; w_2 = A^{-1}z_2,$$

and apply the change of variables

$$w_2 = \frac{re^{i\phi} - i}{re^{i\phi} + i}, \; w_1 = \sqrt{1 - w_2\bar{w}_2}Re^{i\Theta},$$

$$0 < \phi < \pi, \; 0 < r < +\infty, \; 0 < R < 1, \; 0 \leq \Theta < 2\pi.$$

Proposition 4.1. *Any 2-cylinder* $C_{\phi,R} = \{\phi = \text{const}, \; R = \text{const}\}$ *is* γ-*invariant.*

Remark 4.2. The coordinates (r, ϕ) are the polar coordinates on the upper-half plane, (R, Θ) are the polar coordinates in the unit disc, and the coordinates (r, Θ) on the cylinder $C_{\phi,R} \simeq \mathbb{R} \times S^1$ are the axial and the angular coordinates respectively.

Proof. Under the action of γ

$$w_2 \to \frac{aw_2 + b}{bw_2 + a} = \frac{a\frac{re^{i\phi}-i}{re^{i\phi}+i} + b}{b\frac{re^{i\phi}-i}{re^{i\phi}+i} + a} = \frac{r(a+b)e^{i\phi} - i(a-b)}{r(a+b)e^{i\phi} + i(a-b)} = \frac{r\frac{a+b}{a-b}e^{i\phi} - i}{r\frac{a+b}{a-b}e^{i\phi} + i},$$

so

$$r \to r\frac{a+b}{a-b}, \quad \phi \to \phi,$$

also

$$\frac{|w_1|}{\sqrt{1 - w_2\bar{w}_2}} \to \frac{|\frac{w_1}{bw_2+a}|}{\sqrt{1 - |\frac{aw_2+b}{bw_2+a}|^2}} = \frac{|w_1|}{\sqrt{1 - w_2\bar{w}_2}},$$

so $R \to R$. ∎

For a positive integer l consider the following submanifold of $\tilde{P}$:

$$\tilde{T}(l) := \{(w, (-\langle w, w\rangle)^{\frac{3}{2}} e^{i\psi}) \mid w \in \{\phi = \frac{\pi}{2},\ R = \sqrt{\frac{l}{3k+l}}\},\ \psi = -\frac{l}{k}\Theta\}.$$

The natural projection of $\tilde{T}(l)$ to B^2 is the cylinder $C_{\frac{\pi}{2},\sqrt{\frac{l}{3k+l}}}$ whose axis of symmetry is the geodesic $\tilde{C}$. Denote $T(l) = \langle\gamma\rangle \setminus \tilde{T}(l)$, $\tilde{\Lambda}(l) = A\tilde{T}(l)$ and $\Gamma_0 = \langle\gamma_0\rangle$.

Proposition 4.3. $\Lambda(l) := AT(l) = \Gamma_0 \setminus \tilde{\Lambda}(l)$ *is a compact Legendrian submanifold of* P.

Proof. Both $T(l)$ and $\Lambda(l)$ are compact submanifolds of P of real dimension 2.

Let us prove that $\Lambda(l)$ is Legendrian. The restriction of α onto $\tilde{T}(l)$ is

$$\begin{aligned}
&-3i\frac{\langle dw, w\rangle}{\langle w, w\rangle} + i\frac{d\zeta}{\zeta}\\
&= -\frac{3}{2}i\frac{\bar{w}_1 dw_1 + \bar{w}_2 dw_2}{w_1\bar{w}_1 + w_2\bar{w}_2 - 1} + \frac{3}{2}i\frac{w_1 d\bar{w}_1 + w_2 d\bar{w}_2}{w_1\bar{w}_1 + w_2\bar{w}_2 - 1} - d\psi\\
&= -\frac{3}{2}i\frac{(1 - w_2\bar{w}_2)R^2 id\Theta + \sqrt{1 - w_2\bar{w}_2}R^2 d\sqrt{1 - w_2\bar{w}_2} + \bar{w}_2 dw_2}{(1 - w_2\bar{w}_2)R^2 + w_2\bar{w}_2 - 1}\\
&\quad + \frac{3}{2}i\frac{-(1 - w_2\bar{w}_2)R^2 id\Theta + \sqrt{1 - w_2\bar{w}_2}R^2 d\sqrt{1 - w_2\bar{w}_2} + w_2 d\bar{w}_2}{(1 - w_2\bar{w}_2)R^2 + w_2\bar{w}_2 - 1} - d\psi\\
&= -\frac{3}{2}i\frac{(1 - w_2^2)R^2 id\Theta + \sqrt{1 - w_2^2}R^2 d\sqrt{1 - w_2^2} + w_2 dw_2}{(1 - w_2^2)(R^2 - 1)}\\
&\quad + \frac{3}{2}i\frac{-(1 - w_2^2)R^2 id\Theta + \sqrt{1 - w_2^2}R^2 d\sqrt{1 - w_2^2} + w_2 dw_2}{(1 - w_2^2)(R^2 - 1)} - d\psi\\
&= -3i\frac{(1 - w_2^2)R^2 id\Theta}{(1 - w_2^2)(R^2 - 1)} - d\psi = 3\frac{R^2 d\Theta}{R^2 - 1} - d\psi\\
&= 3\frac{\frac{l}{3k+l}d\Theta}{\frac{l}{3k+l} - 1} + \frac{l}{k}d\Theta = 0.
\end{aligned}$$

The form α is $SU(2,1)$-invariant, indeed, under the action of $M \in SU(2,1)$,

$$(z, \zeta) \mapsto (Mz, \zeta \det J(M, z)) = (Mz, \zeta c^3),$$

where $c = c(z) = (m_{31}z_1 + m_{32}z_2 + m_{33})^{-1}$. We have:

$$\begin{aligned}\alpha &= i\frac{d\zeta}{\zeta} - 3i\partial \ln(-\langle z, z\rangle) \to i\frac{d(c^3\zeta)}{c^3\zeta} - 3i\partial \ln(-\langle Mz, Mz\rangle) \\ &= i\frac{c^3 d\zeta + 3c^2\zeta dc}{c^3\zeta} - 3i\partial \ln(-\langle z, z\rangle c\bar{c}) \\ &= i\frac{d\zeta}{\zeta} + 3i\frac{dc}{c} - 3i\partial \ln(-\langle z, z\rangle) - 3i\partial \ln(c\bar{c}) \\ &= i\frac{d\zeta}{\zeta} + 3i\frac{\partial c}{c} - 3i\partial \ln(-\langle z, z\rangle) - 3i\frac{\partial c}{c} \\ &= i\frac{d\zeta}{\zeta} - 3i\partial \ln(-\langle z, z\rangle).\end{aligned}$$

□

The natural projection $\Lambda_0(l)$ of $\Lambda(l)$ onto X is a compact Lagrangian submanifold of X.

Proposition 4.4. *$\Lambda_0(l)$ satisfies a Bohr-Sommerfeld condition.*

Proof. Let $\tilde{T}_0(l)$ be the natural projection of $\tilde{T}(l)$ onto B^2, and let $T_0(l)$ be the natural projection of $T(l)$ onto X, $AT_0(l) = \Lambda_0(l)$. If $C \subset \Lambda_0(l)$ is a closed curve then $A^{-1}C \subset T_0(l)$ is closed too. Let $z \in \Lambda_0(l)$, $w \in T_0(l)$, $c = c(w) = (a_{31}w_1 + a_{32}w_2 + a_{33})^{-1}$, we have:

$$\begin{aligned}-\int_C \theta &= 3i\int_C \partial \ln(-\langle z, z\rangle) = 3i\int_{A^{-1}C} \partial \ln(-\langle Aw, Aw\rangle) \\ &= 3i\int_{A^{-1}C} \partial \ln(-\langle w, w\rangle c\bar{c}) = 3i\int_{A^{-1}C} (\partial \ln(-\langle w, w\rangle) + \partial \ln c) \\ &= 3i\int_{A^{-1}C} (\partial \ln(-\langle w, w\rangle) + d\ln c) = 3i\int_{A^{-1}C} \partial \ln(-\langle w, w\rangle),\end{aligned}$$

so $\int_C \theta$ is A^{-1}-invariant (in fact $SU(2,1)$-invariant) and it is enough to prove that $T_0(l)$ satisfies the Bohr-Sommerfeld condition. The restriction of θ to

$\tilde{T}_0(l)$ is

$$\begin{aligned}
&-3i\frac{\bar{w}_1 dw_1 + \bar{w}_2 dw_2}{w_1\bar{w}_1 + w_2\bar{w}_2 - 1}\\
&= -3i\frac{(1-w_2\bar{w}_2)R^2 id\Theta + \sqrt{1-w_2\bar{w}_2}R^2 d\sqrt{1-w_2\bar{w}_2} + \bar{w}_2 dw_2}{(1-w_2\bar{w}_2)R^2 + w_2\bar{w}_2 - 1}\\
&= -3i\frac{(1-w_2^2)R^2 id\Theta + \sqrt{1-w_2^2}R^2 d\sqrt{1-w_2^2} + w_2 dw_2}{(1-w_2^2)(R^2-1)}\\
&= -3i\frac{(1-w_2^2)R^2 id\Theta + \sqrt{1-w_2^2}R^2 \frac{-2w_2dw_2}{2\sqrt{1-w_2^2}} + w_2 dw_2}{(1-w_2^2)(R^2-1)}\\
&= -3i\left(\frac{R^2 i}{R^2-1}d\Theta - \frac{w_2dw_2}{1-w_2^2}\right) = 3\frac{R^2}{R^2-1}d\Theta - 3i\frac{1}{2}d\ln(1-w_2^2)\\
&= -\frac{l}{k}d\Theta - \frac{3}{2}id\ln(1-w_2^2),
\end{aligned}$$

then

$$\begin{aligned}
&\frac{k}{2\pi}\int_{A^{-1}C}\left(\frac{l}{k}d\Theta + \frac{3}{2}id\ln(1-w_2^2)\right)\\
&= \frac{l}{2\pi}\int_{A^{-1}C} d\Theta = \frac{l}{2\pi}2\pi m = lm \in \mathbb{Z}.
\end{aligned}$$

□

So the torus $\Lambda_0(l)$ is a Lagrangian submanifold satisfying the Bohr-Sommerfeld condition.

Proposition 4.5. *The orthogonal projection of the delta function at* $(w,\eta) \in \tilde{P}$ *into* $\tilde{E}_k$ *is*

$$\Psi_{(w,\eta)}(z,\zeta) := \tilde{\Pi}_k(\delta_{(w,\eta)}) = \frac{(3k-1)(3k-2)}{4\pi^2}\frac{\zeta^k\bar{\eta}^k}{\langle z,w\rangle^{3k}}.$$

Remark 4.6. The orthogonal projection of the delta function at $(w,\eta) \in \tilde{P}$ into $\tilde{E}_k$ is the *coherent state* in $\tilde{E}_k$ associated to the point $(w,\eta) \in \tilde{P}$, by definition $g\Psi_{(w,\eta)} = \Psi_{g(w,\eta)}$ for $g \in SU(2,1)$.

Proof. The fact that $\Psi_{(w,\eta)} = \tilde{\Pi}_k(\delta_{(w,\eta)})$ is equivalent to the reproducing property:

$$F(w,\eta) = \int_{\tilde{P}} \bar{\Psi}_{(w,\eta)}(z,\zeta)F(z,\zeta)dV \wedge d\psi$$

for all $F \in \tilde{E}_k$. Given any orthonormal basis $\{F_{l,k}\}$ for $\tilde{E}_k$, we can write the reproducing kernel as the series $\Psi_{(w,\eta)}(z,\zeta) = \sum_l \bar{F}_{l,k}(w,\eta)F_{l,k}(z,\zeta)$ which converges absolutely and uniformly on compact sets.

Using the basis

$$F_{l,m,k}(z,\zeta) = \frac{1}{2\pi}\sqrt{\frac{(3k+l+m-1)!}{l!m!(3k-3)!}}z_1^l z_2^m \zeta^k,$$

which is orthonormal with respect to the inner product

$$(f(z)\zeta^k, g(z)\zeta^k) = i^2 \int_{B^2} f\bar{g}(-\langle z,z\rangle)^{3k-3} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2,$$

we obtain:

$$\begin{aligned}\Psi_{(w,\eta)}(z,\zeta) &= \sum_{l,m} \bar{F}_{l,m,k}(w,\eta)F_{l,m,k}(z,\zeta)\\ &= \sum_{l,m} \frac{1}{(2\pi)^2}\frac{(3k+l+m-1)!}{l!m!(3k-3)!}\bar{w}_1^l\bar{w}_2^m z_1^l z_2^m \zeta^k\bar{\eta}^k\\ &= \frac{\zeta^k\bar{\eta}^k}{4\pi^2(3k-3)!}\sum_{l,m}\frac{(3k+l+m-1)!}{l!m!}(\bar{w}_1 z_1)^l(\bar{w}_2 z_2)^m.\end{aligned}$$

To calculate

$$\sum_{l,m}\frac{(3k+l+m-1)!}{l!m!}x^l y^m = \sum_m \frac{y^m}{m!}\sum_l \frac{(3k+l+m-1)!}{l!}x^l$$

we notice that

$$\begin{aligned}\sum_l \frac{(N+l)!}{l!}t^l &= \frac{d^N}{dt^N}\sum_{l=0}^{\infty} t^{l+N} = \frac{d^N}{dt^N}\frac{t^N}{1-t}\\ &= \frac{d^N}{dt^N}\left(\frac{t^N-1}{1-t}+\frac{1}{1-t}\right)\\ &= \frac{d^N}{dt^N}\frac{1}{1-t} = \frac{N!}{(1-t)^{N+1}},\end{aligned}$$

hence

$$\sum_m \frac{y^m}{m!}\sum_l \frac{(3k+l+m-1)!}{l!}x^l = \sum_m \frac{y^m}{m!}\frac{(3k+m-1)!}{(1-x)^{3k+m}}$$
$$= \frac{1}{(1-x)^{3k}}\sum_m \frac{(3k+m-1)!}{m!}\left(\frac{y}{1-x}\right)^m$$
$$= \frac{1}{(1-x)^{3k}}\frac{(3k-1)!}{(1-\frac{y}{1-x})^{3k}} = \frac{(3k-1)!}{(1-x-y)^{3k}},$$

so

$$\Psi_{(w,\eta)}(z,\zeta) = \frac{\zeta^k\bar{\eta}^k}{4\pi^2(3k-3)!}\frac{(3k-1)!}{(1-\bar{w}_1z_1-\bar{w}_2z_2)^{3k}}$$
$$= \frac{(3k-1)(3k-2)}{4\pi^2}\zeta^k\bar{\eta}^k\frac{1}{(-\langle z,w\rangle)^{3k}}.$$

□

For $\tilde{E}_{2k}$ we have:

$$\Psi_{(u,\eta)}(z,\zeta) = \tilde{\Pi}_{2k}(\delta_{(u,\eta)}) = \frac{(6k-1)(6k-2)}{4\pi^2}\frac{\zeta^{2k}\bar{\eta}^{2k}}{\langle z,u\rangle^{6k}}.$$

We omit the weight in the notation $\Psi_{(u,\eta)}(z,\zeta)$ but further exposition will be for E_{2k} (i.e., weight $6k$) so this will not lead to any confusion.

To get the orthogonal projection of the delta function at $[(u,\eta)] \in P = \Gamma \setminus \tilde{P}$ (by $[(u,\eta)]$ we denote the equivalence class of (u,η)) into E_{2k} we average over the action of Γ:

$$\Pi_{2k}(\delta_{[(u,\eta)]}) = \sum_{g\in\Gamma} g\Psi_{(u,\eta)}. \tag{7}$$

The series (7) converges absolutely and uniformly on compact sets by Theorem 9.1 [4].

Following the method of [6], to the submanifold $\Lambda(l) \subset P$ we associate a section $u_{2k} \in E_{2k}$ defined as follows:

$$u_{2k} = \int_{\Lambda(l)} \Pi_{2k}(\delta_{[(u,\eta)]})\nu = \sum_{g\in\Gamma}\int_{\Lambda(l)} g\Psi_{(u,\eta)}\nu$$
$$= \sum_{g\in\Gamma/\Gamma_0}\sum_{m=-\infty}^{+\infty}\int_{\Lambda(l)} g\gamma_0^m\Psi_{(u,\eta)}(z,\zeta)\nu$$

$$= \sum_{g\in\Gamma/\Gamma_0} \sum_{m=-\infty}^{+\infty} \int_{\Lambda(l)} g\Psi_{\gamma_0^m(u,\eta)}(z,\zeta)\nu$$

$$= \sum_{g\in\Gamma/\Gamma_0} g \int_{\tilde\Lambda(l)} \Psi_{(u,\eta)}(z,\zeta)\nu = \sum_{g^{-1}\in\Gamma_0\backslash\Gamma} g^{-1} \int_{\tilde\Lambda(l)} \Psi_{(u,\eta)}(z,\zeta)\nu,$$

where

$$\nu = \frac{d(A^{-1}u_1)}{A^{-1}u_1} \wedge \frac{d(A^{-1}u_2)}{1-(A^{-1}u_2)^2}.$$

Proposition 4.7.

(i) $I := \int_{\tilde\Lambda(l)} \Psi_{(u,\eta)}(z,\zeta)\nu = C\frac{\langle z,v\rangle^{2l}}{(\langle z,X\rangle\langle z,Y\rangle)^{3k+l}}\zeta^{2k}$, *where the constant* C *is given by*

$$C = \frac{2^{3k+l-2}}{\pi}\frac{((3k+l-1)!)^2}{(2l)!(6k-3)!}\frac{(3k)^{3k}l^l}{(3k+l)^{3k+l}}(-\langle Y,X\rangle)^{3k+l},$$

(ii) $\int_{\Lambda(l)} |\nu| = 2\pi \ln|\lambda|$.

Remark 4.8. The 2-form ν on $\tilde\Lambda(l)$ is γ_0-invariant and in properly chosen coordinates (r,Θ) on $\tilde\Lambda_0(l)$ it is expressed as $\nu = \frac{i}{2}d\Theta \wedge \frac{dr}{r}$.

Proof. Let $u \in \tilde\Lambda(l)$, $w = A^{-1}u \in \tilde T(l)$, then

$$I = \frac{(6k-1)(6k-2)}{4\pi^2}\zeta^{2k}\int_{\tilde\Lambda(l)} \frac{\bar\eta^{2k}}{\langle z,u\rangle^{6k}}\frac{d(A^{-1}u_1)}{A^{-1}u_1} \wedge \frac{d(A^{-1}u_2)}{1-(A^{-1}u_2)^2}$$

$$= \frac{(6k-1)(6k-2)}{4\pi^2}\zeta^{2k}$$

$$\cdot \int_{\tilde T(l)} \frac{((-\langle w,w\rangle)^{\frac{3}{2}}e^{-i\psi}\det \bar J(A,w))^{2k}}{\langle z,Aw\rangle^{6k}}\frac{dw_1}{w_1}\wedge\frac{dw_2}{1-w_2^2}$$

$$= \frac{(6k-1)(6k-2)}{4\pi^2}\zeta^{2k}\int_{\tilde T(l)} \frac{(-\langle w,w\rangle)^{3k}e^{-i2k\psi}(\det \bar J(A,w))^{2k}}{\langle A^{-1}z,w\rangle^{6k}(\det \bar J(A,w))^{2k}}$$

$$\cdot (\det J(A^{-1},z))^{2k}\frac{dw_1}{w_1}\wedge\frac{dw_2}{1-w_2^2},$$

let $A^{-1}z = \begin{pmatrix} v_1 \\ v_2 \\ 1\end{pmatrix}$, then we get:

$$I = \frac{(6k-1)(6k-2)}{4\pi^2}\zeta^{2k}(\det J(A^{-1},z))^{2k}$$

$$\cdot \int_{\tilde{T}(l)} \frac{(-\langle w, w \rangle)^{3k} e^{-i2k\psi}}{(v_1\bar{w}_1 + v_2\bar{w}_2 - 1)^{6k}} \frac{dw_1}{w_1} \wedge \frac{dw_2}{1 - w_2^2},$$

on $\tilde{T}(l)$

$$w_2 = \frac{r-1}{r+1}, \; w_1 = \sqrt{1 - w_2^2} Re^{i\Theta} = \frac{2\sqrt{r}}{r+1} Re^{i\Theta},$$

$$-\langle w, w \rangle = (1 - R^2)(1 - w_2^2) = (1 - R^2)\frac{4r}{(r+1)^2},$$

so we have:

$$I = \frac{(6k-1)(6k-2)}{4\pi^2} \zeta^{2k} (\det J(A^{-1}, z))^{2k} (4(1-R^2))^{3k}$$

$$\cdot \int_{\tilde{T}(l)} \frac{(\frac{r}{(r+1)^2})^{3k} e^{-i2k\psi}}{(v_1 \frac{2\sqrt{r}}{r+1} Re^{-i\Theta} + v_2 \frac{r-1}{r+1} - 1)^{6k}} \frac{i}{2} d\Theta \wedge \frac{dr}{r}$$

$$= \frac{(6k-1)(6k-2)}{4\pi^2} \zeta^{2k} (\det J(A^{-1}, z))^{2k} (4(1-R^2))^{3k} \frac{i}{2}$$

$$\cdot \int_0^\infty dr \int_0^{2\pi} d\Theta \frac{r^{3k-1} e^{i2l\Theta}}{(v_1 2\sqrt{r} Re^{-i\Theta} + v_2(r-1) - r - 1)^{6k}}.$$

The integral

$$\int_{|w|=1} \frac{1}{(Aw+B)^{6k}} \frac{dw}{w^{2l+1}}, \qquad \left|\frac{B}{A}\right| > 1$$

is equal to

$$\frac{2\pi i}{(2l)!} \frac{d^{2l}}{dw^{2l}} \frac{1}{(Aw+B)^{6k}}|_{w=0} = \frac{2\pi i}{(2l)!} \frac{(6k+2l-1)!}{(6k-1)!} \frac{A^{2l}}{B^{6k+2l}},$$

Let $w = e^{-i\Theta}$, $A = v_1 2\sqrt{r} R$, $B = v_2(r-1) - r - 1$. Let us check that $|\frac{B}{A}| > 1$.

$$\left| \frac{v_2(r-1) - r - 1}{v_1 2\sqrt{r} R} \right| = \left| \frac{v_2 w_2 - 1}{v_1 R \sqrt{1 - w_2^2}} \right| > \frac{|v_2 w_2 - 1|}{\sqrt{1 - v_2\bar{v}_2} R \sqrt{1 - w_2^2}}$$

$$\geq \frac{|v_2 w_2 - 1|}{\sqrt{1 - v_2\bar{v}_2}\sqrt{1 - w_2^2}} \geq 1$$

because

$$0 \leq |v_2 - w_2|^2 = (\bar{v}_2 - w_2)(v_2 - w_2) = v_2\bar{v}_2 - \bar{v}_2 w_2 - w_2 v_2 + w_2^2$$

$$= -\bar{v}_2 w_2 - w_2 v_2 + v_2\bar{v}_2 w_2^2 + 1 + v_2\bar{v}_2 + w_2^2 - v_2\bar{v}_2 w_2^2 - 1$$

$$= (\bar{v}_2 w_2 - 1)(v_2 w_2 - 1) - (1 - v_2\bar{v}_2)(1 - w_2^2).$$

We get:

$$\begin{aligned}
I = &-\frac{(6k-1)(6k-2)}{4\pi^2}\zeta^{2k}(4(1-R^2))^{3k}\frac{i}{2}\frac{2\pi i}{(2l)!}\frac{(6k+2l-1)!}{(6k-1)!} \\
&\cdot (\det J(A^{-1},z))^{2k}\int_0^\infty \frac{r^{3k-1+l}v_1^{2l}2^{2l}R^{2l}}{(v_2(r-1)-r-1)^{6k+2l}}dr \\
= &\frac{4^{3k-1+l}}{\pi}\frac{(6k+2l-1)!}{(2l)!(6k-3)!}R^{2l}(1-R^2)^{3k}\zeta^{2k}(\det J(A^{-1},z))^{2k} \\
&\cdot \frac{v_1^{2l}}{(v_2-1)^{6k+2l}}\int_0^\infty \frac{r^{3k-1+l}}{(r-\frac{v_2+1}{v_2-1})^{6k+2l}}dr.
\end{aligned}$$

Notice that $\frac{v_2+1}{v_2-1}$ can not be a real non-negative number, and the integration by parts gives:

$$\int_0^\infty \frac{r^{3k-1+l}}{(r-\frac{v_2+1}{v_2-1})^{6k+2l}}dr = \frac{((3k+l-1)!)^2}{(6k+2l-1)!}\left(-\frac{v_2-1}{v_2+1}\right)^{3k+l},$$

hence

$$\begin{aligned}
I = &\frac{4^{3k+l-1}}{\pi}\frac{((3k+l-1)!)^2}{(2l)!(6k-3)!}R^{2l}(1-R^2)^{3k} \\
&\cdot (\det J(A^{-1},z))^{2k}\frac{v_1^{2l}\zeta^{2k}}{[(1-v_2)(1+v_2)]^{3k+l}},
\end{aligned}$$

and

$$\begin{aligned}
&\frac{v_1^{2l}}{[(1-v_2)(1+v_2)]^{3k+l}} \\
&= \frac{\langle A^{-1}z, A^{-1}\cdot v\rangle^{2l}}{(\langle A^{-1}z, A^{-1}X\rangle\langle A^{-1}z, A^{-1}Y\rangle)^{3k+l}} \\
&= \frac{\langle z,v\rangle^{2l}}{(\langle z,X\rangle\langle z,Y\rangle)^{3k+l}}(\det J(A^{-1},z))^{-2k}\left(-\frac{\langle Y,X\rangle}{2}\right)^{3k+l},
\end{aligned}$$

therefore

$$I = \frac{2^{3k+l-2}}{\pi}\frac{((3k+l-1)!)^2}{(2l)!(6k-3)!}\frac{(3k)^{3k}l^l}{(3k+l)^{3k+l}}(-\langle Y,X\rangle)^{3k+l}\frac{\langle z,v\rangle^{2l}}{(\langle z,X\rangle\langle z,Y\rangle)^{3k+l}}\zeta^{2k}.$$

Proof of (ii):

$$\int_{\Lambda(l)} \nu = \int_{AT(l)} \left|\frac{d(A^{-1}u_1)}{A^{-1}u_1} \wedge \frac{d(A^{-1}u_2)}{1-(A^{-1}u_2)^2}\right|$$

$$= \int_{T(l)} \left| \frac{dw_1}{w_1} \wedge \frac{dw_2}{1-w_2^2} \right| = \int_{|\lambda|^{-1}}^{|\lambda|} dr \int_0^{2\pi} d\Theta \frac{1}{2r} = 2\pi \ln |\lambda|.$$

□

We got:

$$u_{2k}(z,\zeta) = \zeta^{2k} \sum_{g\in\Gamma_0\backslash\Gamma} q_l(gz)(\det J(g,z))^{2k} \in \tilde{S}_{2(n+1)k}(\Gamma),$$

where

$$q_l(z) = C \frac{\langle z, v\rangle^{2l}}{(\langle z, X\rangle\langle z, Y\rangle)^{3k+l}}$$

and the relative Poincaré series associated to $\Lambda_0(l)$ is

$$\Theta_{\gamma_0,l,k}(z) := \sum_{g\in\Gamma_0\backslash\Gamma} q_l(gz)(\det J(g,z))^{2k} \in S_{2(n+1)k}.$$

From the results of [6] (Theorem 3.2, Corollary 3.3) it follows that for large values of k

$$||u_{2k}||^2 = ||\Theta_{\gamma_0,l,k}||^2 \sim \frac{2k}{\pi} \int_{\Lambda(l)} |\nu| = 4k \ln |\lambda|),$$

and, in particular, the relative Poincaré series $\Theta_{\gamma_0,l,k}$ are not identically zero for large weights.

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Appendix A.

We shall prove the following theorem modifying the proof of convergence of Poincaré series contained in [4] and [3].

Theorem A.1. *Let φ be a $\mathbb{C}$-valued function on $G = SU(n,1)$. Assume that*

1) *φ is $Z(\mathfrak{g})$-finite,*

2) $\varphi \in L^1(\Gamma_0 \setminus G)$,

3) *φ is K-finite on the right.*

Let $p_\varphi(x) = \sum_{\gamma\in\Gamma_0\backslash\Gamma} \varphi(\gamma x)$.
Then p_φ converges absolutely and uniformly on compact sets.

Proof. By Lemma 9.2 [4] there exists $\alpha \in C_c^\infty(G)$ satisfying $\alpha(k^{-1}xk) = \alpha(x)$, $k \in K$, $x \in G$, such that $\varphi = \varphi * \alpha$. Fix a neighborhood U of 1 in G such that $U^{-1} = U$, the closure of U is compact, and $U \supset$ *supp* α. We have:

$$\varphi(\gamma x) = (\varphi * \alpha)(\gamma x) = \int_G \varphi(\gamma x y)\alpha(y^{-1})dy = \int_U \varphi(\gamma x y)\alpha(y^{-1})dy,$$

hence

$$|\varphi(\gamma x)| \leq ||\alpha||_\infty \int_U |\varphi(\gamma x y)|dy = ||\alpha||_\infty \int_{xU} |\varphi(\gamma y)|dy$$

Here $||\alpha||_\infty = \sup_{y\in U} |\alpha(y)|$.

Fix a compact subset C of G. We want to prove absolute and uniform convergence on C. The closure of CU is compact. CU is covered by N copies of a fundamental domain of Γ in G (i.e., a connected set of representatives of $\Gamma\backslash G$), where N is a positive integer. Denote these domains by $F_1,\ldots,F_N$.

Let $x \in C$. Then

$$|\varphi(\gamma x)| \leq ||\alpha||_\infty \int_{xU} |\varphi(\gamma y)|dy \leq ||\alpha||_\infty \int_{CU} |\varphi(\gamma y)|dy$$

and we get

$$\begin{aligned}
&\sum_{\gamma\in\Gamma_0\backslash\Gamma} ||\alpha||_\infty \int_{CU} |\varphi(\gamma y)|dy \\
&= ||\alpha||_\infty \sum_{\gamma\in\Gamma_0\backslash\Gamma} \int_{CU} |\varphi(\gamma y)|dy \\
&\leq ||\alpha||_\infty \sum_{\gamma\in\Gamma_0\backslash\Gamma} \left(\int_{F_1} |\varphi(\gamma y)|dy + \cdots + \int_{F_N} |\varphi(\gamma y)|dy\right) \\
&= ||\alpha||_\infty \left(\sum_{\gamma\in\Gamma_0\backslash\Gamma} \int_{F_1} |\varphi(\gamma y)|dy + \cdots + \sum_{\gamma\in\Gamma_0\backslash\Gamma} \int_{F_N} |\varphi(\gamma y)|dy\right) \\
&= N||\alpha||_\infty \int_{\Gamma_0\backslash G} |\varphi(y)|dy < \infty.
\end{aligned}$$

So we proved that

$$|\varphi(\gamma x)| \le c_\gamma := \|\alpha\|_\infty \int_{CU} |\varphi(\gamma y)| dy$$

and that the numerical series $\sum_{\gamma \in \Gamma_0 \backslash \Gamma} c_\gamma$ converges, hence by Weierstrass theorem the series $\sum_{\gamma \in \Gamma_0 \backslash \Gamma} \varphi(\gamma x)$ converges absolutely and uniformly on C. □

References.

[1] F. Berezin, *General concept of quantization,* Comm. Math. Phys., **40** (1975), 153–174.

[2] M. Bordemann, E. Meinrenken, and M. Schlichenmaier, *Toeplitz quantization of Kähler manifolds and $gl(N)$, $N \to \infty$ limits,* Comm. Math. Phys., **165(2)** (1994), 281–296.

[3] A. Borel, *Automorphic forms on $SL_2(\mathbb{R})$,* Cambridge University Press, 1997.

[4] A. Borel, *Introduction to automorphic forms,* Proc. Symp. Pure Math., **9** (1966), 199–210.

[5] D. Borthwick, A. Lesniewski, and H. Upmeier, *Non-perturbative deformation quantization of Cartan domains,* J. Funct. Anal., **113** (1993), 153–176.

[6] D. Borthwick, T. Paul, and A. Uribe, *Legendrian distributions with applications to relative Poincaré series,* Invent. Math., **122** (1995), 359–402.

[7] L. Boutet de Monvel and V. Guillemin, *The spectral theory of Toeplitz operators,* Princeton University Press, 1981.

[8] D. Epstein, *Complex hyperbolic geometry,* in 'Analytical and geometric aspects of hyperbolic space,' London Math. Soc. Lecture Note Ser., **111**, Cambridge Univ. Press, 1987.

[9] T. Foth and S. Katok, *Spanning sets for automorphic forms and dynamics of the frame flow on complex hyperbolic spaces,* to appear in Ergod. Th. Dyn. Sys.

[10] W. Goldman, *Complex hyperbolic geometry,* Oxford University Press, 1999.

[11] V. Guillemin and S. Sternberg, *Geometric asymptotics,* Math. Surveys, **14**, AMS, 1977.

[12] N. Hurt, *Geometric quantization in action,* D. Reidel Publ. Co., 1983.

[13] S. Klimek and A. Lesniewski, *Quantum Riemann surfaces I. The unit disc,* Comm. Math. Phys., **146** (1992), 103–122.

[14] S. Klimek and A. Lesniewski, *Quantum Riemann surfaces II. The discrete series,* Lett. Math. Phys., **24** (1992), 125–139.

[15] S. Kobayashi, C. Horst, and H. Wu, *Complex differential geometry,* Birkhäuser, 1983.

[16] J. Kollar, *Shafarevich maps and automorphic forms,* Princeton University Press, 1995.

[17] S. Krantz, *Function theory of several complex vaiables,* Wadsworth & Brooks/Cole, 1992.

[18] A. Perelomov, *Generalized cohereht states and applications,* Springer-Verlag, 1986.

[19] J. Rawnsley, *Quantization on Kähler manifolds,* in 'Differential geometric methods in theoretical physics,' Lect. Notes in Phys., **375** (1991), 155–161.

[20] J. Rawnsley, *Deformation quantization of Kähler manifolds,* in 'Symplectic geometry and mathematical physics,' Prog. in Math., **99** (1991), 366–373.

[21] Y. Tai and H. Resnikoff, *On the structure of a graded ring of automorphic forms on the 2-dimensional complex ball, I and II,* Math. Ann., **238** (1978), 97–117; Math. Ann., **258** (1982), 367–382.

[22] J. Sniatycki, *Geometric quantization and quantum mechanics,* Springer-Verlag, 1980.

[23] D. Zhelobenko and A. Shtern, *Representations of Lie groups (Russian),* "Nauka", Moscow, 1983.

[24] N. Woodhouse, *Geometric quantization,* Oxford University Press, 1980.

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