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Hamiltonian stationary Lagrangian surfaces in $\mathbb{C}^2$

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We study Hamiltonian stationary Lagrangian surfaces in $\mathbb{C}^2$, i.e., Lagrangian surfaces in $\mathbb{C}^2$ which are stationary points of the area functional under smooth Hamiltonian variations. Using loop groups, we propose a formulation of the equation as a completely integrable system. We construct a Weierstrass type representation and produce all tori through either the integrable systems machinery or more direct arguments.

1. Introduction.

This paper addresses the study of Hamiltonian stationary oriented Lagrangian surfaces in a symplectic Euclidean vector space of dimension 4, using techniques of completely integrable systems. The ambient space may be seen as $\mathbb{C}^2$ with, using complex coordinates $z^1 = x^1 + iy^1$ and $z^2 = x^2 + iy^2$, the symplectic form $\omega = dx^1 \wedge dy^1 + dx^2 \wedge dy^2$ and the canonical scalar product. The Lagrangian surfaces in $\mathbb{C}^2$ are the immersed surfaces on which the restriction of ω vanishes. On the set of oriented Lagrangian surfaces Σ in $\mathbb{C}^2$, we let the area functional to be

$$\mathcal{A}(\Sigma) = \int_\Sigma dv,$$

where the volume form dv is defined using the induced metric on Σ. A critical point of this functional is a Lagrangian surface such that $\delta\mathcal{A}(\Sigma)(X) = 0$ for any compactly supported smooth vector field X on $\mathbb{C}^2$, satisfying some particular constraint: if X is arbitrary we just say that Σ is *stationary* (it is actually a minimal surface in $\mathbb{C}^2 \simeq \mathbb{R}^4$), if X is Lagrangian, i.e., its flow preserves Lagrangian surfaces, Σ is called *Lagrangian stationary*[1], and lastly if X is Hamiltonian, i.e., $X = -J\nabla h = \frac{\partial h}{\partial y^1}\frac{\partial}{\partial x^1} - \frac{\partial h}{\partial x^1}\frac{\partial}{\partial y^1} + \frac{\partial h}{\partial y^2}\frac{\partial}{\partial x^2} - \frac{\partial h}{\partial x^2}\frac{\partial}{\partial y^2}$, for some $h \in \mathcal{C}_c^\infty(\mathbb{C}^2, \mathbb{R})$, Σ is called *Hamiltonian stationary*[2].

The first variation of the area involves the Lagrangian angle: if m is a point in Σ and if (e_1, e_3) is a direct orthonormal basis of $T_m\Sigma$, $dz^1 \wedge$

[1]called isotropic minimal in [CM].
[2]called E-minimal in [CM] and H-minimal in [O2] and [CU].

$dz^2(e_1, e_3)$ is a complex number of modulus equal to 1, which we can denote $e^{i\beta}$, for some real number β. It builds up a map β from Σ to $\mathbb{R}/2\pi\mathbb{Z}$. This map is a part of the full Gauss map of the immersion of Σ. The mean curvature vector H on Σ is then given by $H = J\nabla\beta$, and thus

$$\delta\mathcal{A}(\Sigma)(X) = \int_\Sigma \langle X, H\rangle dv = \int_\Sigma \langle -J\nabla h, J\nabla\beta\rangle dv = \int_\Sigma \langle \nabla h, \nabla\beta\rangle dv,$$

see [O1] for more details. Hence Hamiltonian stationary surfaces are characterized by the equation $-\Delta\beta = 0$, where Δ is the Laplace operator on Σ, which comes from the induced metric. Surfaces such that β is constant (or $H = 0$) are a particular case, called *special Lagrangian surfaces* by R. Harvey and H.B. Lawson [HaL]: they are actually area minimizing since calibrated by $e^{-i\beta}dz^1 \wedge dz^2$.

Examples of Hamiltonian stationary surfaces are the standard square tori $T_r = \{(z^1, z^2) \in \mathbb{C}^2 / |z^1| = |z^2| = r\sqrt{2}\}$ and "rectangular" variants $T_{r_1,r_2} = \{(z^1, z^2) \in \mathbb{C}^2 / |z^1|/r_1 = |z^2|/r_2 = \sqrt{2}\}$. These are candidates to be area minimizing with respect to Hamiltonian deformations as conjectured by Y.G. Oh [O1, O2]. More recently, I. Castro and F. Urbano [CU] have constructed more exotic examples of Hamiltonian stationary tori. Beside these explicit instances, R. Schoen and J. Wolfson obtained recently various existence and partial regularity results and in particular a proof of the existence of a smooth solution to the Plateau problem in $\mathbb{C}^2$ [ScW].

A motivation to study Hamiltonian stationary surfaces is for instance the following model of incompressible elasticity. If (ϕ, ψ) is a diffeomorphism between two two-dimensional domains U and U', which is incompressible, i.e., $\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial x} = 1$ everywhere, and which minimizes the area of the graph functional $\int_U \sqrt{2 + |\nabla\phi|^2 + |\nabla\psi|^2}dxdy$ among all possible incompressible diffeomorphisms with the same boundary data, then its graph $\Sigma = \{(x, y, \phi(x, y), -\psi(x, y))/(x, y) \in U\}$ is Hamiltonian stationary Lagrangian and conversely. Such a problem has been considered by J. Wolfson in [W]. Also Hamiltonian stationary surfaces offer a nice generalization of the minimal surface theory. The conjecture of Y.G. Oh above is an interesting generalization of isoperimetric inequalities. Such an inequality would be related to many questions in symplectic geometry, as illustrated by C. Viterbo [V], who also gave a lower bound for the area functional of a torus. Also special Lagrangian surfaces has appeared in recent developments in mathematical Physics, in M-theory [AFS], and about Mirror symmetry for Calabi-Yau manifolds: see for example [SYZ], where A. Strominger, S.T. Yau and E. Zaslow proposed that the moduli space of special Lagrangian surfaces in a Calabi-Yau is related to the mirror of the manifold.

Our aim here is to show that the set of Hamiltonian stationary Lagrangian surfaces in $\mathbb{C}^2$ forms a completely integrable system, and to use ideas from the Adler-Kostant-Symes theory in a similar way as it was done by F. Burstall, D. Ferus, F. Pedit, U. Pinkall [BFPP] and J. Dorfmeister, F. Pedit, H. Wu [DPW] for harmonic maps between a surface and a homogeneous manifold, or by F. Hélein [H2] for Willmore surfaces. (See also [U, SWi, Hi, FP, FPPS, DH] about previous results.) Our main results are: a formulation of the Hamiltonian stationary surfaces problem in terms of a family depending on a complex parameter of curvature free connections (a characteristic feature in integrable systems); a correspondence between conformal immersions of Hamiltonian stationary surfaces in $\mathbb{C}^2$ and holomorphic maps into $\mathbb{C}^3$ (similar to [DPW]); a proof that all Hamiltonian stationary tori in $\mathbb{C}^2$ are obtained by a finite type construction (this is similar to [BFPP]); lastly a construction of all such tori by integrating linear elliptic equations.

From the point of view of the theory of completely integrable systems, we obtain an original (at least for us!) example of situation where:

- the family of curvature free connections has the form $\alpha_\lambda = \lambda^{-2}\alpha_2' + \lambda^{-1}\alpha_{-1}' + \alpha_0 + \lambda\alpha_1'' + \lambda^2\alpha_2''$ instead of $\lambda^{-1}\alpha_1' + \alpha_0 + \lambda\alpha_1''$ as in many integrable systems,

- the situation is almost linear and, in some situations, simplifies in such a way that we could present the results without these techniques.

However we choose to expose the full machinery in our situation since this is the way we obtained all the constructions here and it seems to illuminate how completely integrable systems work.

Our paper is organized as follows. In section 2 we present the symmetry group of affine isometries of $\mathbb{R}^4$ preserving the symplectic form and the description of conformal immersions of Hamiltonian stationary Lagrangian surfaces using moving frames. A Cartan decomposition of the Lie algebra appears to be the key of the formulation. In section 3 we show that the construction of conformal immersions of Hamiltonian stationary simply connected surfaces is equivalent to solving three simple linear PDE's as follows: let β to be a real harmonic map on a simply connected domain Ω; we solve on Ω the linear equation

$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\frac{\partial \beta}{\partial z} J\bar{u},$$

for $u = {}^t(a/2, b/2, -ia/2, ib/2)$ and a and b complex valued functions. Then we integrate the equation

$$dX = e^{\beta J/2}(udz + \bar{u}d\bar{z})$$

to obtain a map X to $\mathbb{R}^4$. Then X is a weakly conformal Hamiltonian stationary Lagrangian map. We use these ideas to deduce explicit parametrizations of all tori and we identify known examples: the standard torus and the surfaces of I. Castro and F. Urbano and we show other examples. In section 4 we introduce loop groups and twisted loop groups and we prove various Riemann-Hilbert and Birkhoff-Grothendieck decomposition results. These constructions have been revisited in [HR3] and [H3], where the quaternionic structure has been put in evidence. In section 5 we use the previous results to establish a Weierstrass type representation. In section 6 we use the finite gap ideas in integrable systems and prove that, for Hamiltonian stationary conformal immersions of tori, the set of solutions splits into a countable union of vector spaces ("finite type" solutions). Lastly we link this formulation with the one in section 3.

We point out that our results could be generalized to Hamiltonian stationary Lagrangian conformal immersions in $\mathbb{CP}^2$ (or isotropic surfaces in higher dimensional Kähler manifolds). This is the subject of [HR3].

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2. Moving frames and groups.

2.1. Symmetry groups for symplectic Euclidean affine 4-spaces.

Let E^4 be an affine oriented Euclidean symplectic space and $\vec{E}^4$ the associated oriented Euclidean vector space. We denote by $\langle .,. \rangle$ the scalar product and ω the symplectic form on $\vec{E}^4$. There exists a unique complex structure J on $\vec{E}^4$, such that $\omega(x,y) = \langle Jx, y \rangle$, $\forall x, y \in \vec{E}^4$. We denote by $\mathcal{F}$, the set of all orthonormal bases $e = (e_1, e_2, e_3, e_4)$ of $\vec{E}^4$, such that $e_2 = Je_1$ and $e_4 = Je_3$. We choose an origin O in E^4 and an orthonormal basis of $\vec{E}^4$, $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \in \mathcal{F}$. In the corresponding coordinates (x^1, x^2, x^3, x^4), the symplectic form reads

$$\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4.$$

And the complex structure J has the matrix

$$L_i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(the meaning of that notation will become clear below).

The relevant symmetry groups here are

- $\mathcal{G}$, the group of affine transformations of E^4 which preserve $\langle .,. \rangle$ and ω (or alternatively which preserve $\langle .,. \rangle$ and J)
- $\vec{\mathcal{G}}$, the group of linear transformations of $\vec{E}^4$ which preserve $\langle .,. \rangle$ and ω (or $\langle .,. \rangle$ and J), which we may view as a subgroup of $\mathcal{G}$, namely the isotropy group at 0.

Let us analyze first $\vec{\mathcal{G}}$. A first description of $\vec{\mathcal{G}}$ is obtained by the identification of $\vec{E}^4$ through the quaternions $\mathbb{H}$:

$$\begin{array}{llcl} Q: & \vec{E}^4 \simeq \mathbb{R}^4 & \longrightarrow & \mathbb{H} \\ & x^1\epsilon_1 + x^2\epsilon_2 + x^3\epsilon_3 + x^4\epsilon_4 \simeq (x^1, x^2, x^3, x^4) & \longmapsto & x^1 + ix^2 + jx^3 + kx^4. \end{array}$$

Let $S^3_{\mathbb{H}} = \{p \in \mathbb{H}/|p| = 1\}$. To each pair $(p,q) \in S^3_{\mathbb{H}} \times S^3_{\mathbb{H}}$ corresponds a rotation $G_{(p,q)} \in SO(4)$ defined by: $\forall x \in \mathbb{R}^4$,

$$Q \circ G_{(p,q)}(x) = pQ(x)\bar{q}.$$

The surjective map $\begin{array}{ccc} S^3_{\mathbb{H}} \times S^3_{\mathbb{H}} & \longrightarrow & SO(4) \\ (p,q) & \longmapsto & G_{(p,q)} \end{array}$ is a 2-sheeted covering map (since $G_{(-p,-q)} = G_{(p,q)}$). Explicitly we have,

$$G_{(p,q)}x = L_p R_{\bar{q}} x = R_{\bar{q}} L_p x,$$

where, denoting $p = p^1 + ip^2 + jp^3 + kp^4$ and $q = q^1 + iq^2 + jq^3 + kq^4$,

$$L_p = p^1 1\!\!1_4 + p^2 L_i + p^3 L_j + p^4 L_k$$

is the left multiplication by p in $\mathbb{H}$,

$$R_{\bar{q}} = q^1 1\!\!1_4 - q^2 R_i - q^3 R_j - q^4 R_k$$

is the right multiplication by $\bar{q}$ in $\mathbb{H}$ (notice that $R_{\bar{q}}R_{\bar{q}'} = R_{\overline{qq'}}$), and

$$L_i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad L_j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$L_k = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad R_i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$R_j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad R_k = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then, from $\vec{\mathcal{G}} \simeq \{G \in SO(4)/[G, L_i] = 0\}$, we obtain

$$\vec{\mathcal{G}} \simeq \mathcal{G}_0.\mathcal{G}_2,$$

where $\mathcal{G}_0 = \{R_{\bar{q}} = q^1 1\!\!1_4 - q^2 R_i - q^3 R_j - q^4 R_k / q \in S^3_{\mathbb{H}}\}$ and $\mathcal{G}_2 = \{L_p = p^1 1\!\!1_4 + p^2 L_i / p \in S^1_{\mathbb{C}}\}$. Notice that, for any $G \in \vec{\mathcal{G}}$, there exists $(G_0, G_2) \in \mathcal{G}_0 \times \mathcal{G}_2$, such that $G = G_0 G_2 = G_2 G_0$, and (G_0, G_2) is unique up to change of sign.

Alternatively, we can describe $\vec{\mathcal{G}}$ using the isomorphism

$$\begin{array}{cccc} C: & \vec{E}^4 \simeq \mathbb{R}^4 & \longrightarrow & \mathbb{C}^2 \\ & x^1\epsilon_1 + x^2\epsilon_2 + x^3\epsilon_3 + x^4\epsilon_4 \simeq (x^1, x^2, x^3, x^4) & \longmapsto & (x^1 + ix^2, x^3 + ix^4), \end{array}$$

which is holomorphic from $(\vec{E}^4, J)$ to $\mathbb{C}^2$. Through that identification, $\vec{\mathcal{G}}$ corresponds to $U(2)$, $\mathcal{G}_0$ to $SU(2)$ and $\mathcal{G}_2$ to

$$\left\{ \begin{pmatrix} e^{\frac{i\theta}{2}} & 0 \\ 0 & e^{\frac{i\theta}{2}} \end{pmatrix} / \theta \in \mathbb{R} \right\} \simeq U(1).$$

It is useful to keep in mind these representations. However, we shall mostly represent $\vec{\mathcal{G}}$ as a subgroup of the 4×4 matrices $\mathcal{M}(4, \mathbb{R})$ (which we can also identify with a subgroup of $\mathcal{M}(5, \mathbb{R})$, see below), since several complex structures will be involved.

The group $\mathcal{G}$ is just the semidirect product $\vec{\mathcal{G}} \ltimes \mathbb{R}^4$. If $G, G' \in \vec{\mathcal{G}}$ and $T, T' \in \mathbb{R}^4$, the product is $(G, T).(G', T') = (GG', GT' + T)$. This group is embedded in $\mathcal{M}(5,\mathbb{R})$ through

$$(G,T) \longmapsto \begin{pmatrix} G & T \\ 0 & 1 \end{pmatrix}.$$

We shall call G the *rotation component* of (G,T) and T the *translation component* of (G,T). (Notice that we also have the representation $\mathcal{G} \simeq U(2) \ltimes \mathbb{C}^2$.)

The Lie algebra of $\mathcal{G}$ will be identified with

$$\mathfrak{g} = \left\{(aL_i + b^1R_i + b^2R_j + b^3R_k, t)/a, b^1, b^2, b^3 \in \mathbb{R}, t \in \mathbb{R}^4\right\}.$$

The Lie bracket of two elements (η, t), $(\tilde{\eta}, \tilde{t}) \in \mathfrak{g}$ is

$$[(\eta, t), (\tilde{\eta}, \tilde{t})] = (\eta\tilde{\eta} - \tilde{\eta}\eta, \eta\tilde{t} - \tilde{\eta}t).$$

We denote by $\mathfrak{g}_0$ the Lie algebra of $\mathcal{G}_0$, generated by $(R_i, 0)$, $(R_j, 0)$ and $(R_k, 0)$, and $\mathfrak{g}_2$ the Lie algebra of $\mathcal{G}_2$, generated by $(L_i, 0)$. Then the Lie algebra of $\vec{\mathcal{G}}$ is $\vec{\mathfrak{g}} = \mathfrak{g}_0 \oplus \mathfrak{g}_2$.

2.2. Action on the Lagrangian Stiefel manifold and on the Lagrangian Grassmannian.

Let us define the Lagrangian Grassmannian Gr_{lag} to be the set of all oriented 2-dimensional Lagrangian subspaces of $\vec{E}^4$, and the Lagrangian Stiefel manifold by

$$Stief_{lag} = \{(e_1, e_3) \in \vec{E}^4 \times \vec{E}^4 / |e_1| = |e_3| = 1, \langle e_1, e_3 \rangle = 0, \omega(e_1, e_3) = 0\}.$$

Notice that $Stief_{lag}$ is nothing but the set of oriented orthonormal bases of planes in Gr_{lag}. Actually, we may identify $Stief_{lag}$ with $\mathcal{F}$ by the following: to each basis $(e_1, e_2, e_3, e_4) \in \mathcal{F}$, we associate (e_1, e_3) in $Stief_{lag}$. Conversely, we associate to each $(e_1, e_3) \in Stief_{lag}$ the frame (e_1, e_2, e_3, e_4) such that $e_2 = L_ie_1$ and $e_4 = L_ie_3$. (Through the identification $\vec{E}^4 \simeq \mathbb{C}^2$, it just amounts to say that (e_1, e_3) is a Hermitian basis of $\mathbb{C}^2$ over $\mathbb{C}$ if and only if (e_1, ie_1, e_3, ie_3) is an orthonormal basis of $\mathbb{C}^2$ over $\mathbb{R}$.)

Now the group $\vec{\mathcal{G}}$ acts freely and transitively on $\mathcal{F}$, i.e., for any $(e_1, e_2, e_3, e_4) \in \mathcal{F}$, there exists a unique $G \in \vec{\mathcal{G}}$ such that $(e_1, e_2, e_3, e_4) = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)G$. To prove that, it suffices to realize that the columns of G are the components of each vector e_i in the basis $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$. Hence $\vec{\mathcal{G}}$ acts freely and transitively on $Stief_{lag}$ as well, and transitively on Gr_{lag}: if $(e_1, e_3) \in Stief_{lag}$ and $G \in \vec{\mathcal{G}}$ we shall denote (Ge_1, Ge_3) its image by G.

An important object for the study of Hamiltonian stationary surfaces is the *Lagrangian angle map* $\Theta : Stief_{lag} \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$. For any $(e_1, e_3) \in Stief_{lag}$, let G be the unique element in $\vec{\mathcal{G}}$ such that $G\epsilon_1 = e_1$ and $G\epsilon_3 = e_3$, i.e., $(e_1, L_ie_1, e_3, L_ie_3) = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)G$. Then, viewing G as a matrix in $U(2)$, we may compute its determinant: it is a complex number of modulus one, which we denote $e^{i\Theta(e_1,e_3)}$. It builds up the *Lagrangian angle map* Θ. Alternatively, Θ is defined implicitly by the decomposition $G = G_0G_2$, where $G_0 \in \mathcal{G}_0$ and $G_2 = e^{\frac{\Theta(e_1,e_3)L_i}{2}} \in \mathcal{G}_2$ [3]. A last possible definition is given by:

$$(dx^1 + idx^2) \wedge (dx^3 + idx^4)(e_1, e_3) = e^{i\Theta(e_1,e_3)}.$$

One can check easily that $\Theta(e_1, e_3)$ does not change if we replace (e_1, e_3) by another direct orthonormal basis of the oriented Lagrangian plane spanned by (e_1, e_3). Hence it defines a map from Gr_{lag} to $\mathbb{R}/2\pi\mathbb{Z}$ which we shall also denote Θ.

Lastly, in the following we shall abuse notations and identify vectors $x = x^1\epsilon_1 + x^2\epsilon_2 + x^3\epsilon_3 + x^4\epsilon_4$ in $\vec{E}^4$ with column matrices $\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}$. In particular we let $\epsilon_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\epsilon_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\epsilon_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $\epsilon_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$,

$\epsilon = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -i \\ 0 \end{pmatrix}$ and $\bar{\epsilon} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}$, $L_i\epsilon = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -i \end{pmatrix}$ and $L_i\bar{\epsilon} = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ i \end{pmatrix}$.

2.3. Moving frames for conformal Lagrangian immersions.

Let us consider a smooth conformal Lagrangian immersion of a simply connected open domain Ω of $\mathbb{C} \simeq \mathbb{R}^2$, $X : \Omega \longrightarrow E^4$. We shall denote $z = x + iy \simeq (x, y)$ the coordinates on $\mathbb{R}^2$. We let $f : \Omega \longrightarrow \mathbb{R}$, such that $e^{f(z)} = |\frac{\partial X}{\partial x}| = |\frac{\partial X}{\partial y}|$ and we set $e_1(z) = e^{-f(z)}\frac{\partial X}{\partial x}(z)$ and $e_3(z) = e^{-f(z)}\frac{\partial X}{\partial y}(z)$, so that

$$dX = e^f(e_1dx + e_3dy),$$

[3] In particular it proves that $\Theta(GK\epsilon_1, GK\epsilon_3) = \Theta(G\epsilon_1, G\epsilon_3)$, $\forall G \in \vec{\mathcal{G}}$, $\forall K \in \mathcal{G}_0$.

and then, X is a conformal Lagrangian immersion if and only if for $z \in \Omega$, $(e_1(z), e_3(z))$ is in $Stief_{lag}$. Without loss of generality, we will normalize X by assuming that $X(z_0) = 0$ and $(e_1(z_0), e_3(z_0)) = (\epsilon_1, \epsilon_3)$, for some fixed point $z_0 \in \Omega$. We let $\mathcal{X}$ to be the set of such conformal Lagrangian immersions. We denote $e_2(z) := L_i e_1(z)$ and $e_4(z) := L_i e_3(z)$. Therefore the system $e(z) := (e_1(z), e_2(z), e_3(z), e_4(z))$ belongs to $\mathcal{F}$: let $F_X(z) \in \vec{\mathcal{G}}$ such that $e(z) = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)F_X(z)$ and (abusing notations) let $X(z)$ be the column vector of the components of X in the basis $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$. Then we construct a map $\tilde{X} : \Omega \longrightarrow \mathcal{G}$ lifting X, defined by

$$\tilde{X}(z) = \begin{pmatrix} F_X(z) & X(z) \\ 0 & 1 \end{pmatrix} \simeq (F_X(z), X(z)).$$

We shall call $\tilde{X}$ the *fundamental lift* of X. According to our normalization, we have $\tilde{X}(z_0) = (\mathbb{1}, 0)$. The Maurer-Cartan form of $\tilde{X}$ is

$$\tilde{X}^{-1} d\tilde{X} = (F_X^{-1} dF_X, F_X^{-1} dX).$$

It is a 1-form with coefficients in $\mathfrak{g}$, with the property that its translation component has the form

$$F_X^{-1} dX = e^f (\epsilon_1 dx + \epsilon_3 dy) = e^f (\epsilon dz + \bar{\epsilon} d\bar{z}). \tag{1}$$

The key idea in the following will be to study suitably defined lifts of conformal Lagrangian immersions instead of immersions themselves - which has the effect of decreasing by one the order of the PDE. One could use the fundamental lift. We shall however enlarge the possibilities as follows:

Definition 1. A *lifted conformal Lagrangian immersion* (LCLI) is a map $U = (F, X) : \Omega \longrightarrow \mathcal{G}$, satisfying one of the three following equivalent hypotheses.

a) $U(z) = (F_X(z), X(z)).(K(z)^{-1}, 0) = (F_X(z)K(z)^{-1}, X(z))$ where X is a conformal Lagrangian immersion, (F_X, X) is its fundamental lift and $K \in \mathcal{C}^\infty_\star(\Omega, \mathcal{G}_0) = \{K \in \mathcal{C}^\infty(\Omega, \mathcal{G}_0) / K(z_0) = \mathbb{1}\}$.

b) $U(z_0) = (\mathbb{1}, 0)$ and the translation component of the Maurer-Cartan form $U^{-1}dU$ has the form

$$t = F^{-1} dX = e^f K(\epsilon_1 dx + \epsilon_3 dy) = e^f K(\epsilon dz + \bar{\epsilon} d\bar{z}),$$

where $K \in \mathcal{C}^\infty(\Omega, \mathcal{G}_0)$ and $f \in \mathcal{C}^\infty(\Omega, \mathbb{R})$.

c) $U(z_0) = (\mathbb{1}, 0)$ and X is a conformal Lagrangian immersion and, $\forall z \in \Omega$,

$$\Theta\left(\frac{\partial X}{\partial x}(z), \frac{\partial X}{\partial y}(z)\right) = \Theta(F(z)\epsilon_1, F(z)\epsilon_3),$$

where we let $\Theta\left(\frac{\partial X}{\partial x}(z), \frac{\partial X}{\partial y}(z)\right)$ to be the value of Θ on the oriented Lagrangian plane spanned by $\frac{\partial X}{\partial x}(z)$ and $\frac{\partial X}{\partial y}(z)$.

We shall denote by $\mathcal{G}X$ the set of all LCLI's.

Proof of the equivalence between a), b) *and* c).

a) $\Rightarrow$ b): it is a direct computation.

b) $\Rightarrow$ c): from b), it follows that $\frac{\partial X}{\partial x} = e^f FK\epsilon_1$ and $\frac{\partial X}{\partial y} = e^f FK\epsilon_3$, therefore, using the remark in the footnote, section 2.2, $\Theta\left(\frac{\partial X}{\partial x}(z), \frac{\partial X}{\partial y}(z)\right) = \Theta(FK\epsilon_1, FK\epsilon_3) = \Theta(F\epsilon_1, F\epsilon_3)$.

c) $\Rightarrow$ a): let (F_X, X) be the fundamental lift of X and let $K = F^{-1}F_X \in \mathcal{C}^\infty(\Omega, \vec{\mathcal{G}})$. Then a computation shows that the relation $\Theta\left(\frac{\partial X}{\partial x}(z), \frac{\partial X}{\partial y}(z)\right) = \Theta(F\epsilon_1, F\epsilon_3)$ is equivalent to $\Theta(F_X\epsilon_1, F_X\epsilon_3) = \Theta(F_XK^{-1}\epsilon_1, F_XK^{-1}\epsilon_3)$ and using the remark in the footnote of section 2.3, this implies that $K \in \mathcal{C}^\infty_\star(\Omega, \mathcal{G}_0)$. □

For any simply connected domain Ω and for any conformal Lagrangian immersion $X : \Omega \longrightarrow E^4$, we shall lift the Lagrangian angle map and define a map $\beta : \Omega \longrightarrow \mathbb{R}$, such that $\forall z \in \Omega$, $\Theta\left(\frac{\partial X}{\partial x}(z), \frac{\partial X}{\partial y}(z)\right) = \beta(z)$ modulo 2π. The (lifted) Lagrangian angle map β of a LCLI U is characterized by the - unique up to sign - decomposition $U(z) = (e^{\beta(z)L_i/2}M_0(z), X(z))$, where $M_0 \in \mathcal{C}^\infty(\Omega, \mathcal{G}_0)$. In the following, for any $X \in \mathcal{X}$, we shall choose β to be the unique Lagrangian angle map such that $\beta(z_0) = 0$.

Remark 1. It is clear that the gauge group $\mathcal{C}^\infty_\star(\Omega, \mathcal{G}_0)$ acts on the right on $\mathcal{G}X$ and that the quotient of $\mathcal{G}X$ under this gauge action coincides with $\mathcal{X}$. Furthermore, in a given gauge orbit, there are three special LCLI's: the fundamental lift $\tilde{X}$ and a pair of lifts such that $F \in \mathcal{C}^\infty(\Omega, \mathcal{G}_2)$, namely

$$U_+(z) = (e^{\beta L_i/2}, X) \text{ and } U_-(z) = (-e^{\beta L_i/2}, X).$$

(Note that by a change $\beta \to \beta + 2\pi$ of the choice of the determination of β, U_+ and U_- would be exchanged.) We call U_+ and U_- the *spinor lifts.*

We have the following characterization of Hamiltonian stationary surfaces (see [O1], [ScW]).

Theorem 1. *Let $X : \Omega \longrightarrow E^4$ be a conformal Lagrangian immersion, then, X is Hamiltonian stationary if and only if the Lagrangian angle map is a harmonic function on the surface (with the induced metric), i.e.,*

$$\Delta\beta = 0 \text{ on } \Omega.$$

Thus we are led to study LCLI's with harmonic Lagrangian angle map. Alternatively, we can isolate the differential $d\beta$ by a decomposition of the Maurer-Cartan form of $U = (e^{\beta L_i/2}M_0, X)$ (according to $\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_0 \oplus (0, \mathbb{R}^4)$),

$$\alpha = U^{-1}dU = \frac{d\beta}{2}(L_i, 0) + (M_0^{-1}dM_0, 0) + (0, t). \tag{2}$$

Therefore, we may study connection 1-forms $\alpha \in \mathcal{C}^\infty(\Omega, T^\star\mathbb{R}^2 \otimes \mathfrak{g})$ on a simply connected domain Ω which satisfy relation (2) with coclosed $d\beta$, and the zero curvature equation

$$d\alpha + \alpha \wedge \alpha = 0, \tag{3}$$

a necessary and sufficient condition for the existence of a map $U : \Omega \longrightarrow \mathcal{G}$ such that $dU = U.\alpha$; furthermore, U is unique, if we assume also the condition

$$U(z_0) = (\mathbb{1}, 0), \text{ for some fixed point } z_0 \in \Omega. \tag{4}$$

We shall concentrate in the following on this last characterization.

Remark 2. The gauge action of $\mathcal{C}^\infty_\star(\Omega, \mathcal{G}_0)$ on $\mathcal{G}X$ induces an action on Maurer-Cartan 1-forms described by

$$(\eta, t) \ \mapsto \ (K\eta K^{-1} - dKK^{-1}, Kt).$$

In any orbit of this gauge action, the fundamental lift $\tilde{X} = (F_X, X) = (e^{\beta L_i/2}M_X, X)$ has the Maurer-Cartan form

$$\begin{aligned}
\tilde{X}^{-1}d\tilde{X} &= (F_X^{-1}dF_X, 0) + (0, e^f(\epsilon dz + \bar{\epsilon}d\bar{z})) \\
&= \frac{d\beta}{2}(L_i, 0) + (M_X^{-1}dM_X, 0) + (0, e^f(\epsilon dz + \bar{\epsilon}d\bar{z}))
\end{aligned}$$

i.e., with "simplest" translation component, whereas the spinor lifts has the Maurer-Cartan forms

$$U_\pm^{-1}dU_\pm = \frac{d\beta}{2}(L_i, 0) + (0, 0) + (0, \pm e^{-\beta L_i/2}dX),$$

i.e., with zero $\mathfrak{g}_0$ component.

2.4. Splitting the Lie algebra.

Our aim will be to refine the decomposition given in (2). We introduce the following automorphism τ acting on $\mathcal{G}$ through conjugation by $(-L_j, 0)$, i.e.,

$$\tau(G,T) = (-L_j,0)(G,T)(-L_j,0)^{-1} = (-L_jGL_j, -L_jT).$$

It induces a linear action on $\mathfrak{g}$, which diagonalizes on $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$, with eigenvalues i^{-1}, i^0, i^1 and i^2, since $\tau^4 = \mathbb{1}$. For $k = -1, 0, 1, 2$, we denote by $\mathfrak{g}_k^{\mathbb{C}}$ the eigenspace of τ for the eigenvalue i^k, and we have

- for the eigenvalue $-i$, $\mathfrak{g}_{-1}^{\mathbb{C}} = (0, \mathbb{C}\epsilon \oplus \mathbb{C}L_i\bar{\epsilon})$ (notice that $\mathbb{C}\epsilon \oplus \mathbb{C}L_i\bar{\epsilon}$ is the $(-i)$-eigenspace of $-L_j$),
- for the eigenvalue 1, $\mathfrak{g}_0^{\mathbb{C}} = \mathfrak{g}_0 \otimes \mathbb{C}$, where $\mathfrak{g}_0$ is the Lie algebra of $\mathcal{G}_0$,
- for the eigenvalue i, $\mathfrak{g}_1^{\mathbb{C}} = (0, \mathbb{C}\bar{\epsilon} \oplus \mathbb{C}L_i\epsilon)$ (notice that $\mathbb{C}\bar{\epsilon} \oplus \mathbb{C}L_i\epsilon$ is the i-eigenspace of $-L_j$),
- for the eigenvalue -1, $\mathfrak{g}_2^{\mathbb{C}} = \mathfrak{g}_2 \otimes \mathbb{C}$, where $\mathfrak{g}_2$ is the Lie algebra of $\mathcal{G}_2$.

We also have the following characterization of the $\pm i$-eigenspaces.

Lemma 1. *The group $\mathbb{R}_+^* \times \mathcal{G}_0$ acts freely and transitively on the $\pm i$-eigenspaces of L_j (minus the origin); in particular the i-eigenspace of L_j, $\mathbb{C}\epsilon \oplus \mathbb{C}L_i\bar{\epsilon}$, coincides with the orbit of ϵ and the $-i$-eigenspace of L_j, $\mathbb{C}\bar{\epsilon} \oplus \mathbb{C}L_i\epsilon$, coincides with the orbit of $\bar{\epsilon}$.*

Proof. Since $\mathcal{G}_0$ commutes with L_j, it preserves its eigenspaces. We now prove the freeness and transitivity of the action. Let $\xi = a\epsilon + bL_i\bar{\epsilon} = \frac{1}{2}(a, b, -ia, ib)$ be an eigenvector associated to the eigenvalue i (for the other eigenspace use conjugation). If $H \in \mathbb{R}_+^* \times \mathcal{G}_0$ maps ϵ to ξ, then we infer necessary conditions: $H\epsilon = \frac{1}{2}(H\epsilon_1 - iH\epsilon_3) = \xi$; thus $H\epsilon_1 = \frac{1}{2}\mathrm{Re}[\xi]$ and $H\epsilon_3 = -\frac{1}{2}\mathrm{Im}[\xi]$. Since we want H to commute with L_i, $H\epsilon_2 = HL_i\epsilon_1 = \frac{1}{2}L_i\mathrm{Re}[\xi] = \frac{1}{2}\mathrm{Re}[L_i\xi]$, and $H\epsilon_4 = -\frac{1}{2}\mathrm{Im}[L_i\xi]$. So H is uniquely determined. Check easily that $(H\epsilon_1, H\epsilon_2, H\epsilon_3, H\epsilon_4)$ is a conformal basis of $\mathbb{R}^4$, and write $H = rK$ for some isometry K and some $r \in \mathbb{R}_+^*$. By construction K commutes with L_i, so K belongs in $\vec{\mathcal{G}}$. It cannot have any nontrivial component in $\mathcal{G}_2$ otherwise ξ would not be a eigenvector of L_j (whose eigenspaces are not stable under L_i). Thus K belongs to $\mathcal{G}_0$. Therefore there exists a unique $H \in \mathbb{R}_+^* \times \mathcal{G}_0$ sending ϵ to ξ.

Notice that the action of $\mathbb{R}_+^* \times \mathcal{G}_0$ coincides with the right action of $\mathbb{H}^\star$ on $\mathbb{C}\epsilon \oplus \mathbb{C}L_i\bar{\epsilon}$ as a subset of $\mathbb{H} \otimes \mathbb{C}$. □

Using the decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_{-1}^{\mathbb{C}} \oplus \mathfrak{g}_0^{\mathbb{C}} \oplus \mathfrak{g}_1^{\mathbb{C}} \oplus \mathfrak{g}_2^{\mathbb{C}}$, we define the projection mapping $[.]_k : \mathfrak{g}^{\mathbb{C}} \longrightarrow \mathfrak{g}_k^{\mathbb{C}}$. Then denoting $\alpha_k := [\alpha]_k$, we have

$$\alpha = \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2. \tag{5}$$

We now substitute (5) in (3). We use the relations $[\mathfrak{g}_k^{\mathbb{C}}, \mathfrak{g}_l^{\mathbb{C}}] \subset \mathfrak{g}_{(k+l) \bmod 4}^{\mathbb{C}}$ and $[\mathfrak{g}_{\pm 1}^{\mathbb{C}}, \mathfrak{g}_{\pm 1}^{\mathbb{C}}] = [\mathfrak{g}_0, \mathfrak{g}_2] = [\mathfrak{g}_2, \mathfrak{g}_2] = 0$. The projection of the resulting equation on each eigenspace gives us four relations

$$\left\{ \begin{array}{ll} d\alpha_{-1} + [\alpha_{-1} \wedge \alpha_0] + [\alpha_1 \wedge \alpha_2] & = 0, \\ d\alpha_0 + \frac{1}{2}[\alpha_0 \wedge \alpha_0] & = 0, \\ d\alpha_1 + [\alpha_0 \wedge \alpha_1] + [\alpha_{-1} \wedge \alpha_2] & = 0, \\ d\alpha_2 & = 0, \end{array} \right. \tag{6}$$

where $[\alpha_a \wedge \alpha_b] = \alpha_a \wedge \alpha_b + \alpha_b \wedge \alpha_a$. We further decompose each form α_k as $\alpha_k = \alpha_k' + \alpha_k''$, with $\alpha_k' = \alpha_k(\frac{\partial}{\partial z})dz$ and $\alpha_k'' = \alpha_k(\frac{\partial}{\partial \bar{z}})d\bar{z}$. We remark that, because α derives from a LCLI, $\alpha_{-1} + \alpha_1 = (0, e^{f(z)}K(z)(\epsilon dz + \bar{\epsilon} d\bar{z}))$ and hence, Lemma 1 implies that

$$\alpha_{-1} = (0, e^{f(z)}K(z)\epsilon dz) = \alpha_{-1}' \text{ and } \alpha_{-1}'' = 0, \tag{7}$$

and similarly,

$$\alpha_1 = \alpha_1'' \text{ and } \alpha_1' = 0. \tag{8}$$

Thus,

$$\alpha = \alpha_2' + \alpha_{-1}' + \alpha_0 + \alpha_1'' + \alpha_2''.$$

Now we exploit (7) and (8) in (6) and we obtain

$$\left\{ \begin{array}{lll} d\alpha_{-1}' + [\alpha_{-1}' \wedge \alpha_0] + [\alpha_1'' \wedge \alpha_2'] & = & 0, \\ d\alpha_0 + \frac{1}{2}[\alpha_0 \wedge \alpha_0] & = & 0, \\ d\alpha_1'' + [\alpha_0 \wedge \alpha_1''] + [\alpha_{-1}' \wedge \alpha_2''] & = & 0, \\ d\alpha_2 & = & 0. \end{array} \right.$$

A convenient way to rewrite this system is to introduce a complex parameter $\lambda \in \mathbb{C}^\star$ and to let

$$\alpha_\lambda := \lambda^{-2}\alpha_2' + \lambda^{-1}\alpha_{-1}' + \alpha_0 + \lambda\alpha_1'' + \lambda^2\alpha_2'',$$

and then

$$\begin{array}{rcl} d\alpha_\lambda + \alpha_\lambda \wedge \alpha_\lambda & = & \lambda^{-2} d\alpha_2' \\ & + & \lambda^{-1}(d\alpha_{-1}' + [\alpha_{-1}' \wedge \alpha_0] + [\alpha_1'' \wedge \alpha_2']) \\ & + & (d\alpha_0 + \frac{1}{2}[\alpha_0 \wedge \alpha_0]) \\ & + & \lambda(d\alpha_1'' + [\alpha_0 \wedge \alpha_1''] + [\alpha_{-1}' \wedge \alpha_2'']) \\ & + & \lambda^2 d\alpha_2'' \\ & = & \lambda^{-2} d\alpha_2' + \lambda^2 d\alpha_2''. \end{array}$$

We now are in position to prove the

Theorem 2. *Assume that Ω is a simply connected domain of $\mathbb{C} \simeq \mathbb{R}^2$. Let α be in $\mathcal{C}^\infty(\Omega, T^\star\mathbb{R}^2 \otimes \mathfrak{g})$. Then*

- *α is the Maurer-Cartan form of a LCLI if and only if $d\alpha + \alpha \wedge \alpha = 0$, $\alpha_{-1}'' = \alpha_1' = 0$ and $\alpha_{-1}' \neq 0$, $\alpha_1'' \neq 0$,*
- *furthermore, α corresponds to some Hamiltonian stationary immersion if and only if the* extended Maurer-Cartan form $\alpha_\lambda = \lambda^{-2}\alpha_2' + \lambda^{-1}\alpha_{-1}' + \alpha_0 + \lambda\alpha_1'' + \lambda^2\alpha_2''$ *satisfies*

$$d\alpha_\lambda + \alpha_\lambda \wedge \alpha_\lambda = 0, \ \forall \lambda \in \mathbb{C}^\star. \tag{9}$$

Proof. First, according to Definition 1, *b)*, α will be the Maurer-Cartan form of a LCLI if and only if $d\alpha + \alpha \wedge \alpha = 0$ and

$$(\alpha_{-1} + \alpha_1)\left(\frac{\partial}{\partial z}\right) = (0, e^{f(z)}K(z)\epsilon) \qquad \text{and}$$
$$(\alpha_{-1} + \alpha_1)\left(\frac{\partial}{\partial \bar{z}}\right) = (0, e^{f(z)}K(z)\bar{\epsilon}).$$

But, from Lemma 1, this is equivalent to

$$(\alpha_{-1} + \alpha_1)\left(\frac{\partial}{\partial z}\right) \in \mathfrak{g}_{-1}^{\mathbb{C}} \setminus \{0\} \text{ and } (\alpha_{-1} + \alpha_1)\left(\frac{\partial}{\partial \bar{z}}\right) \in \mathfrak{g}_1^{\mathbb{C}} \setminus \{0\},$$

or $\alpha_{-1}'' = \alpha_1' = 0$ and $\alpha_{-1}' \neq 0$, $\alpha_1'' \neq 0$.

Second, the previous computation shows that

$$d\alpha_\lambda + \alpha_\lambda \wedge \alpha_\lambda = \frac{1}{2}\frac{\partial^2 \beta}{\partial z \partial \bar{z}}(\lambda^{-2} - \lambda^2)(L_i, 0) d\bar{z} \wedge dz,$$

which vanishes if and only if the immersion is Hamiltonian stationary, according to Theorem 1. □

Notice that it suffices to check Relation (9) for $\lambda \in S^1 \subset \mathbb{C}^\star$, or even for one value of λ different from ± 1 to ensure the Hamiltonian stationary condition.

Corollary 1. *Assume that Ω is simply connected. Let α be in $\mathcal{C}^\infty(\Omega, T^\star\mathbb{R}^2 \otimes \mathfrak{g})$, a Maurer-Cartan form of a Hamiltonian stationary LCLI and $z_0 \in \Omega$. Then for any $\lambda \in S^1$, there exists a unique LCLI $U_\lambda \in \mathcal{C}^\infty(\Omega, \mathcal{G})$ such that*

$$dU_\lambda = U_\lambda \alpha_\lambda \text{ and } U_\lambda(z_0) = \mathbb{1}. \tag{10}$$

Thus there is a S^1-family of Hamiltonian stationary Lagrangian conformal immersions X_λ given by $U_\lambda = (F_\lambda, X_\lambda)$.

Proof. First the condition $\lambda \in S^1$ ensures that α_λ is $\mathfrak{g}$-valued (and not $\mathfrak{g}^{\mathbb{C}}$-valued). Then equation (9) is the necessary and sufficient condition for the existence of a unique solution to (10). □

Recovering U_λ from α_λ can be done in two steps (this is due to the semiproduct structure of $\mathcal{G}$), namely: the rotation term can be obtained by solving $(F_\lambda^{-1} dF_\lambda, 0) = [\alpha_\lambda]_2 + \alpha_0$; recall however that F_λ is defined only up to gauge transformation, which leaves $[\alpha_\lambda]_2$ invariant but changes all the other components. The immersion X_λ is obtained by solving $(0, F_\lambda^{-1} dX_\lambda) = \lambda^{-1}\alpha'_{-1} + \lambda^1 \alpha''_1$.

The family of solutions $(X_\lambda)_{\lambda \in S^1}$ is quite similar to the conjugate family of minimal surfaces, also obtained by varying a parameter in S^1. As in the classical minimal case, this family is in general not well-defined if the parameter domain is not simply connected; there may be period problem (in our setting: non trivial monodromy). Notice a big difference though: the group involved in the classical minimal surface theory is simply $\mathbb{R}^3$, which unlike $\mathcal{G}$ is commutative.

3. An associated linear problem.

In this section we show how a particular choice of gauge (or equivalently a particular moving frame) reduces the problem to solving successively three surprisingly simple linear PDEs, the two first involving the conformal structure, the third being simply the integration procedure from the connection

1-form to the immersion. We are then in the position to describe explicitly all weakly conformal Hamiltonian stationary Lagrangian tori.

3.1. Using the spinor lift.

In this section we still assume that the immersion X is defined on some simply connected domain Ω. Hence there is no problem in considering a spinor lift $U = (e^{\beta L_i/2}, X)$, whose Maurer-Cartan form is

$$\alpha = U^{-1}dU = \left(\frac{1}{2}\frac{\partial \beta}{\partial z}(L_i, 0) + (0, u)\right) dz + \left((0, \bar{u}) + \frac{1}{2}\frac{\partial \beta}{\partial \bar{z}}(L_i, 0)\right) d\bar{z}$$

where $u = a\epsilon + bL_i\bar{\epsilon}$ for some smooth complex valued functions a, b (recall Lemma 1). Equation (9) yields the condition

$$\Delta\beta = 0. \tag{11}$$

Using the fact that in our connection form $\alpha_0 = 0$, the only other condition in (9) is another linear PDE:

$$\left(0, \frac{\partial u}{\partial \bar{z}}\right) + \left[(0, \bar{u}), \frac{1}{2}\frac{\partial \beta}{\partial z}(L_i, 0)\right] = 0;$$

written more simply:

$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\frac{\partial \beta}{\partial z} L_i \bar{u}. \tag{12}$$

Finally, once β and u are found, X is obtained by integrating

$$dX = e^{\beta L_i/2}(u dz + \bar{u} d\bar{z}). \tag{13}$$

Notice that the set of solutions of (12) is a real vector space; thus the set of solutions for X is the orbit under $\mathcal{G}$ of a vector space. Beware also that solving (12) does not guarantee that u (and hence the induced metric) will never vanish; so we may actually obtain weakly conformal solutions. Therefore the conclusion:

Theorem 3. *The Hamiltonian stationary conformal Lagrangian immersions from a simply connected domain Ω into E^4 are given by solving successively three linear partial differential equations* (11), (12) *and* (13). *Then for given conformal structure and Lagrangian angle map β, the set of weakly conformal solutions is the $\mathcal{G}$-orbit of a vector space.*

3.2. Hamiltonian stationary Lagrangian tori.

We specialize to the case of Hamiltonian stationary Lagrangian tori. Let us fix some notations: Γ is a lattice in $\mathbb{C}$, with dual lattice $\Gamma^* = \{\gamma \in \mathbb{C}, \langle \gamma, \Gamma \rangle \subset \mathbb{Z}\}$ (here $\langle .,. \rangle$ is the usual dot product in $\mathbb{C} \simeq \mathbb{R}^2$); then any torus $\mathbb{T}$ is conformally equivalent to some $\mathbb{C}/\Gamma$. We want to classify Hamiltonian stationary conformal Lagrangian maps X from $\mathbb{T}$ to $\mathbb{R}^4$, or equivalently their Γ-periodic lift to the universal cover, that we will also write abusively $X : \mathbb{C} \to \mathbb{R}^4$. This amounts to finding solutions of Equations (11), (12) and (13) which give Γ-periodic maps on $\mathbb{C}$.

First one should notice that the rotation part in the spinor lift $U = (e^{\beta L_i/2}, X)$, is not Γ-periodic but only 2Γ-periodic a priori[4]; so is the translation part of the corresponding Maurer-Cartan form and in particular the complex vector $u = e^{-\beta L_i/2}\frac{\partial X}{\partial z}$. So we will distinguish the "truly periodic" solutions, for which $e^{\beta L_i/2}$ is Γ-periodic, from the "anti-periodic" ones; still, given an anti-periodic solution, its fourfold[5] cover is truly periodic. This detail will become important when we restrict the solutions obtained on the universal cover $\mathbb{C}$ to the torus $\mathbb{T}$.

The solutions of equation (11) are particularly simple: since $e^{i\beta}$ is periodic, i.e., $\beta(z + \Gamma) \equiv \beta(z) \mod 2\pi$, we have

$$\beta(z) = 2\pi \langle \beta_0, z - z_0 \rangle \tag{14}$$

for some $z_0 \in \Omega$ and $\beta_0 \in \Gamma^*$. Up to a translation in z we may suppose that $z_0 = 0$. We see that $e^{\beta L_i/2}$ is Γ-periodic if and only if $\beta_0/2$ belongs to Γ^*; otherwise $e^{\beta L_i/2}$ is just anti-periodic (we will give examples of both cases). Now, setting $u = a\epsilon + bL_i\bar{\epsilon}$, equation (12) is equivalent to

$$\frac{\partial a}{\partial \bar{z}} = -\frac{\pi \bar{\beta}_0}{2}\bar{b} \quad \text{and} \quad \frac{\partial b}{\partial \bar{z}} = \frac{\pi \bar{\beta}_0}{2}\bar{a}. \tag{15}$$

A necessary condition for (a, b) to be a solution of (15), is that a and b solve the eigenvalue problem:

$$\Delta\psi + \pi^2 |\beta_0|^2 \psi = 0. \tag{16}$$

Since a is 2Γ-periodic, it has the Fourier expansion $a = \sum_{\gamma \in \Gamma^*/2} \hat{a}_\gamma e^{2i\pi\langle \gamma, z \rangle}$, and a is a solution of (16) if and only if all coefficient $\hat{a}_\gamma$ vanish unless $|\gamma| = |\beta_0/2|$. Notice that except for $\pm\beta_0/2$, existence of such lattice points is far

[4]recall that the dual lattice to 2Γ is just $\frac{1}{2}\Gamma^*$.

[5]i.e., twofold in each lattice direction.

from obvious and depends strongly on the conformal structure together with the choice of the lattice point β_0. We remark now that if a is a solution of (16) and if b is given by the first equation of (15), then a and b are automatically solutions of the second equation in (15). We deduce the following (all sums being taken with $\gamma \in \frac{1}{2}\Gamma^*$)

$$a(z) = \sum_{|\gamma|=|\beta_0|/2} \hat{a}_\gamma e^{2i\pi\langle\gamma,z\rangle} ,$$

$$b(z) = \frac{2}{i\beta_0} \sum_{|\gamma|=|\beta_0|/2} \bar{\gamma}\overline{\hat{a}_{-\gamma}} e^{2i\pi\langle\gamma,z\rangle} = -\frac{2}{i\beta_0} \sum_{|\gamma|=|\beta_0|/2} \bar{\gamma}\overline{\hat{a}_\gamma} e^{-2i\pi\langle\gamma,z\rangle}.$$

We conclude that, for β given by (14), any solution to (12) has the form $u = \sum_{|\gamma|=|\beta_0|/2} u_\gamma$, where each $u_\gamma = \hat{a}_\gamma e^{2i\pi\langle\gamma,z\rangle}\epsilon + \frac{2i\bar{\gamma}}{\beta_0}\overline{\hat{a}_\gamma} e^{-2i\pi\langle\gamma,z\rangle} L_i\bar{\epsilon}$. In other words, the set of solutions to (12) is a finite real vector space with basis vectors

$$v_\gamma = e^{2i\pi\langle\gamma,z\rangle}\epsilon + \frac{2i\bar{\gamma}}{\beta_0} e^{-2i\pi\langle\gamma,z\rangle} L_i\bar{\epsilon} \text{ and } w_\gamma = ie^{2i\pi\langle\gamma,z\rangle}\epsilon + \frac{2\bar{\gamma}}{\beta_0} e^{-2i\pi\langle\gamma,z\rangle} L_i\bar{\epsilon},$$

for $\gamma \in \{\gamma \in \frac{1}{2}\Gamma^* / |\gamma| = |\beta_0|/2\}$.

The last step is finding X, by integrating (13). Again, assuming that $X(0) = 0$, the set of solutions is a vector space, with the basis vectors

$$A_\gamma(z) = \int_0^z e^{\pi\langle\beta_0,\xi\rangle L_i} (v_\gamma(\xi) d\xi + \overline{v_\gamma(\xi)} d\bar{\xi})$$

and

$$B_\gamma(z) = \int_0^z e^{\pi\langle\beta_0,\xi\rangle L_i} (w_\gamma(\xi) d\xi + \overline{w_\gamma(\xi)} d\bar{\xi}),$$

and any solution has the form

$$X = \sum_{|\gamma|=|\beta_0|/2} \mathrm{Re}(\hat{a}_\gamma) A_\gamma + \mathrm{Im}(\hat{a}_\gamma) B_\gamma.$$

In the computation of A_γ and B_γ, two cases occur: either $\gamma = \pm\frac{1}{2}\beta_0$, and then A_γ and B_γ cannot be periodic, but only pseudo-periodic (and both periods cannot compensate). If $\gamma \neq \pm\frac{1}{2}\beta_0$, then A_γ and B_γ have frequencies

$\gamma \pm \frac{1}{2}\beta_0$ and more precisely:

$$A_\gamma = \frac{4e^{\pi\langle\beta_0,z\rangle L_i}}{\pi}\mathrm{Re}\left[\frac{e^{-2i\pi\langle\gamma,z\rangle}}{\beta_0^2-4\gamma^2}\begin{pmatrix}-i\gamma\\ -\beta_0/2\\ \gamma\\ -i\beta_0/2\end{pmatrix}\right],$$

and

$$B_\gamma = \frac{4e^{\pi\langle\beta_0,z\rangle L_i}}{\pi}\mathrm{Im}\left[\frac{e^{-2i\pi\langle\gamma,z\rangle}}{\beta_0^2-4\gamma^2}\begin{pmatrix}-i\gamma\\ -\beta_0/2\\ \gamma\\ -i\beta_0/2\end{pmatrix}\right].$$

A necessary and sufficient condition for $X = \sum_{|\gamma|=|\beta_0|/2}\mathrm{Re}(\hat{a}_\gamma)A_\gamma + \mathrm{Im}(\hat{a}_\gamma)B_\gamma$ to be Γ-periodic is obviously $\gamma - \frac{1}{2}\beta_0 \in \Gamma^*$ (then $\gamma+\frac{1}{2}\beta_0$, $-\gamma+\frac{1}{2}\beta_0$, $-\gamma-\frac{1}{2}\beta_0$ automatically belong to Γ^*). So we define the set

$$\Gamma^*_{\beta_0} = \left\{\gamma \in \frac{\beta_0}{2} + \Gamma^* \text{ such that } |\gamma|^2 = \left|\frac{\beta_0}{2}\right|^2 \text{ and } \gamma^2 \neq \left(\frac{\beta_0}{2}\right)^2\right\}.$$

Remark 3. In the truly periodic case, $\frac{1}{2}\beta_0$ belongs to Γ^*, thus $\Gamma^*_{\beta_0}$ is just the intersection of the dual lattice with the circle through $\frac{1}{2}\beta_0$, minus $\pm\frac{1}{2}\beta_0$.

Remark 4. As noted above, multiplication of the solution by a constant matrix in $\mathcal{G}_2$ is equivalent to a translation in z-space; such a change of variable in turn amounts to multiplying each $\hat{a}_\gamma$ by a constant (depending on γ). Furthermore, the group action of $\mathbb{R}^*_+ \times \mathcal{G}_0$ on X descends to a free action on the couples $(\hat{a}_\gamma, \hat{a}_{-\gamma})$, so that all solutions are obtained once for each choice of the $\hat{a}_\gamma$, up to the obvious $\beta(0) = 0$ assumption.

We may now conclude by the following classification theorem:

Theorem 4. *The Hamiltonian stationary weakly conformal Lagrangian immersions from $\mathbb{C}/\Gamma$ into E^4 are characterized by their Lagrangian angle β in as much as $\beta_0 = \frac{1}{\pi}\partial\beta/\partial\bar{z}$ belongs to the dual lattice Γ^*. The set of solutions for a chosen β_0 is the orbit under $\mathcal{G}_2$ of the vector space generated by the A_γ, B_γ and translations in 4-space, as γ ranges over the (possibly empty) set $\Gamma^*_{\beta_0}$. Its dimension – if not empty – is* $2\mathrm{Card}(\Gamma^*_{\beta_0}) + 5$ *(counting the $\mathcal{G}_2$ action and the translations), or* $2\mathrm{Card}(\Gamma^*_{\beta_0}) - 3$ *if one identifies solutions in the same $\mathcal{G}$-orbit.*

Remark 5. A torus constructed with this method may well be a multiple cover of another torus (with a potentially different conformal type): while X is by hypothesis Γ-periodic, its lattice of periods Δ may strictly contain Γ. Indeed the relevant dual lattice Δ^* is obtained as the one generated by all $\gamma - \frac{1}{2}\beta_0$, $\gamma + \frac{1}{2}\beta_0$ for $\gamma \in \Gamma^*_{\beta_0}$ and $\hat{a}_\gamma \neq 0$, so $\Delta^* \subset \Gamma^*$ and equality is not always true. For instance, truly periodic examples are not always covers of antiperiodic ones, but that is the case for square tori:

Proposition 1. *Let $\mathbb{T} = \mathbb{C}/\Gamma$ be a square torus; then a truly periodic solution X is always a twofold cover of some simpler (i.e. less periodic) solution.*

Proof. For simplicity assume $\Gamma = \Gamma^*$ is just $\mathbb{Z} \oplus \mathbb{Z}$. By hypothesis β_0 belongs to $2\Gamma^*$. We denote Δ^*, dual to Δ, the lattice generated by the $\gamma - \frac{1}{2}\beta_0$ and $\gamma + \frac{1}{2}\beta_0$ for all relevant[6] $\gamma \in \Gamma^*_{\beta_0}$. We claim that Δ^* is a subgroup of $\Delta_0^* = \{(n,m) \in \Gamma; n + m \equiv 0 \bmod 2\} = (1-i)\mathbb{Z} \oplus (1+i)\mathbb{Z}$. Indeed let $\gamma = (p,q)$ and $\frac{1}{2}\beta_0 = (p_0, q_0)$ – both in Γ –, then $p^2 + q^2 = p_0^2 + q_0^2$ implies

$$p - p_0 \equiv (p - p_0)(p + p_0) \equiv (q - q_0)(q + q_0) \equiv q - q_0 \bmod 2$$

Now $\Delta^* \subset \Delta_0^*$ is equivalent to $\Delta_0 = \frac{1-i}{2}\mathbb{Z} \oplus \frac{1+i}{2}\mathbb{Z} \subset \Delta$, so X is Δ_0-periodic, and $\mathbb{C}/\Gamma$ is a double cover of $\mathbb{C}/\Delta_0$. Notice that Δ_0 is again square. □

Remark 6. Examples with arbitrarily many frequencies can be constructed by taking for instance the square lattice $\mathbb{Z} \oplus \mathbb{Z}$ and $\beta_0 = 2(p+iq)^n$ where p, q are integers. Then the set $\Gamma^*_{\beta_0}$ contains all the $\pm(p+iq)^{n-k}(p-iq)^k$ for k ranging from 1 to n.

3.3. Some toric examples.

A truly periodic example on a rhombic torus.
Set $\omega = e^{i\pi/3}$, and $\Gamma^* = \mathbb{Z} \oplus \omega\mathbb{Z}$. Taking $\beta_0 = 2 \in 2\Gamma^*$ and non zero coefficients $\hat{a}_\omega$, $\hat{a}_{\omega^2}$, we construct a Γ-periodic Hamiltonian stationary (weakly conformal) Lagrangian immersion X. Let Δ be the lattice of periods of X, and Δ^* its dual. Then Δ^* contains $1 = (\omega - \frac{1}{2}\beta_0) - (\omega^2 - \frac{1}{2}\beta_0)$ and $\omega = (\omega + \frac{1}{2}\beta_0) - 1$. So $\Delta^* = \Gamma^*$ and X does not cover another torus. Taking

[6] that is $\hat{a}_\gamma \neq 0$.

for instance $\hat{a}_\omega = \hat{a}_{\omega^2} = 1$, we have $X(z) = A_\omega(z) + A_{\omega^2}(z)$:

$$
\begin{aligned}
X(z) = \frac{e^{2\pi x L_i}}{\pi\sqrt{3}} & \left(\begin{pmatrix} \cos(\pi(x + y\sqrt{3})) \\ -\sin(\pi(x + y\sqrt{3} + \frac{1}{3})) \\ \sin(\pi(x + y\sqrt{3})) \\ \cos(\pi(x + y\sqrt{3} + \frac{1}{3})) \end{pmatrix} \right. \\
& \left. + \begin{pmatrix} \cos(\pi(-x + y\sqrt{3})) \\ \sin(\pi(-x + y\sqrt{3} - \frac{1}{3})) \\ \sin(\pi(-x + y\sqrt{3})) \\ -\cos(\pi(-x + y\sqrt{3} - \frac{1}{3})) \end{pmatrix} \right) \\
= \frac{2}{\pi\sqrt{3}} & \begin{pmatrix} \cos(\pi y\sqrt{3}) \left(\cos(2\pi x)\cos(\pi x) + \sin(2\pi x)\sin(\pi x + \frac{\pi}{3})\right) \\ \cos(\pi y\sqrt{3}) \left(\sin(2\pi x)\cos(\pi x) - \cos(2\pi x)\sin(\pi x + \frac{\pi}{3})\right) \\ \sin(\pi y\sqrt{3}) \left(\cos(2\pi x)\cos(\pi x) + \sin(2\pi x)\sin(\pi x + \frac{\pi}{3})\right) \\ \sin(\pi y\sqrt{3}) \left(\sin(2\pi x)\cos(\pi x) - \cos(2\pi x)\sin(\pi x + \frac{\pi}{3})\right) \end{pmatrix}.
\end{aligned}
$$

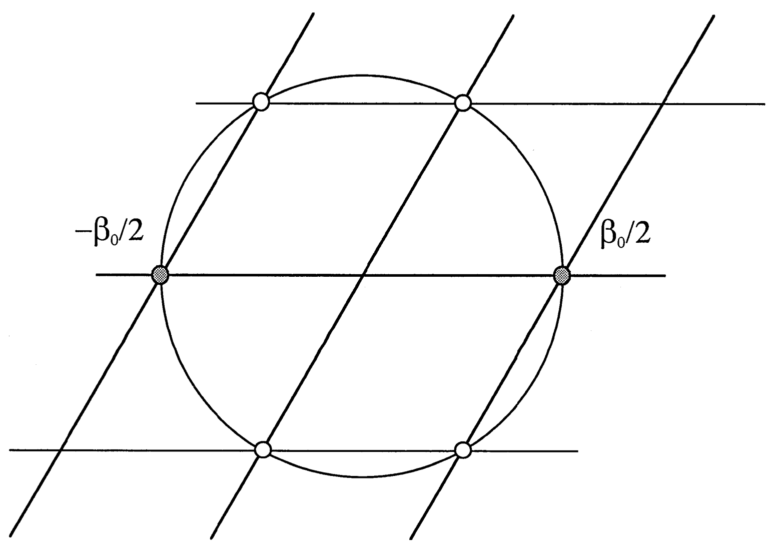

Figure 1: A rhombic torus.

The standard torus and its rectangular counterparts.
The simplest – and until recently (cf [CU]) – only known tori were the product of circles $S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C} = \mathbb{R}^4$; more precisely define on the

rectangular torus $\mathbb{T} = \mathbb{C}/\omega_1\mathbb{Z} \oplus i\omega_2\mathbb{Z}$ $(\omega_1, \omega_2 \in \mathbb{R})$

$$X(x+iy) = \begin{pmatrix} \omega_1 \sin(2\pi x/\omega_1) \\ -\omega_1 \cos(2\pi x/\omega_1) \\ \omega_2 \sin(2\pi y/\omega_2) \\ -\omega_2 \cos(2\pi y/\omega_2) \end{pmatrix}$$

When $\omega_1 = \omega_2 = 1$, $X(\mathbb{T})$ is the standard square torus.

The Lagrangian angle of X is $\beta(z) = 2\pi(x/\omega_1 + y/\omega_2)$ so $\beta_0 = \omega_1^{-1} + i\omega_2^{-1}$ which belongs to $\Gamma^* = \omega_1^{-1}\mathbb{Z} \oplus i\omega_2^{-1}\mathbb{Z}$ but not to $\frac{1}{2}\Gamma^*$, so we are in the antiperiodic case. Then $\Gamma^*_{\beta_0} = \{\frac{1}{2}\bar{\beta}_0, -\frac{1}{2}\bar{\beta}_0\}$ [7]. That torus corresponds exactly to $\hat{a}_{\bar{\beta}_0/2} = \hat{a}_{-\bar{\beta}_0/2} = \pi$). As noted in remark 4, other choices of those coefficients amount to multiplying X by an element in $\mathbb{R}^*_+ \times \mathcal{G}_0$.

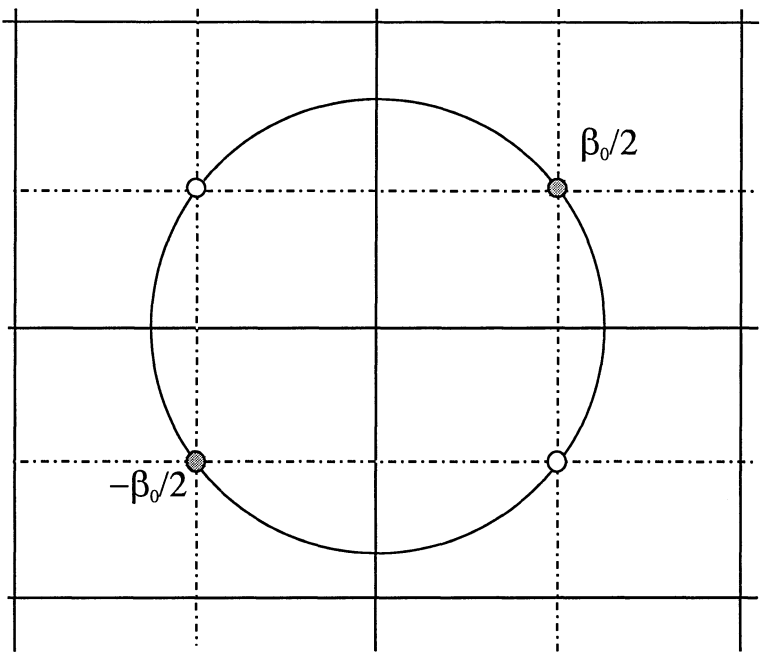

Figure 2: The standard torus.

The examples of I. Castro and F. Urbano.
In a recent article a new 3-parameter family of Hamiltonian stationary La-

[7]except for some particular lattices: if $1 + \omega_k^2/\omega_\ell^2 = m^2$ for some $m \in \mathbb{Z}$ then $\pm m\omega_k^{-1}$ also belongs to $\Gamma^*_{\beta_0}$.

grangian surfaces was described, some of them giving rise to tori (when the periodicity conditions were satisfied). The construction was based on geometric properties of parallel lifts on the 3-sphere, instead of the more analytic methods used here. These examples also satisfied the rigidity property of being invariant under a one-parameter group of isometries, which characterizes them among tori without parallel mean curvature vector (thus excluding the above rectangular tori). Though we will not describe explicitly the examples (we refer the interested Reader to [CU]), we indicate how they fit in our classification and how their properties are linked with a special lattice structure. From now on we will use their notations.

Let $X^{\alpha}_{\theta,\beta}$ be an immersion with real parameters $\alpha, \beta, \theta \in [0, \pi/2) \times (0, \pi/2) \times (-\pi/2, \pi/2)$ satisfying $\theta, |\alpha| < \beta$. The double periodicity condition amounts to $\frac{\sin\alpha}{\sin\beta}$ and $\frac{\cos\alpha}{\cos\beta}$ being rational[8], so we write $\frac{\sin\alpha}{\sin\beta} = \frac{r}{s}$ and $\frac{\cos\alpha}{\cos\beta} = \frac{p}{q}$. The lattice of periods Γ is $\frac{q\pi}{\cos\beta}\mathbb{Z} \oplus i\frac{s\pi}{\sin\beta}\mathbb{Z}$, and the dual lattice is

$$\Gamma^* = \frac{\cos\beta}{q\pi}\mathbb{Z} \oplus i\frac{\sin\beta}{s\pi}\mathbb{Z} = \frac{\cos\alpha}{p\pi}\mathbb{Z} \oplus i\frac{\sin\alpha}{r\pi}\mathbb{Z} .$$

Using the expression for the mean curvature vector[9] is (in conformal coordinates)

$$H = \frac{e^{-2f}}{2}\left(\frac{\partial\phi}{\partial x}L_i\frac{\partial X^{\alpha}_{\theta,\beta}}{\partial x} + \frac{\partial\phi}{\partial y}L_i\frac{\partial X^{\alpha}_{\theta,\beta}}{\partial y}\right),$$

where ϕ denotes here the Lagrangian angle, together with $\phi(z) = 2\pi\langle\phi_0, z\rangle +$ *constant*[10], we deduce that $\phi_0 = \frac{1}{\pi}\frac{\partial\phi}{\partial\bar{z}} = \frac{e^{i\alpha}}{\pi}$ has lattice coordinates (p, r) (in Γ^*). The periodicity condition above translates as the geometric property that the circle of radius $|\phi_0| = 1/\pi$ possesses 8 lattice points (instead of the generic 4), namely: $\pm\frac{e^{i\alpha}}{\pi}, \pm\frac{e^{-i\alpha}}{\pi}, \pm\frac{e^{i\beta}}{\pi}, \pm\frac{e^{-i\beta}}{\pi}$. It may be that $\alpha = 0$, but the property still remains that there are 4 extra points more than usual. These are exactly the points that come into play. Denoting $\gamma = \frac{e^{i\beta}}{2\pi} \in \frac{1}{2}\Gamma^*$,

$$\Gamma^*_{\phi_0} = \left\{\gamma, -\gamma, \bar{\gamma}, -\bar{\gamma}, \frac{\bar{\phi}_0}{2}, -\frac{\bar{\phi}_0}{2}\right\}$$

where the two last points are removed if ϕ_0 is real (i.e. $\alpha = 0$). It also comes naturally that the limit case $\alpha = \beta$ corresponds to the previous (and simpler) rectangular tori; if furthermore $\beta = \pi/4$ the lattice structure is exactly that of the (square) torus.

[8] take for instance $\beta \in (\pi/4, \pi/2)$ such that $\tan\beta$ is rational and $\alpha = \pi/2 - \beta$.

[9] i.e., the half trace of the second fundamental form.

[10] a careful computation shows that the constant is π.

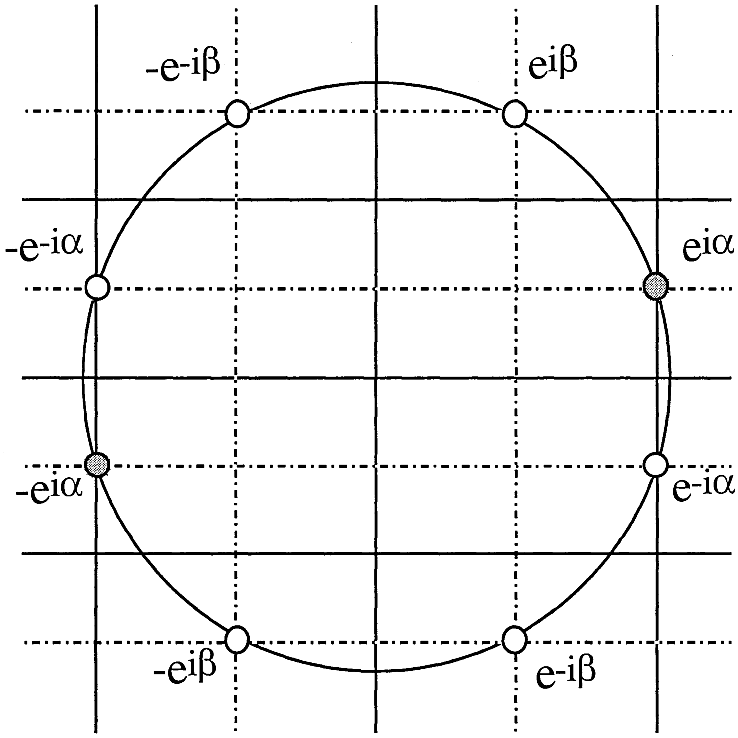

Figure 3: A Castro and Urbano torus.

The isometry described by I. Castro and F. Urbano is

$$X^{\alpha}_{\theta,\beta}(z+it)=e^{t(\sin\alpha L_i-\sin\beta R_i)}X(z).$$

This property implies that the only dual lattice elements in the Fourier expansion of $u=e^{\phi L_i/2}\frac{\partial X^{\alpha}_{\theta,\beta}}{\partial z}$ are precisely $\gamma,-\gamma,\bar{\gamma}$ and $-\bar{\gamma}$; moreover opposite elements vanish simultaneously and we have the conditions:

$$\hat{a}_{-\gamma}=-e^{-i(\beta+\alpha)}\overline{\hat{a}_{\gamma}}\quad,\quad\hat{a}_{-\bar{\gamma}}=e^{i(\beta-\alpha)}\overline{\hat{a}_{\bar{\gamma}}}$$

Using this the coefficients can be computed in terms of the functions defined in [CU].

4. Introducing loop groups.

4.1. Twisted loop groups.

We introduce loop groups, sets of maps $\lambda \mapsto G_\lambda$ from the circle $S^1 = \{\lambda \in \mathbb{C}/|\lambda| = 1\}$ to some Lie groups (here various subgroups of $\mathcal{G}^{\mathbb{C}}$), with a multiplication law given as follows: the product of two elements $\lambda \mapsto G_\lambda$ and $\lambda \mapsto G'_\lambda$ is just $\lambda \mapsto G_\lambda G'_\lambda$. We denote

$$\Lambda\mathcal{G} := \{[\lambda \mapsto G_\lambda]; S^1 \to \mathcal{G}\} \text{ and } \Lambda\mathcal{G}^{\mathbb{C}} := \{[\lambda \mapsto G_\lambda]; S^1 \to \mathcal{G}^{\mathbb{C}}\}.$$

We endow these groups with the H^s topology for some $s > 1/2$: if $G_\lambda = \sum_{k\in\mathbb{Z}} \hat{G}_k \lambda^k$ is the Fourier expansion of G_λ, its H^s norm is

$$||G_\lambda||_s = \left(\sum_{k\in\mathbb{Z}} |\hat{G}_k|^2(1+k^2)^{s/2}\right)^{1/2}.$$

Other topologies can be used (for instance the $\mathcal{C}^\infty$ topology), for more details, see [PS]. We define the twisted loop groups

$$\begin{aligned}
\Lambda\mathcal{G}_\tau &= \{[\lambda \mapsto G_\lambda] \in \Lambda\mathcal{G}/G_{i\lambda} = \tau(G_\lambda)\} \qquad \text{and}\\
\Lambda\mathcal{G}^{\mathbb{C}}_\tau &= \{[\lambda \mapsto G_\lambda] \in \Lambda\mathcal{G}^{\mathbb{C}}/G_{i\lambda} = \tau(G_\lambda)\},
\end{aligned}$$

twisted meaning equivariant with respect to τ. Also

$$\begin{aligned}
\Lambda^-_\star\mathcal{G}^{\mathbb{C}}_\tau &= \{[\lambda \mapsto G_\lambda] \in \Lambda\mathcal{G}^{\mathbb{C}}_\tau/G_\lambda \text{ extends holomorphically}\\
&\qquad \text{to the complement of the unit disk and } G_\infty = \mathbb{1}\}\\
\Lambda^+\mathcal{G}^{\mathbb{C}}_\tau &= \{[\lambda \mapsto G_\lambda] \in \Lambda\mathcal{G}^{\mathbb{C}}_\tau/G_\lambda \text{ extends holomorphically}\\
&\qquad \text{to the unit disk}\}\\
\Lambda^+_{\mathcal{B}}\mathcal{G}^{\mathbb{C}}_\tau &= \{[\lambda \mapsto G_\lambda] \in \Lambda\mathcal{G}^{\mathbb{C}}_\tau/G_\lambda \text{ extends holomorphically}\\
&\qquad \text{to the unit disk and } G_0 \in (\mathcal{B}, 0)\}
\end{aligned}$$

where $\mathcal{B}$ is some subgroup of $\mathcal{G}^{\mathbb{C}}_0$. In an analogous way define the corresponding Lie algebras $\Lambda\mathfrak{g}_\tau$, $\Lambda\mathfrak{g}^{\mathbb{C}}_\tau$, $\Lambda^-_\star\mathfrak{g}^{\mathbb{C}}_\tau$, $\Lambda^+\mathfrak{g}^{\mathbb{C}}_\tau$ and $\Lambda^+_{\mathfrak{b}}\mathfrak{g}^{\mathbb{C}}_\tau$ where $\mathfrak{b}$ is the Lie algebra of $\mathcal{B}$.

$$\begin{aligned}
\Lambda\mathfrak{g}^{\mathbb{C}}_\tau &= \{[\lambda \mapsto \gamma_\lambda]; S^1 \to \mathfrak{g}^{\mathbb{C}}/\gamma_{i\lambda} = \tau(\gamma_\lambda)\}\\
\Lambda\mathfrak{g}_\tau &= \{[\lambda \mapsto \gamma_\lambda] \in \Lambda\mathfrak{g}^{\mathbb{C}}_\tau/\gamma_\lambda \in \mathfrak{g}, \forall\lambda \in S^1\}\\
\Lambda^-_\star\mathfrak{g}^{\mathbb{C}}_\tau &= \{[\lambda \mapsto \gamma_\lambda] \in \Lambda\mathfrak{g}^{\mathbb{C}}_\tau/\gamma_\lambda \text{ extends holomorphically}\\
&\qquad \text{to the complement of the unit disk and } \gamma_\infty = 0\}\\
\Lambda^+\mathfrak{g}^{\mathbb{C}}_\tau &= \{[\lambda \mapsto \gamma_\lambda] \in \Lambda^{\mathbb{C}}_{\mathfrak{g}}\tau/\gamma_\lambda \text{ extends holomorphically}\\
&\qquad \text{to the unit disk}\}\\
\Lambda^+_{\mathfrak{b}}\mathfrak{g}^{\mathbb{C}}_\tau &= \{[\lambda \mapsto \gamma_\lambda] \in \Lambda\mathfrak{g}^{\mathbb{C}}_\tau/\gamma_\lambda \text{ extends holomorphically}\\
&\qquad \text{to the unit disk and } \gamma_0 \in (\mathfrak{b}, 0)\}
\end{aligned}$$

An analysis of the relation $\gamma_{i\lambda} = \tau(\gamma_\lambda)$, for any $\gamma_\lambda \in \Lambda\mathfrak{g}_\tau^{\mathbb{C}}$, shows that, writing $\gamma_\lambda = \sum_{k\in\mathbb{Z}} \hat{\gamma}_k \lambda^k$, this twisting condition is equivalent to $\hat{\gamma}_k \in \mathfrak{g}^{\mathbb{C}}_{k \bmod 4}$.

We remark in particular that $\Lambda\mathfrak{g}_\tau^{\mathbb{C}} = \Lambda_\star^- \mathfrak{g}_\tau^{\mathbb{C}} \oplus \Lambda^+ \mathfrak{g}_\tau^{\mathbb{C}}$, thus defining a projection $[.]_{\Lambda_\star^- \mathfrak{g}_\tau^{\mathbb{C}}} : \Lambda\mathfrak{g}_\tau^{\mathbb{C}} \to \Lambda_\star^- \mathfrak{g}_\tau^{\mathbb{C}}$. Using this language, we can state the

Corollary 2. *To each* $\mathfrak{g}$*-valued* 1*-form* α *giving rise to a Hamiltonian stationary conformal Lagrangian immersion corresponds a* $\Lambda\mathfrak{g}_\tau$*-valued* 1*-form* α_λ (*extended* 1*-form*) *satisfying relation* (9) *and such that*

$$\left[\alpha_\lambda\left(\frac{\partial}{\partial z}\right)\right]_{\Lambda_\star^- \mathfrak{g}_\tau^{\mathbb{C}}} = \lambda^{-2}\hat{\alpha}_{-2}\left(\frac{\partial}{\partial z}\right) + \lambda^{-1}\hat{\alpha}_{-1}\left(\frac{\partial}{\partial z}\right) \qquad \textit{and} \tag{17}$$

$$\left[\alpha_\lambda\left(\frac{\partial}{\partial \bar{z}}\right)\right]_{\Lambda_\star^- \mathfrak{g}_\tau^{\mathbb{C}}} = 0,$$

and

$$\hat{\alpha}_{-1}\left(\frac{\partial}{\partial z}\right) \neq 0, \tag{18}$$

and conversely. Moreover there exists a unique map $U_\lambda : \Omega \longrightarrow \Lambda\mathcal{G}_\tau$ *such that* $dU_\lambda = U_\lambda \alpha_\lambda$ *and* $U_\lambda(z_0) = \mathbb{1}$. U_λ *is called an* extended lift. *If* Ω *is not simply connected,* α_λ *is still well defined but* U_λ *will be multivalued in general.*

Proof. On one hand, Theorem 2 implies obviously that each $\mathfrak{g}$-valued 1-form α associated with a Hamiltonian stationary conformal Lagrangian immersion can be deformed into such a α_λ. On the other hand, any $\Lambda\mathfrak{g}_\tau$ valued 1-form α_λ satisfying (17) and (18) should satisfy

$$\alpha_\lambda = \lambda^{-2}\hat{\alpha}_{-2} + \lambda^{-1}\hat{\alpha}_{-1} + \hat{\alpha}_0 + \lambda^1\hat{\alpha}_1 + \lambda^2\hat{\alpha}_2,$$

with $\hat{\alpha}_1(\frac{\partial}{\partial z}) = \hat{\alpha}_2(\frac{\partial}{\partial z}) = 0$ and $\hat{\alpha}_1\left(\frac{\partial}{\partial \bar{z}}\right) \neq 0$, because of the reality condition $\overline{\hat{\alpha}_k} = \hat{\alpha}_{-k}$ contained in the definition of $\Lambda\mathfrak{g}_\tau$. If furthermore α_λ satisfies (9), then we conclude by using Theorem 2. The existence of U_λ is just a reformulation of Corollary 1. □

Remark 7.
a) If U is a LCLI and U_λ is an extended LCLI such that $U_1 = U$, then the gauge action of $\mathcal{C}_\star^\infty(\Omega, \mathcal{G}_0)$ on U extends in a natural way on U_λ. Precisely if $K \in \mathcal{C}_\star^\infty(\Omega, \mathcal{G}_0)$ and if we denote $(KU)_\lambda$ the extended LCLI constructed

from $U(K^{-1},0)$, then $(KU)_\lambda = U_\lambda(K^{-1},0)$. To prove it, since we know that $(KU)_\lambda(z_0) = U_\lambda(K^{-1},0)(z_0) = \mathbb{1}$, it suffices to check that both functions have the same Maurer-Cartan form, namely

$$\lambda^{-2}\alpha_2' + \lambda^{-1}(K,0)\alpha_{-1}' + (K,0)\alpha_0(K,0)^{-1} \\ - (dK.K^{-1},0) + \lambda(K,0)\alpha_1'' + \lambda^2(K,0)\alpha_2''.$$

b) The extended LCLI of the fundamental lift has the Maurer-Cartan form

$$\begin{aligned}\tilde{X}_\lambda^{-1}d\tilde{X}_\lambda &= \frac{\lambda^{-2}}{2}\frac{\partial\beta}{\partial z}(L_i,0)dz + \lambda^{-1}e^f(0,\epsilon)dz + (M_X^{-1}dM_X,0)\\ &\quad +\lambda e^f(0,\bar{\epsilon})d\bar{z} + \frac{\lambda^2}{2}\frac{\partial\beta}{\partial\bar{z}}(L_i,0)d\bar{z},\end{aligned}$$

which implies that $\tilde{X}_\lambda = (M_X e^{\frac{\beta_\lambda L_i}{2}}, X_\lambda)$, where $d\beta_\lambda = \lambda^{-2}\frac{\partial\beta}{\partial z}dz + \lambda^2\frac{\partial\beta}{\partial\bar{z}}d\bar{z}$. Denoting γ the harmonic conjugate function of β, i.e., such that $\frac{1}{2}(\beta + i\gamma)$ vanishes at z_0 and is holomorphic, $\beta_\lambda = \frac{1}{2}(\lambda^{-2}+\lambda^2)\beta + \frac{i}{2}(\lambda^{-2}-\lambda^2)\gamma$. The extended LCLI's of the spinors lifts have the Maurer-Cartan form

$$\begin{aligned}U_{\pm,\lambda}^{-1}dU_{\pm,\lambda} &= \frac{\lambda^{-2}}{2}\frac{\partial\beta}{\partial z}(L_i,0)dz \pm \lambda^{-1}\left(0, e^{-\frac{\beta L_i}{2}}\frac{\partial X}{\partial z}\right)dz\\ &\quad \pm\lambda\left(0, e^{-\frac{\beta L_i}{2}}\frac{\partial X}{\partial\bar{z}}\right)d\bar{z} + \frac{\lambda^2}{2}\frac{\partial\beta}{\partial\bar{z}}(L_i,0)d\bar{z},\end{aligned}$$

and hence $U_{\pm,\lambda} = (\pm e^{\frac{\beta_\lambda L_i}{2}}, X_\lambda)$.

4.2. Group decompositions.

The main tool for Weierstrass representations, as those proven in [DPW], are loop groups decompositions. They are infinite dimensional analogs of Iwasawa decompositions such as $SU(n)^{\mathbb{C}} = SL(n,\mathbb{C}) = SU(n).\mathcal{B}$, where $\mathcal{B}$ is a solvable (Borel) subgroup of $SL(n,\mathbb{C})$. For the convenience of the Reader, we first give here the proof of this splitting for the case $n=2$ (recall that in our language, $SU(2) \simeq \mathcal{G}_0$).

Proposition 2. *Let $\mathcal{B}_0$ be the subgroup of matrices in $\mathcal{G}_0^{\mathbb{C}}$ leaving $\mathbb{R}_+^*\epsilon$ invariant, then $\mathcal{G}_0^{\mathbb{C}} = \mathcal{G}_0.\mathcal{B}_0$. More precisely the map*

$$\begin{aligned}\mathcal{G}_0 \times \mathcal{B}_0 &\longrightarrow \mathcal{G}_0^{\mathbb{C}}\\ (K,B) &\longmapsto KB\end{aligned}$$

is a diffeomorphism.

Proof. We use essentially Lemma 1 and recall that $\mathbb{R}_+^\star \times \mathcal{G}_0$ acts freely and transitively on $\mathbb{C}\epsilon \oplus \mathbb{C}L_i\bar{\epsilon} \setminus \{0\}$ which is the pointed i eigenspace of L_j. Since $\mathcal{G}_0$ commutes with L_j, so does $\mathcal{G}_0^{\mathbb{C}}$, hence $G\epsilon$ belongs to $\mathbb{C}\epsilon \oplus \mathbb{C}L_i\bar{\epsilon}$ for any $G \in \mathcal{G}_0^{\mathbb{C}}$. By Lemma 1, there exist unique $K \in \mathcal{G}_0$ and $r \in \mathbb{R}_+^\star$ such that $G\epsilon = rK\epsilon$. Just set $B = K^{-1}G \in \mathcal{B}_0$. Notice that we might as well use $\bar{\mathcal{B}}_0$ to construct our Iwasawa decomposition. □

Before stating the main results of this section, we shall establish a preliminary one. We set

$$\Lambda\vec{\mathcal{G}}_\tau^{\mathbb{C}} = \{[\lambda \mapsto G_\lambda] \in \Lambda\vec{\mathcal{G}}^{\mathbb{C}} / \tau(G_\lambda) = G_{i\lambda}, \forall \lambda \in S^1\}.$$

Notice that, since $\forall G \in \vec{\mathcal{G}}^{\mathbb{C}}$, $\tau^2(G) = G$, any $G_\lambda \in \Lambda\vec{\mathcal{G}}_\tau^{\mathbb{C}}$ satisfies $G_\lambda = \tau^2(G_\lambda) = G_{-\lambda}$. Also we denote

$$\begin{aligned}\Lambda\mathcal{G}_{2,\tau}^{\mathbb{C}} &= \Lambda\mathcal{G}_2^{\mathbb{C}} \cap \Lambda\mathcal{G}_\tau^{\mathbb{C}} \\ &= \left\{[\lambda \mapsto K_\lambda] \in \Lambda\mathcal{G}_2^{\mathbb{C}} / K_\lambda = \sum_{k\in\mathbb{Z}} \hat{K}_{2k}\lambda^{2k}, \hat{K}_{4k} \in \mathbb{C}\mathbb{1}, \hat{K}_{4k+2} \in \mathbb{C}L_i\right\},\end{aligned}$$

and

$$\Lambda\mathcal{G}_{0,\tau}^{\mathbb{C}} = \Lambda\mathcal{G}_0^{\mathbb{C}} \cap \Lambda\mathcal{G}_\tau^{\mathbb{C}} = \{[\lambda \mapsto F_\lambda] \in \Lambda\mathcal{G}_0^{\mathbb{C}} / F_\lambda = \sum_{k\in\mathbb{Z}} \hat{F}_{4k}\lambda^{4k}\}. \tag{19}$$

Lemma 2. *For any $\lambda \mapsto G_\lambda \in \Lambda\vec{\mathcal{G}}_\tau^{\mathbb{C}}$, there exists $(K_\lambda, M_\lambda) \in \Lambda\mathcal{G}_2^{\mathbb{C}} \times \Lambda\mathcal{G}_0^{\mathbb{C}}$, unique up to sign, such that*

$$G_\lambda = K_\lambda M_\lambda.$$

Moreover

(i) *either $K_\lambda \in \Lambda\mathcal{G}_{2,\tau}^{\mathbb{C}}$ and $M_\lambda \in \Lambda\mathcal{G}_{0,\tau}^{\mathbb{C}}$,*

(ii) *or $K_\lambda = L_i\tilde{K}_\lambda$ and $M_\lambda = \left(\frac{1}{2}(\lambda^2+\lambda^{-2})\mathbb{1} + \frac{1}{2i}(\lambda^2-\lambda^{-2})R_i\right)\tilde{M}_\lambda$, with $(\tilde{K}_\lambda, \tilde{M}_\lambda) \in \Lambda\mathcal{G}_{2,\tau}^{\mathbb{C}} \times \Lambda\mathcal{G}_{0,\tau}^{\mathbb{C}}$.*

In other words, setting $\pi_\lambda := L_i\left(\frac{1}{2}(\lambda^2+\lambda^{-2})\mathbb{1} + \frac{1}{2i}(\lambda^2-\lambda^{-2})R_i\right) \in \Lambda\vec{\mathcal{G}}_\tau$,

$$\Lambda\vec{\mathcal{G}}_\tau^{\mathbb{C}} = \Lambda\mathcal{G}_{2,\tau}^{\mathbb{C}}.\Lambda\mathcal{G}_{0,\tau}^{\mathbb{C}} \sqcup \pi_\lambda.\Lambda\mathcal{G}_{2,\tau}^{\mathbb{C}}.\Lambda\mathcal{G}_{0,\tau}^{\mathbb{C}}.$$

Proof. Let $\lambda \mapsto G_\lambda \in \Lambda\vec{\mathcal{G}}^{\mathbb{C}}_\tau$ and consider a lift $g : \mathbb{R} \longrightarrow \vec{\mathcal{G}}^{\mathbb{C}}$ such that $g(\theta) = G_{e^{i\theta}}, \forall\theta \in \mathbb{R}$. For any $\theta \in \mathbb{R}$, there exists $k(\theta) \in \mathcal{G}_2$ and $m(\theta) \in \mathcal{G}_0$ such that $g(\theta) = k(\theta)m(\theta)$, and $k(\theta)$ and $m(\theta)$ are unique up to sign. Moreover we can choose $k(\theta)$ and $m(\theta)$ to be continuous functions of θ. Since $G_{-\lambda} = G_\lambda$, we have $k(\theta+\pi)m(\theta+\pi) = k(\theta)m(\theta)$, and therefore $k(\theta+\pi) = \pm k(\theta)$ and $m(\theta+\pi) = \pm m(\theta)$. Hence $k(\theta+2\pi) = k(\theta)$ and $m(\theta+2\pi) = m(\theta)$ and we may define $(K_\lambda, M_\lambda) \in \Lambda\mathcal{G}_2^{\mathbb{C}} \times \Lambda\mathcal{G}_0^{\mathbb{C}}$ by $k(\theta) = K_{e^{i\theta}}$ and $m(\theta) = M_{e^{i\theta}}$. This proves the first assertion of the Lemma.

Notice that $\tau(K_\lambda)\tau(M_\lambda) = \tau(G_\lambda) = G_{i\lambda} = K_{i\lambda}M_{i\lambda}$, which implies that $(K_{i\lambda})^{-1}\tau(K_\lambda) = M_{i\lambda}(\tau(M_\lambda))^{-1} \in \mathcal{G}_2^{\mathbb{C}} \cap \mathcal{G}_0^{\mathbb{C}} = \{\pm\mathbb{1}\}$. Hence

$$(20) \qquad \tau(K_\lambda) = sK_{i\lambda} \text{ and } \tau(M_\lambda) = sM_{i\lambda}, \text{ with } s = \pm 1.$$

Moreover because of the parity of G_λ, $K_{-\lambda}M_{-\lambda} = K_\lambda M_\lambda$, which leads to the alternatives

a) $K_{-\lambda} = K_\lambda$ and $M_{-\lambda} = M_\lambda$,

b) $K_{-\lambda} = -K_\lambda$ and $M_{-\lambda} = -M_\lambda$.

If b) occurs, K_λ has the Fourier decomposition $K_\lambda = \sum_{k\in\mathbb{Z}} \hat{K}_{2k+1}\lambda^{2k+1}$. Then equation (20) implies that $\tau(\hat{K}_{2k+1}) = si(-1)^k\hat{K}_{2k+1}$, which is possible only if all the $\hat{K}_{2k+1}$'s vanish, because the eigenvalues of the action of τ on $\mathbb{C}\mathbb{1} + \mathbb{C}L_i$ are 1 and -1. We exclude that since $K_\lambda \in \mathcal{G}_2^{\mathbb{C}}$. Hence only case a) may occur.

To conclude we inspect the consequence of (20). If $s = 1$, case (i) of the Lemma occurs. If $s = -1$, we define $\tilde{K}_\lambda$ in $\Lambda\mathcal{G}_2^{\mathbb{C}}$ and $\tilde{M}_\lambda$ in $\Lambda\mathcal{G}_0^{\mathbb{C}}$ by $K_\lambda = L_i\tilde{K}_\lambda$ and $M_\lambda = \left(\frac{1}{2}(\lambda^2+\lambda^{-2})\mathbb{1} + \frac{1}{2i}(\lambda^2-\lambda^{-2})R_i\right)\tilde{M}_\lambda$. Then we check that $\tau(\tilde{K}_\lambda) = \tilde{K}_{i\lambda}$ and $\tau(\tilde{M}_\lambda) = \tilde{M}_{i\lambda}$ which shows that we are in case (ii). $\square$

We recall results in [PS]: let $\mathfrak{G}$ be a compact Lie group and $\mathfrak{G}^{\mathbb{C}}$ its complexification, and assume that the Iwasawa decomposition $\mathfrak{G}^{\mathbb{C}} = \mathfrak{G}.\mathcal{B}_{\mathfrak{G}}$ holds, for some solvable subgroup $\mathcal{B}_{\mathfrak{G}}$ of $\mathfrak{G}^{\mathbb{C}}$. Define as before the loop groups $\Lambda\mathfrak{G}^{\mathbb{C}}$, $\Lambda\mathfrak{G}$, $\Lambda^+\mathfrak{G}^{\mathbb{C}}$, $\Lambda^+_{\mathcal{B}_{\mathfrak{G}}}\mathfrak{G}^{\mathbb{C}}$ and $\Lambda^-_\star\mathfrak{G}^{\mathbb{C}}$.

Theorem 5 (Pressley-Segal).

a) *The product mapping*

$$\begin{array}{ccc} \Lambda\mathfrak{G} \times \Lambda^+_{\mathcal{B}_{\mathfrak{G}}}\mathfrak{G}^{\mathbb{C}} & \longrightarrow & \Lambda\mathfrak{G}^{\mathbb{C}} \\ (\phi_\lambda, \beta_\lambda) & \longmapsto & \phi_\lambda.\beta_\lambda \end{array}$$

is a diffeomorphism.

b) *There exists an open subset* $\mathcal{C}_{\mathfrak{G}}$ *of* $\Lambda\mathfrak{G}^{\mathbb{C}}$, *called the* big cell, *such that the product mapping*

$$\begin{array}{ccc} \Lambda_\star^-\mathfrak{G}^{\mathbb{C}} \times \Lambda^+\mathfrak{G}^{\mathbb{C}} & \longrightarrow & \mathcal{C}_{\mathfrak{G}} \\ (\gamma_\lambda^-, \gamma_\lambda^+) & \longmapsto & \gamma_\lambda^-.\gamma_\lambda^+ \end{array}$$

is a diffeomorphism.

We now use these results for proving the following decomposition theorems, adapted to our situation.

Theorem 6. *We have the decomposition* $\Lambda\mathcal{G}_\tau^{\mathbb{C}} = \Lambda\mathcal{G}_\tau.\Lambda_{\mathcal{B}_0}^+\mathcal{G}_\tau^{\mathbb{C}}$, *i.e., the map*

$$\begin{array}{ccc} \Lambda\mathcal{G}_\tau \times \Lambda_{\mathcal{B}_0}^+\mathcal{G}_\tau^{\mathbb{C}} & \longrightarrow & \Lambda\mathcal{G}_\tau^{\mathbb{C}} \\ (F_\lambda, B_\lambda) & \longmapsto & F_\lambda.B_\lambda \end{array}$$

is a diffeomorphism.

Theorem 7. *There exists an open subset* $\mathcal{C}$ *of* $\Lambda\mathcal{G}_\tau^{\mathbb{C}}$, *called the* big cell, *such that* $\mathcal{C} = \Lambda_\star^-\mathcal{G}_\tau^{\mathbb{C}}.\Lambda^+\mathcal{G}_\tau^{\mathbb{C}}$, *i.e., the product mapping*

$$\begin{array}{ccc} \Lambda_\star^-\mathcal{G}_\tau^{\mathbb{C}} \times \Lambda^+\mathcal{G}_\tau^{\mathbb{C}} & \longrightarrow & \mathcal{C} \\ (G_\lambda^-, G_\lambda^+) & \longmapsto & G_\lambda^-.G_\lambda^+ \end{array}$$

is a diffeomorphism.

Proof of Theorem 6.
Step 1. We prove the decomposition $\Lambda\vec{\mathcal{G}}_\tau^{\mathbb{C}} = \Lambda\vec{\mathcal{G}}_\tau.\Lambda_{\mathcal{B}_0}^+\vec{\mathcal{G}}_\tau^{\mathbb{C}}$.

Let $G_\lambda \in \Lambda\vec{\mathcal{G}}_\tau^{\mathbb{C}}$. By Lemma 2, $\exists (K_\lambda, M_\lambda) \in \Lambda\mathcal{G}_{2,\tau}^{\mathbb{C}} \times \Lambda\mathcal{G}_{0,\tau}^{\mathbb{C}}$ such that either (i) $G_\lambda = K_\lambda.M_\lambda$, or (ii) $G_\lambda = \pi_\lambda.K_\lambda.M_\lambda$.

We use Theorem 5 a) for $\mathfrak{G} = SU(2) \simeq \mathcal{G}_0$. Let $\tilde{M}_\lambda \in \Lambda\mathcal{G}_0^{\mathbb{C}}$ such that $M_\lambda = \tilde{M}_{\lambda^4}$. Then there exists a unique $(\tilde{\phi}_\lambda, \tilde{\beta}_\lambda) \in \Lambda\mathcal{G}_0 \times \Lambda_{\mathcal{B}_0}^+\mathcal{G}_0^{\mathbb{C}}$ such that $\tilde{M}_\lambda = \tilde{\phi}_\lambda.\tilde{\beta}_\lambda$. Setting $\phi_\lambda = \tilde{\phi}_{\lambda^4} \in \Lambda\mathcal{G}_{0,\tau}$ (recall (19)) and $\beta_\lambda = \tilde{\beta}_{\lambda^4} \in \Lambda_{\mathcal{B}_0}^+\mathcal{G}_{0,\tau}^{\mathbb{C}}$, we obtain $M_\lambda = \phi_\lambda\beta_\lambda$.

Similarly, we apply Theorem 5 a) for $\mathfrak{G} = U(1) \simeq \mathcal{G}_2$: since $K_\lambda \in \Lambda\mathcal{G}_2^{\mathbb{C}}$ there exists a unique $(\psi_\lambda, \gamma_\lambda) \in \Lambda\mathcal{G}_2 \times \Lambda_{\mathcal{B}_2}^+\mathcal{G}_2^{\mathbb{C}}$ such that $K_\lambda = \psi_\lambda\gamma_\lambda$ (here we set $\mathcal{B}_2 = \{e^{itL_i}/t \in \mathbb{R}\}$.) Thus $\tau(\psi_\lambda)\tau(\gamma_\lambda) = \tau(K_\lambda) = K_{i\lambda} = \psi_{i\lambda}\gamma_{i\lambda}$, which implies that $\tau(\psi_\lambda)\psi_{i\lambda}^{-1} = \tau(\gamma_\lambda)\gamma_{i\lambda}^{-1} \in \Lambda\mathcal{G}_2 \cap \Lambda_{\mathcal{B}_2}^+\mathcal{G}_2^{\mathbb{C}} = \{\mathbb{1}\}$. (Here we used the fact that $\mathcal{B}_2$ is stable under the action of τ and therefore $\tau(\Lambda_{\mathcal{B}_2}^+\mathcal{G}_2^{\mathbb{C}}) \subset \Lambda_{\mathcal{B}_2}^+\mathcal{G}_2^{\mathbb{C}}$.) Hence $\tau(\psi_\lambda) = \psi_{i\lambda}$ and $\tau(\gamma_\lambda) = \gamma_{i\lambda}$, meaning that $\psi_\lambda \in \Lambda\mathcal{G}_{2,\tau}$ and $\gamma_\lambda \in \Lambda_{\mathcal{B}_2}^+\mathcal{G}_{2,\tau}^{\mathbb{C}}$. Lastly we remark that $\Lambda_{\mathcal{B}_2}^+\mathcal{G}_{2,\tau}^{\mathbb{C}} = \Lambda_\star^+\mathcal{G}_{2,\tau}^{\mathbb{C}}$ and thus $\gamma_0 = \mathbb{1}$.

Hence we conclude that

$$G_\lambda = F_\lambda B_\lambda,$$

where in case (i),

$$F_\lambda = \psi_\lambda \phi_\lambda \in \Lambda\mathcal{G}_{2,\tau}.\Lambda\mathcal{G}_{0,\tau} \subset \Lambda\vec{\mathcal{G}}_\tau, \qquad \text{and}$$
$$B_\lambda = \gamma_\lambda \beta_\lambda \in \Lambda^+_\star \mathcal{G}^{\mathbb{C}}_{2,\tau}.\Lambda^+_{\mathcal{B}_0}\mathcal{G}_{0,\tau} \subset \Lambda^+_{\mathcal{B}_0}\vec{\mathcal{G}}_\tau.$$

And, in case (ii),

$$F_\lambda = \pi_\lambda \psi_\lambda \phi_\lambda \in \Lambda\vec{\mathcal{G}}_\tau, \text{ and } B_\lambda = \gamma_\lambda \beta_\lambda \in \Lambda^+_{\mathcal{B}_0}\vec{\mathcal{G}}_\tau.$$

The diffeomorphism property of the decomposition is easy to check.

Step 2. We prove the decomposition $\Lambda\mathcal{G}^{\mathbb{C}}_\tau = \Lambda\mathcal{G}_\tau.\Lambda^+_{\mathcal{B}_0}\mathcal{G}^{\mathbb{C}}_\tau$. Let $(G_\lambda, T_\lambda) \in \Lambda\mathcal{G}^{\mathbb{C}}_\tau$. We want to prove that there exist unique $(F_\lambda, X_\lambda) \in \Lambda\mathcal{G}_\tau$ and $(B_\lambda, b_\lambda) \in \Lambda^+_{\mathcal{B}_0}\mathcal{G}^{\mathbb{C}}_\tau$, such that

$$(F_\lambda, X_\lambda)(B_\lambda, b_\lambda) = (F_\lambda B_\lambda, F_\lambda b_\lambda + X_\lambda) = (G_\lambda, T_\lambda). \tag{21}$$

Since $G_\lambda \in \Lambda\vec{\mathcal{G}}^{\mathbb{C}}_\tau$, the equation $G_\lambda = F_\lambda B_\lambda$ has a unique solution $(F_\lambda, B_\lambda) \in \Lambda\vec{\mathcal{G}}_\tau \cap \Lambda^+_{\mathcal{B}_0}\vec{\mathcal{G}}^{\mathbb{C}}_\tau$, according to Step 1. The other equation, $F_\lambda b_\lambda + X_\lambda = T_\lambda$, is equivalent to

$$F_\lambda^{-1} T_\lambda = b_\lambda + F_\lambda^{-1} X_\lambda. \tag{22}$$

Let us denote $\Lambda\mathbb{C}^4_\tau = \{[\lambda \mapsto V_\lambda]; S^1 \to \mathbb{C}^4 / -L_j V_\lambda = V_{i\lambda}\}$, $\Lambda\mathbb{R}^4_\tau = \{[\lambda \mapsto V_\lambda]; S^1 \to \mathbb{R}^4 / -L_j V_\lambda = V_{i\lambda}\}$ and $\Lambda^+\mathbb{C}^4_\tau = \{[\lambda \mapsto V_\lambda] \in \Lambda\mathbb{C}^4_\tau / V_\lambda$ extends holomorphically to the unit disk $\}$. We have the following splitting

$$\Lambda\mathbb{C}^4_\tau = \Lambda\mathbb{R}^4_\tau \oplus \Lambda^+\mathbb{C}^4_\tau.$$

We define $P : \Lambda\mathbb{C}^4_\tau \longrightarrow \Lambda\mathbb{R}^4_\tau$ to be the projection on the first factor. Explicitly,

$$P\left(\sum_{n\in\mathbb{Z}} \hat{V}_{2n+1}\lambda^{2n+1}\right) = \sum_{n\leq 0} \hat{V}_{2n-1}\lambda^{2n-1} + \sum_{n\geq 0} \overline{\hat{V}_{-2n-1}}\lambda^{2n+1}.$$

Then the solution of (22) is given by $X_\lambda = F_\lambda P\left(F_\lambda^{-1}T_\lambda\right)$ and $b_\lambda = F_\lambda^{-1}T_\lambda - P\left(F_\lambda^{-1}T_\lambda\right)$. $\square$

Proof of Theorem 7. We use Theorem (5) b) with $\mathfrak{G} = \vec{\mathcal{G}} \simeq U(2)$: let $\vec{\mathcal{C}} = \Lambda_\star^- \vec{\mathcal{G}}^{\mathbb{C}}.\Lambda^+ \vec{\mathcal{G}}^{\mathbb{C}}$ and $\mathcal{C} = \{(G_\lambda, T_\lambda) \in \Lambda \mathcal{G}_\tau^{\mathbb{C}} / G_\lambda \in \vec{\mathcal{C}}\}$. The latter is clearly an open subset of $\Lambda \mathcal{G}_\tau^{\mathbb{C}}$. For any $(G_\lambda, T_\lambda) \in \mathcal{C}$, we look for $(G_\lambda^-, T_\lambda^-) \in \Lambda_\star^- \mathcal{G}_\tau^{\mathbb{C}}$ and $(G_\lambda^+, T_\lambda^+) \in \Lambda^+ \mathcal{G}_\tau^{\mathbb{C}}$ such that $(G_\lambda^- G_\lambda^+, G_\lambda^- T_\lambda^+ + T_\lambda^-) = (G_\lambda, T_\lambda)$. Let $(G_\lambda^-, G_\lambda^+)$ be the unique element in $\Lambda_\star^- \vec{\mathcal{G}}^{\mathbb{C}} \times \Lambda^+ \vec{\mathcal{G}}^{\mathbb{C}}$ such that $G_\lambda = G_\lambda^- G_\lambda^+$. Since $G_\lambda \in \Lambda \vec{\mathcal{G}}_\tau^{\mathbb{C}}$, $\tau(G_\lambda^-)\tau(G_\lambda^+) = \tau(G_\lambda) = G_{i\lambda} = G_{i\lambda}^- G_{i\lambda}^+$, which implies

$$(G_{i\lambda})^{-1}\tau(G_\lambda^-) = G_{i\lambda}^+ \tau(G_\lambda^+)^{-1} \in \Lambda_\star^- \vec{\mathcal{G}}^{\mathbb{C}} \cap \Lambda^+ \vec{\mathcal{G}}^{\mathbb{C}} = \{\mathbb{1}\}.$$

Hence $(G_\lambda^-, G_\lambda^+) \in \Lambda_\star^- \vec{\mathcal{G}}_\tau^{\mathbb{C}} \times \Lambda^+ \vec{\mathcal{G}}_\tau^{\mathbb{C}}$. To conclude, we look at the equation

$$G_\lambda^- T_\lambda^+ + T_\lambda^- = T_\lambda \Longleftrightarrow T_\lambda^+ + (G_\lambda^-)^{-1} T_\lambda^- = (G_\lambda^-)^{-1} T_\lambda. \tag{23}$$

We let $\Lambda^- \mathbb{C}_\tau^4 = \{[\lambda \mapsto V_\lambda] \in \Lambda \mathbb{C}_\tau^4 / V_\lambda$ extends holomorphically to the complement of the unit disk in $\mathbb{C} \cup \{\infty\}\}$ and use the linear splitting $\Lambda \mathbb{C}_\tau^4 = \Lambda^- \mathbb{C}_\tau^4 \oplus \Lambda^+ \mathbb{C}_\tau^4$. Let $Q^- : \Lambda \mathbb{C}_\tau^4 \longrightarrow \Lambda^- \mathbb{C}_\tau^4$ and $Q^+ : \Lambda \mathbb{C}_\tau^4 \longrightarrow \Lambda^+ \mathbb{C}_\tau^4$ be the projection maps on each factor, namely $Q^- \left(\sum_{n\in\mathbb{Z}} \hat{V}_{2n+1} \lambda^{2n+1}\right) = \sum_{n\leq 0} \hat{V}_{2n-1} \lambda^{2n-1}$ and $Q^+ \left(\sum_{n\in\mathbb{Z}} \hat{V}_{2n+1} \lambda^{2n+1}\right) = \sum_{n\geq 0} \hat{V}_{2n+1} \lambda^{2n+1}$. Then the unique solution to (23) is given by

$$T_\lambda^- = G_\lambda^- Q^- \left((G_\lambda^-)^{-1} T_\lambda\right) \text{ and } T_\lambda^+ = Q^+ \left((G_\lambda^-)^{-1} T_\lambda\right).$$

Thus we obtained the right decomposition. □

5. Weierstrass representations.

5.1. From Hamiltonian stationary surfaces to holomorphic potentials.

First we shall here sketch how to use ideas from [DPW] in order to construct Weierstrass type data, starting from a Hamiltonian stationary Lagrangian conformal immersion. Then we will revisit the obtained results and see how it simplifies in our situation.

Let $U = (F, X) : \Omega \longrightarrow \mathcal{G}$ be a Hamiltonian stationary LCLI. Then it follows from Corollary 2, that U extends to a map $U_\lambda = (F_\lambda, X_\lambda) : \Omega \longrightarrow \Lambda \mathcal{G}_\tau$ satisfying (17), (18) and $U_1 = U$.

5.1.1. A family of holomorphic potentials. There exists a holomorphic map $H_\lambda : \Omega \longrightarrow \Lambda \mathcal{G}_\tau^{\mathbb{C}}$ and a map $B_\lambda : \Omega \longrightarrow \Lambda_{\mathcal{B}_0}^+ \mathcal{G}_\tau^{\mathbb{C}}$ such that

$$U_\lambda(z) = H_\lambda(z) B_\lambda(z), \ \forall \lambda \in S^1, \forall z \in \Omega.$$

The construction if $H_\lambda(z)$ and $B_\lambda(z)$ is done as follows: one looks for a map $B_\lambda : \Omega \longrightarrow \Lambda^+_{\mathcal{B}_0}\mathcal{G}^{\mathbb{C}}_\tau$ such that $H_\lambda(z) = U_\lambda(z)B_\lambda(z)^{-1}$ is holomorphic, i.e.,

$$0 = \frac{\partial(U_\lambda B_\lambda^{-1})}{\partial \bar{z}} = U_\lambda\left(\alpha_\lambda\left(\frac{\partial}{\partial\bar{z}}\right) - B_\lambda^{-1}\frac{\partial B_\lambda}{\partial\bar{z}}\right)B_\lambda^{-1},$$

which is equivalent to

$$\frac{\partial B_\lambda}{\partial\bar{z}} = B_\lambda\left(\alpha_0\left(\frac{\partial}{\partial\bar{z}}\right) + \lambda\alpha_1\left(\frac{\partial}{\partial\bar{z}}\right) + \lambda^2\alpha_2\left(\frac{\partial}{\partial\bar{z}}\right)\right).$$

The existence of a solution B_λ to this equation is first obtained locally (see [DPW] or [H2]), then one can glue the local solutions into a global one [DPW]. Then we write

$$H_\lambda^{-1}dH_\lambda = B_\lambda\left(\alpha_\lambda - B_\lambda^{-1}dB_\lambda\right)B_\lambda^{-1},$$

and using the fact that B_λ takes its values in $\Lambda^+_{\mathcal{B}_0}\mathcal{G}^{\mathbb{C}}_\tau$ and that $z \longmapsto H_\lambda(z)$ is holomorphic, we deduce that

$$H_\lambda^{-1}dH_\lambda := \mu_\lambda = \sum_{n\geq -2}\hat{\mu}_n\lambda^n,$$

where each $\hat{\mu}_n$ is a closed (1,0)-form (i.e., holomorphic). As we shall see, in 4.2, we can reconstruct U_λ from μ_λ. J. Dorfmeister, F. Pedit and H.Y. Wu call the form μ_λ a *holomorphic potential.* Notice that μ_λ is far from being uniquely defined, so we associate to U_λ a whole family of holomorphic potentials.

5.1.2. A single meromorphic potential. We can refine the above result as follows. First one can show that there exists a non accumulating set of points $a_1, a_2, \ldots$ in Ω such that $U_\lambda(z)$ belongs to the big cell $\mathcal{C}$ (see Theorem 7), for all $z \in \Omega\setminus\{a_1, a_2, \ldots\}$. The proof of that is delicate and uses in particular the result of 5.1.1. Thus applying Theorem 7, we deduce that $\forall z \in \Omega\setminus\{a_1, a_2, \ldots\}$, $\exists!(U_\lambda^-(z), U_\lambda^+(z)) \in \Lambda^-_\star\mathcal{G}^{\mathbb{C}}_\tau \times \Lambda^+\mathcal{G}^{\mathbb{C}}_\tau$ such that

$$U_\lambda(z) = U_\lambda^-(z)U_\lambda^+(z), \tag{24}$$

and then

$$\mu_\lambda := \left(U_\lambda^-\right)^{-1}dU_\lambda^- = U_\lambda^+\left(\alpha_\lambda - (U_\lambda^+)^{-1}dU_\lambda^+\right)\left(U_\lambda^+\right)^{-1}. \tag{25}$$

We analyze equation (25): the right hand side tells us that $\hat{\mu}_n = 0$ for $n < -2$ and the left hand side that $\hat{\mu}_n = 0$ for $n \geq 0$. Hence

$$\mu_\lambda = \lambda^{-2}\hat{\mu}_{-2} + \lambda^{-1}\hat{\mu}_{-1}. \tag{26}$$

Moreover, by writing the Fourier expansion of the right hand side of (25), one shows that $\mu_\lambda\left(\frac{\partial}{\partial \bar{z}}\right) = 0$. Hence $z \longmapsto U_\lambda^-(z)$ is holomorphic on $\Omega \setminus \{a_1, a_2, \ldots\}$. The analysis in [DPW] shows furthermore that $z \longmapsto U_\lambda^-(z)$ extends as a meromorphic map on Ω: the potential μ_λ is a uniquely defined meromorphic potential.

5.1.3. Explicit description. We shall now revisit the previous facts. Since the 1-form μ_λ defined in (25) has his coefficients in $\Lambda_\star^- \mathfrak{g}_\tau^{\mathbb{C}}$, we may write it as

$$\mu_\lambda = \lambda^{-2}(cL_i, 0)dz + \lambda^{-1}(0, a\epsilon + bL_i\bar{\epsilon})dz, \tag{27}$$

where a, b, c are *a priori* meromorphic functions on Ω. Moreover, it follows from (25) that

$$\begin{aligned}(cL_i, 0)dz = U_0^+ \alpha_2' \left(U_0^+\right)^{-1} &= U_0^+ \left(\frac{1}{2}\frac{\partial \beta}{\partial z}L_i, 0\right)\left(U_0^+\right)^{-1} dz \\ &= \left(\frac{1}{2}\frac{\partial \beta}{\partial z}L_i, 0\right) dz,\end{aligned}$$

where we used the fact that $U_0^+ \in \mathcal{G}_0^{\mathbb{C}}$. Thus $c = \frac{1}{2}\frac{\partial \beta}{\partial z}$. Hence, letting $U_\lambda^- = (G_\lambda^-, T_\lambda^-)$ and using $dU_\lambda^- = U_\lambda^- \mu_\lambda$, we obtain

$$d(G_\lambda^-, T_\lambda^-) = \left(\lambda^{-2}G_\lambda^- \frac{1}{2}\frac{\partial \beta}{\partial z}L_i dz, \lambda^{-1}G_\lambda^-(a\epsilon + bL_i\bar{\epsilon})dz\right),$$

from which we deduce

$$G_\lambda^-(z) = e^{\frac{\lambda^{-2}}{4}(\beta(z) + i\gamma(z))L_i},$$

where $\gamma : \Omega \longrightarrow \mathbb{C}$ is such that $\gamma(z_0) = 0$ and $d(\frac{1}{2}(\beta + i\gamma)) = \frac{\partial \beta}{\partial z}dz$ [11], and also $dT_\lambda^- = \lambda^{-1}e^{\frac{\lambda^{-2}}{4}(\beta + i\gamma)L_i}(a\epsilon + bL_i\bar{\epsilon})dz$. Thus

$$\begin{aligned}&(G_\lambda^-(z), T_\lambda^-(z)) \\ &= \left(e^{\frac{\lambda^{-2}}{4}(\beta(z)+i\gamma(z))L_i}, \lambda^{-1}\int_{z_0}^{z} e^{\frac{\lambda^{-2}}{4}(\beta(v)+i\gamma(v))L_i}(a(v)\epsilon + b(v)L_i\bar{\epsilon})dv\right).\end{aligned}$$

[11] recall that $\frac{1}{2}(\beta + i\gamma)$ is the only holomorphic function vanishing at z_0 with a real part equal to $\frac{\beta}{2}$.

Now, letting $U_\lambda^+ = (G_\lambda^+, T_\lambda^+)$,we can write (24) as

$$\left(e^{-\frac{\lambda^{-2}}{4}(\beta(z)+i\gamma(z))L_i}F_\lambda(z), e^{-\frac{\lambda^{-2}}{4}(\beta(z)+i\gamma(z))L_i}X_\lambda(z)\right)$$
$$= \Big(G_\lambda^+(z), T_\lambda^+(z)$$
$$+ e^{-\frac{\lambda^{-2}}{4}(\beta(z)+i\gamma(z))L_i}\lambda^{-1}\int_{z_0}^{z} e^{\frac{\lambda^{-2}}{4}(\beta(v)+i\gamma(v))L_i}(a(v)\epsilon + b(v)L_i\bar{\epsilon})dv\Big).$$

We conclude that

$$(G_\lambda^-, T_\lambda^-) = \left(e^{\frac{\lambda^{-2}}{4}(\beta+i\gamma)L_i}, e^{\frac{\lambda^{-2}}{4}(\beta+i\gamma)L_i}Q^-\left(e^{-\frac{\lambda^{-2}}{4}(\beta+i\gamma)L_i}X_\lambda\right)\right)$$
$$(G_\lambda^+, T_\lambda^+) = \left(e^{-\frac{\lambda^{-2}}{4}(\beta+i\gamma)L_i}F_\lambda, Q^+\left(e^{-\frac{\lambda^{-2}}{4}(\beta+i\gamma)L_i}X_\lambda\right)\right).$$

Moreover we have obtained two different expressions of T_λ^- which imply the relation

$$(a + bL_i\bar{\epsilon})dz = \lambda e^{-\frac{\lambda^{-2}}{4}(\beta+i\gamma)L_i} d\left(e^{\frac{\lambda^{-2}}{4}(\beta+i\gamma)L_i}Q^-\left(e^{-\frac{\lambda^{-2}}{4}(\beta+i\gamma)L_i}X_\lambda\right)\right),$$

from which we deduce *a posteriori* that a and b are holomorphic. After this analysis, we are led to the following

Theorem 8. *For any Hamiltonian stationary LCLI $U_\lambda = (F_\lambda, X_\lambda)$, there exist unique $U_\lambda^- = (F_\lambda^-, X_\lambda^-) \in \Lambda_\star^- \mathcal{G}_\tau^{\mathbb{C}}$ and $U_\lambda^+ = (F_\lambda^+, X_\lambda^+) \in \Lambda^+ \mathcal{G}_\tau^{\mathbb{C}}$ such that $U_\lambda = U_\lambda^- U_\lambda^+$, defined explicitly by*

$$U_\lambda^- = \left(e^{\frac{\lambda^{-2}}{4}(\beta+i\gamma)L_i}, e^{\frac{\lambda^{-2}}{4}(\beta+i\gamma)L_i}Q^-\left(e^{-\frac{\lambda^{-2}}{4}(\beta+i\gamma)L_i}X_\lambda\right)\right)$$
$$U_\lambda^+ = \left(e^{-\frac{\lambda^{-2}}{4}(\beta+i\gamma)L_i}F_\lambda, Q^+\left(e^{-\frac{\lambda^{-2}}{4}(\beta+i\gamma)L_i}X_\lambda\right)\right),$$

for γ solution of $\gamma(z_0) = 0$ and $d(\frac{1}{2}(\beta + i\gamma)) = \frac{\partial\beta}{\partial z}dz$. Moreover,

$$\mu_\lambda = \left(U_\lambda^-\right)^{-1} dU_\lambda^- = \left(\frac{\lambda^{-2}}{2}L_i\frac{\partial\beta}{\partial z}dz, \lambda^{-1}(a\epsilon + bL_i\bar{\epsilon})dz\right)$$

for some holomorphic functions a, b.

Proof. The uniqueness of the decomposition follows from Theorem 7. One checks easily that $U_\lambda = U_\lambda^- U_\lambda^+$ and $U_\lambda^- \in \Lambda_\star^- \mathcal{G}^{\mathbb{C}}$. For the verification of $U_\lambda^+ \in \Lambda^+ \mathcal{G}^{\mathbb{C}}$, we assume first that U_λ corresponds to the fundamental lift: then (see Remark 3) $F_\lambda = M_X e^{\frac{\beta_\lambda L_i}{2}}$, which implies

$$G_\lambda^+ = e^{-\frac{\lambda^{-2}}{2} B L_i} F_\lambda = M_X e^{\frac{\beta_\lambda - \lambda^{-2}\frac{1}{2}(\beta + i\gamma)}{2} L_i} = M_X e^{\frac{\lambda^2}{4}(\beta - i\gamma) L_i}.$$

Therefore G_λ^+ belongs to $\Lambda^+ \vec{\mathcal{G}}^{\mathbb{C}}$. Thus obviously $U_\lambda^+ \in \Lambda^+ \mathcal{G}^{\mathbb{C}}$. If U_λ corresponds to an arbitrary lift, then, according to Remark 3, there exists $K \in \mathcal{C}^\infty(\Omega, \mathcal{G}_0)^\star$ such that $F_\lambda = M_X e^{\frac{\beta_\lambda L_i}{2}} K^{-1}$, and thus $G_\lambda^+ = M_X K^{-1} e^{\frac{\lambda^2}{4}(\beta - i\gamma) L_i}$ and we obtain the same conclusion.

Lastly repeating the argument of Theorem 7, we can deduce that $U_\lambda^- \in \Lambda_\star^- \mathcal{G}_\tau^{\mathbb{C}}$ and $U_\lambda^+ \in \Lambda^+ \mathcal{G}_\tau^{\mathbb{C}}$. The computation of μ_λ was done before. □

The data $a, b, c = \frac{1}{2}\frac{\partial \beta}{\partial z}$ are called the *Weierstrass data of* U_λ.

5.2. From a Weierstrass data to a Hamiltonian stationary conformal immersion.

We shall now see that the construction of the previous section has a converse. As above, we first sketch how to adapt the strategy of [DPW] and then we explore in more details what it means in our context.

Let $\mu_\lambda = \sum_{n \geq -2} \hat{\mu}_n \lambda^n$ be a holomorphic potential; it is a 1-form on Ω with coefficients in $\Lambda \mathfrak{g}_\tau^{\mathbb{C}}$ which is holomorphic, i.e., which satisfies $\mu_\lambda\left(\frac{\partial}{\partial \bar{z}}\right) = 0$ and $d\mu_\lambda = 0$. Then, $\mu_\lambda\left(\frac{\partial}{\partial \bar{z}}\right) = 0$ implies in particular that $\mu_\lambda \wedge \mu_\lambda = 0$. Thus

$$d\mu_\lambda + \mu_\lambda \wedge \mu_\lambda = 0,$$

and there exists a unique map $H_\lambda \in \Lambda \mathcal{G}_\tau^{\mathbb{C}}$ such that $H_\lambda(z_0) = 0$ and

$$dH_\lambda = H_\lambda \mu_\lambda. \tag{28}$$

For any $z \in \Omega$, we use Theorem 6 with $H_\lambda(z)$: there exists a unique $(U_\lambda(z), V_\lambda(z)) \in \Lambda \mathcal{G}_\tau \times \Lambda_{\mathcal{B}_0}^+ \mathcal{G}_\tau^{\mathbb{C}}$ such that $H_\lambda(z) = U_\lambda(z) V_\lambda(z)$. A straightforward computation using (28) shows that

$$U_\lambda^{-1} dU_\lambda = V_\lambda \left(\mu_\lambda - V_\lambda^{-1} dV_\lambda\right) V_\lambda^{-1}. \tag{29}$$

Let us denote $\alpha_\lambda := U_\lambda^{-1} dU_\lambda$. Again the right hand side of (29) tells us that α_λ should be of the form

$$\alpha_\lambda = \sum_{n \geq -2} \hat{\alpha}_n \lambda^n,$$

but the left hand side says that $\overline{\hat{\alpha}_n} = \hat{\alpha}_{-n}$ and thus

$$\alpha_\lambda = \lambda^{-2}\hat{\alpha}_{-2} + \lambda^{-1}\hat{\alpha}_{-1} + \hat{\alpha}_0 + \lambda\hat{\alpha}_1 + \lambda^2\hat{\alpha}_2.$$

Moreover, using $V_\lambda^{-1} = \hat{V}_0^{-1} + \lambda\hat{V}_1^{-1} + \cdots = \hat{V}_0^{-1} - \lambda\hat{V}_0^{-1}\hat{V}_1\hat{V}_0^{-1} + \cdots$, it follows also from (29) that

$$\hat{\alpha}_{-2} = \hat{V}_0\hat{\mu}_{-2}\hat{V}_0^{-1} \text{ and } \hat{\alpha}_{-1} = \hat{V}_0\hat{\mu}_{-1}\hat{V}_0^{-1} + [\hat{V}_1, \hat{\mu}_{-2}]\hat{V}_0^{-1}$$

are (1,0)-forms. Hence, since α_λ satisfies condition (9) automatically, Corollary 2 implies that - provided that we can prove the condition $\hat{\alpha}_{-1} \neq 0$ - U_λ is an extended lift of a Hamiltonian stationary conformal immersion. Lastly, by the relation $U_\lambda = H_\lambda V_\lambda^{-1}$, we see that μ_λ is a holomorphic potential for U_λ in the sense of the above section.

Let us now look at the particular case where μ_λ has the form

$$\mu_\lambda = \left(\frac{\lambda^{-2}}{2}\frac{\partial\beta}{\partial z}L_i, \lambda^{-1}(a\epsilon + bL_i\bar{\epsilon})\right)dz,$$

for some holomorphic β, a, b. We integrate the equation $dH_\lambda = H_\lambda\mu_\lambda$. Denoting $H_\lambda = (h_\lambda, \eta_\lambda)$, it gives

$$(dh_\lambda, d\eta_\lambda) = \left(\frac{\lambda^{-2}}{2}\frac{\partial\beta}{\partial z}h_\lambda L_i dz, \lambda^{-1}h_\lambda(a\epsilon + bL_i\bar{\epsilon})dz\right).$$

It has the following solution

$$\begin{aligned} H_\lambda(z) &= (h_\lambda(z), \eta_\lambda(z)) \\ &= \left(e^{\frac{\lambda^{-2}}{4}(\beta(z)+i\gamma(z))L_i}, \int_{z_0}^{z} e^{\frac{\lambda^{-2}}{4}(\beta(v)+i\gamma(v))L_i}\lambda^{-1}(a(v)\epsilon + b(v)L_i\bar{\epsilon})dv\right), \end{aligned}$$

where γ is the harmonic conjugate function of β vanishing at z_0. We now look for $U_\lambda = (F_\lambda, X_\lambda) \in \Lambda\mathcal{G}_\tau$ and $V_\lambda = (B_\lambda, b_\lambda) \in \Lambda^+_{\mathcal{B}_0}\mathcal{G}^{\mathbb{C}}_\tau$ such that $H_\lambda = U_\lambda V_\lambda$. We first use that $\beta_\lambda = \frac{1}{2}(\lambda^{-2}+\lambda^2)\beta + \frac{i}{2}(\lambda^{-2}-\lambda^2)\gamma$ (see the proof of Theorem 6) and thus

$$h_\lambda = e^{\frac{\lambda^{-2}}{4}(\beta+i\gamma)L_i} = e^{\frac{1}{2}\beta_\lambda L_i}e^{-\frac{\lambda^2}{4}(\beta-i\gamma)L_i},$$

meaning that we have $F_\lambda = e^{\frac{1}{2}\beta_\lambda L_i}$ and $B_\lambda = e^{-\frac{\lambda^2}{4}(\beta-i\gamma)L_i}$. Now we need to solve

$$\begin{aligned} \eta_\lambda(z) &= \int_{z_0}^{z} e^{\frac{\lambda^{-2}}{4}(\beta(v)+i\gamma(v))L_i}\lambda^{-1}(a(v)\epsilon + b(v)L_i\bar{\epsilon})dv \\ &= F_\lambda(z)b_\lambda(z) + X_\lambda(z), \end{aligned}$$

or

$$
\begin{aligned}
&e^{-\frac{1}{2}\beta_\lambda(z)L_i}\int_{z_0}^{z} e^{\frac{\lambda^{-2}}{4}(\beta(v)+i\gamma(v))L_i}\lambda^{-1}(a(v)\epsilon+b(v)L_i\bar{\epsilon})dv \\
&= b_\lambda(z)+e^{-\frac{1}{2}\beta_\lambda(z)L_i}X_\lambda(z).
\end{aligned}
$$

We deduce that

$$
\begin{aligned}
&e^{-\frac{1}{2}\beta_\lambda(z)L_i}X_\lambda(z) \\
&= P\left(e^{-\frac{1}{2}\beta_\lambda(z)L_i}\int_{z_0}^{z} e^{\frac{\lambda^{-2}}{4}(\beta(v)+i\gamma(v))L_i}\lambda^{-1}(a(v)\epsilon+b(v)L_i\bar{\epsilon})dv\right).
\end{aligned}
$$

Hence we proved

Theorem 9. *For any harmonic β and holomorphic data a, b, the potential $\mu_\lambda = \left(\frac{\lambda^{-2}}{2}\frac{\partial\beta}{\partial z}L_i, \lambda^{-1}(a\epsilon + bL_i\bar{\epsilon})\right)dz$ leads to construct the map $U_\lambda : \Omega \longrightarrow \Lambda\mathcal{G}_\tau$ by*

$$
\begin{aligned}
U_\lambda(z) &= (F_\lambda(z), X_\lambda(z)) \\
&= \left(e^{\frac{1}{2}\beta_\lambda(z)L_i}, e^{\frac{1}{2}\beta_\lambda(z)L_i}P\left(e^{-\frac{1}{2}\beta_\lambda(z)L_i}\int_{z_0}^{z} e^{\frac{\lambda^{-2}}{4}(\beta(v)+i\gamma(v))L_i}\right.\right. \\
&\qquad\left.\left.\cdot\,\lambda^{-1}(a(v)\epsilon+b(v)L_i\bar{\epsilon})dv\right)\right),
\end{aligned}
$$

where $\beta_\lambda = \frac{1}{2}(\lambda^{-2}+\lambda^2)\beta + \frac{i}{2}(\lambda^{-2}-\lambda^2)\gamma$ and γ is the harmonic conjugate map to β vanishing at z_0 (i.e. $\partial(\beta+i\gamma)/\partial\bar{z} = 0$). And U_λ is an extended lift of a Hamiltonian stationary conformal immersion if and only if X_λ is an immersion.

6. Tori and finite type solutions.

Going back to the torus, we will apply the concept of holomorphic potential defined in the previous section to the study of Hamiltonian stationary Lagrangian tori (in conformal coordinates). What makes the torus specific is that we can define – intrinsically – a notion of *constant* potential, i.e. $\mu = \eta dz$, where dz is any globally defined holomorphic 1-form and η is a constant twisted loop of Lie-algebra elements. Indeed two globally defined holomorphic 1-forms on a torus $\mathbb{T}$ differ by a multiplicative constant. We may further restrict to those potentials having only a finite number of nonzero terms in their Fourier expansion (known as *polynomial loops*). While such conditions may seem (i) far-fetched and (ii) too restrictive, it turns out that

- integrating potentials that are constant (in z) and polynomial (in λ) is equivalent to integrating commuting flows, which in our case leads to a much simpler integration process than the one described in section 5.2 (also known as the Adler-Kostant-Symes (AKS) scheme); the corresponding Hamiltonian stationary immersions are called *finite type solutions*;

- all immersed tori are finite type solutions.

Notice also that in the toric case, the considerations below prove the existence of potential (without resorting to the previous section). Such ideas originate in the theory of completely integrable systems, however we will not explain here the link between commuting flows and finite type solutions, and refer the Reader to [BFPP] for a good description of both sides of the AKS scheme. Finally we will see how this new description relates to the one given in section 3.

6.1. Construction of finite type solutions.

Throughout the section, dz will denote some fixed global holomorphic 1-form on a torus $\mathbb{T}$, or its universal cover $\mathbb{C}$. Then for any $d \in \mathbb{N}$ define

$$\Lambda^d \mathfrak{g}_\tau = \left\{ [\lambda \mapsto \xi_\lambda] \in \Lambda \mathfrak{g}_\tau ; \xi_\lambda = \sum_{-d}^{d} \hat{\xi}_n \lambda^n \right\}$$

the space of real polynomial loops of degree d.

Proposition 3. *Let $d \in 4\mathbb{N}+2$ and $\eta_\lambda \in \Lambda^d \mathfrak{g}_\tau$ be a polynomial loop. Then the extended 1-form α_λ obtained through the AKS scheme from the constant potential $\lambda^{d-2}\eta_\lambda dz$ on $\mathbb{C}$ (with starting point z_0) is exactly the projection $\pi_{\Lambda \mathfrak{g}_\tau}(\lambda^{d-2}\xi_\lambda dz)$ of the solution ξ_λ to the following differential equation:*

$$(30) \qquad \left\{ \begin{array}{l} d\xi_\lambda = [\xi_\lambda, \pi_{\Lambda \mathfrak{g}_\tau}(\lambda^{d-2}\xi_\lambda dz)] \\ \xi_\lambda(z_0) = \eta_\lambda \end{array} \right.$$

where $\pi_{\Lambda \mathfrak{g}_\tau}$ denotes the projection on $\Lambda \mathfrak{g}_\tau$ in the direct sum $\Lambda \mathfrak{g}_\tau^{\mathbb{C}} = \Lambda \mathfrak{g}_\tau \oplus \Lambda_{\mathfrak{b}}^{+} \mathfrak{g}_\tau^{\mathbb{C}}$. Reciprocally, the solution exists and is defined for all $z \in \mathbb{C}$.

Proof. First notice that $\lambda^{d-2}\eta_\lambda$ is a constant real polynomial loop with lowest Fourier coefficient $\lambda^{-2}\hat{\eta}_{-d}$, thus also a holomorphic potential, that we integrate on $\mathbb{C}$. Let $M_\lambda \in \Lambda\mathcal{G}_\tau^{\mathbb{C}}$ be such that $M_\lambda(z_0) = \mathbb{1}$ and $\mu_\lambda = M_\lambda^{-1}dM_\lambda = \lambda^{d-2}\eta_\lambda dz$. Use the Iwasawa decomposition (Theorem 6) as in section 5 to write $M_\lambda = H_\lambda B_\lambda$ and by definition $\alpha_\lambda = H_\lambda^{-1}dH_\lambda$. Set $\xi_\lambda(z) = H_\lambda^{-1}(z)\eta_\lambda H_\lambda$, which is well-defined on all $\mathbb{C}$. By construction ξ_λ is real (i.e., belongs to $\Lambda\mathcal{G}_\tau$) since η_λ and $H_\lambda(z)$ are. Using the fact that η_λ commutes with M_λ we write

$$\eta_\lambda = M_\lambda \eta_\lambda M_\lambda^{-1} = H_\lambda B_\lambda \eta_\lambda B_\lambda^{-1} H_\lambda^{-1}$$

so $\xi_\lambda = B_\lambda \eta_\lambda B_\lambda^{-1}$, which proves that ξ_λ has no Fourier coefficient with exponent lower than $-d$ (simply write the Fourier expansions). Being real, ξ_λ is $\Lambda^d\mathfrak{g}_\tau$ valued. To prove that it solves the differential equation above, we write $d\xi_\lambda = [\xi_\lambda, H_\lambda^{-1}dH_\lambda]$; but

$$\begin{aligned} H_\lambda^{-1}dH_\lambda &= \pi_{\Lambda\mathfrak{g}_\tau}\left(H_\lambda^{-1}dH_\lambda\right) = \pi_{\Lambda\mathfrak{g}_\tau}\left(B_\lambda \mu_\lambda B_\lambda^{-1} - dB_\lambda B_\lambda^{-1}\right) \\ &= \pi_{\Lambda\mathfrak{g}_\tau}\left(B_\lambda(\lambda^{d-2}\eta_\lambda dz)B_\lambda^{-1}\right) = \pi_{\Lambda\mathfrak{g}_\tau}\left(\lambda^{d-2}\xi_\lambda dz\right) \end{aligned}$$

□

Any Hamiltonian stationary conformal Lagrangian immersion so obtained, either by integrating a constant polynomial loop as above, or by solving the differential equation (30), is called a *finite type solution.* Equation (30) can be written more explicitly, thus showing how to derive from ξ_λ the extended 1-form α_λ. Indeed writing $\xi_\lambda = \sum_{-d}^{d} \lambda^n \hat{\xi}_n$, the projection $\pi_{\Lambda\mathfrak{g}_\tau}(\lambda^{d-2}\xi_\lambda dz)$ is

$$\begin{aligned} \pi_{\Lambda\mathfrak{g}_\tau}(\lambda^{d-2}\xi_\lambda dz) = \lambda^{-2}\hat{\xi}_{-d}dz + \lambda^{-1}\hat{\xi}_{-d+1}dz + \pi_{\mathfrak{g}_0}(\hat{\xi}_{-d+2}dz) \\ + \lambda\overline{\hat{\xi}_{-d+1}}d\bar{z} + \lambda^2\overline{\hat{\xi}_{-d}}d\bar{z} \end{aligned}$$

where $\pi_{\mathfrak{g}_0}$ is the projection in the direct sum $\mathfrak{g}_0^{\mathbb{C}} = \mathfrak{g}_0 \oplus \mathfrak{b}$; define on $\mathfrak{g}_0^{\mathbb{C}}$ the operator

$$r : \zeta \mapsto \frac{\pi_{\mathfrak{g}_0}(\zeta) - i\pi_{\mathfrak{g}_0}(i\zeta)}{2}$$

satisfying $\pi_{\mathfrak{g}_0}(\zeta dz) = r(\zeta)dz + \overline{r(\zeta)}d\bar{z}$. Then we rewrite equation (30) as

$$\frac{\partial \xi_\lambda}{\partial z} = [\xi_\lambda, \lambda^{-2}\hat{\xi}_{-d} + \lambda^{-1}\hat{\xi}_{-d+1} + r(\hat{\xi}_{-d+2})] \tag{31}$$

plus the initial condition; the conjugate equation is implied by the reality of ξ_λ.

Since we aim at constructing solutions on a torus $\mathbb{T}$, we ought to notice that our construction, while valid on $\mathbb{C}$, does not necessarily give an immersion of the torus. To produce an actual torus we need to verify period conditions (a.k.a. monodromy conditions) obtained by integrating α_λ. Also recall that the regularity of the immersion is equivalent to $\hat{\xi}_{-d+1}$ being non zero, otherwise the solution is only weakly conformal.

6.2. A finiteness result.

Theorem 10. *Let $\mathbb{T}$ be a 2-torus; then any Hamiltonian stationary conformal Lagrangian immersion in $\mathbb{R}^4$ is of finite type.*

This may seem surprising, especially if we think how restrictive the finite type condition is; however this result is almost classical in the theory of infinite dimensional integrable systems. As a consequence the space of solutions is a countable union of finite dimensional spaces.

Proof. We will adapt here an idea found in [BFPP]. Let X be a Hamiltonian stationary conformal Lagrangian immersion, α an associated Maurer-Cartan form (for some LCLI) and α_λ its extended 1-form. We first consider all quantities as being defined on the universal cover $\mathbb{C}$ of $\mathbb{T}$. We also choose a global holomorphic 1-form dz on $\mathbb{T}$ (and $\mathbb{C}$). A necessary and sufficient condition for X to be of finite type is the existence of $\xi_\lambda : \mathbb{C} \to \Lambda^d \mathfrak{g}_\tau$ such that both equations hold

$$d\xi_\lambda = [\xi_\lambda, \alpha_\lambda] \tag{32}$$

and

$$\alpha_\lambda \left(\frac{\partial}{\partial z} \right) = \lambda^{-2} \hat{\xi}_{-d} + \lambda^{-1} \hat{\xi}_{-d+1} + r(\hat{\xi}_{-d+2}) \tag{33}$$

Before we step into the proof, notice that finite type imposes a condition obviously not satisfied in generality: consider the first Fourier term (multiple of λ^{-d}) in equation (32), then

$$d\hat{\xi}_{-d} = [\hat{\xi}_{-d}, (r(\hat{\xi}_{-d+2}) - \hat{\xi}_{-d+2})dz - \overline{r(\hat{\xi}_{-d+2})}d\bar{z}] \in [\mathfrak{g}_2, \mathfrak{g}_0^{\mathbb{C}}] = 0$$

by the commutations properties of $\mathfrak{g}$; so $\hat{\xi}_{-d}$ is constant. Using condition (33), we see that $\hat{\xi}_{-d} = \frac{1}{2} \frac{\partial \beta}{\partial z}(L_i, 0)$, so that the Lagrangian angle is an affine function of x and y. On a torus this condition is trivially satisfied since $\frac{\partial \beta}{\partial z}$ is holomorphic hence constant. This constant cannot vanish, otherwise the immersed torus would be special-Lagrangian (or minimal); but there are no compact minimal tori.

The proof is divided in two steps: we first prove the existence of a formal solution to (32) and (33), then extract a polynomial solution from these solutions, using the property that each Fourier coefficient of the formal solution satisfies an elliptic equation on the torus. By taking a proper combination we infer the existence of ξ_λ.

Step 1: existence of adapted formal Killing fields. A formal Killing field ζ_λ is a formal Laurent series in λ verifying (32). Such a field is said *adapted* if its first three terms are respectively equal to $\alpha_{-2}\left(\frac{\partial}{\partial z}\right)$, $\alpha_{-1}\left(\frac{\partial}{\partial z}\right)$ and $r\left(\alpha_0\left(\frac{\partial}{\partial z}\right)\right)$. Using the gauge action, we may suppose without loss of generality that $\alpha_0 = 0$, so that $\alpha\left(\frac{\partial}{\partial z}\right) = \alpha_{-2}\left(\frac{\partial}{\partial z}\right) + \alpha_{-1}\left(\frac{\partial}{\partial z}\right) = (aL_i, u)$ for some nonzero complex constant a, and $u(z) \in \mathbb{C}\epsilon \oplus \mathbb{C}L_i\bar{\epsilon}$. Let us look for ζ_λ of the following form (typical of a gauge change used in [FT]): $\zeta_\lambda = (\mathbb{1}, w_\lambda)^{-1}(aL_i, u)(\mathbb{1}, w_\lambda)$ where w_λ has non negative Fourier exponents (beware: $(\mathbb{1}, w_\lambda)$ is a matrix in $\mathcal{G}^{\mathbb{C}}$, not in $\mathfrak{g}^{\mathbb{C}}$). The expression of ζ_λ simplifies here to give $\zeta_\lambda = (aL_i, u + aL_i w_\lambda)$, and we solve the (1,0) part of equation (32):

$$\begin{aligned}\left(0, \frac{\partial u}{\partial z} + aL_i \frac{\partial w_\lambda}{\partial z}\right) &= [(aL_i, u + aL_i w_\lambda), (\lambda^{-2} aL_i, \lambda^{-1} u)] \\ &= (0, \lambda^{-1} aL_i u - \lambda^{-2} aL_i u + \lambda^{-2} a^2 w_\lambda)\end{aligned}$$

so, using $a \neq 0$

$$w_\lambda = \lambda^2 a^{-1} L_i \frac{\partial w_\lambda}{\partial z} + \lambda^2 a^{-2} \frac{\partial u}{\partial z} + a^{-1} L_i u - \lambda a^{-1} L_i u$$

Writing the hypothesis $w_\lambda = \sum_{n \geq 0} \lambda^n \hat{w}_n$, we obtain w_λ by simple recurrence (hence the formal series):

$$\begin{cases} \hat{w}_0 = a^{-1} L_i u \\ \hat{w}_1 = -a^{-1} L_i u \\ \hat{w}_2 = a^{-1} L_i \dfrac{\partial \hat{w}_0}{\partial z} + a^{-2} \dfrac{\partial u}{\partial z} = 0 \\ \hat{w}_n = a^{-1} L_i \dfrac{\partial \hat{w}_{n-2}}{\partial z} \quad \text{for } n > 2 \end{cases}$$

So $w_\lambda = a^{-1}L_i u - \sum_{n\geq 0} \lambda^{2n+1}(a^{-1}L_i)^{n+1}\frac{\partial^n u}{\partial z^n}$. Finally

$$\zeta_\lambda = \left(aL_i, \sum_{n\geq 0} \lambda^{2n+1} a^{-n} L_i^n \frac{\partial^n u}{\partial z^n} \right).$$

We now check that the (0,1) equation holds. Using the same idea as in section 3, $d\alpha'_{-1} + [\alpha''_1 \wedge \alpha'_2] + [\alpha'_{-1} \wedge \alpha''_0] = 0$, which in our notations yields $\frac{\partial u}{\partial \bar z} = aL_i \bar u$. Then

$$\begin{aligned} \frac{\partial \zeta_\lambda}{\partial \bar z} &= \left(0, \sum_{n\geq 0} \lambda^{2n+1} a^{-n} L_i^n \frac{\partial^{n+1} u}{\partial \bar z \partial z^n} \right) \\ &= \left(0, aL_i \lambda \bar u - \lambda^2 \bar a L_i \sum_{n\geq 0} \lambda^{2n+1} a^{-n} L_i^n \frac{\partial^n u}{\partial z^n} \right) = [\zeta_\lambda, (\lambda^2 \bar a L_i, \lambda \bar u)] \end{aligned}$$

Finally we verify that ζ_λ is adapted: $\hat\zeta_0 = (aL_i, 0) = \alpha'_{-2}\left(\frac{\partial}{\partial z}\right)$, $\hat\zeta_1 = (0, u) = \alpha'_{-1}\left(\frac{\partial}{\partial z}\right)$ and $\hat\zeta_2 = 0$. It should be noted that for any $n \in \mathbb{Z}$, $\lambda^n \zeta_\lambda$ is still an adapted formal Killing field.

Step 2: elliptic equation and polynomial Killing fields. The possibility of reducing formal Killing fields to polynomial ones relies on the following property: all coefficients of the formal series ζ_λ satisfy the elliptic equation

$$\left(\frac{\partial}{\partial z} \frac{\partial}{\partial \bar z} + |a|^2 \right) \hat\zeta_n = 0 \tag{34}$$

Recall that u being doubly periodic (since originally defined on $\mathbb{T}$), so are all these coefficients. The space of solutions to an elliptic equation on a compact Riemann surface is finite dimensional. Consider now the sequence $(\zeta_\lambda^m)_{m\geq 0}$ of (linearly independent) truncated formal Killing fields: $\zeta_\lambda^m = (\lambda^{-4m-2}\zeta_\lambda)_-$, truncated meaning that we keep only negative powers of λ. The image of the sequence by the operator $d + \mathrm{ad}_{\alpha_\lambda}$ is finite dimensional, indeed for any $\phi_\lambda = \lambda^{-4m-2}\zeta_\lambda = \sum_{n\geq -4m-2} \lambda^n \hat\phi_n$

$$\begin{aligned} (d + \mathrm{ad}_{\alpha_\lambda})(\phi_\lambda)_- &= d(\phi_\lambda)_- + [\alpha_\lambda, (\phi_\lambda)_-] = (d\phi_\lambda)_- - [(\phi_\lambda)_-, \alpha_\lambda] \\ &= [\phi_\lambda, \alpha_\lambda]_- - [(\phi_\lambda)_-, \alpha_\lambda] \\ &= [\lambda^{-2}\hat\phi_{-2} + \lambda^{-1}\hat\phi_{-1} + \hat\phi_0 + \lambda\hat\phi_1 + \lambda^2\hat\phi_2, \alpha_\lambda]_- \\ &\quad - [\lambda^{-2}\hat\phi_{-2} + \lambda^{-1}\hat\phi_{-1} + \hat\phi_0, \alpha_\lambda] \end{aligned}$$

All other terms either vanish in the truncation or compensate between the two brackets. Since each coefficient $\hat{\phi}_k$ belongs to a finite dimensional space, we have our claim. So a finite combination of the ζ_λ^m, call it ϕ, lies in the kernel of $d + \mathrm{ad}_{\alpha_\lambda}$, and is automatically adapted. Then $\xi_\lambda = \phi_\lambda + \bar{\phi}_\lambda$ is adapted, satisfies equation (32) and is real; that is ξ_λ belongs to some $\Lambda^d \mathfrak{g}_\tau$. □

Remark 8. The solutions of (32) we have constructed have the following property: ξ_λ has no term of Fourier exponent equal to 0 (mod 4), except the first and last ones; $\hat{\xi}_n = 0$ but for $\hat{\xi}_{-d} = (aL_i, 0)$ and $\hat{\xi}_d = \overline{\hat{\xi}_{-d}}$.

Remark 9. So far we have used a fixed complex coordinate z on $\mathbb{T}$ (or $\mathbb{C}$), but we might want to switch to another coordinate say $w = \mu z$. A quick look at (30) shows that a solution in the z variable is usually not valid in the w variable (that can also be seen on (31)). However the immersion stays of finite type whatever the coordinate may be; so an Hamiltonian stationary immersion can be of finite type d for some variable z and $d' \neq d$ for another variable w. The type itself is not a well-defined invariant, and an example of this will be given in the next section.

6.3. Finite type and lattice properties.

We will now use the information given by section 3 together with the finite type point of view. Let Γ be a lattice, with dual lattice Γ^*, and set $\mathbb{T} = \mathbb{C}/\Gamma$. For any Hamiltonian stationary conformal Lagrangian immersion X there exists $p \in \mathbb{N}$ and ξ_λ in $\Lambda^{4p+2}\mathfrak{g}_\tau$ solution of (32) projecting to the extended 1-form α_λ associated with X (more precisely with one of the spinor lifts of X). As mentioned in remark 8 above, we may assume that ξ_λ has no term of Fourier exponent equal to 0 mod 4. So we can write

$$\begin{aligned}\xi_\lambda = \lambda^{-4p-2}&\left(\frac{\pi\bar{\beta}_0}{2}L_i, 0\right)\\ &+ \sum_{q=-p}^{p}\left(\lambda^{4q-1}(0, u_q) + \lambda^{4q+1}(0, v_q) + \lambda^{4q+2}\left(\pi\frac{\pi\bar{\beta}_0 c_q}{2}L_i, 0\right)\right)\end{aligned}$$

with $\beta_0 \in \Gamma^* - 0$, $u_q(z) \in \mathfrak{g}_{-1}^{\mathbb{C}}$ and $v_q(z) \in \mathfrak{g}_1^{\mathbb{C}}$. Equation (31) can be rewritten using the Fourier expansion,

$$\frac{\partial \hat{\xi}_n}{\partial z} = [\hat{\xi}_{n+2}, \hat{\xi}_{-4p-2}] + [\hat{\xi}_{n+1}, \hat{\xi}_{-4p-1}]. \tag{35}$$

Commutation properties $[\mathfrak{g}_0^{\mathbb{C}}, \mathfrak{g}_2^{\mathbb{C}}] = [\mathfrak{g}_2^{\mathbb{C}}, \mathfrak{g}_2^{\mathbb{C}}] = [\mathfrak{g}_1^{\mathbb{C}}, \mathfrak{g}_{-1}^{\mathbb{C}}] = 0$ show that terms with even Fourier exponent – hence the c_q's – are constant (and given by the initial condition in (30)). We can then use (35) or its conjugate to derive a recurrence relation:

$$v_q = \frac{2}{\pi\bar{\beta}_0} L_i \frac{\partial u_q}{\partial z}$$

$$u_{q+1} = c_q u_{-p} + \frac{2}{\pi\bar{\beta}_0} L_i \frac{\partial v_q}{\partial z} = c_q u_{-p} - \left(\frac{2}{\pi\bar{\beta}_0}\right)^2 \frac{\partial^2 u_q}{\partial z^2} .$$

Conjugating equation (35) (or recalling (34)) one easily derives the second order equation $\Delta u_q + \pi^2 |\beta_0|^2 u_q = 0$, so all terms have Fourier frequencies $\gamma \in \frac{1}{2}\Gamma^*$ such that $|\gamma| = \frac{1}{2}|\beta_0|$. Thus we write $u_q = \sum_\gamma (\hat{a}_{\gamma,q}\epsilon + \hat{b}_{\gamma,q} L_i \bar{\epsilon}) e^{2i\pi\langle\gamma,z\rangle}$ and the recurrence relation yields

$$\hat{a}_{\gamma,q+1} = c_q \hat{a}_{\gamma,-p} + \left(\frac{2\bar{\gamma}}{\bar{\beta}_0}\right)^2 \hat{a}_{\gamma,q}$$

(same equation for $\hat{b}_{\gamma,q}$). Taking the first and last $\mathfrak{g}_{-1}^{\mathbb{C}}$ terms:

$$\hat{a}_{\gamma,p} = \hat{a}_{\gamma,-p} \left(\sum_{q=0}^{2p} c_{p-q-1} \left(\frac{2\bar{\gamma}}{\bar{\beta}_0}\right)^{2q} \right)$$

with the convention that $c_{-p-1} = 1$. Now we may compare both ends of the chain using the reality of ξ_λ: $v_p = \bar{u}_{-p}$, while

$$v_p = \frac{2}{\pi\bar{\beta}_0} L_i \frac{\partial u_p}{\partial z}$$

which yields (recalling from section 3 the expression for u_{-p}): $\gamma \hat{a}_{\gamma,-p} = -\bar{\gamma}\hat{a}_{\gamma,p}$. So for each frequency γ such that $\hat{a}_{\gamma,-p} \neq 0$

$$\sum_{q=0}^{2p} c_{p-q-1} \left(\frac{2\bar{\gamma}}{\bar{\beta}_0}\right)^{2q} = -\frac{\gamma}{\bar{\gamma}} .$$

We may rewrite this condition as a polynomial equation of degree exactly $d = 4p + 2$ in γ:

$$\gamma^{4p+2} + \frac{\bar{\beta}_0}{2} \left(\frac{\beta_0}{2}\right)^{4p+1} \sum_{q=0}^{2p} c_{q-p-1} \left(\frac{2\gamma}{\beta_0}\right)^{2q} = 0 . \tag{36}$$

We conclude with the following result:

Theorem 11. *A finite type solution with type $d \in 4\mathbb{N}+2$ and Lagrangian angle $\beta = 2\pi\langle \beta_0, z\rangle$ + constant has all its Fourier frequencies in $\Gamma^*_{\beta_0}$ and satisfying a polynomial equation of degree d, depending only on β_0 and the initial value in* (32) *(i.e., the constant potential). As a consequence* $\mathrm{Card}(\Gamma^*_{\beta_0}) \leq d$.

Genus zero solutions. We conclude with the study of the simplest case of type $d = 2$, also called genus zero solutions (the genus here being the genus of the associated *spectral curve*, see [Hi]). Then condition (36) implies that $\gamma = \pm i|\frac{1}{2}\beta_0|$. There are only two possibilities, and if the lattice is rectangular, we find – after a change of variable – the rectangular generalizations of the standard torus. It should be noted that the condition on γ implies that – *in the original complex coordinate* – the term $e^{-\beta L_i/2}\frac{\partial X}{\partial z}$ depends only on y. This is clearly variable-dependent, and shows that the type d may well change if one changes the variable.

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