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Covariant Poisson Structures on Complex Projective Spaces

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In this paper, we describe a one-parameter family of nonstandard $SU(n+1)$-covariant (also known as $SU(n+1)$-homogeneous) Poisson structures τ_c on the projective space $\mathbb{C}P^n$ that represents the classical counterpart of the quantum family $C\left(\mathbb{C}P^n_{q,c}\right)$, and show that the standard Poisson $\mathbb{S}^{2n-1}$ is indeed embedded in each of these nonstandard Poisson $\mathbb{C}P^n$. We explicitly describe the Lagrangian subalgebra associated with such a Poisson homogeneous space. We also give an elementary proof of the statement that (non-zero) $SU(n)$-invaraint contravariant alternating 2-tensors on $\mathbb{S}^{2n-1}$ with $n \neq 3$ (or $\mathbb{C}P^n$) are unique, up to a constant factor.

Introduction.

One of the most intriguing aspects of the theory of quantum groups and quantum spaces [Dr1, RTF, So, Wo1, Wo2, Po, Ri, VaSo] is the close interplay between the geometric structure of the underlying Poisson Lie group (or Poisson space) [We2, LuWe1] and the algebraic structure on the corresponding quantum group (or quantum space). For example, Soibelman's classification [So] of all irreducible *-representations of the quantum algebra $C(G_q)$ for compact Poisson simple Lie groups G gives a one-to-one correspondence between irreducible *-representations of $C(G_q)$ and the symplectic leaves [We2] on G. This leads to a groupoid C*-algebraic [Re] approach to study the structure of the algebra $C(G_q)$ [Sh2] which shows that the decomposition of $SU(n)$ (or $\mathbb{S}^{2n+1}$) by symplectic leaves of various dimensions corresponds to a compatible decomposition of $C\left(SU(n)_q\right)$ (or $C\left(\mathbb{S}_q^{2n+1}\right)$) by its (closed) ideals in the spirit of noncommutative geometry.

Given a Poisson Lie group G with Poisson structure π, we call a Poisson structure τ on a homogeneous space M of G a (G,π)-covariant [LuWe2] (also known as (G,π)-homogeneous [Dr2]) Poisson structure on M if the

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G-action map $G \times M \to M$ is a Poisson map with respect to the product Poisson structure $\pi \oplus \tau$ on $G \times M$ and the Poisson structure τ on M. It is well known that the standard multiplicative Poisson structure on $SU(n+1)$ induces a standard covariant Poisson structure on the homogeneous spaces $\mathbb{S}^{2n+1} = SU(n+1)/SU(n)$ and $\mathbb{C}P^n = SU(n+1)/U(n)$ determined by the Poisson Lie subgroups $SU(n)$ and $U(n)$, respectively. On the other hand, Lu and Weinstein [LuWe2] described explicitly all $SU(2)$-covariant Poisson structures on $\mathbb{S}^2 = \mathbb{C}P^1$ including a one-parameter family of nonstandard $SU(2)$-covariant Poisson structures on $\mathbb{S}^2$, and showed that each nonstandard covariant Poisson sphere $\mathbb{S}^2_c$ contains a copy of the trivial Poisson 1-sphere $\mathbb{S}^1$ (consisting of a circle family of 0-dimensional symplectic leaves) and exactly two 2-dimensional symplectic leaves. This geometric structure is reflected faithfully in the algebraic structure of the algebra $C(\mathbb{S}^2_{qc})$ of the nonstandard quantum spheres $\mathbb{S}^2_{qc}$ [Sh1].

Dijkhuizen and Noumi studied in great detail [DiNo] a one-parameter family of nonstandard quantum projective spaces $\mathbb{C}P^n_{q,c}$ with quantum algebras $C(\mathbb{C}P^n_{q,c})$. In [Sh3], the structure of $C(\mathbb{C}P^n_{q,c})$ is studied and analyzed as a groupoid C*-algebra, and an algebraic decomposition of $C(\mathbb{C}P^n_{q,c})$ by a closed ideal indicates that the underlying nonstandard projective space Poisson $\mathbb{C}P^n$ should contain an embeded copy of the standard Poisson $\mathbb{S}^{2n-1}$. Using the result of [KhRaRu], one can classify all $SU(n+1)$-covariant Poisson structures on $\mathbb{C}P^n$ into a one-parameter family. In this paper, we describe the part of this one-parameter family of Poisson structures on $\mathbb{C}P^n$ that represents the classical counterpart of the quantum family $C(\mathbb{C}P^n_{q,c})$, and show that the standard Poisson $\mathbb{S}^{2n-1}$ is indeed embedded in each of these nonstandard Poisson $\mathbb{C}P^n$. Furthermore we find explicitly the Lagrangian subalgebras [Dr2] associated with these nonstandard $SU(n+1)$-covariant Poisson structures on $\mathbb{C}P^n$. We also give an elementary proof of the fact that (non-zero) $SU(n)$-invaraint (contravariant alternating) 2-tensors on $\mathbb{S}^{2n-1}$ with $n \neq 3$ (or on $\mathbb{C}P^n$) are unique, up to a constant factor. We remark that in [KhRaRu], Khoroshkin, Radul, and Rubtsov obtained interesting results about covariant Poisson structures on coadjoint orbits, including $\mathbb{C}P^n$. Our approach, motivated by Dijkhuizen and Noumi's work [DiNo], is different from theirs and the embedding of the standard Poisson $\mathbb{S}^{2n-1}$ in the nonstandard Poisson $\mathbb{C}P^n$ is new.

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1. Poisson structure on Lie groups.

In this section, we discuss some basic properties of affine Poisson structures in the form needed later. We recall that an affine Poisson structure on a Lie group G is given by a Poisson 2-tensor $\pi \in \Gamma\left(\wedge^2 TG\right)$, such that

$$\pi(gh) = L_g(\pi(h)) + R_h(\pi(g)) - L_g R_h(\pi(e))$$

for any $g, h \in G$ [We3], or equivalently,

$$\pi_l(g) := \pi(g) - L_g(\pi(e))$$

for $g \in G$ defines a multiplicative Poisson 2-tensor on G [Lu, DaSo], where L_g and R_g are the left and the right actions by $g \in G$, respectively, and e is the identity element of G. For an affine Poisson 2-tensor π on a Lie group G, the left action of the Poisson-Lie group (G, π_l) on the Poisson manifold (G, π) by left translation is a Poisson action, i.e., the multiplication map $G \times G \to G$ is a Poisson map, where $G \times G$ and G are endowed with the Poisson structures $\pi_l \times \pi$ and π, respectively. In another word, π on G (as a homogeneous space of G) is a (left) (G, π_l)-covariant Poisson structure.

A typical example of an affine Poisson structure on a Poisson-Lie group G with multiplicative Poisson 2-tensor π is provided by a right translation π_σ of π by an element $\sigma \in G$, i.e.,

$$\pi_\sigma(g) := R_\sigma\left(\pi\left(g\sigma^{-1}\right)\right)$$

for $g \in G$. Since the right translation by σ on G is a diffeomorphism on G, the 'push-forward' π_σ of π by R_σ is clearly a Poisson 2-tensor on G. Furthermore,

$$\begin{aligned}
(\pi_\sigma)_l(g) &= R_\sigma\left(\pi\left(g\sigma^{-1}\right)\right) - L_g\left(R_\sigma\left(\pi\left(\sigma^{-1}\right)\right)\right) \\
&= R_\sigma\left(L_g\left(\pi\left(\sigma^{-1}\right)\right) + R_{\sigma^{-1}}(\pi(g))\right) - L_g\left(R_\sigma\left(\pi\left(\sigma^{-1}\right)\right)\right) \\
&= R_\sigma L_g\left(\pi\left(\sigma^{-1}\right)\right) + \pi(g) - R_\sigma L_g\left(\pi\left(\sigma^{-1}\right)\right) = \pi(g)
\end{aligned}$$

which is a multiplicative Poisson 2-tensor on G. So π_σ on G (as a homogeneous space of G) is a (left) (G, π)-covariant Poisson structure, for any $\sigma \in G$. Note that

$$\pi_\sigma = \pi + (X_\sigma)^l$$

where $X_\sigma := \pi_\sigma(e) = R_\sigma\left(\pi\left(\sigma^{-1}\right)\right) \in \mathfrak{g} \wedge \mathfrak{g}$ and $X^l(g) := L_g(X)$ is the left-invariant 2-tensor generated by $X \in \mathfrak{g} \wedge \mathfrak{g}$, since $(\pi_\sigma)_l = \pi$.

When a closed subgroup H of a Lie group G is coisotropic [Wel] with respect to a Poisson structure ρ on G, i.e.,

$$\rho(gh) - R_h(\rho(g)) \in L_{gh}(\mathfrak{h} \wedge \mathfrak{g})$$

for all $g \in G$ and $h \in H$, it is easy to see that the Poisson bracket $\{f_1, f_2\} := (df_1 \wedge df_2)(\rho)$ of $f_1, f_2 \in C^\infty(G/H) \subset C^\infty(G)$ is still in $C^\infty(G/H)$ and hence induces a Poisson structure on G/H, or equivalently, a Poisson 2-tensor $\tilde{\rho}$ on the homogeneous space G/H is well defined by

$$\tilde{\rho}([gH]) := [\rho(g)] \in L_g\left(\wedge^2(\mathfrak{g}/\mathfrak{h})\right) = \wedge^2 T_{[gH]}(G/H).$$

Given a Poisson-Lie group (G, π) and $\sigma \in G$, if a closed subgroup H of G is coisotropic with respect to π, then H is coisotropic with respect to the affine Poisson structure π_σ on G if and only if

$$(X_\sigma)^l(gh) - R_h\left((X_\sigma)^l(g)\right) \in L_{gh}(\mathfrak{h} \wedge \mathfrak{g}),$$

since $\pi_\sigma = \pi + (X_\sigma)^l$ and $\pi(gh) - R_h(\pi(g)) \in L_{gh}(\mathfrak{h} \wedge \mathfrak{g})$. Now

$$\begin{aligned}
(X_\sigma)^l(gh) - R_h\left((X_\sigma)^l(g)\right) &= L_{gh}(X_\sigma) - R_h(L_g(X_\sigma)) \\
&= L_g(L_h X_\sigma - R_h X_\sigma) \\
&= L_g L_h(X_\sigma - L_{h^{-1}} R_h X_\sigma) \\
&= L_{gh}(\mathrm{id} - \mathrm{Ad}_{h^{-1}})(X_\sigma).
\end{aligned}$$

So $(X_\sigma)^l(gh) - R_h\left((X_\sigma)^l(g)\right) \in L_{gh}(\mathfrak{h} \wedge \mathfrak{g})$ for all $(g, h) \in G \times H$ if and only if

$$(\mathrm{id} - \mathrm{Ad}_{h^{-1}})(X_\sigma) \in \mathfrak{h} \wedge \mathfrak{g}$$

for all $h \in H$, or equivalently,

$$\mathrm{ad}_\mathfrak{h}(X_\sigma) \subset \mathfrak{h} \wedge \mathfrak{g}.$$

Thus we get the following result.

Proposition 1. *Given a Poisson-Lie group (G, π) and a closed subgroup H of G that is coisotropic with respect to π, the subgroup H is coisotropic with respect to π_σ for $\sigma \in G$, if and only if* $\mathrm{ad}_\mathfrak{h}(X_\sigma) \subset \mathfrak{h} \wedge \mathfrak{g}$, *where* $X_\sigma := R_\sigma\left(\pi\left(\sigma^{-1}\right)\right) \in \mathfrak{g} \wedge \mathfrak{g}$.

In case the multiplicative Poisson structure π on G is given by an r-matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$ (satisfying the modified Yang-Baxter equation), i.e.,

$$\pi(g) = L_g r - R_g r,$$

we have

$$X_\sigma = R_\sigma\left(\pi\left(\sigma^{-1}\right)\right) = R_\sigma\left(L_{\sigma^{-1}} r - R_{\sigma^{-1}} r\right) = \mathrm{Ad}_{\sigma^{-1}}(r) - r.$$

A closed subgroup H being coisotropic with respect to π is equivalent to

$$\mathrm{ad}_{\mathfrak{h}}(r) \subset \mathfrak{h} \wedge \mathfrak{g},$$

since

$$\begin{aligned}
\pi(gh) - R_h(\pi(g)) &= L_{gh} r - R_{gh} r - R_h\left(L_g r - R_g r\right) \\
&= L_{gh} r - R_{gh} r - L_g R_h r + R_h R_g r = L_{gh} r - L_g R_h r \\
&= L_{gh}\left(r - L_{h^{-1}} R_h r\right) = L_{gh}\left(\mathrm{id} - \mathrm{Ad}_{h^{-1}}\right)(r)
\end{aligned}$$

and $L_{gh}\left(\mathrm{id} - \mathrm{Ad}_{h^{-1}}\right)(r) \in L_{gh}(\mathfrak{h} \wedge \mathfrak{g})$ for all $(g,h) \in G \times H$ if and only if

$$\left(\mathrm{id} - \mathrm{Ad}_{h^{-1}}\right)(r) \in \mathfrak{h} \wedge \mathfrak{g}$$

for all $h \in H$.

Corollary 2. *Given a Poisson-Lie group (G, π) with π defined by an r-matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$, a π-coisotropic closed subgroup H of G is coisotropic with respect to π_σ for $\sigma \in G$, if and only if* $\mathrm{ad}_{\mathfrak{h}}\left(\mathrm{Ad}_{\sigma^{-1}}(r)\right) \subset \mathfrak{h} \wedge \mathfrak{g}$.

2. Non-standard Poisson $\mathbb{C}P^n$.

In this section, we construct via affine Poisson structures on $SU(n)$ some $SU(n)$-covariant Poisson structures on $\mathbb{C}P^{n-1}$ that represent the classical counterpart of the nonstandard quantum projective spaces $\mathbb{C}P^n_{q,c}$ [DiNo].

Recall that the standard Poisson $SU(n)$ is defined (up to a constant multiple) by the Poisson 2-tensor $\pi(u) = \pi^{(n)}(u) := L_u r - R_u r$ determined by the r-matrix

$$r := \sum_{1 \leq i < j \leq n} X_{ij}^+ \wedge X_{ij}^-$$

where $X_{ij}^+ = e_{ij} - e_{ji}$, $X_{ij}^- = i\left(e_{ij} + e_{ji}\right)$, and e_{ij} are the matrix units.

It is well known that $SU(n-1) \cong \{1\} \oplus SU(n-1)$ (or $SU(n-1) \oplus \{1\}$) and

$$U(n-1) \cong \left\{ \det(u)^{-1} \oplus u : u \in U(n-1) \right\}$$

are Poisson-Lie subgroups of $SU(n)$ and hence induce the 'standard' $SU(n)$-covariant Poisson structures $\rho = \rho^{(n)}$ (also called Bruhat Poisson structure [LuWe1]) and $\tau = \tau^{(n-1)}$ on the sphere

$$\mathbb{S}^{2n-1} \cong SU(n) / [\{1\} \oplus SU(n-1)]$$

and the complex projective space

$$\mathbb{C}P^{n-1} \cong SU(n) / U(n-1),$$

respectively.

Theorem 3. *The closed subgroup $U(n-1)$ of $SU(n)$ is coisotropic with respect to the (left) $SU(n)$-covariant affine Poisson structure π_{σ_c} on $SU(n)$ defined by*

$$\sigma_c := \left(\sqrt{c} e_{11} + \sqrt{1-c} e_{n1} - \sqrt{1-c} e_{1n} + \sqrt{c} e_{nn} \right) + \sum_{i=2}^{n-1} e_{ii} \in SU(n)$$

with $c \in [0,1]$. Hence π_{σ_c} induces a (left) $SU(n)$-covariant Poisson structure τ_c on $\mathbb{C}P^{n-1} \cong SU(n) / U(n-1)$.

Proof. We set $\sigma = \sigma_c$ for simplicity. It is easy to see that if the Poisson structure τ_c induced by π_{σ_c} on $\mathbb{C}P^{n-1} \cong SU(n) / U(n-1)$ is well defined, then it is automatically (left) $SU(n)$-covariant since π_{σ_c} is. Now since $U(n-1)$ is a Poisson-Lie subgroup and hence coisotropic with respect to π, we have

$$\mathrm{ad}_{\mathfrak{u}(n-1)}(r) \subset \mathfrak{u}(n-1) \wedge \mathfrak{su}(n),$$

and hence only need to show that

$$\mathrm{ad}_{\mathfrak{u}(n-1)}(\mathrm{Ad}_{\sigma^{-1}}(r)) \subset \mathfrak{u}(n-1) \wedge \mathfrak{su}(n).$$

From

$$\begin{cases} \mathrm{Ad}_{\sigma^{-1}}\left(X_{ij}^{+}\right) = X_{ij}^{+}, & \text{if } 1 < i < j < n \\ \mathrm{Ad}_{\sigma^{-1}}\left(X_{1j}^{+}\right) = \sqrt{c} X_{1j}^{+} + \sqrt{1-c} X_{jn}^{+}, & \text{if } 1 < j < n \\ \mathrm{Ad}_{\sigma^{-1}}\left(X_{in}^{+}\right) = -\sqrt{1-c} X_{1i}^{+} + \sqrt{c} X_{in}^{+}, & \text{if } 1 < i < n \\ \mathrm{Ad}_{\sigma^{-1}}\left(X_{1n}^{+}\right) = X_{1n}^{+}, & \end{cases}$$

and

$$\begin{cases} \mathrm{Ad}_{\sigma^{-1}}\left(X_{ij}^{-}\right) = X_{ij}^{-}, & \text{if } 1 < i < j < n \\ \mathrm{Ad}_{\sigma^{-1}}\left(X_{1j}^{-}\right) = \sqrt{c}X_{1j}^{-} - \sqrt{1-c}X_{jn}^{-}, & \text{if } 1 < j < n \\ \mathrm{Ad}_{\sigma^{-1}}\left(X_{in}^{-}\right) = \sqrt{1-c}X_{1i}^{-} + \sqrt{c}X_{in}^{-}, & \text{if } 1 < i < n \\ \mathrm{Ad}_{\sigma^{-1}}\left(X_{1n}^{-}\right) = (2c-1)\, X_{1n}^{-} & \\ \qquad +2i\sqrt{c\,(1-c)}\,(e_{11} - e_{nn})\,. & \end{cases}$$

we get

$$\begin{aligned} &\mathrm{Ad}_{\sigma^{-1}}(r) \\ &= \sum_{1\le i<j\le n} \mathrm{Ad}_{\sigma^{-1}}\left(X_{ij}^{+}\right) \wedge \mathrm{Ad}_{\sigma^{-1}}\left(X_{ij}^{-}\right) \\ &= \sum_{1<i<j<n} X_{ij}^{+} \wedge X_{ij}^{-} + X_{1n}^{+} \wedge \left[(2c-1)\, X_{1n}^{-} + 2\sqrt{c\,(1-c)}i\,(e_{11} - e_{nn})\right] \\ &\quad + \sum_{1<i<n} \left(\sqrt{c}X_{1i}^{+} + \sqrt{1-c}X_{in}^{+}\right) \wedge \left(\sqrt{c}X_{1i}^{-} - \sqrt{1-c}X_{in}^{-}\right) \\ &\quad + \sum_{1<i<n} \left(-\sqrt{1-c}X_{1i}^{+} + \sqrt{c}X_{in}^{+}\right) \wedge \left(\sqrt{1-c}X_{1i}^{-} + \sqrt{c}X_{in}^{-}\right) \\ &= 2\,(1-c) \sum_{1<i<j<n} X_{ij}^{+} \wedge X_{ij}^{-} + 2\sqrt{c\,(1-c)}X_{1n}^{+} \wedge i\,(e_{11} - e_{nn}) \\ &\quad + (2c-1)\, r + 2\sqrt{c\,(1-c)} \sum_{1<i<n} \left(X_{in}^{+} \wedge X_{1i}^{-} - X_{1i}^{+} \wedge X_{in}^{-}\right) \end{aligned}$$

which is in $(2c-1)\,r + (\mathfrak{u}\,(n-1) \wedge \mathfrak{su}\,(n))$, since

$$X_{ij}^{+}, X_{ij}^{-}, X_{in}^{+}, X_{in}^{-}, i\,(e_{11} - e_{nn}) \in \mathfrak{u}\,(n-1)$$

for all $1 < i < j < n$. So we get

$$\begin{aligned} &\mathrm{ad}_{\mathfrak{u}(n-1)}\left(\mathrm{Ad}_{\sigma^{-1}}(r)\right) \\ &\quad \subset (2c-1)\,\mathrm{ad}_{\mathfrak{u}(n-1)}(r) + \mathrm{ad}_{\mathfrak{u}(n-1)}\left(\mathfrak{u}\,(n-1) \wedge \mathfrak{su}\,(n)\right) \\ &\quad \subset \mathfrak{u}\,(n-1) \wedge \mathfrak{su}\,(n)\,, \end{aligned}$$

because $\mathrm{ad}_{\mathfrak{u}(n-1)}(r) \subset \mathfrak{u}\,(n-1) \wedge \mathfrak{su}\,(n)$ and

$$\mathrm{ad}_{\mathfrak{u}(n-1)}\left(\mathfrak{u}\,(n-1) \wedge \mathfrak{su}\,(n)\right) \subset \mathfrak{u}\,(n-1) \wedge \mathfrak{su}\,(n)\,.$$

□

The Poisson manifold $\left(\mathbb{C}P^{n-1}, \tau_c\right)$ with $c \in (0,1)$ is referred to as a nonstandard Poisson $\mathbb{C}P^{n-1}$. Note that $\tau_1 = \tau^{(n-1)}$ the standard Poisson structure on $\mathbb{C}P^{n-1}$ since $\sigma_1 = 1 \in SU(n)$ and hence $\pi_{\sigma_1} = \pi$. On the other hand, R_{σ_0} simply swaps the first column with the n-th column and hence τ_0 is the standard Poisson structure on $\mathbb{C}P^{n-1} \cong SU(n) / \left\{u \oplus \det(u)^{-1} : u \in U(n-1)\right\}$ induced by π.

We remark that the above nonstandard Poisson structures on $\mathbb{C}P^{n-1}$ form only a part of the one-parameter family of all $SU(n)$-covariant Poisson structures on $\mathbb{C}P^{n-1}$ that can be classified using the results of [KhRaRu] and [Kos2] as follows. (We thank the referee for providing this argument.) Indeed it is easy to see, as in [LuWe2], that the difference of two covariant Poisson 2-tensors is an invariant alternating 2-tensor. Applying the known fact that the p-th de Rham cohomology of the Grassmannian G_n^k of k-dimensional subspaces in $\mathbb{C}^n$ is isomorphic to the space of $SU(n)$-invariant p-forms on G_n^k [Kos2] to the case of $k=1$ and $p=2$, we see that the known de Rham cohomology group $H^2_{DR}\left(\mathbb{C}P^{n-1}\right) \cong \mathbb{R}$ (for $n \geq 2$) is isomorphic to the space of all $SU(n)$-invariant 2-forms on $\mathbb{C}P^{n-1}$. So every $SU(n)$-invariant 2-form, or equivalently, every $SU(n)$-invariant 2-tensor (by duality via the standard $U(1)$-invariant Euclidean structure of $\mathbb{C}^n$) on $\mathbb{C}P^{n-1}$ is a scalar multiple of the $SU(n)$-invariant Fubini-Study symplectic form on the compact Kähler manifold $\mathbb{C}P^{n-1}$, or equivalently, the corresponding nondegenerate $SU(n)$-invariant 2-tensor $\tilde{\tau}$ on $\mathbb{C}P^{n-1}$. (In section 4, we give an elementary direct proof of this fact.) So every $SU(n)$-covariant Poisson 2-tensor on $\mathbb{C}P^{n-1}$ belongs to the family $\tau_1 + \mathbb{R}\tilde{\tau}$. On the other hand, it is easy to see that the sum of an $SU(n)$-covariant 2-tensor and an $SU(n)$-invariant 2-tensor is $SU(n)$-covariant. Furthermore it is known [KhRaRu] that the Schouten-Nijenhuis bracket $[[\tilde{\tau}, \tau_1]] = 0$ and hence every 2-tensor in $\tau_1 + \mathbb{R}\tilde{\tau}$ is a Poisson 2-tensor. (In fact, the covariance of τ_1 implies that the 3-tensor $[[\tilde{\tau}, \tau_1]]$ is $SU(n)$-invariant. However since $H^3_{DR}\left(\mathbb{C}P^{n-1}\right) = 0$, there is no non-zero $SU(n)$-invariant 3-tensor on $\mathbb{C}P^{n-1}$.) Thus $\tau_1 + \mathbb{R}\tilde{\tau}$ consists of all $SU(n)$-covariant Poisson 2-tensors on $\mathbb{C}P^{n-1}$.

In [Dr2], Drinfeld associated a Lagrangian subalgebra of the double Lie algebra $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$ with each G-covariant Poisson structure τ on a homogeneous space M of G. More precisely,

$$\mathfrak{i}_m := \left\{(x, \xi) \in \mathfrak{g} \times (\mathfrak{g}_m)^{\perp} : x + \mathfrak{g}_m = \xi \lrcorner \tau(m)\right\}$$

[Dr2, ELu] is the Lagrangian subalgebra associated with (M, τ) at a point $m \in M$, where $\mathfrak{g}_m \subset \mathfrak{g}$ is the Lie algebra of the stabilizer subgroup of G at m, the coset space $\mathfrak{g}/\mathfrak{g}_m$ is identified with the tangent space $T_m M$ via the differential of the action $g \in G \mapsto gm \in M$ at the neutral element e, and $(\mathfrak{g}_m)^{\perp} \subset \mathfrak{g}^*$ consisting of linear functionals of $\mathfrak{g}$ that annihilate $\mathfrak{g}_m$ is identified with the cotangent space $T_m^* M$. Since the correspondence $m \mapsto \mathfrak{i}_m$ is G-equivariant [Dr2], $\mathfrak{i}_{m_0}$ for any specific $m_0 \in M$ determines the other $\mathfrak{i}_m$. Interesting results have been obtained recently on this topic. For example, Karolinsky classified all Lagrangian subalgebras of the double Lie algebra $\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{su}(n) \bowtie \mathfrak{su}(n)^*$ [Ka], and Evens and Lu studied the geometry of the variety of all Lagrangian subalgebras of $\mathfrak{sl}(n, \mathbb{C})$ [ELu]. In the following, we identify explicitly the Lagrangian subalgebra associated with the $SU(n)$-covariant nonstandard Poisson $\left(\mathbb{C}P^{n-1}, \tau_c\right)$ at the point $[(1,0,..,0)]$, and we denote by $\left\{X_+^{ij}, X_-^{ij}\right\}_{1\le i<j\le n} \cup \left\{E^k\right\}_{k=2}^n \subset \mathfrak{su}(n)^*$ the basis dual to the basis $\left\{X_{ij}^+, X_{ij}^-\right\}_{1\le i<j\le n} \cup \{i(e_{11} - e_{kk})\}_{k=2}^n \subset \mathfrak{su}(n)$.

Proposition 4. *The Lagrangian subalgebra* $\mathfrak{i}_{[(1,0,..,0)]}$ *associated with the* $SU(n)$*-covariant nonstandard Poisson homogeneous space* $\left(\mathbb{C}P^{n-1}, \tau_c\right)$ *at the point* $[(1,0,..,0)] \in \mathbb{C}P^{n-1}$ *is the linear span of* $\mathfrak{u}(n-1) \times \{0\}$ *and the vectors* $\left(\pm 2(c-1) X_{1j}^{\mp}, X_{\pm}^{1j}\right)$ *with* $1 < j \le n$ *in* $\mathfrak{su}(n) \bowtie \mathfrak{su}(n)^*$.

Proof. The stabilizer subgroup of $G = SU(n)$ at $m = [(1,0,\ldots,0)] \in \mathbb{C}P^{n-1}$ is $U(n-1)$, and the differential η at $e \in SU(n)$ of the action map $g \in SU(n) \mapsto g[(1,0,\ldots,0)] \in \mathbb{C}P^{n-1}$ sends e to $[(1,0,\ldots,0)]$ and identifies $\mathfrak{su}(n)/\mathfrak{u}(n-1)$ with $T_{[(1,0,\ldots,0)]}\mathbb{C}P^{n-1}$. Since $\tau_c([(1,0,\ldots,0)])$ is the projection of $\pi_{\sigma_c}(e)$ under η, the Lagrangian subalgebra $\mathfrak{i}_{[(1,0,\ldots,0)]}$ associated with $\left(\mathbb{C}P^{n-1}, \tau_c\right)$ at the point $[(1,0,\ldots,0)]$ equals

$$\begin{aligned}&\left\{(x,\xi) \in \mathfrak{su}(n) \times \mathfrak{u}(n-1)^{\perp} : x \equiv \xi \lrcorner \pi_{\sigma_c}(e) \pmod{\mathfrak{u}(n-1)}\right\} \\ &= \left\{(\xi \lrcorner \pi_{\sigma_c}(e), \xi) : \xi \in \mathfrak{u}(n-1)^{\perp}\right\} + \mathfrak{u}(n-1) \times \{0\},\end{aligned}$$

which is the linear span of $\mathfrak{u}(n-1) \times \{0\}$ and the vectors $\left(X_{\pm}^{1j} \lrcorner \pi_{\sigma_c}(e), X_{\pm}^{1j}\right)$ with $1 < j \le n$ in $\mathfrak{su}(n) \bowtie \mathfrak{su}(n)^*$ because $\left\{X_+^{1j}, X_-^{1j}\right\}_{j=2}^n$ is a basis of $\mathfrak{u}(n-1)^{\perp}$. From the proof of the above theorem, we have

$$\begin{aligned}\pi_{\sigma_c}(e) &= R_{\alpha_c}\left(\pi\left(\sigma_c^{-1}\right)\right) = \mathrm{Ad}_{\sigma_c^{-1}}(r) - r \\ &= 2(1-c)\sum_{1<i<j<n} X_{ij}^{+}\wedge X_{ij}^{-} + 2\sqrt{c(1-c)}X_{1n}^{+}\wedge i(e_{11}-e_{nn}) \\ &\quad + 2(c-1)r + 2\sqrt{c(1-c)}\sum_{1<i<n}\left(X_{in}^{+}\wedge X_{1i}^{-} - X_{1i}^{+}\wedge X_{in}^{-}\right) \\ &= 2(c-1)\left(\sum_{1<j\le n} X_{1j}^{+}\wedge X_{1j}^{-} + \sum_{1<i<n} X_{in}^{+}\wedge X_{in}^{-}\right) \\ &\quad + 2\sqrt{c(1-c)}X_{1n}^{+}\wedge i(e_{11}-e_{nn}) \\ &\quad + 2\sqrt{c(1-c)}\sum_{1<i<n}\left(X_{in}^{+}\wedge X_{1i}^{-} - X_{1i}^{+}\wedge X_{in}^{-}\right),\end{aligned}$$

and hence

$$X_{\pm}^{1j} \lrcorner\, \pi_{\sigma_c}(e) \equiv \pm 2(c-1)\, X_{1j}^{\mp}\ (\mathrm{mod}\ \mathfrak{u}(n-1)).$$

Thus the Lagrangian subalgebra $\mathfrak{i}_{[(1,0,..,0)]}$ is the linear span of $\mathfrak{u}(n-1)\times\{0\}$ and the vectors $\left(\pm 2(c-1)X_{1j}^{\mp}, X_{\pm}^{1j}\right)$ with $1<j\le n$ in $\mathfrak{su}(n)\bowtie\mathfrak{su}(n)^{*}$. □

3. Standard Poisson sphere in $\mathbb{C}P^n$.

In this section, we show that the nonstandard Poisson $\left(\mathbb{C}P^{n-1},\tau_c\right)$ contains a copy of the standard Poisson $\mathbb{S}^{2n-3}$ generalizing the result of [LuWe2] for $n=1$.

We first remark that $X_{ij}^{+}, X_{ij}^{-}, X_{in}^{+}, X_{in}^{-} \in \mathfrak{su}(n-1)$ but $i(e_{11}-e_{nn}) \notin \mathfrak{su}(n-1)$, so $\mathrm{ad}_{\mathfrak{su}(n-1)}\left(\mathrm{Ad}_{\sigma^{-1}}(r)\right) \nsubseteq \mathfrak{su}(n-1)\wedge\mathfrak{su}(n)$ and hence π_{σ_c} does not induce a Poisson structure on $\mathbb{S}^{2n-1}\cong SU(n)/SU(n-1)$. On the other hand, as a generalization of Lu and Weinstein's result on covariant Poisson spheres $\mathbb{S}^2=\mathbb{C}P^1$ [LuWe2], we can show that $\left(\mathbb{C}P^{n-1},\tau_c\right)$ contains a copy of the standard Poisson sphere $\left(\mathbb{S}^{2n-3},\rho^{(n-1)}\right)$. Here it is understood that $\rho^{(1)}=0$ on $\mathbb{S}^1$ by definition.

Theorem 5. *The standard Poisson sphere* $\left(\mathbb{S}^{2n-3},\rho^{(n-1)}\right)$ *is embedded in* $\left(\mathbb{C}P^{n-1},\tau_c\right)$ *for* $c\in(0,1)$ *and* $n\ge 2$.

Proof. Note that the quotient map $\phi : SU(n) \to \mathbb{C}P^{n-1}$ can be viewed as the composition of the quotient map

$$\phi_1 : u \in SU(n) \mapsto u_1 \in \mathbb{S}^{2n-1} \cong SU(n)/SU(n-1)$$

and the quotient map

$$\phi_2 : v \in \mathbb{S}^{2n-1} \mapsto [v] \in \mathbb{C}P^{n-1} \cong \mathbb{S}^{2n-1}/\mathbb{T},$$

where the circle group $\mathbb{T}$ acts diagonally on $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ and

$$u_1 := (u_{11}, u_{21}, \ldots, u_{n1}) \in \mathbb{S}^{2n-1} \subset \mathbb{C}^n$$

is the first column of $u \in SU(n)$. It is well known that ϕ_2 is a diffeomorphism from the submanifold

$$S_+ := \left\{ v \in \mathbb{S}^{2n-1} : v_1 > 0 \right\} \subset \mathbb{S}^{2n-1}$$

onto its image $\phi_2(S_+) \subset \mathbb{C}P^{n-1}$, and

$$\phi_3 : v \in S_c \mapsto \phi_3(v) := \frac{1}{\sqrt{1-c}}(v_2, \ldots, v_n) \in \mathbb{S}^{2n-3}$$

is a diffeomorphism identifying

$$S_c := \left\{ v \in \mathbb{S}^{2n-1} : v_1 = \sqrt{c} \right\} \subset S_+$$

with $\mathbb{S}^{2n-3}$. We denote by $\psi : u \in SU(n) \mapsto u_n \in \mathbb{S}^{2n-1}$ the projection to the last column. Functions similar to ϕ_1, ϕ_2, and ψ, for other dimensions than n, will be denoted by the same symbols for the simplicity of notation. First we assume that $n > 2$. For each $v \in S_c$, we can find some $u' \in SU(n)$ with the first column $u'_1 = (1, 0, \ldots, 0)$ and the last column $u'_n = \sqrt{1-c}^{-1}(0, v_2, \ldots, v_n)$. Note that the first row of u' has to be $(1, 0, \ldots, 0)$, and hence

$$u' = 1 \oplus u'' \in \{1\} \oplus SU(n-1)$$

for some $u'' \in SU(n-1)$ with

$$(u'')_{n-1} = \sqrt{1-c}^{-1}(v_2, \ldots, v_n) = \phi_3(v).$$

Furthermore since $\{1\} \oplus SU(n-1)$ is a Poisson-Lie subgroup of $SU(n)$,

$$\pi(u') = 0 \oplus \pi^{(n-1)}(u'') \in \{0\} \oplus \wedge^2 T_{u''} SU(n-1) \subset \wedge^2 T_{u'} SU(n)$$

where $\pi^{(n-1)}$ is the standard multiplicative Poisson structure on $SU(n-1)$. Note that

$$\rho^{(n-1)}(\phi_3(v)) = (D\psi)_{u''}\left(\pi^{(n-1)}(u'')\right)$$

for the standard Poisson 2-tensor $\rho^{(n-1)}$ on $\mathbb{S}^{2n-3}$. Here we take

$$\mathbb{S}^{2n-3} = SU(n-1)/[SU(n-2)\oplus\{1\}].$$

For

$$u := R_{\sigma_c}(u') = u'\sigma_c \in SU(n),$$

we have

$$\phi_1(u) = u_1 = (u'\sigma_c)_1 = v \in S_c,$$

and in $\wedge^2 T_v S_+$,

$$\begin{aligned}(D\phi_1)_u(\pi_{\sigma_c}(u)) &= (D\phi_1)_u(R_{\sigma_c}(\pi(u'))) = (D\phi_1)_u(\pi(u')\sigma_c)\\ &= \sqrt{1-c}(D\psi)_{u'}(\pi(u')) \in \wedge^2 T_v S_c \subset \wedge^2 T_v S_+\end{aligned}$$

because the first columns of the component matrices in the 2-tensor $\pi(u') = 0\oplus\pi^{(n-1)}(u'')$ are all zero. Note that

$$\begin{aligned}\tau_c([v]) &= \tau_c(\phi_2(v)) = \tau_c(\phi(u))\\ &= (D\phi)_u(\pi_{\sigma_c}(u)) = (D\phi_2)_{\phi_1(u)}((D\phi_1)_u(\pi_{\sigma_c}(u)))\end{aligned}$$

is a well-defined 2-tensor at $[v] \in \phi_2(S_c) \subset \mathbb{C}P^{n-1}$ and ϕ_2 is a diffeomorphism on S_+. So

$$\pi' : v \in S_c \mapsto (D\phi_1)_u(\pi_{\sigma_c}(u)) \in \wedge^2 T_v S_c$$

is a well-defined Poisson 2-tensor on S_c and $\phi_2(S_c)$ is a Poisson submanifold of $(\mathbb{C}P^{n-1}, \tau_c)$ that is Poisson isomorphic to (S_c, π'). Under the diffeomorphism $\phi_3 : S_c \to \mathbb{S}^{2n-3}$ identifying $v \in S_c$ with $\phi_3(v) \in \mathbb{S}^{2n-3}$, the 2-tensor $\pi'(v)$ is identified with

$$\begin{aligned}(D\phi_3)_v((D\phi_1)_u(\pi_{\sigma_c}(u))) &= (D\phi_3)_v\left(\sqrt{1-c}(D\psi)_{u'}(\pi(u'))\right)\\ &= (D\phi_3)_v\left(\sqrt{1-c}\left(0\oplus(D\psi)_{u'}\left(\pi^{(n-1)}(u'')\right)\right)\right)\\ &= (D\phi_3)_v\left(\sqrt{1-c}\left(0\oplus\rho^{(n-1)}(\phi_3(v))\right)\right)\\ &= \rho^{(n-1)}(\phi_3(v)) \in \wedge^2 T_{\phi_3(v)}\mathbb{S}^{2n-3}.\end{aligned}$$

Thus (S_c, π') or $(\phi_2(S_c), \tau_c)$ is Poisson isomorphic to the standard Poisson sphere $\left(\mathbb{S}^{2n-3}, \rho^{(n-1)}\right)$. When $n = 2$, for $v \in S_c$ with $v_2 \neq \sqrt{1-c}$, we cannot find a $u' \in SU(2)$ with the first column $u_1' = (1,0)$ and the last column $u_2' = \sqrt{1-c}^{-1}(0, v_2)$. But for $v_0 = (\sqrt{c}, \sqrt{1-c})$, such a u_0' exists, namely, $u_0' = I_2$ the 2×2 identity matrix, and the above argument essentially works. More precisely, it is well known that $\pi(u_0') = 0$ since $u_0' \in U(1) \subset SU(2)$, and hence for $u_0 = u_0'\sigma_c = \sigma_c$,

$$(D\phi_1)_{u_0}(\pi_{\sigma_c}(u_0)) = (D\phi_1)_{u_0}(R_{\sigma_c}(\pi(u_0'))) = (D\phi_1)_{u_0}(0) = 0.$$

So

$$\tau_c([v_0]) = (D\phi_2)_{v_0}(D\phi_1)_{u_0}(\pi_{\sigma_c}(u_0)) = 0.$$

On the other hand, since τ_c on $\mathbb{C}P^1 \approx \mathbb{S}^2$ is $SU(2)$-covariant and $U(1) \subset SU(2)$ consists of 0-dimensional leaves, the action of any

$$t = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in U(1)$$

on $\mathbb{C}P^1$ preserves the Poisson structure τ_c. In particular, $\tau_c([tv_0]) = 0$ for any $t \in U(1)$. Since any $v = (\sqrt{c}, \sqrt{1-c}e^{i\theta}) \in S_c$ is equivalent to a tv_0 with $t \in U(1)$ under the diagonal $\mathbb{T}$-action, namely,

$$\begin{aligned}[v] &= \left[\begin{pmatrix} e^{-i\theta/2}\sqrt{c} \\ e^{-i\theta/2}\sqrt{1-c}e^{i\theta} \end{pmatrix}\right] = \left[\begin{pmatrix} e^{-i\theta/2}\sqrt{c} \\ e^{i\theta/2}\sqrt{1-c} \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}\begin{pmatrix} \sqrt{c} \\ \sqrt{1-c} \end{pmatrix}\right] = \left[\begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} v_0\right]\end{aligned}$$

in $\mathbb{C}P^1$, we have $\tau_c([v]) = 0$ for all $v \in S_c$, i.e., $\tau_c = 0$ on $\phi_2(S_c) \subset \mathbb{C}P^1$. Since $\phi_2(S_c)$ is diffeomorphic to S_c and hence to $\mathbb{S}^1$, we get the standard (trivial) Poisson $(\mathbb{S}^1, \rho^{(1)})$ embedded in $(\mathbb{C}P^1, \tau_c)$. □

4. Invariant 2-tensor on $\mathbb{S}^{2n-1}$.

In this section, we first classify the $SU(n)$-invariant (contravariant alternating) 2-tensor on $\mathbb{S}^{2n-1}$, and then we conclude that the canonical $SU(n)$-invariant symplectic structure on $\mathbb{C}P^{n-1}$ gives the only, up to a constant factor, $SU(n)$-invariant (contravariant alternating) 2-tensor on $\mathbb{C}P^{n-1}$.

For each $p \in \mathbb{S}^{2n-1}$, we have $ip \in T_p\mathbb{S}^{2n-1}$, and the orthogonal complement $E_p := \{p, ip\}^\perp \subset T_p\mathbb{S}^{2n-1}$ is a complex subspace of $\mathbb{C}^n = T_p\mathbb{C}^n$ endowed with a canonical symplectic structure $\tilde{\Omega}_p$ determined by the complex hermitian structure on $\mathbb{C}^n$. Indeed $(d\omega)_p = \tilde{\Omega}_p$ on E_p for the unique 1-form ω, the standard contact structure, on $\mathbb{S}^{2n-1}$ such that $\omega_p(ip) = 1$ and

$\omega_p(E_p) = \{0\}$ at each $p \in \mathbb{S}^{2n-1}$. The contact manifold $(\mathbb{S}^{2n-1}, \omega)$ with the diagonal $\mathbb{T}$-action on $\mathbb{S}^{2n-1}$ is the standard prequantization [Kos1, Wel] of the canonical $SU(n)$-invariant symplectic structure on $\mathbb{C}P^{n-1} \cong \mathbb{S}^{2n-1}/\mathbb{T}$.

Since the vector fields $p \mapsto p$ and $p \mapsto ip$ on $\mathbb{S}^{2n-1}$ are invariant under the $U(n)$-action, so is the distribution $p \mapsto E_p$ of tangent subspaces. Furthermore, since the $U(n)$-action preserves the complex hermitian structure on $\mathbb{C}^n$ (and on E_p), the field $p \mapsto \tilde{\Omega}_p$ of symplectic forms on $\mathbb{S}^{2n-1}$ is also invariant under the $U(n)$-action. Thus the contravariant 2-tensor $\tilde{\pi}$ on $\mathbb{S}^{2n-1}$ uniquely determined by the form $\tilde{\Omega}$ on $E \subset TSU(n)$ is $U(n)$-invariant. Note that this 2-tensor $\tilde{\pi}$ on $\mathbb{S}^{2n-1}$, invariant under the diagonal $\mathbb{T}$-action, induces the cnanonical symplectic structure on $\mathbb{C}P^{n-1} \cong \mathbb{S}^{2n-1}/\mathbb{T}$ determined by its complex hermitian structure.

Given an $SU(n)$-invariant contravriant 2-tensors $\pi \neq 0$ on $\mathbb{S}^{2n-1}$ with $n \geq 2$, we show that $\pi = \tilde{\pi}$ after a suitable normalization if $n \neq 3$ or if π is $U(n)$-invariant. Through the standard Euclidean structure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$, we identify the $SU(n)$-invariant contravriant 2-tensors $\pi \neq 0$ on $\mathbb{S}^{2n-1}$ with $SU(n)$-invariant 2-forms $\Omega \neq 0$ on $\mathbb{S}^{2n-1}$.

First we show that the tangent vector

$$e_1' := ie_1 \in T_{e_1}\mathbb{S}^{2n-1} = i\mathbb{R} \oplus \mathbb{C}^{n-1}$$

at $e_1 \in \mathbb{S}^{2n-1}$ is in

$$\ker \Omega_p := \left\{ v \in T_p\mathbb{S}^{2n-1} : \Omega_p(v, \cdot) = 0 \right\}.$$

If not, then we can find an orthonormal set $\{e_i'\}_{i=2}^{n-1} \cup \{\eta_i'\}_{i=1}^{n} \subset 0 \oplus \mathbb{C}^{n-1}$ such that $\Omega_{e_1}\left(e_i', \eta_j'\right) = \delta_{ij} a_{ii}$ and $\Omega_{e_1}\left(e_i', e_j'\right) = \Omega_{e_1}\left(\eta_i', \eta_j'\right) = 0$ with $a_{ii} \in \mathbb{R}$ and $a_{11} \neq 0$. Now since Ω is $SU(n)$-invariant, we have

$$\Omega_{e_1}\left(e_1', u\left(\eta_1'\right)\right) = \Omega_{u(e_1)}\left(u\left(e_1'\right), u\left(\eta_1'\right)\right) = \Omega_{e_1}\left(e_1', \eta_1'\right) = a_{11}$$

for any $u \in \{1\} \oplus SU(n-1) \subset SU(n)$. This cannot be true, since by a suitable choice of u, $u(\eta_1')$ can be any unit vector in $0 \oplus \mathbb{C}^{n-1}$, for example, η_n'. Thus $e_1' = ie_1 \in \ker \Omega_p$.

Now with respect to the standard orthonormal $\mathbb{R}$-linear basis of

$$i\mathbb{R} \oplus \mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1} \cong i\mathbb{R} \oplus \mathbb{C}^{n-1} = T_{e_1}\mathbb{S}^{2n-1},$$

the 2-form Ω_{e_1} can be represented by a block diagonal matrix $0 \oplus B$ where $B \in M_{2(n-1)}(\mathbb{R})$ is a skew symmetric matrix. The $SU(n)$-invariance of $\Omega \neq 0$ implies that $\Omega_{e_1} \neq 0$ and

$$uBu^{-1} = uBu^t = B \neq 0,$$

or $uB = Bu$, for any $u \in SU(n-1) \subset O_{2n-2}(\mathbb{R})$ since $1 \oplus u \in SU(n)$ and $(1 \oplus u)(e_1) = e_1$. (If Ω is $U(n)$-invariant, then $uB = Bu$ for any $u \in U(n-1)$ since $1 \oplus u \in U(n)$.)

We claim that B must be conformal, i.e., $\|B(v)\| = \|B\| \|v\|$ for all $v \in \mathbb{R}^{2n-2}$ where $\|B\| := \sup_{\|v\|=1} \|B(v)\| > 0$. Let w be a unit vector with $\|B(w)\| = \|B\|$. Since $SU(n-1)$ acts on $\mathbb{S}^{2n-3} \subset \mathbb{R}^{2n-2}$ transitively, for any unit vector $v \in \mathbb{R}^{2n-2}$, we can find $u \in SU(n-1)$ with $u^{-1}(v) = w$, and hence

$$\|B(v)\| = \left\|uBu^{-1}(v)\right\| = \|u(B(w))\| = \|B(w)\| = \|B\|.$$

Thus $B/\|B\|$ is a skew-symmetric isometry on $\mathbb{R}^{2n-2}$ and so $B/\|B\| \in O_{2n-2}(\mathbb{R})$.

If $n = 2$, then any skew symmetric $0 \neq B/\|B\| \in O_2(\mathbb{R})$ determines the same 2-form Ω_{e_1} on $0 \oplus \mathbb{R}^2$ and hence on $i\mathbb{R} \oplus \mathbb{R}^2$, up to a constant multiple. So $\Omega = \tilde{\Omega}$ after normalized.

If $n \geq 4$, then the commutativity of $\mathbb{T}^{n-2} \subset SU(n-1)$ with B implies that B is complex linear on $\mathbb{R}^{2n-2} = \mathbb{C}^{n-1}$ and so $B/\|B\| \in U(n-1)$. In fact, since for any $1 \leq j \neq k \leq n-1$, $t_{jk\theta}B = Bt_{jk\theta}$ for all $\theta \in \mathbb{R}$ implies that $B_{jj}, B_{kk} \in \mathbb{C}$ and $B_{kl} = 0$ for any $j \neq l \neq k$, where $B = (B_{jk})_{1 \leq j,k \leq n-1}$ with $B_{jk} \in M_2(\mathbb{R})$, and

$$t_{jk\theta} := e^{i\theta} e_{jj} + e^{-i\theta} e_{kk} + \sum_{\substack{1 \leq l \leq n-1 \\ j \neq l \neq k}} e_{ll} \in \mathbb{T}^{n-2} \subset SU(n-1).$$

It is well known that only scalar matrices in $M_{n-1}(\mathbb{C})$ commute with $SU(n-1)$, so we get $B/\|B\| \in \mathbb{T}$ with $-B/\|B\| = (B/\|B\|)^* = (B/\|B\|)^{-1}$, i.e., $(B/\|B\|)^2 = -1$ or $B = \pm i \|B\|$. Thus

$$\Omega_{e_1} = \pm \|B\| \tilde{\Omega}_{e_1}$$

a (real) constant multiple of the standard symplectic form. Hence we get $\pi = \tilde{\pi}$ after a suitable normalization.

If Ω is $U(n)$-invariant, then the commutativity of $\mathbb{T}^{n-1} \subset U(n-1)$ with B implies that B is complex linear and hence $B/\|B\| \in U(n-1)$ and as above, $\Omega = \pm \|B\| \tilde{\Omega}$. In fact, $t'_{k\theta}B = Bt'_{k\theta}$ for all $\theta \in \mathbb{R}$ implies that $B_{kk} \in \mathbb{C}$ and $B_{kl} = 0$ for any $l \neq k$, where

$$t'_{k\theta} := e^{i\theta} e_{kk} + \sum_{\substack{1 \leq l \leq n-1 \\ l \neq k}} e_{ll} \in \mathbb{T}^{n-1} \subset U(n-1).$$

We observe that the quotient map $\phi : \mathbb{S}^{2n-1} \to \mathbb{C}P^{n-1}$ and its differential $D\phi : T\mathbb{S}^{2n-1} \to T\mathbb{C}P^{n-1}$ are $U(n)$-equivariant since the diagonal $\mathbb{T}$-action commutes with the $U(n)$-action. Furthermore, the restriction

$$(D\phi)|_E : E \to T\mathbb{C}P^{n-1}$$

of $D\phi$ to the $U(n)$-equivariant subbundle E defined above is a bundle isomorphism. So any $SU(n)$-invariant (and hence $U(n)$-invariant) 2-tensor $\tau \in \Gamma\left(\wedge^2 T\mathbb{C}P^{n-1}\right)$ on $\mathbb{C}P^{n-1}$ can be 'pulled back' to an $U(n)$-invariant 2-tensor

$$\pi = (D\phi)|_E^{-1}(\tau) \in \Gamma\left(\wedge^2 E\right) \subset \Gamma\left(\wedge^2 T\mathbb{S}^{2n-1}\right)$$

on $\mathbb{S}^{2n-1}$ which must be, up to a constant factor, equal to $\tilde{\pi}$. Thus $\tau = \tilde{\tau} := (D\phi)(\tilde{\pi})$ which is the standard symplectic 2-tensor on $\mathbb{C}P^{n-1}$.

References.

[DaSo] P. Dazord and D. Sondaz, *Groupes de Poisson affines*, in 'Symplectic Geometry, Groupoids, and Integrable Systems', P. Dazord and A. Weinstein (Eds.), Springer-Verlag, 1991.

[DiNo] M.S. Dijkhuizen and M. Noumi, *A family of quantum projective spaces and related q-hypergeometric orthogonal polynomials*, preprint.

[Dr1] V.G. Drinfeld, *Quantum groups*, Proc. I.C.M. Berkeley 1986, **1**, 789–820, Amer. Math. Soc., Providence, 1987.

[Dr2] V.G. Drinfeld, *On Poisson homogeneous spaces of Poisson Lie groups*, Theo. Math. Phys., **95** (1993), 226–227.

[ELu] S. Evens and J.H. Lu, *On the variety of Lagrangian subalgebras*, preprint, math.DG/9909005.

[Ka] E. Karolinsky, *The classification of Poisson homogeneous spaces of compact Poisson Lie groups*, Mathematical Physics, Analysis, and Geometry, **3** (1996), 272–289.

[KhRaRu] S. Khoroshkin, A. Radul and V. Rubtsov, *A family of Poisson structures on hermitian symmetric spaces*, Comm. Math. Phys., **152** (1993), 299–315.

[KoVa] L.I. Korogodsky and L.L. Vaksman, *Quantum G-spaces and Heisenberg algebra*, in 'Quantum Groups', P.P. Kulish (Ed.), Lecture Notes in Mathematics, **1510**, Springer-Verlag, Berlin, (1992), 56–66.

[Kos1] B. Kostant, *Quantization and unitary representations. I: Prequantization*, in 'Lectures in Modern Analysis and Applications, III', E.T. Taam (Ed.), Lecture Notes in Mathematics, **170**, Springer-Verlag, Berlin and New York, (1970), 87–208.

[Kos2] B. Kostant, *Lie algebra cohomology and generalized Schubert cells*, Ann. of Math., **77** (1963), 72–144.

[Lu] J.H. Lu, *Multiplicative and affine Poisson structures on Lie groups*, Ph. D. thesis, Univ. of California, Berkeley, 1990.

[LuWe1] J.H. Lu and A. Weinstein, *Poisson Lie groups, dressing transformations and Bruhat decompositions*, J. Diff. Geom., **31** (1990), 501–526.

[LuWe2] J.H. Lu and A. Weinstein, *Classification of $SU(2)$-covariant Poisson structures on $\mathbb{S}^2$*, Comm. Math. Phys., **135** (1991), 229–231.

[Po] P. Podles, *Quantum spheres*, Letters Math. Phys., **14** (1987), 193–202.

[Re] J. Renault, *A Groupoid Approach to C*-algebras*, Lecture Notes in Mathematics, **793**, Springer-Verlag, New York, 1980.

[RTF] N.Yu. Reshetikhin, L.A. Takhtadzhyan and L.D. Faddeev, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J., **1** (1990), 193–225.

[Ri] M.A. Rieffel, *Non-compact quantum groups associated with abelian subgroups*, Comm. Math. Phys., **171** (1995), 181–201.

[Sh1] A.J.L. Sheu, *Quantization of the Poisson $SU(2)$ and its Poisson homogeneous space – the 2-sphere*, Comm. Math. Phys., **135** (1991), 217–232.

[Sh2] A.J.L. Sheu, *Compact quantum groups and groupoid C^*-algebras*, J. Func. Anal., **144** (1997), 371–393.

[Sh3] A.J.L. Sheu, *Groupoid approach to quantum projective spaces*, preprint.

[So] Ya.S. Soibelman, *The algebra of functions on a compact quantum group, and its representations*, Algebra Analiz., **2** (1990), 190–221 (Leningrad Math. J., **2** (1991), 161–178).

[VaSo] L.L. Vaksman and Ya.S. Soibelman, *The algebra of functions on the quantum group $SU(n+1)$, and odd-dimensional quantum spheres*, Leningrad Math. J., **2** (1991), 1023–1042.

[We1] A. Weinstein, *Lectures on Symplectic Manifolds*, CBMS Regional Conference Series in Mathematics, **29**, Amer. Math. Soc., Providence, 1977.

[We2] A. Weinstein, *The local structure of Poisson manifolds*, J. Diff. Geom., **18** (1983), 523–557.

[We3] A. Weinstein, *Affine Poisson structures*, International J. Math., **1** (1990), 343–360.

[Wo1] S.L. Woronowicz, *Twisted $SU(2)$ group: an example of a non-commutative differential calculus*, Publ. RIMS., **23** (1987), 117–181.

[Wo2] S.L. Woronowicz, *Compact matrix pseudogroups*, Comm. Math. Phys., **111** (1987), 613–665.

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