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A counterexample to unique continuation in dimension two

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We construct a non-zero solution $u \in C_0^\infty(\mathbb{C})$ of the equation $\overline{\partial} u = Vu$ for a certain V which belongs to L^p for any $p < 2$. The same is done in arbitrary dimension $d \geq 2$ for the Laplace equation with a first order term $\Delta v + W \cdot \nabla v = 0$ and for the Dirac equation $\mathcal{D}w + Ww = 0$, with $W \in L^p$ for any $p < 2$. The construction is based on a Weierstrass product in the unit ball. Although its poles accumulate at the boundary, it is flat at the boundary if we remove small disjoint discs around the poles.

1. Introduction.

Let $\Omega \subset \mathbb{R}^n$ be a connected open set. We say that an equation $P(x, \partial/\partial x)u = 0$ has the unique continuation property in Ω if any solution u which vanishes in a non-empty open subset vanishes identically. We are interested in the equation

$$\overline{\partial} u = Vu \quad \text{in } \mathbb{C} = \mathbb{R}^2, \tag{1.1}$$

where $\overline{\partial}$ is the Cauchy-Riemann operator, $\overline{\partial} = \frac{1}{2}(\partial/\partial x_1 + i\partial/\partial x_2)$ and $V \in L^1_{\mathrm{loc}}(\mathbb{C})$. We will also consider here the Laplace equation with a first order term:

$$\Delta v + W \cdot \nabla v = 0, \tag{1.2}$$

and the Dirac equation:

$$\mathcal{D}w + Ww = 0, \tag{1.3}$$

with the (zero mass) Dirac operator $\mathcal{D} = \sum_{i=1}^d \alpha_i \partial/\partial x_i$. The α_i are matrices of order m satisfying $\alpha_i^* = -\alpha_i$ and the Clifford commutation relations: $\alpha_i\alpha_j + \alpha_j\alpha_i = -2\delta_{jk}$. m can be chosen $2^{[d/2]}$.

There is extensive literature on unique continuation and strong unique continuation properties (in the later, one supposes the solution to vanish to infinite order at a point instead of vanishing on an open set), see [6]. Although the problem of strong unique continuation has been almost settled, it is a long standing problem whether equations (1.1), (1.2) and (1.3) have the unique continuation property when the coefficients are L^1_{loc}. We give here a negative answer, constructing solution with compact support for (1.1),(1.2) and (1.3) with coefficients in L^p for any $p < 2$. This is optimal in dimension 2. The best positive result belongs to Wolff [5]: in dimension d, if $W \in L^d$ then (1.2) and (1.3) have the unique continuation property.

While this paper was awaiting publication, a new way of constructing counterexamples to unique continuation was found by Kenig and Nadirashvili [2]. Their result was improved and extended from dimension two to arbitrary dimension by Koch and Tataru [3].

We fix the function $F(t) = t^2 \ln^{-3}(t+2)$ on $[0,\infty)$. It has the property that $F(at) \leq a^2 F(t)$ for $a > 1$ and $F(t)/\tilde{F}(t)$ is bounded, $\tilde{F}$ being the greatest convex function with $\tilde{F}(t) \leq F(t)$ for any $t \geq 0$. Therefore the set

$$L^F(\mathbb{R}^d) = \left\{ f : f \text{ measurable on } \mathbb{R}^d, \int F(|f|)\, dx < \infty \right\}$$

coincides with the Orlicz space associated with $\tilde{F}$ on $(\mathbb{R}^d, dx)$ (see e.g., [1]). It is a Banach space such that $L^p \subset L^F_{loc}$ for $p < 2$.

Theorem 1.1. *There are u, V which satisfy* (1.1) *such that $u \in C_0^\infty(\mathbb{C})$,* $\operatorname{supp} u = B(0,1)$ *and $V \in L^F(\mathbb{C})$. There are a real-valued [both v and W are real valued] $v \in C_0^\infty(\mathbb{C})$ and $W \in L^F(\mathbb{C})$ satisfying* (1.2), *such that* $\operatorname{supp} v = B(0,1)$.

Corollary 1.2. *For any $d \geq 2$, there are non-zero smooth solutions v real-valued and w, for the equations* (1.3) *and* (1.2) *respectively, with compact support in $\mathbb{R}^d$ and such that the coefficients W belong to $L^F(\mathbb{R}^d)$.*

Notations. C, C', C'' will stand for absolute positive constants, not necessarily the same in different formulae. The constants which will be carried from a formula to another will be numbered $C_1, C_2, \ldots$. We write $f(x) \sim g(x)$ in M if there are absolute constants $0 < c < C$ such that $cf(x) \leq g(x) \leq Cf(x)$ for all $x \in M$.

We use the variable $z = x_1 + ix_2$, identifying $\mathbb{C}$ with $\mathbb{R}^2$. If $\beta \in \mathbb{N}^2$ is a multi-index, $\partial^\beta = (\partial/\partial x_1)^{\beta_1}(\partial/\partial x_2)^{\beta_2}$ and $|\beta| = \beta_1 + \beta_2$. $B(a,r)$ is the ball of center a and radius r and $\overset{\circ}{B}(a,r)$ is its interior. If x is a real number, we denote by $[x]$ the greatest integer in $(-\infty, x]$.

2. Proofs.

We start by constructing a meromorphic function in the unit disk $B(0,1)$ as a Weierstrass product. It vanishes to infinite order at the boundary, in a sense which is made precise by Lemma 2.1 below:

$$f(z) = \prod_{j=2}^{\infty} \frac{1}{1 - \left(\frac{z}{1-1/j}\right)^{j^2}}. \tag{2.1}$$

Let us denote the poles of f by

$$z_{k,l} = \frac{k-1}{k} \exp(2\pi i l/k^2), \quad \text{for } k \geq 2,\ 0 \leq l < k^2.$$

Then each of the sets

$$M_{k,l} = \left\{ z : \begin{array}{c} |z| \in \left[\frac{2k-3}{2k-1}, \frac{2k-1}{2k+1}\right), \\ \arg z \in [(2l-1)\pi/k^2, (2l+1)\pi/k^2) \end{array} \right\},\ k \geq 2,\ 0 \leq l \leq k^2,$$

contains the corresponding pole of f. Moreover, these sets form a partition of the annulus $\{z : |z| \in [1/3, 1)\}$. We will also need slightly bigger sets:

$$\tilde{M}_{k,l} = \left\{ z : \begin{array}{c} |z| \in \left[\frac{4k-7}{4k-3}, \frac{4k-1}{4k+3}\right], \\ \arg z \in [(4l-3)\pi/2k^2, (4l+3)\pi/2k^2] \end{array} \right\}.$$

We have then

$$M_{k,l} + B(0, k^{-2}/8) \subset \tilde{M}_{k,l}. \tag{2.2}$$

$$B(z_{k,l}, k^{-2}/8) \subset M_{k,l}. \tag{2.3}$$

Lemma 2.1. *The infinite product* (2.1) *converges to a meromorphic function in the unit open disk, and it converges to* 0 *elsewhere. Let*

$$g_{k,l}(z) = f(z)(z - z_{k,l})$$

be the analytic function in $M_{k,l}$ obtained by multiplicatively removing the pole in this region. Then the following estimate holds:

(2.4)

$$|g_{k,l}(z)| \sim k^{-2} \prod_{j=2}^{k-1} (1 - 1/j)^{j^2} |z|^{-k^3/3 + k^2/2} \leq \exp(-k^2/6 + 2k),\ z \in \tilde{M}_{k,l}.$$

Proof of Lemma 2.1. We divide the product (2.1) in three parts: the factors with index $j < k$, the factors with $j > k$ and the factor with $j = k$.

We estimate first the modulus of $\left(\frac{z}{1-1/j}\right)^{j^2}$ for $z \in \tilde{M}_{k,l}$, for $2 \le j \le k-1$. Since $\frac{4k-7}{4k-3} \le |z|$ we have $\frac{4j-3}{4j+1} \le |z|$, so

$$\left|\left(\frac{z}{1-1/j}\right)^{j^2}\right| \ge \left(\frac{1-1/(j+1/4)}{1-1/j}\right)^{j^2} \to e^{1/4} \text{ as } j \to \infty, \text{ hence:}$$

$$\left|\left(\frac{z}{1-1/j}\right)^{j^2}\right| \ge C_1 > 1. \tag{2.5}$$

In the same way we obtain for $|z| \le \frac{4k-1}{4k+3}$:

$$\left|\left(\frac{z}{1-1/j}\right)^{j^2}\right| \le C_2 < 1 \text{ for } j \ge k+1. \tag{2.6}$$

Since $|\ln(|1/(1-a)|\cdot|a|)| = |\ln(|1-1/a|)| \le C_3|1/a|$ for $|a| \ge C_1 > 1$, putting $a = (z/(1-1/j))^{j^2}$, we obtain from (2.5), using again $|z| \ge (4k-7)/(4k-3)$ as $z \in \tilde{M}_{k,l}$:

$$\begin{aligned}
\left|\ln\left|\frac{\prod_{j=2}^{k-1}\frac{1}{1-\left(\frac{z}{1-1/j}\right)^{j^2}}}{\prod_{j=2}^{k-1}\left(\frac{1-1/j}{z}\right)^{j^2}}\right|\right| &\le C_3\sum_{j=2}^{k-1}\left|\frac{1-1/j}{(4k-7)/(4k-3)}\right|^{j^2} \\
&\le C_3\sum_{j=2}^{k-1}(1-1/j)^{j^2}(1+1/(k-7/4))^{j^2} \\
&\le C_3\sum_{j=2}^{k-1}\exp-\frac{j(k-7/4-j)}{k-7/4} \\
&\le C_3e^3 + C_3\sum_{j=2}^{k-2}\exp(-\min(j,k-7/4-j)/2) \\
&\le C_3e^3 + 2C_3\sum_{j=1}^{\infty}\exp(-j/2+3/8) = C_4.
\end{aligned}$$

Since $|z|^k \sim 1$ for $\frac{4k-7}{4k-3} \le |z| \le 1$, the above gives in this region:

$$\left|\prod_{j=2}^{k-1}\frac{1}{1-\left(\frac{z}{1-1/j}\right)^{j^2}}\right| \sim \prod_{j=2}^{k-1}\left|\frac{1-1/j}{z}\right|^{j^2} \sim |z|^{-k^3/3+k^2/2}\prod_{j=2}^{k-1}(1-1/j)^{j^2}. \tag{2.7}$$

We have the following upper bound for the right hand side above:

$$
\begin{aligned}
(2.8)\qquad & |z|^{-k^3/3+k^2/2}\prod_{j=2}^{k-1}(1-1/j)^{j^2} \\
& \le \left(\frac{4k-7}{4k-3}\right)^{-k^3/3+k^2/2} \exp\sum_{j=2}^{k-1} j^2(-1/j) \\
& \le \exp\left((-k^3/3+k^2/2)\frac{-1}{k-7/4} - \sum_{j=2}^{k-1} j\right) \\
& \le \exp(k^2/3+k-k(k-1)/2+1) \\
& \le \exp(-k^2/6+2k), \quad \text{for } |z| \ge \tfrac{4k-7}{4k-3}.
\end{aligned}
$$

This proves the inequality in (2.4). Since for (2.5) to hold we only need $|z| \ge \frac{4k-7}{4k-3}$, we infer from (2.7) and (2.8) that the product (2.1) converges to 0 if $|z| \ge 1$.

Now, for the factors with index greater than k in the definition of f, we use that $|\ln(1-a)| \le C_5|a|$ for $|a| \le C_2 < 1$. Via the relation (2.6) this can be applied for $a = (\frac{z}{1-1/j})^{j^2}$ and we obtain for $|z| \le \frac{4k-1}{4k+3}$:

$$
\begin{aligned}
\sum_{j=k+1}^{\infty}\left|\ln\frac{1}{1-\left(\frac{z}{1-1/j}\right)^{j^2}}\right| &\le C_5\sum_{j=k+1}^{\infty}\left|\frac{1-\frac{1}{k+3/4}}{1-1/j}\right|^{j^2} \\
&\le C_5\sum_{j=k+1}^{\infty}\left(1-\tfrac{1}{k+1}\right)^{j^2}\left(1+\tfrac{1}{j-1}\right)^{j^2} \\
&\le C_5\sum_{j=k+1}^{\infty}\exp\left(\tfrac{j^2}{j-1}-\tfrac{j^2}{k+1}\right) \\
&\le C_5e^{3/2}+C_5\sum_{j=k+2}^{\infty}\exp(k+2-j)=C_6.
\end{aligned}
$$

Hence the product $\prod_{j=k+1}^{\infty}\frac{1}{1-\left(\frac{z}{1-1/j}\right)^{j^2}}$ is uniformly absolutely convergent for $|z| \le \frac{4k-1}{4k+3}$, and

$$
(2.9)\qquad e^{-C_6} \le \left|\prod_{j=k+1}^{\infty}\frac{1}{1-\left(\frac{z}{1-1/j}\right)^{j^2}}\right| \le e^{C_6}.
$$

This implies that the product (2.1) is convergent everywhere in $\overset{\circ}{B}(0,1)$. Indeed, for $|z| \leq \frac{4k-1}{4k+3}$ the product up to the factor k is a meromorphic function and the rest of the product is absolutely convergent.

For the middle factor with $j = k$ in (2.1), notice first that the logarithm is a quasi-isometry in the region $0 < c \leq |z| \leq C$ and $-3\pi/4 \leq \arg z \leq 3\pi/4$. More generally, $|z - z'| \sim |\ln \frac{z}{z'}|$ for $|z|, |z'| \in [c, C]$ and $\arg \frac{z}{z'} \in [-3\pi/4, 3\pi/4]$. Since $z_{k,l}^{k^2} = (1-1/k)^{k^2}$ we have:

$$\left| \frac{1}{1 - \left(\frac{z}{1-1/k}\right)^{k^2}} \right| \sim \frac{1}{k^2 \left|\ln \frac{z}{z_{k,l}}\right|} \sim \frac{1}{k^2} \frac{1}{|z - z_{k,l}|} \quad \text{for } z \in \tilde{M}_{k,l}. \tag{2.10}$$

It remains to multiply the relations (2.7), (2.9) and (2.10) to obtain the first part of (2.4). □

Proof of Theorem 1.1. We choose a function $\phi \in C^\infty(\mathbb{C})$ such that:

$$\phi(z) = \phi(|z|) = \begin{cases} 1 & \text{for } |z| \geq 1, \\ |z|^2 & \text{for } |z| \leq 1/2. \end{cases}$$

We require ϕ to be increasing on $[0, \infty)$ and $\phi(r)/r$ to have only one critical point in $(0, \infty)$ and that to be non-degenerate. For a choice of the quantities ε_k to be performed later, under the constraint $0 < \varepsilon_k \leq k^{-2}/8$, we define our function solution of equation (1.1):

$$u(z) = f(z) \prod_{k=2}^{\infty} \prod_{l=0}^{k^2-1} \phi\left(\frac{z - z_{k,l}}{\varepsilon_k}\right). \tag{2.11}$$

Notice first that the product is well defined, since by (2.3) the sets of points where each of the factors is different from 1 are disjoint. Since the poles $z_{k,l}$ of f are simple and $\phi(z) = z\bar{z}$ in a neighborhood of 0, u is smooth in $\overset{\circ}{B}(0,1)$. For $z \in M_{k,l}$ we have

$$\begin{aligned} \bar{\partial} u(z) = \bar{\partial}\left(f(z)\phi\left(\frac{z - z_{k,l}}{\varepsilon_k}\right)\right) &= f(z)\bar{\partial}\phi\left(\frac{z - z_{k,l}}{\varepsilon_k}\right) \\ &= f(z)\frac{1}{\varepsilon_k}(\bar{\partial}\phi)\left(\frac{z - z_{k,l}}{\varepsilon_k}\right) \end{aligned}$$

and we obtain that the equation (1.1) is satisfied with a potential

$$V = \sum_{k=2}^{\infty} \sum_{l=0}^{k^2-1} \frac{1}{\varepsilon_k}\left(\frac{\bar{\partial}\phi}{\phi}\right)\left(\frac{z - z_{k,l}}{\varepsilon_k}\right).$$

The function $\overline{\partial}\phi/\phi$ has modulus $|z|^{-1}$ for $|z| \leq 1/2$ and vanishes for $|z| > 1$. Since ϕ does not vanish outside 0, we have that $|z|\overline{\partial}\phi(z)/\phi(z)$ is bounded in $B(0,1)$. Then

$$\begin{aligned}\int F(|V|)dx_1dx_2 &\leq C\sum_{k=2}^{\infty}\sum_{l=0}^{k^2-1}\int_{B(0,\varepsilon_k)}|z|^{-2}\ln^{-3}(|z|^{-1}+2)dx_1dx_2 \\ &\leq C\sum_{k=2}^{\infty}k^2 2\pi\int_0^{\varepsilon_k}-x^{-1}\ln^{-3}x\,dx \\ &\leq C'\sum_{k=2}^{\infty}k^2\ln^{-2}\varepsilon_k.\end{aligned}$$

We choose

$$\varepsilon_k = \min(k^{-2}/8, e^{-k^{1.7}}), \tag{2.12}$$

and obtain that the series above is convergent, so $V \in L^F$. It remains to check that the solution (2.11) is smooth.

Take $\tilde{g}_{k,l}(z) = z^{k(k-1)(2k-1)/6}g_{k,l}(z)$. The relation (2.4) from Lemma 2.1 implies that $|\tilde{g}_{k,l}(z)| \sim |\tilde{g}_{k,l}(z_{k,l})|$ for $z \in \tilde{M}_{k,l}$. From (2.2), each point $z \in M_{k,l}$ is the center of a ball of radius $k^{-2}/8$ contained in $\tilde{M}_{k,l}$. Since $\tilde{g}_{k,l}$ is holomorphic in $\tilde{M}_{k,l}$, we can apply the Cauchy representation formula using as contour the boundary of this ball, to obtain $|\partial^\alpha\tilde{g}_{k,l}(z)| \leq C_\alpha(k^{2|\alpha|})|\tilde{g}_{k,l}(z_{k,l})|$ for $z \in M_{k,l}$. Then differentiating $g_{k,l}(z) = \tilde{g}_{k,l}(z)z^{-k(k-1)(2k-1)/6}$ we obtain for a suitable choice of the constants C'_α:

$$|\partial^\alpha g_{k,l}(z)| \leq C'_\alpha k^{3|\alpha|}|g_{k,l}(z)| \quad \text{for } z \in M_{k,l}\,. \tag{2.13}$$

Let us define

$$\tilde{\phi}(z) = \frac{\phi(z)}{z}\,. \tag{2.14}$$

Since $\tilde{\phi}$ is a symbol of order -1, all its derivatives are bounded. Then, with the notation of Lemma 2.1, $u(z) = \frac{g_{k,l}(z)}{\varepsilon_k}\tilde{\phi}\left(\frac{z-z_{k,l}}{\varepsilon_k}\right)$ for $z \in \tilde{M}_{k,l}$. Using (2.13), the definition of ε_k and then (2.4) we obtain for k big enough:

$$\begin{aligned}|\partial^\alpha u(z)| &\leq C''_\alpha(\max(\varepsilon_k^{-1}, k^3))^{|\alpha|}\varepsilon_k^{-1}|g_{k,l}(z)| \\ &\leq C''_\alpha\exp((1+|\alpha|)k^{1.7})C\exp(-k^2/6+2k), z \in M_{k,l},\end{aligned} \tag{2.15}$$

so all the derivatives of u tend to zero as $|z| \to 1$, hence u is smooth in $\mathbb{C}$.

For the second part of Theorem 1.1, we take $v = \operatorname{Re} u$, where u is given by (2.11), but with one more condition on ε_k, which will change the choice (2.12) for finitely many values of k. In order to obtain a lower bound for $|\nabla v|$ we will need the following

Lemma 2.2. *Let $h_0(z) = \operatorname{Re} \tilde{\phi}(z)$, where $\tilde{\phi}$ is given by (2.14). Then there are positive constants c, C such that any real valued function $h \in C^2(\overset{o}{B}(0,1))$, with $||h - h_0||_{C^2} \leq c$, has exactly two critical points z_1, z_2 and we have*

$$|\nabla h(z)| \geq C \min(|z - z_1|, |z - z_2|).$$

Proof. We have $h_0(z) = \frac{\phi(|z|)}{|z|} \cos(\arg z)$, so ∇h_0 has two zeroes $(r_\phi, 0)$ and $(-r_\phi, 0)$, where r_ϕ is, by the choice of ϕ, the unique zero of $(r^{-1}\phi(r))'$ in $(0, \infty)$ and $\frac{d^2}{dr^2}(r^{-1}\phi(r))\big|_{r=r_\phi} \neq 0$. Since $0, \pi$ are non-degenerate critical points of the function $\theta \to \cos\theta$, we can choose open sets $V_1 \ni (r_\phi, 0)$ and $V_2 \ni (-r_\phi, 0)$ such that $\nabla h_0 | V_i$ is a diffeomorphism and the tangent mapping of its inverse is bounded. We still get these properties after replacing h_0 by h if $||h - h_0||_{C^2} \leq c_2$ and c_2 is small enough. Taking c_2 even smaller, we can ensure that h has exactly one critical point in each of V_1, V_2; let these be z_1, z_2. Then $|\nabla h|^{-1}|z - z_i|$ is uniformly bounded in V_i with respect to different choices of h.

On the other hand, outside $V = V_1 \cup V_2$ the gradient $|\nabla h_0|$ is bounded away from zero and so is $|\nabla h|$ if $||h - h_0||_{C^1} \leq c_1$ with $c_1 = \frac{1}{2}\inf\{|\nabla h(z)| : z \in B(0,1) \setminus V\}$. We choose then $c = \min(c_2, c_1)$. □

Notice that Lemma 2.2 remains valid if we replace $\tilde{\phi}$ by $a\tilde{\phi}$, with $|a| = 1$, since this is equivalent to composing h_0 with a rotation. This allows us to apply it to $h(z) = |g_{k,l}(z_{k,l})|^{-1}\varepsilon_k v(z_{k,l} + \varepsilon_k z)$ which is a perturbation of $h_0 = \operatorname{Re} \frac{g_{k,l}(z_{k,l})}{|g_{k,l}(z_{k,l})|}\tilde{\phi}$. From (2.13) we infer

$$\frac{|\partial^\alpha g_{k,l}(z_{k,l} + \varepsilon_k z)|}{|g_{k,l}(z_{k,l} + \varepsilon_k z)|} \leq C'_\alpha (\varepsilon_k k^3)^{|\alpha|}, \tag{2.16}$$

so making $\varepsilon_k k^3$ arbitrarily small we can make

$$\left|\left| \frac{g_{k,l}(z_{k,l} + \varepsilon_k z)}{|g_{k,l}(z_{k,l})|} - \frac{g_{k,l}(z_{k,l})}{|g_{k,l}(z_{k,l})|} \right|\right|_{C^2(B(0,1))}$$

arbitrarily small. Indeed, if $\varepsilon_k k^3 \le C$ then (2.16) implies that $|g_{k,l}(z)| \sim |g_{k,l}(z_{k,l})|$ for $z \in B(z_{k,l},\varepsilon_k)$, so we can replace the denominator in (2.16) with $|g_{k,l}(z_{k,l})|$. Applying Lemma 2.2 we obtain that there is a constant $c_1 > 0$ such that for every k and l, if $\varepsilon_k \le c_1 k^{-3}$ then there are two points $z_{k,l}^{(1)}, z_{k,l}^{(2)} \in B(z_{k,l},\varepsilon_k)$ and $C > 0$ such that for $z \in B(z_{k,l},\varepsilon_k)$ we have:

$$(2.17) \qquad |g_{k,l}(z_{k,l})|^{-1}\varepsilon_k |\nabla v(z)| \ge C\varepsilon_k^{-1} \min\left(\left|\frac{z - z_{k,l}^{(1)}}{\varepsilon_k}\right|, \left|\frac{z - z_{k,l}^{(2)}}{\varepsilon_k}\right|\right).$$

We take $\varepsilon_k = \min(k^{-2}/8, c_1 k^{-3}, e^{-k^{1.7}})$ and $W(z) = -(\Delta v)\frac{\nabla v}{|\nabla v|^2}$. We use again that $|g_{k,l}(z)| \sim |g_{k,l}(z_{k,l})|$ in $B(z_{k,l},\varepsilon_k)$, to obtain in this ball from the first line of (2.15) and from (2.17):

$$|W(z)| = \frac{|\Delta v(z)|}{|\nabla v(z)|} \le \frac{(C''_{(2,0)} + C''_{(0,2)})\varepsilon_k^{-3}|g_{k,l}(z)|}{C\varepsilon_k^{-3}|g_{k,l}(z_{k,l})| \min\limits_{i=1,2} |z - z_{k,l}^{(i)}|} \le C'\left(\frac{1}{|z-z_{k,l}^{(1)}|} + \frac{1}{|z-z_{k,l}^{(2)}|}\right)$$

Since v is harmonic in $M_{k,l} \setminus B(z_{k,l},\varepsilon_k)$ we obtain that $W \in L^F$:

$$\begin{aligned}
\int F(|W|)dx_1dx_2 &\le C\sum_{k=2}^{\infty} \sum_{\substack{0\le l<k^2 \\ j=1,2}} \int_{B(z_{k,l}^{(j)},2\varepsilon_k)} F(|z - z_{k,l}^{(j)}|^{-1})\, dx_1dx_2 \\
&\le C\sum_{k=2}^{\infty} 2k^2 2\pi \int_0^{2\varepsilon_k} -x^{-1}\ln^{-3} x\, dx \\
&\le C'\sum_{k=2}^{\infty} k^2 \ln^{-2}\varepsilon_k < \infty.
\end{aligned}$$

□

Proof of Corollary 1.2. The method is a standard one (see e.g., Theorem 2 in [4]): we wrap the solution around a compact submanifold to make it have compact support. Here the support will be a tube around a sphere of dimension $d-2$. We decompose $\mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1}$, and use the notation $x = (x_1, x')$ with $x' = (x_2, \dots x_d) \in \mathbb{R}^{d-1}$. Let $(W_1(x_1,x_2), W_2(x_1,x_2))$ and v be the coefficient of the gradient and the solution constructed in Theorem 1.1. Then in dimension $d \ge 3$ we take

$$\begin{aligned}
\tilde{v}(x) &= v(x_1, |x'| - 2) \\
\tilde{W}(x) &= \left(W_1(x_1, |x'| - 2), \left(W_2(x_1, |x'| - 2) - \frac{d-2}{|x'|}\right)\frac{x'}{|x'|}\right)
\end{aligned}$$

and using polar coordinates in $\mathbb{R}^{d-1}$ it is easy to check that (1.2) is satisfied.

For the equation (1.3), we take $w(x) = \mathcal{D}\tilde{v}\mathbf{v_0} = \sum_{i=1}^{d} \partial_i \tilde{v}(x)\alpha_i \mathbf{v_0}$, where $\mathbf{v_0} \in \mathbb{C}^m$ is a fixed non-zero vector, and $W_{\mathcal{D}} = \frac{\tilde{W}\cdot\nabla\tilde{v}}{|\nabla\tilde{v}|^2} \sum_{i=1}^{d} \partial_i \tilde{v}\alpha_i$, with the above $\tilde{v}, \tilde{W}$ (in dimension 2 we use v, W provided by Theorem 1.1). $\square$

Remark. It is possible to obtain the above results by the standard procedure going back to Pliś (e.g., [4]). It consists of constructing a basic 'brick' of the solution and then gluing infinitely many of them. However, the closure under multiplication of the set of solutions of the $\overline{\partial}$ equation allowed to avoid this.

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