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Some New Behaviour in the Deformation Theory of Kleinian Groups

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Anderson and Canary ([1]) have constructed examples of 3-manifolds M for which the space of convex co-compact representations of $\pi_1(M)$ in $PSL_2(\mathbb{C})$ is disconnected but has connected closure (in the topology of algebraic convergence). More precisely, they showed that for their examples the intersection of the closures of any two path components of the space of convex co-compact representations is non-empty. We study these examples and show that there is a connected, uncountable set of geometrically finite representations which is contained in the closure of every path component of the space of convex co-compact representations.

Introduction.

Suppose M is an irreducible, orientable, atoroidal, compact 3-manifold whose boundary is non-empty. Thurston's Hyperbolization Theorem ([14]) guarantees that the interior of M may be imbued with the structure of a geometrically finite hyperbolic manifold. Moreover, if ∂M contains no torus components this structure may be taken to be convex co-compact. But in particular there is a discrete, faithful representation of the fundamental group of M into $PSL_2(\mathbb{C})$, the group of orientation preserving isometries of hyperbolic 3-space, $\mathbb{H}^3$, modelled as the upper half space in $\mathbb{R}^3$. If Γ is the image of this representation (so that $\text{int}\,(M) \cong \mathbb{H}^3/\Gamma$) then conjugating Γ by an element of $PSL_2(\mathbb{C})$ produces a group whose quotient manifold is isometric to $\mathbb{H}^3/\Gamma$. Hence if we are to understand all the possible hyperbolic structures on the interior of M up to isometry we should study the set $H(\pi_1(M))$ of equivalence classes of faithful representations of $\pi_1(M)$ in $PSL_2(\mathbb{C})$ whose images are discrete, where two representations are in the same equivalence class if they are conjugate in $PSL_2(\mathbb{C})$.

$H(\pi_1(M))$ is a quotient of $D(\pi_1(M)) = \{\rho \in \text{Hom}\,(\pi_1(M), PSL_2(\mathbb{C}))|\rho$ is faithful and has discrete image$\}$ and so we can topologize $H(\pi_1(M))$ by giving $D(\pi_1(M))$ the compact-open topology and $H(\pi_1(M))$ the quotient topology. This is called the algebraic topology on $H(\pi_1(M))$.

Call $H(\pi_1(M))$ with the algebraic topology $AH(\pi_1(M))$. An element of $AH(\pi_1(M))$ does not necessarily give a hyperbolic structure to the interior of M; it may give a hyperbolic structure to a compact manifold homotopy equivalent to M or the quotient of its image may not admit a manifold compactification, though it is a conjecture of Marden's that the latter does not occur. Let $\mathcal{A}(M)$ denote the set of marked homeomorphism classes of compact, oriented, atoroidal, irreducible 3-manifolds homotopy equivalent to M. It follows from work of Ahlfors, Bers, Kra, Marden, Maskit, Sullivan and Thurston that the components of the interior of $AH(\pi_1(M))$ are enumerated by the elements of $\mathcal{A}(M)$ (see [5]). In particular the homeomorphism type is constant on each component of the interior. In the case when $\pi_1(M)$ is the fundamental group of a surface Bers ([3]) conjectured that any B-group is contained in the closure of some Bers slice, and later Thurston ([18]) and Sullivan ([17]) extended this to conjecture that the interior of $AH(\pi_1(M))$ is dense in $AH(\pi_1(M))$.

We specialize to the case when ∂M contains no tori (so that the interior of $AH(\pi_1(M))$ is $CC(\pi_1(M))$, the subspace consisting of all those representations whose image is convex co-compact, see [11] and [17]). Anderson and Canary ([1]) have constructed a family of examples for which there are finitely many components of $CC(\pi_1(M))$ and the closures of any two such components intersect. They exhibit, for each integer $k \geq 3$, a manifold M_k such that $\mathcal{A}(M_k)$ has $(k-1)!$ elements and they construct for each $[(M,h)] \in \mathcal{A}(M_k)$ representations ρ_i in the component of $CC(\pi_1(M_k))$ indexed by $[(M_k, id)] \in \mathcal{A}(M_k)$ so that ρ_i converges to ρ and ρ is in the closure of the component of $CC(\pi_1(M_k))$ indexed by $[(M,h)]$ (in fact ρ uniformizes M). This is to say, the homeomorphism type can change in the limit and the closure of $CC(\pi_1(M_k))$ is connected. (Recently Anderson, Canary and McCullough ([2]) have generalized this result to give necessary and sufficient conditions for the closures of two components of the interior of $AH(\pi_1(M))$ to intersect, where now M is any compact, orientable, irreducible, atoroidal 3-manifold with incompressible boundary.)

In this paper we examine Anderson and Canary's examples and show that for these examples there is a set of geometrically finite representations which is contained in the closure of every component of $CC(\pi_1(M_k))$. The construction shows that in some sense this set can be chosen to be "large", in that given any $K \geq 1$ the set contains all of the K-quasiconformal deformations of some geometrically finite representation of $\pi_1(M)$ (this representation depending on K).

In the first section we will describe the construction of the examples and state the theorems that will be proven in Section 3. In Section 2 we will present the versions of the Klein-Maskit combination theorems that we will be using, a statement of a generalization of the Hyperbolic Dehn Surgery theorem due to Comar (upon which the proofs in Section 3 are largely based) and a few technical lemmas that we will use in Section 3.

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1. The examples.

Choose an integer $k \geq 3$. Let $F(j)$ be a surface of genus j with one boundary component and set $B(j) = F(j) \times [0,1]$, $j = 1, \ldots, k$. Let $V = D^2 \times S^1$ be a solid torus and let $A(j)$ ($j = 1, \ldots, k$) be the annulus on ∂V given by $A(j) = [e^{2\pi i(4j-1)/4k}, e^{2\pi i(4j+1)/4k}] \times S^1$. Set $\partial_0 B(j) = \partial F(j) \times [0,1]$. Form M_k by identifying $\partial_0 B(j)$ with $A(j)$, $j = 1, \ldots, k$, with orientation-reversing homeomorphisms. In other words, M_k is obtained by attaching, in order, the I-bundles $B(1), \ldots, B(k)$ to a solid torus V by identifying the annuli $\partial_0(B(1)), \ldots, \partial_0(B(k))$ with a collection of k disjoint, parallel, longitudinal annuli $A(1), \ldots, A(k) \subset \partial V$.

Suppose $\tau \in S_k$ is a permutation of the integers $1, \ldots, k$. Then we may obtain a manifold homotopy equivalent to M_k by performing the above sort of construction but sewing $\partial_0 B(\tau(j))$ to $A(j)$. Denote the manifold so obtained by M_k^τ. M_k^τ is homeomorphic to $M_k^{\tau'}$ if an only if τ and τ' belong to the same right coset of D_k, the dihedral subgroup of $2k$ elements of S_k.

Let $C_1 = \{e^{3\pi i/4k}\} \times S^1 \subset \partial V$ and $C_2 = \{e^{5\pi i/4k}\} \times S^1 \subset \partial V$ be two parallel curves in $\partial V \cap \partial M_k^\tau$. Form $\widehat{M_k^\tau}$ by attaching $S^1 \times [0,1] \times [0,1]$ to M_k^τ by an embedding $h : S^1 \times [0,1] \times \{0,1\} \longrightarrow \partial V$ such that $h(S^1 \times \{\frac{1}{2}\} \times \{0\}) = C_1$ and $h(S^1 \times \{\frac{1}{2}\} \times \{1\}) = C_2$. $\widehat{M_k^\tau}$ is homeomorphic to the manifold obtained by removing an open neighbourhood of the core curve of V from M_k^τ.

Let $h_\tau : M_k \longrightarrow M_k^\tau$ be a fixed homotopy equivalence that is the identity when restricted to the solid torus V. Anderson and Canary have proven that $\{(M_k^\tau, h_\tau) | \tau \in S_k/\mathbb{Z}_k\}$ is a complete set of representatives for $\mathcal{A}(M_k)$. For our purposes it is more convenient (and more natural in light of Johannson's

Deformation Theorem([9])) to require that h_τ be a homotopy equivalence between M_k and M_k^τ which is the identity off of the solid torus V. Using essentially the same proof as in [1] we have the following:

Lemma 3.2 in [1]. *Let* $\{\tau_1, \ldots, \tau_N\}$ *be a set of right coset representatives of* $\mathbb{Z}_k = \langle(123\cdots k)\rangle$ *in* S_k $(N = (k-1)!)$. *Then* $\{(M_k^{\tau_1}, h_{\tau_1}), \ldots, (M_k^{\tau_N}, h_{\tau_N})\}$ *is a complete indexing set for the set of path components of* $CC(\pi_1(M_k))$.

In this paper we prove the following two results:

Theorem A. *For any integer* $k \geq 3$ *there is a representation which lies in the closure of every component of* $CC(\pi_1(M_k))$.

Theorem B. *For any integer* $k \geq 3$ *and* $K \geq 1$ *there is a representation* ρ_K *such that the set of representations of* $\pi_1(M_k)$ *induced by* K*-quasiconformal deformations of* ρ_K *is contained in the closure of every component of* $CC(\pi_1(M_k))$.

2. Preliminaries.

Definitions.

In this article a *Kleinian group* is a discrete, torsion-free subgroup of the group of isometries of hyperbolic 3-space, $\mathbb{H}^3$. Realizing hyperbolic space as the upper half space $\{(z,t) \mid t > 0,\ z \in \mathbb{C}\}$ allows us to view a Kleinian group as a subgroup of $PSL_2(\mathbb{C})$. If Γ is a Kleinian group then the orbit space $\mathbb{H}^3/\Gamma$ is an orientable 3-manifold with constant sectional curvature equal to -1. By fixing an orientation on $\mathbb{H}^3$ and passing this orientation to the quotient we may assume that the orbit space of a Kleinian group is an oriented 3-manifold.

If M is a compact, oriented 3-manifold and Γ is a Kleinian group such that $\mathbb{H}^3/\Gamma$ is homeomorphic to $int(M)$ via an orientation preserving homeomorphism we say that M is *uniformized* by Γ. If Γ is also the image of a representation $\rho : \pi_1(M) \longrightarrow PSL_2(\mathbb{C})$ then we sometimes say that ρ uniformizes M.

The *limit set* of a Kleinian group Γ is the set of accumulation points in $\widehat{\mathbb{C}}$ for the action of Γ on $\mathbb{H}^3$. The limit set of Γ is denoted by $\Lambda(\Gamma)$. When Γ is non-abelian $\Lambda(\Gamma)$ is the smallest, non-empty, closed, Γ-invariant subspace of $\widehat{\mathbb{C}}$. The complement of $\Lambda(\Gamma)$ in $\widehat{\mathbb{C}}$ is the *domain of discontinuity* for Γ and is denoted by $\Omega(\Gamma)$. As is implied by the name Γ acts properly discontinuously on $\Omega(\Gamma)$.

A *Fuchsian group* is a Kleinian group whose limit set is a circle and the stabilizer of any component of its domain of discontinuity is the whole group. (Sometimes such a group is called Fuchsian of the *first kind.*) For our purposes a Fuchsian group has the extended real line as its limit set. That is, for our purposes a Fuchsian group is a subgroup of $PSL_2(\mathbb{R})$.

The *convex core* of a hyperbolic 3-manifold $N = \mathbb{H}^3/\Gamma$ is the smallest convex sub-manifold of N whose inclusion map is a homotopy equivalence. The convex core of N is denoted by $C(N)$. Equivalently, $C(N)$ is the quotient of the convex hull of $\Lambda(\Gamma)$ in $\mathbb{H}^3$ by Γ. If $C(N)$ is compact, then we say that N (or Γ) is *convex co-compact.* If $C(N)$ has finite volume and $\pi_1(N)$ is finitely generated then we say that N is *geometrically finite.*

For a hyperbolic 3-manifold $N = \mathbb{H}^3/\Gamma$ the *conformal extension* of N is the manifold with boundary given by $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$.

Given a Kleinian group Γ and a subgroup $J \subset \Gamma$ a subset $B \subset \widehat{\mathbb{C}}$ is *precisely J-invariant in* Γ (or, equivalently *precisely invariant under J in* Γ), if B is invariant under J and if $g \in \Gamma$ is such that $g(B) \cap B \neq \emptyset$ then $g \in J$.

A *quasiconformal deformation* of a representation ρ is a representation ρ' such that there is a quasiconformal map f of $\widehat{\mathbb{C}}$ with $\rho' = f \circ \rho \circ f^{-1}$. A *quasifuchsian group* is a quasiconformal deformation of a Fuchsian group.

For a compact, atoroidal, irreducible, orientable 3-manifold M we have the set $\mathcal{A}(M)$ of homeomorphism classes of marked, compact, atoroidal, oriented, irreducible 3-manifolds homotopy equivalent to M. Explicitly $\mathcal{A}(M)$ consists of equivalence classes of pairs $[(M', h)]$, where $h : M \longrightarrow M'$ is a homotopy equivalence and two pairs (M_0, h_0) and (M_1, h_1) are in the same equivalence class if and only if there exists an orientation preserving homeomorphism $j : M_0 \longrightarrow M_1$ such that $j \circ h_0$ is homotopic to h_1.

M_0 is a *compact core* for an irreducible 3-manifold M if M_0 is a compact, co-dimension 0 submanifold of M whose inclusion map is a homotopy equivalence. By Scott [16] any 3-manifold with finitely generated fundamental group has a compact core. It is a theorem of McCullough, Miller and Swarup ([13]) that if $i_1 : M_1 \to M$ and $i_2 : M_2 \to M$ are two compact cores for an oriented 3-manifold M then $i_1 \circ \overline{i_2}$ is homotopic to an orientation preserving homeomorphism, where $\overline{i_2}$ is a homotopy inverse for i_2.

We have a map $\Theta : AH(\pi_1(M)) \longrightarrow \mathcal{A}(M)$ defined as follows. Let $\rho \in AH(\pi_1(M))$ and let M_ρ be a compact core for $\mathbb{H}^3/\rho(\pi_1(M))$. Let i_ρ be inclusion of M_ρ into $\mathbb{H}^3/\rho(\pi_1(M))$. Since $(i_\rho)_*^{-1} \circ \rho : \pi_1(M) \longrightarrow \pi_1(M_\rho)$ is an isomorphism between aspherical manifolds there is a homotopy equivalence $h_\rho : M \longrightarrow M_\rho$ such that $(h_\rho)_*$ is conjugate to $(i_\rho)_*^{-1} \circ \rho$. Then Θ maps ρ to the marked homeomorphism class $[(M_\rho, h_\rho)]$. By the result of McCullough, Miller and Swarup above Θ is well-defined. Marden's Isomorphism Theorem and Stability Theorem [11] and a result of Sullivan's [17] implies that two convex co-compact representations are in the same component of $int(AH(\pi_1(M)))$ if and only if they have the same image under Θ.

The Klein-Maskit combination theorems.

Very roughly, the Klein-Maskit combination theorems give sufficient conditions for a group generated by two Kleinian groups to be Kleinian and tells us the topology of the manifold associated to the result. The versions we state here are special cases of those given in [12].

Theorem 1 (The First Klein-Maskit Combination Theorem). *Let $J = \langle z \mapsto z + 1 \rangle$ be a rank 1 parabolic subgroup of discrete groups G_1 and G_2 with $J \neq G_i$, $i = 1, 2$. Assume that there is a horizontal line W which separates $\mathbb{C}$ into two closed disks B_1 and B_2 with $\Lambda(G_j) \subset B_j$, and with the property that B_j is precisely invariant under J in G_{3-j}. Set $G = \langle G_1, G_2 \rangle$. Then*

1. *G is discrete;*
2. *$G = G_1 *_J G_2$;*
3. *G is geometrically finite if and only if both G_1 and G_2 are geometrically finite;*
4. *Let C be the plane spanned by W in $\mathbb{H}^3$ and let B_i^3 be the half space bounded by C and B_i in $\mathbb{H}^3$, $i = 1, 2$. Then $\mathbb{H}^3/G$ is isometric to the manifold obtained by removing B_1^3/J from $\mathbb{H}^3/G_2$ and removing B_2^3/J from $\mathbb{H}^3/G_1$ and identifying the resulting manifolds along their common boundary C/J.*
5. *$(\mathbb{H}^3 \cup \Omega(G))/G$ is orientation preserving homeomorphic to the manifold obtained by gluing $(\mathbb{H}^3 \cup \Omega(G_1))/G_1$ to $(\mathbb{H}^3 \cup \Omega(G_2))/G_2$ by identifying the punctured disk $B_1/J \subset \Omega(G_1)/G_1$ with the punctured disk $B_2/J \subset \Omega(G_2)/G_2$ via an orientation reversing homeomorphism.*

The Second Combination Theorem deals with HNN-extensions of Kleinian groups.

Theorem 2 (The Second Klein-Maskit Combination Theorem). *Let $J = \langle z \mapsto z+1 \rangle$ be a subgroup of a group H. Suppose f is a parabolic transformation $f(z) = z + c$ with c not completely real. Let $A = \{z \mid a < \Im z < b\}$ with $|a-b| < |\Im c|$ and suppose that each component of $\mathbb{C} - A$ is precisely invariant under J in H. Set $G = \langle H, f \rangle$. Then*

1. *G is discrete;*
2. *$G = H*_f$;*
3. *G is geometrically finite if and only if H is;*
4. *Let B_1^3 be the totally geodesic half-space whose closure meets $\widehat{\mathbb{C}}$ exactly in $B_1 = \{z \mid \Im z \geq b\}$, and B_2^3 the totally geodesic half-space whose closure meets $\widehat{\mathbb{C}}$ exactly in $B_2 = \{z \mid \Im z \leq a\}$. Let P_1 be the plane in $\mathbb{H}^3$ $\{(z,t) \mid \Im z = b\}$ and let P_2 be the plane $\{(z,t) \mid \Im z = a\}$. Then $\mathbb{H}^3/G$ is isometric to the manifold obtained by removing B_1^3/J and B_2^3/J from $\mathbb{H}^3/H$ and identifying the resulting boundaries P_1/J and P_2/J via f.*
5. *$(\mathbb{H}^3 \cup \Omega(G))/G$ is orientation preserving homeomorphic to the manifold obtained from $(\mathbb{H}^3 \cup \Omega(H))/H$ by identifying the punctured disk B_1/J with the punctured disk B_2/J in $\Omega(H)/H$ via the map induced by f.*

The Hyperbolic Dehn Surgery Theorem.

Let $\widehat{M}$ be a compact, irreducible, oriented 3-manifold whose boundary contains a single torus T. There may be other surfaces in the boundary, but only one torus. Choose a meridian m and a longitude l for the torus T and consider them as a basis for $\pi_1(T)$. For a pair of relatively prime integers (p,q) denote by $\widehat{M}(p,q)$ the manifold obtained by performing (p,q) Dehn surgery on $\widehat{M}$. That is, $\widehat{M}(p,q)$ is obtained by sewing a solid torus V to $\widehat{M}$ along T via an orientation reversing homeomorphism which maps the meridian of V to a simple closed curve in the homotopy class of $m^p l^q$ on T. The following generalization of Thurston's Hyperbolic Dehn Surgery Theorem is due to T. Comar. See also Bonahon and Otal [4].

Theorem 3 (The Hyperbolic Dehn Surgery Theorem [6]). *Let $\widehat{M}$ be a compact, oriented 3-manifold with one toroidal boundary component T. Let $\widehat{N} = \mathbb{H}^3/\widehat{\Gamma}$ be a geometrically finite hyperbolic 3-manifold and $\phi : \operatorname{int}(\widehat{M}) \longrightarrow \widehat{N}$ an orientation preserving homeomorphism between the interior of M and $\widehat{N}$. Further assume that every parabolic element of $\widehat{\Gamma}$ is conjugate to an element of $\phi_*(\pi_1(T))$. Let $\{(p_n, q_n)\}$ be a sequence of distinct pairs of relatively prime integers.*

Then, for all sufficiently large n, there exists a representation $\beta_n : \widehat{\Gamma} \longrightarrow PSL_2(\mathbb{C})$ with discrete image such that:

1. *$\beta_n(\widehat{\Gamma})$ is convex co-compact and uniformizes $\widehat{M}(p_n, q_n)$;*

2. *the kernel of β_n is normally generated by $m^{p_n} l^{q_n}$; and*

3. *$\{\beta_n\}$ converges to the identity representation of $\widehat{\Gamma}$.*

Moreover, if we let i_n denote the inclusion of $\widehat{M}$ into $\widehat{M}(p_n, q_n)$, then there exists an orientation-preserving homeomorphism ϕ_n : $\operatorname{int}(\widehat{M}(p_n, q_n)) \longrightarrow \mathbb{H}^3/\beta_n(\widehat{\Gamma})$ such that $\beta_n \circ \phi_*$ is conjugate to $(\phi_n)_* \circ (i_n)_*$.

Some Notation: Let A_c^b denote the horizontal strip $\{z \in \mathbb{C} \mid c \leq \Im z \leq b\}$.

Let H_a denote the half-plane $\{z \in \mathbb{C} \mid \Im z \geq a\}$ and let H_a^* denote the lower half-plane $\{z \in \mathbb{C} \mid \Im z \leq a\}$.

For $a \in \mathbb{C}$ let ξ_a be the conformal transformation given by $\xi_a(z) = z + a$.

Lemma 1. *Let $K \geq 1$. Let G be a Fuchsian group containing ξ_1 as a primitive element. Then there is a constant $c = c(K, G)$ with the property that if f is any K-quasiconformal deformation of G fixing $0, 1$ and ∞ then H_c and H_{-c}^* are precisely J-invariant in $f \circ G \circ f^{-1}$.*

Proof.
Suppose there does not exist a c such that H_c is precisely J-invariant in $f \circ G \circ f^{-1}$ for every normalised K-quasiconformal deformation f of G. Then there is a sequence of normalized K-quasiconformal maps $\{f_n\}$ inducing quasiconformal deformations of G and a sequence of group elements $\{g_n\}$ of $G - J$ such that $f_n \circ g_n \circ f_n^{-1}(H_n) \cap H_n \neq \emptyset$. Equivalently $g_n \circ f_n^{-1}(H_n) \cap f_n^{-1}(H_n) \neq \emptyset$. Since G is Fuchsian then Prop. VI.A.6 in [12] gives us that the half-spaces H_1 and H_{-1}^* are precisely J-invariant in G. Hence $f_n^{-1}(H_n)$ must intersect A_{-1}^1.

For each n, let z_n be a point in H_n such that $f_n^{-1}(z_n)$ is in A_{-1}^1. Since each f_n commutes with ξ_1 we may assume that the $f_n^{-1}(z_n)$ all lie in the (compact) rectangle $R = \{z \mid 0 \le \Re z \le 1,\ -1 \le \Im z \le 1\}$. $\{f_n^{-1}(z_n)\}$ has an accumulation point $p \in R$, so we pass to a subsequence and reindex so that $f_n^{-1}(z_n) \longrightarrow p$.

The space of all K-quasiconformal maps normalized to fix $0, 1$ and ∞ is compact with respect to the topology of uniform convergence on compact subsets ([10], Theorems 2.1, 2.2). Hence there is a normalized K-quasiconformal map f such that, up to subsequence, $f_n \longrightarrow f$ uniformly on R. Thus $f(p) = \lim_{n\to\infty} f_n(f_n^{-1}(z_n)) = \lim_{n\to\infty} z_n = \infty$, supplying us with our contradiction.

By repeating the above argument or by appealing to symmetry we have that there is also a c' such that $H_{-c'}^*$ is precisely J-invariant in $f \circ G \circ f^{-1}$ for any normalized K-quasiconformal deformation f of G. The desired constant $c(K, G)$ is the maximum of the two constants. □

Corollary 2. *With the above assumptions* $\Lambda(f \circ G \circ f^{-1}) \subset A_{-c}^c \cup \{\infty\}$.

Lemma 3. *Let* $G_1,\dots,G_N$ *be quasifuchsian groups containing* $J = \langle \xi_1 \rangle$ *and suppose* $\exists c > 0$ *such that* H_{-c}^* *and* H_c *are precisely* J*-invariant in each* G_i.

Set Γ *to be the group generated by* $\{\xi_{a_j i} G_j \xi_{a_j i}^{-1}\}_{j=1}^N$, *where* $0 \le a_1 < a_2 < \dots < a_N$ *are any real numbers with* $a_{i+1} - a_i \ge 2c\ \forall\ i = 1, \dots, N-1$.

Then

1 $\Gamma = \xi_{a_1 i} G_1 \xi_{a_1 i}^{-1} *_J \xi_{a_2 i} G_2 \xi_{a_2 i}^{-1} *_J \cdots *_J \xi_{a_N i} G_N \xi_{a_N i}^{-1}$;

2 Γ *is discrete and geometrically finite;*

3 $H_{a_N + c} \cup H_{a_1 - c}^*$ *is precisely* J*-invariant in* Γ;

4 *Let* S_j *be a surface with boundary such that* G_j *uniformizes* $S_j \times I$ *and for each* j *let* Δ_j *be a component of* ∂S_j. *Represent a solid torus* V *as* $D^2 \times S^1$ *and form a manifold* M *by attaching each* $S_j \times I$ *to* V *by identifying* $\Delta_j \times I$ *with* $[e^{2\pi i(4j-1)/4N}, e^{2\pi i(4j+1)/4N}] \times S^1$ *via an orientation reversing homeomorphism.*

Then Γ *uniformizes* M, *and if* $\partial S_j = \Delta_j\ \forall j$ *then* $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ *is orientation preserving homeomorphic to* $M - \delta$, *where* δ *is the simple closed curve* $\{(1,0)\} \times S^1 \subset \partial M$ *adjacent to* S_1 *and* S_N.

Note. Since both H_{a_N+c} and $H^*_{a_1-c}$ are J-invariant, 2) implies that H_{a_N+c} and $H^*_{a_1-c}$ are precisely J-invariant in Γ.

Proof.
We proceed by induction on N. Suppose that $N = 1$, so that $\Gamma = \xi_{a_1 i} G_1 \xi_{a_1 i}^{-1}$. $\Lambda(\Gamma)$ is a Jordan curve contained in $A_{a_1-c}^{a_1+c} \cup \{\infty\}$ and the complement of this curve consists of two simply connected components each of which is stabilized by Γ. Hence for $g \in \Gamma - J$, $g(H_{a_1+c}) \subset H_{a_1-c}$ and $g(H^*_{a_1-c}) \subset H^*_{a_1+c}$. Since both H_{a_1+c} and $H^*_{a_1-c}$ are precisely J-invariant in Γ this implies that for $g \in \Gamma - J$, $g(H_{a_1+c}) \subset H_{a_1-c} \cap H^*_{a_1+c} = A_{a_1-c}^{a_1+c}$ and $g(H^*_{a_1-c}) \subset H^*_{a_1+c} \cap H_{a_1-c} = A_{a_1-c}^{a_1+c}$ and hence that $g(H_{a_1+c} \cup H^*_{a_1-c}) \cap (H_{a_1+c} \cup H^*_{a_1-c}) = \emptyset$ proving that part 3) holds and hence, since 1), 2) and 4) are clearly true, the base step of our induction.

Suppose now that $G = \langle \xi_{a_1 i} G_1 \xi_{a_1 i}^{-1}, \ldots, \xi_{a_{k-1} i} G_{k-1} \xi_{a_{k-1} i}^{-1} \rangle$ and $H = \xi_{a_k i} G_k \xi_{a_k i}^{-1}$. We prove that the lemma is true for $\Gamma = \langle G, H \rangle$.

To prove parts 1) and 2) consider the line $W = \xi_{(a_{k-1}+c)i}(\mathbb{R})$. W divides $\widehat{\mathbb{C}}$ into two disks $H_{a_{k-1}+c}$ and $H^*_{a_{k-1}+c}$. By our inductive hypotheses $H_{a_{k-1}+c}$ is precisely J-invariant in G and by assumption $H^*_{a_{k-1}+c}$ is precisely J-invariant in H since $H^*_{a_k-c} \supset H^*_{a_{k-1}+c}$ and $H^*_{a_k-c}$ is precisely J-invariant in H. Parts 1) and 2) now follow with an application of Theorem 1.

To prove part 3) first observe the following:

for $g \in G - J$, $g(H_{a_k+c} \cup H^*_{a_1-c}) \subset g(H_{a_{k-1}+c} \cup H^*_{a_1-c}) \subset A_{a_1-c}^{a_{k-1}+c}$ by our inductive assumption;

for $h \in H - J$, $h(H_{a_k+c} \cup H^*_{a_1-c}) \subset h(H_{a_k+c} \cup H^*_{a_k-c}) \subset A_{a_k-c}^{a_k+c}$;

for $g \in G - J$, $g(A_{a_k-c}^{a_k+c}) = g(H_{a_k-c} \cap H^*_{a_k+c}) \subset g(H_{a_{k-1}+c}) \subset A_{a_1-c}^{a_{k-1}+c}$;

and

for $h \in H - J$, $h(A_{a_1-c}^{a_{k-1}+c}) \subset h(H^*_{a_k-c}) \subset A_{a_k-c}^{a_k+c}$.

Let $\mathcal{T}_G$ be a complete set of right coset representatives for J in G containing the identity and similarly let $\mathcal{T}_H$ be a complete set of right coset representatives for J in H containing the identity. Then any element g of Γ can be expressed as $g = g_1 h_1 \cdots g_n h_n \xi^t$ where $g_i \in \mathcal{T}_G$, $h_i \in \mathcal{T}_H$ and $t \in \mathbb{Z}$. Using the above observations it is clear that $g(H_{a_k+c} \cup H^*_{a_1-c}) \subset A_{a_1-c}^{a_k+c}$.

Hence it follows that for $g \in \Gamma - J$, $g(H_{a_k+c} \cup H^*_{a_1-c}) \cap (H_{a_k+c} \cup H^*_{a_1-c}) = \emptyset$ and part 2) of the lemma follows.

It only remains to prove part 4).

Let C_i be the totally geodesic plane meeting $\widehat{\mathbb{C}}$ in $\{z \mid \Im z = a_i + c\}$. Let B_i^- be the half-space bounded by C_i and $H_{a_{i-1}+c}$ and let B_i^+ be the half-space bounded by C_i and $H^*_{a_{i-1}+c}$. Then by Theorem 1 applied inductively,

$$\mathbb{H}^3/\Gamma \cong (\mathbb{H}^3 - B_1^-)/\xi_{a_1 i} G_1 \xi_{a_1 i}^{-1} \cup (\mathbb{H}^3 - (B_1^+ \cup B_2^-))/\xi_{a_2 i} G_2 \xi_{a_2 i}^{-1} \cup \cdots$$
$$\cdots \cup (\mathbb{H}^3 - (B_{N-1}^+ \cup B_N^-))/\xi_{a_{N-1} i} G_{N-1} \xi_{a_{N-1} i}^{-1} \cup (\mathbb{H}^3 - B_N^+)/\xi_{a_N i} G_N \xi_{a_N i}^{-1}.$$

For each i let A_i be an annulus in S_i with Δ_i as one of its boundary components. Then by Theorem 1, part 5) the right hand side of the above expression is the interior of

$$M' = (S_1 \times I) \cup (S_2 \times I) \cup \cdots \cup (S_N \times I)$$

where in the union $A_i \times \{1\}$ is identified with $A_{i+1} \times \{0\}$, $i = 1, \ldots, N-1$ by orientation reversing homeomorphisms.

We claim that M' is orientation preserving homeomorphic to M. Consider the solid torus $\bigcup A_i \times I$, where again the appropriate identifications are made as above. The removal of this solid torus from the above manifold is orientation preserving homeomorphic to the disjoint union $\bigsqcup_{i=1}^N (S_i \times I - \Delta_i \times I)$. Hence M' is obtained by attaching $\bigsqcup_{i=1}^N S_i \times I$ to a solid torus along annuli $\Delta_i \times I$. In other words, M' is orientation preserving homeomorphic to M.

A similar reasoning shows that $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ is of the desired form. □

3. The Proof of Theorem A.

Our goal is to prove the following result:

Theorem A. *There exists a geometrically finite representation* $\rho \in AH(\pi_1(M_k))$ *with* $\Theta(\rho) = [(M_k, id)]$ *such that for each* $[(M_k^\tau, h_\tau)] \in \mathcal{A}(M)$ *there is a sequence of convex co-compact representations* ρ_n^τ *converging to* ρ *with* $\Theta(\rho_n^\tau) = [(M_k^\tau, h_\tau)]$.

Remark 1. It is clear that Theorem A follows from Theorem B. For expository reasons we prove Theorem A first and then generalize the result in Theorem B. Theorem B states that if you choose a quasiconformal constant $K \geq 1$ then there is a geometrically finite representation such that its

K-quasiconformal deformations are contained in the closure of every component of $CC(\pi_1(M_k))$. Theorem A furnishes us with this representation. So while there is no mention of a K in the statement of Theorem A the proof makes use of the K of the statement of Theorem B.

Proof.
The idea is that we find a "nice" uniformization for M_k, call it Γ_k, and "shuffling parabolics" g_τ so that the group $\widehat{\Gamma_k^\tau}$ generated by Γ_k and g_τ is a geometrically finite uniformization of $\widehat{M_k^\tau}$. For each τ we have an inclusion $j_\tau : \Gamma_k \longrightarrow \widehat{\Gamma_k^\tau}$. Using The Hyperbolic Dehn Surgery Theorem we obtain convex co-compact representations $\beta_n^\tau : \widehat{\Gamma_k^\tau} \longrightarrow PSL_2(\mathbb{C})$ with image uniformizing M_k^τ in the correct homeomorphism class and converging to the identity representation. Suppose $\phi : \text{int}\,(M_k) \longrightarrow \mathbb{H}^3/\Gamma_k$ is an orientation preserving homeomorphism. We will consider the representations $\rho_n^\tau : \pi_1(M_k) \longrightarrow PSL_2(\mathbb{C})$ given by

$$\rho_n^\tau = \beta_n^\tau \circ j_\tau \circ \phi_*$$

ρ_n^τ converges to $\rho_K = j_\tau \circ \phi_* = \phi_*$ with image Γ_k, a geometrically finite uniformization of M_k.

Step 1: *The uniformization of M_k and $\widehat{M_k^\tau}$.*

For each $j \in 1, 2, \ldots, k$ let $G_0(j)$ be a Fuchsian model for $B(j)$ containing ξ_1 as a primitive element. Fix $K \geq 1$ and let $c_i = c(K, G_0(j))$ be the constant given by Lemma 1, $i = 1, \ldots, k$. Set $C = \max\{c_1, \ldots, c_k\}$. Choose, for each τ a prime p_τ so that $p_\tau = p_{\tau'}$ if and only if $\tau = \tau'$ and so that $p_\tau > 4Ck$, $\forall \tau$.

By appealing to the Chinese Remainder Theorem or simply by constructing them, we find integers d_τ, one for each τ, with the following properties:

$$\begin{aligned} d_\tau &\equiv 1 \mod p_\tau \text{ and} \\ d_\tau &\equiv 0 \mod p_{\tau'} \text{ for } \tau \neq \tau'. \end{aligned}$$

Define integers a_j by

$$a_j = \sum_\tau d_\tau (j p_\tau + 4C\tau^{-1}(j))$$

Set $G(j) = \xi_{a_j i} G_0(j) \xi_{a_j i}^{-1}$ and define Γ_k to be the group generated by $G(1), \ldots, G(k)$. By Lemma 3 Γ_k is discrete, geometrically finite and uniformizes M_k.

Set $g_\tau = \xi_{p_\tau i}$ and $\widehat{\Gamma_k^\tau} = \langle \Gamma_k, g_\tau \rangle$. We wish to show that $\widehat{\Gamma_k^\tau}$ is a geometrically finite uniformization of $\widehat{M_k^\tau}$.

Observe now that $a_j \equiv 4C\tau^{-1}(j) \mod p_\tau$ and so that $G(j)$ is conjugate by powers of g_τ in $\widehat{\Gamma_k^\tau}$ to $\xi_{4C\tau^{-1}(j)i} G_0(j) \xi^{-1}_{4C\tau^{-1}(j)i}$ The latter is equal to $\xi_{4Cli} G_0(\tau(l)) \xi^{-1}_{4Cli}$ for some l. Hence g_τ has "shuffled" the limit sets of the $G(j)$'s in the manner that τ shuffles $1, 2, \ldots, k$ (see figure 2). Denote the group generated by $\xi_{4Ci} G_0(\tau(1)) \xi^{-1}_{4Ci}$, $\xi_{8Ci} G_0(\tau(2)) \xi^{-1}_{8Ci}, \ldots,$ $\xi_{4Cki} G_0(\tau(k)) \xi^{-1}_{4Cki}$ by Γ_k^τ. Then $\widehat{\Gamma_k^\tau}$ is generated by Γ_k^τ and g_τ. Applying Lemma 3 we find that Γ_k^τ is geometrically finite and uniformizes M_k^τ.

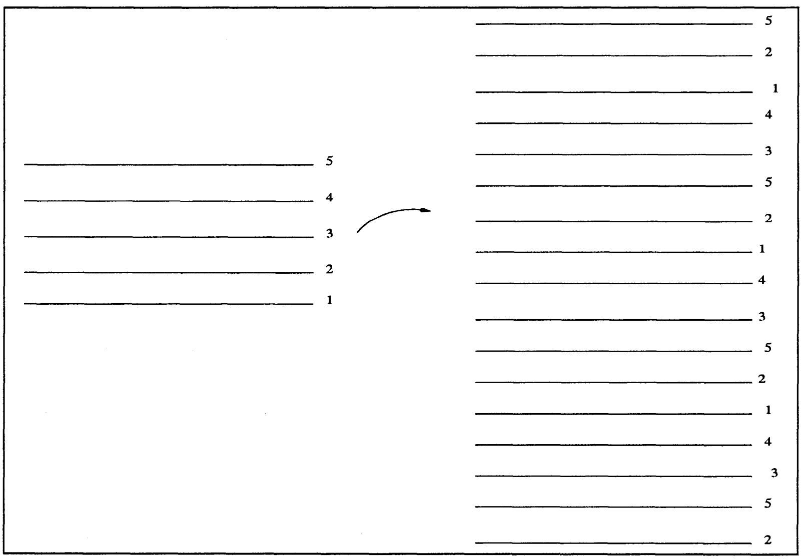

Figure 1.
The shuffling depicted here is for $\tau = (13)(24)$ with $k = 5$.

Lemma 3 gives us that both H_{4Ck+3C} and H^*_{3C} are precisely J-invariant in Γ_k^τ and $\Lambda(\Gamma_k^\tau) \subset A_{3C}^{4Ck+C} \subset A_{3C}^{4Ck+3C}$. Since $p_\tau > 4Ck$ we can apply Theorem 2 to conclude that $\widehat{\Gamma_k^\tau}$ is discrete, torsion free and geometrically finite.

We shall see that $\widehat{\Gamma_k^\tau}$ uniformizes $\widehat{M_k^\tau}$. $(\mathbb{H}^3 \cup \Omega(\Gamma_k^\tau))/\Gamma_k^\tau$ is orientation preserving homeomorphic to $M_k^\tau - \delta$ where $\delta \subset \partial V$ is a longitudinal simple closed curve adjacent to both $B(\tau(1))$ and $B(\tau(k))$. Take two distinct annuli in $(\partial M_k^\tau - \delta) \cap \partial V$ each having δ as a boundary component. By Theorem 2, $(\mathbb{H}^3 \cup \Omega(\widehat{\Gamma_k^\tau}))/\widehat{\Gamma_k^\tau}$ is orientation preserving homeomorphic to the manifold obtained from $M_k^\tau - \delta$ by identifying these two annuli. This resulting manifold is clearly orientation preserving homeomorphic to $M_k^\tau - \alpha$, where α is a core curve for V. Hence

$$\mathbb{H}^3/\widehat{\Gamma_k^\tau} \cong \mathrm{int}\,(\mathbb{H}^3 \cup \Omega(\widehat{\Gamma_k^\tau})/\widehat{\Gamma_k^\tau} \cong M_k^\tau - \mathcal{N}(\alpha) \cong \widehat{M_k^\tau}$$

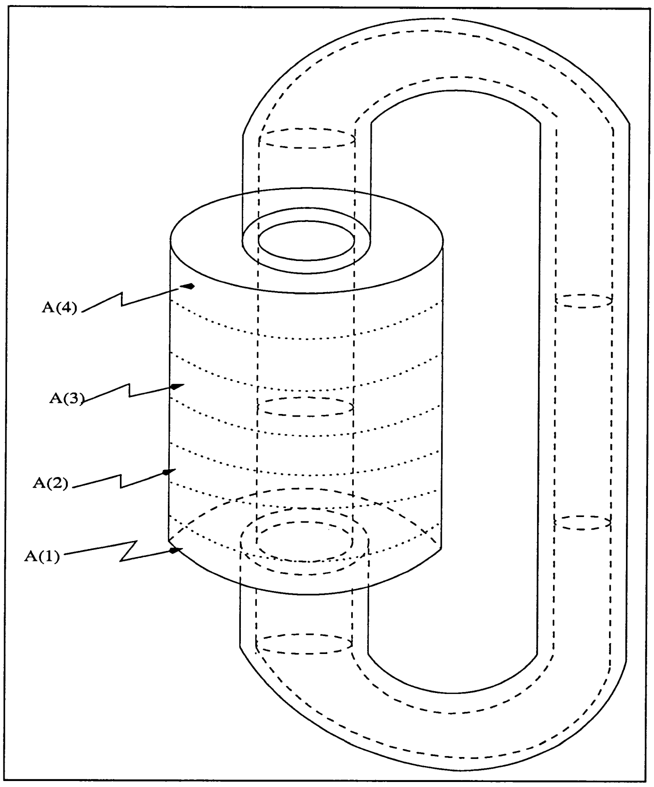

Figure 2.

Step 2: *Constructing the algebraic limits.*

Let α be the core curve of V and consider $M_k^\tau - \mathcal{N}(\alpha)$, which is homeomorphic to $\widehat{M_k^\tau}$. Choose a meridian m and a longitude l on the resulting torus boundary component so that l is homotopic to α in V and m bounds a disk in $\mathcal{N}(\alpha)$. By Lemma 10.3 of [2] the manifold $\widehat{M_k^\tau}(1,n)$ obtained by performing $(1,n)$ Dehn surgery on $\widehat{M_k^\tau}$ is homeomorphic to M_k^τ, $\forall n$. Moreover, if $q_n : \widehat{M_k^\tau}(1,n) \longrightarrow M_k^\tau$ is the homeomorphism furnished by Lemma 10.3 of [2] then q_n can be chosen to be the identity on each $B(j) \subset \widehat{M_k^\tau}(1,n)$.

Choose an orientation preserving homeomorphism $\phi^\tau : \mathrm{int}\,(\widehat{M_k^\tau}) \longrightarrow \mathbb{H}^3/\widehat{\Gamma}_k^\tau$. The Hyperbolic Dehn Surgery Theorem supplies us with representations $\beta_n^\tau : \widehat{\Gamma}_k^\tau \longrightarrow PSL_2(\mathbb{C})$ such that for large enough n

i. $\beta_n^\tau(\widehat{\Gamma}_k^\tau)$ is convex co-compact;

ii. there exist orientation preserving homeomorphisms $\phi_n^\tau : \mathrm{int}\,(\widehat{M_k^\tau}) \longrightarrow \mathbb{H}^3/\beta_n^\tau(\widehat{\Gamma}_n^\tau)$ such that if $i_n : \widehat{M_k^\tau} \longrightarrow \widehat{M_k^\tau}(1,n)$ is inclusion then $\beta_n^\tau \circ (\phi^\tau)_*$ is conjugate to $(\phi_n^\tau)_* \circ (i_n)_*$.

Choose an orientation preserving homeomorphism $\phi : \mathrm{int}\,(M_k) \longrightarrow \mathbb{H}^3/\Gamma_k$. Let $j_\tau : \Gamma_k \longrightarrow \widehat{\Gamma}_k^\tau$ be inclusion and set

$$\rho_n^\tau = \beta_n^\tau \circ j_\tau \circ \phi_*.$$

The kernel of β_n^τ is normally generated by $g_\tau \xi_1^n$. Hence it is the group generated by the set of all elements of the form $h g_\tau \xi_1^n h^{-1}$ with $h \in \widehat{\Gamma_k^\tau}$. Γ_k has trivial intersection with this group, and hence ρ_n^τ is faithful.

For ease of notation denote $\xi_{4C_\tau^{-1}(j)i} G(j) \xi^{-1}_{4C_\tau^{-1}(j)i}$ by $G^\tau(j)$, $j = 1, \ldots, k$. $j_\tau(\xi_{a_j i} G(j) \xi^{-1}_{a_j i}) = g_\tau^{n_j} G^\tau(j) g_\tau^{-n_j}$ where $n_j = \frac{a_j - 4C_\tau^{-1}(j)}{p_\tau}$. Since $\beta_n^\tau(g_\tau^{n_j} G^\tau(j) g_\tau^{-n_j}) = \beta_n^\tau(G^\tau(j))$ and the image of these $G^\tau(j)$'s generates $\beta_n^\tau(\widehat{\Gamma_k^\tau})$ we have that the image of ρ_n^τ is that of β_n^τ and hence is discrete, convex co-compact, and uniformizes M_k^τ.

Recall the map $\Theta : AH(\pi_1(M_k)) \longrightarrow \mathcal{A}(M_k)$ defined previously.

We wish to show that $\Theta(\rho_n^\tau) = [(M_k^\tau, h_\tau)]$ where h_τ is a homotopy equivalence between M_k and M_k^τ which is the identity off of V. This will imply that ρ_n^τ belongs to the component of $CC(\pi_1(M_k))$ corresponding to $[(M_k^\tau, h_\tau)]$. Let $q_n : \widehat{M_k^\tau}(1,n) \longrightarrow M_k^\tau$ be an orientation preserving homeomorphism and $p_\tau : \mathbb{H}^3/\Gamma_k \longrightarrow \mathbb{H}^3/\widehat{\Gamma}_k^\tau$ an orientation preserving covering map with $(p_\tau)_* = j_\tau$. We claim that $r_\tau = q_n \circ i_n \circ (\phi^\tau)^{-1} \circ p_\tau \circ \phi$ is a homotopy equivalence between $\mathrm{int}\,(M_k)$ and $\mathrm{int}\,(M_k^\tau)$. Since ρ_n^τ is conjugate to

$(\phi_n^\tau \circ (q_n)^{-1})_* \circ (r_\tau)_*$ with ρ_n^τ faithful and both ϕ_n^τ and q_n homeomorphisms, $(r_\tau)_*$ is injective. By similar reasoning $(r_\tau)_*$ is surjective.

Denote the homotopy inverse of h_τ by $\overline{h_\tau}$. The composition $h = r_\tau \circ \overline{h_\tau}$ is a homotopy equivalence from M_k^τ to itself. If $K(M)$ denotes the characteristic submanifold of M (see [8] or [9] for details) then by Theorem 24.2 of [9] h is homotopic to a map (again called h) such that $h : K(M_k^\tau) \longrightarrow K(M_k^\tau)$ is homotopy equivalence and $h : \overline{M_k^\tau - K(M_k^\tau)} \longrightarrow \overline{M_k^\tau - K(M_k^\tau)}$ is a homeomorphism. The characteristic submanifold of M_k^τ is $M_k^\tau - \bigcup \mathcal{N}(A(i))$ and so consists of the $B(j)$ together with a solid torus $V_0 \subset V$ such that $\partial V_0 \cap \partial V \subset \partial M_k^\tau \cap \partial V$ (see Section 4 of [7]). Applying Proposition 28.4 of [9] we can homotope h to a map (again, called h) that is a homeomorphism when restricted to each $B(j)$ and to V_0 by a homotopy that preserves the $\partial_0(B(j))$ and the lids of $B(j)$, for $j = 1, \ldots, k$. There is the possibility that this new map reverses the orientation on $\partial_0(B(j))$, for each j. If this were the case then $h_*(\alpha)$ would not be conjugate to α in $\pi_1(M_k^\tau)$, where α generates $\pi_1(V)$. However, $(r_\tau)_* \circ (\overline{h_\tau})_*(\alpha)$ is conjugate to α and hence it is not the case that h reverses the orientation on $\partial_0(B(j))$. Hence h may be assumed to be the identity on $\partial_0(B(j))$ $\forall j$. Since h is homotopic to the identity on V_0, h is an orientation preserving homeomorphism from M_k^τ to itself. In particular $[(M_k^\tau, r_\tau)] = [(M_k^\tau, h_\tau)]$.

Observe that since $i : M_k^\tau \longrightarrow \widehat{M_k^\tau}$ is an embedding and $i_n \circ i$ is homotopic to an orientation preserving homeomorphism $i_n \circ i(M_k^\tau)$ is a compact core for $\widehat{M_k^\tau}(1, n)$. Homotoping $i_n \circ i$ slightly we may assume that the image of M_k^τ lies in the interior of $\widehat{M_k^\tau}(1, n)$. Then $\phi_n^\tau \circ i_n \circ i(M_k^\tau)$ is a compact core for $\mathbb{H}^3/\rho_n^\tau(\pi_1(M_k))$. Hence

$$\begin{aligned}
\Theta(\rho_n^\tau) &= [(\phi_n^\tau \circ i_n \circ i(M_k^\tau), \phi_n^\tau \circ q_n^{-1} \circ r_\tau)] \\
&= [(i_n \circ i(M_k^\tau), q_n^{-1} \circ r_\tau)] \\
&= [(q_n \circ i_n \circ i(M_k^\tau), r_\tau)] \\
&= [(M_k^\tau, r_\tau)] \\
&= [(M_k^\tau, h_\tau)]
\end{aligned}$$

and this second to last equality holds because $q_n \circ i_n \circ i$ is homotopic to an orientation preserving homeomorphism.

Hence ρ_n^τ belongs to the component of $CC(\pi_1(M_k))$ indexed by $[(M_k^\tau, h_\tau)]$ as claimed.

Finally, ρ_n^τ converges to $\rho_K = \phi_*$ which can be seen to belong to the closure of the component of $CC(\pi_1(M_k))$ indexed by $[(M_k, id)]$ by either setting $\tau = id$ and using the proof above, or by a direct application of the Hyperbolic Dehn Surgery Theorem (see Remark(1) following the proof of Theorem B in [2]), or by appealing to Corollary 6 of Ohshika ([15]).

We have completed the proof. □

Theorem B (Intersection Contains a Connected, Uncountable Set). *For every $K \geq 1$ there is a geometrically finite representation ρ_K of $\pi_1(M_k)$ such that the set $O_K \subset QC(\rho_K)$ consisting of all of the K-quasiconformal deformations of ρ_K is contained in the closure of every component of $CC(\pi_1(M_k))$ and $\Theta(\rho_K) = [(M_k, id)]$.*

Proof.
Fix ρ' in O_K and suppose that f is the K-quasiconformal map inducing the isomorphism between ρ' and $\rho = \rho_K$, where ρ_K is the representation constructed in Theorem A. By conjugating if necessary we assume that f fixes $0, 1$ and ∞, so that J is a primitive subgroup of $\rho'(\pi_1(M_k))$.

Set $G(j)' = \rho'(\rho^{-1}(G(j)))$. Then $G(j)'$ is quasi-Fuchsian for each j and, by Corollary 2, $\Lambda(G(j)') \subset A_{a_j-C}^{a_j+C}$ (these a_j are the a_j that appeared in the construction of Γ_k). Let $\Gamma'_k = \rho'(\pi_1(M_k))$ and for each $\tau \in S_k/\mathbb{Z}_k$ set $\widehat{\Gamma}'_\tau = \langle \Gamma'_k, g_\tau \rangle$. We will show that $\widehat{\Gamma}'_\tau$ is discrete, torsion-free, geometrically finite and that the quotient of $\mathbb{H}^3$ by $\widehat{\Gamma}'_\tau$ is homeomorphic (via an orientation preserving homeomorphism) to the interior of $\widehat{M_k^\tau}$.

As in Theorem A, define $n_j = \frac{a_j - 4C\tau^{-1}(j)}{p_\tau}$ and set $G^\tau(j)' = g_\tau^{-n_j} G(j)' g_\tau^{n_j}$. Set Γ'_τ to be the group generated by the $G^\tau(j)'$. Then $\widehat{\Gamma}'_\tau$ is generated by Γ'_τ and g_τ. By Lemma 3 Γ'_τ uniformizes M_k^τ. Just as in Theorem A we can apply Lemma 1, Lemma 3 and Theorem 2 to show that $\widehat{\Gamma}'_\tau$ is geometrically finite and uniformizes $\widehat{M_k^\tau}$.

We now proceed exactly as in Theorem A. To begin, appealing to Theorem 3 (The Hyperbolic Dehn Surgery Theorem) gives a sequence of representations α_n^τ of $\widehat{\Gamma}'_\tau$ in $PSL_2(\mathbb{C})$ converging to the identity, so that the image of each α_n^τ is convex co-compact uniformization of M_k^τ. Then, with $j'_\tau : \Gamma'_k \longrightarrow \widehat{\Gamma}'_\tau$ being inclusion, we set

$$\rho_n^{\tau'} = \alpha_n^\tau \circ j'_\tau \circ \rho'.$$

Then, as in Theorem A, for n sufficiently large each $\rho_n^{\tau'}$ is faithful and has discrete image uniformizing M_k^τ convex co-compactly. Moreover, $\rho_n^{\tau'}$ converges to ρ'. □

Observing that we could have easily done the above proof substituting M_k^σ whenever we saw $M_k = M_k^{id}$, for some σ in S_k, we have the following corollary to the proof.

Corollary C (There are many sets in the intersection). *For every $K \geq 1$ and $[(M_k^\sigma, h_\sigma)] \in \mathcal{A}(M_k)$ there is a geometrically finite representation ρ_K^σ and a set $O_K^\sigma \subset QC(\rho_K^\sigma)$ consisting of the K-quasiconformal deformations of ρ_K^σ such that O_K^σ is contained in the closure of every component of $CC(\pi_1(M_k))$ and $\Theta(\rho_K^\sigma) = [(M_k^\sigma, h_\sigma)]$.*

In a subsequent paper we will prove the analogous results to Theorems A and B for a general irreducible, orientable, atoroidal, compact 3-manifold with boundary, and also prove a theorem that, as a special case, gives us that the components of the intersection of the closures of every component of $CC(\pi_1(M_k))$ are indexed by their marked homeomorphism type.

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