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Brill-Noether theory on singular curves and torsion-free sheaves on surfaces

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Let C be a smooth curve of genus g. Let $W^r_d(C)$ be the Brill-Noether locus of line bundles of degree d and with $r+1$ independent sections. The expected dimension of $W^r_d(C)$ is $\rho(r,d) = g-(r+1)(g-d+r)$. If $\rho(r,d) > 0$ then Fulton and Lazarsfeld have proved that $W^r_d(C)$ is connected. We prove that this is still true if C is a singular irreducible curve lying on a regular surface S with $-K_S$ generated by global sections.
We use this result to give a short new proof of the irreducibility of the moduli space of rank 2 semistable torsion-free sheaves (with a generic polarization and low value of c_2) on a K3 surface (this result was recently proved by a different method by O'Grady).

Introduction.

Let C be a smooth curve of genus g (we will always assume that the base field is $\mathbb{C}$), $J(C)$ its Jacobian, and $W^r_d(C)$ the Brill-Noether locus corresponding to line bundles L of degree d and $h^0(L) \geq r+1$ (see [ACGH]). The expected dimension of this subvariety is $\rho(r,d) = g-(r+1)(g-d+r)$. Fulton and Lazarsfeld [F-L] proved that $W^r_d(C)$ is connected when $\rho > 0$. We are going to generalize this result for certain singular curves, but before stating our result (Theorem I), we need to recall some concepts.

Let C be an integral curve (not necessarily smooth). We still have a generalized Jacobian $J(C)$, defined as the variety parametrizing line bundles, but it will not be complete in general. Define the degree of a rank one torsion-free sheaf on C to be

$$\deg(A) = \chi(A) + p_a - 1,$$

where p_a is the arithmetic genus of C. One can define a scheme $\overline{J}^d(C)$ parametrizing rank one torsion-free sheaves on C of degree d (see [AIK], [D], [R]). If C lies on a surface, then $\overline{J}^d(C)$ is integral, and furthermore the generalized Jacobian $J(C)$ is an open set in $\overline{J}^d(C)$, and then $\overline{J}^d(C)$ is a natural compactification of $J(C)$.

We can define the generalized Brill-Noether locus $\overline{W}^r_d(C)$ as the set of points in $\overline{J}^d(C)$ corresponding to sheaves A with $h^0(A) \geq r+1$ (note that it is complete because of the upper semicontinuity of $h^0(\cdot)$). There is also a determinantal description that gives a scheme structure. This description is a straightforward generalization of the description for smooth curves (see [ACGH]), but we are only interested in the connectivity of $\overline{W}^r_d(C)$, so we can give it the reduced scheme structure.

We will consider curves that lie on a surface S with the following property:

$$(*) \qquad h^1(\mathcal{O}_S) = 0, \ \ \textit{and} \ -K_S \ \ \textit{is generated by global sections.}$$

We will need this condition to prove Proposition 2.5. For instance, S can be a K3 surface. Now we can state the theorem that we are going to prove.

Theorem I. *Let C be a reduced irreducible curve of arithmetic genus p_a that lies in a surface S satisfying $(*)$. Let $\overline{J}^d(C)$, $d>0$, be the compactification of the generalized Jacobian. Then for any $r \geq 0$ such that $\rho(r,d) = p_a - (r+1)(p_a - d + r) > 0$, the generalized Brill-Noether subvariety $\overline{W}^r_d(C)$ is nonempty and connected.*

Remark 1. If $r \leq d - p_a$, by Riemann-Roch inequality we have $\overline{W}^r_d(C) = \overline{J}^d(C)$, and this is connected. Then, in order to prove theorem I we can assume $r > d - p_a$. Note that if A corresponds to a point in $\overline{W}^r_d(C)$ with $r > d - p_a$, then by Riemann-Roch theorem $h^1(A) > 0$.

Let S be a K3 complex surface and H an ample line bundle. Let $\mathfrak{M}_H(c_1, c_2)$ be the moduli space of rank two torsion-free sheaves that are Gieseker semistable with respect to H, with Chern classes equal to c_1 and c_2. As an application of theorem I we give a new short proof of the following.

Theorem II. *With the previous notation, if L is a primitive big and nef line bundle, $c_2 \leq \frac{1}{2}L^2 + 3$, and H is an (L, c_2)-generic polarization, then $\mathfrak{M}_H(L, c_2)$ is irreducible.*

For the definition of (L, c_2)-generic and for the proof of Theorem II see Section 5. Mukai [M] has proved irreducibility when the dimension is 0 or 2. In general for any surface, it is known that for a fixed polarization and c_1, the moduli space is irreducible for high enough second Chern class c_2 ([G-L],

[O1], and [O2]). The case of a K3 surface has also been studied by O'Grady [O3], that has proved irreducibility (for any c_2 and for any rank) as well as results about the Hodge structure, using a different method. Göttsche and Huybrechts [G-H] have studied the Hodge numbers of this moduli space. Yoshioka also has a paper [Yo] on bundles on K3 surfaces in which he proves irreducibility of the moduli space among other things.

To prove Theorem I for a curve C satisfying $(*)$ we will construct a deformation of C into a smooth curve and we will use the fact that Fulton-Lazarsfeld's theorem holds for smooth curves to show that it also holds for C. Given a family of curves we will need to construct a corresponding family of generalized Jacobians and Brill-Noether loci. All this can be done using a relative version of $\overline{J}^d(C)$, but we will proceed in a different way. We will use the fact that all these curves are going to lie on a fixed surface S. Then we will think of the coherent sheaves on C as torsion sheaves on S (all sheaves in this paper will be coherent). To define precisely which sheaves we will consider we need some notation. For any sheaf F on S, let $d(F)$ be the dimension of its support. We say that F has pure dimension n if for any subsheaf E of F we have $d(E) = d(F) = n$. Note that if the support is irreducible, then having pure dimension n is equivalent to being torsion-free when considered as a sheaf on its support. The following theorem follows from [S, Theorem 1.21].

Theorem (Simpson). *Let C be an integral curve on a surface S. Let $\overline{\mathcal{J}}^d_{|C|}$ be the functor which associates to any scheme T the set of equivalence classes of sheaves $\mathcal{A}$ on $S \times T$ with*

(a) *$\mathcal{A}$ is flat over T.*

(b) *The induced sheaf A_t on each fiber $S \times \{t\}$ has pure dimension 1, and its support is an integral curve in the linear system $|C|$.*

(c) *If we consider A_t as a sheaf on its support, it is torsion-free and has rank one and degree d.*

Sheaves $\mathcal{A}$ and $\mathcal{B}$ are equivalent if there exists a line bundle L on T such that $\mathcal{A} \cong \mathcal{B} \otimes p_T^* L$, where $p_T : S \times T \to T$ is the projection on the second factor.

Then there is a coarse moduli space that we also denote by $\overline{\mathcal{J}}^d_{|C|}$. I.e., the points of $\overline{\mathcal{J}}^d_{|C|}$ correspond to isomorphism classes of sheaves, and for any family $\mathcal{A}$ of such sheaves parametrized by T, there is a morphism

$$\phi : T \to \overline{\mathcal{J}}^d_{|C|}$$

such that $\phi(t)$ corresponds to the isomorphism class of A_t.

Note that $\overline{\mathcal{J}}^d_{|C|}$ parametrizes pairs (C', A) with C' an integral curve linearly equivalent to C and A a torsion-free rank one sheaf on C.

We denote by $\pi : \overline{\mathcal{J}}^d_{|C|} \to U \subset |C|$ the obvious projection giving the support of each sheaf, where U is the open subset of $|C|$ corresponding to integral curves.

A family of curves on a surface S parametrized by a curve T is a subvariety $\mathcal{C} \subset S \times T$, flat over T, such that the fiber $\mathcal{C}|_t = C_t$ over each $t \in T$ is a curve on S. Analogously, a family of sheaves on a surface S parametrized by a curve T is a sheaf $\mathcal{A}$ on $S \times T$, flat over T. For each $t \in T$ we will denote the corresponding member of the family by $A_t = \mathcal{A}|_t$.

Altman, Iarrobino and Kleiman [AIK] proved the following theorem

Theorem (Altman–Iarrobino–Kleiman). *With the same notation as before, $\overline{\mathcal{J}}^d_{|C|}$ is flat over U and its geometric fibers are integral. The subset of $\overline{\mathcal{J}}^d_{|C|}$ corresponding to line bundles (i.e., the relative generalized Jacobian) is open and dense in $\overline{\mathcal{J}}^d_{|C|}$.*

We also consider the family of generalized Brill-Nother loci $\overline{\mathcal{W}}^r_{d,|C|} \subset \overline{\mathcal{J}}^d_{|C|}$, and the projection $q : \overline{\mathcal{W}}^r_{d,|C|} \to U$.

Outline of the proof of Theorem I.

Note that $\overline{W}^r_d(C)$ is the fiber of q over the point $u_0 \in |C|$ corresponding to the curve C. Let U be the open subset of $|C|$ corresponding to integral curves, and V the subset of smooth curves. Define $(\overline{\mathcal{W}}^r_d)_V$ to be the Brill-Noether locus of sheaves with smooth support, i.e., $(\overline{\mathcal{W}}^r_d)_V = q^{-1}(V)$. By [F-L], the restriction $q_V : (\overline{\mathcal{W}}^r_d)_V \to V$ has connected fibers. We want to use this fact to show that $\overline{W}^r_d(C)$ is connected. Let A be a rank one torsion-free sheaf on C corresponding to a point in $\overline{W}^r_d(C)$, and assume that it is generated by global sections. We think of A as a torsion sheaf on S. Then we have a short exact sequence on S

$$0 \to E \xrightarrow{f_0} H^0(A) \otimes \mathcal{O}_S \to A \to 0,$$

where the map on the right is evaluation. This sequence has already appeared in the literature (see [L], [Ye]). Our idea is to deform f_0 to a family f_t. The cokernel of f_t will define a family of sheaves A_t with $h^0(A_t) \geq h^0(A)$ (because $h^0(E) = 0$), and then for each t the point in $\overline{\mathcal{J}}^d_{|C|}$ corresponding to A_t lies in $\overline{\mathcal{W}}^r_{d,|C|}$. Assume that there are 'enough' homomorphisms from E to $H^0 \otimes \mathcal{O}_S$ and the family f_t can be chosen general enough, so that for a general t, the support of A_t is smooth (the details of this construction are in Section 2). The family A_t shows that the point in $\overline{W}^r_d(C)$ corresponding to A is in the closure of $(\overline{\mathcal{W}}^r_d)_V$ in $\overline{\mathcal{J}}^d_{|C|}$. It can be shown that this closure has connected fibers. Let X be the fiber over u_0 of this closure. Then all sheaves for which this construction works are in the connected component X of $\overline{W}^r_d(C)$. If this could be done for all sheaves in $\overline{W}^r_d(C)$ this would finish the proof, but there are sheaves for which this construction doesn't work. For these sheaves we show in section 3 that they can be deformed (keeping the support C unchanged) to a sheaf for which a refinement of this construction works. This shows that all points in $\overline{W}^r_d(C)$ are in the connected component X.

1. The main lemma.

The precise statement that we will use to prove Theorem I is the following lemma.

Lemma 1.1. *Let C be an integral complete curve in a surface S. Assume that for each rank one torsion-free sheaf A on C with $h^0(A) = r+1 > 0$ and* $\deg(A) = d > 0$ *such that $\rho(r,d) > 0$ we have the following data:*

(a) *A family of curves $\mathcal{C}$ in S parametrized by an irreducible curve T (not necessarily complete).*

(b) *A connected curve T' (not necessarily irreducible nor complete) with a map $\psi : T' \to T$.*

(c) *A rank one torsion-free sheaf $\mathcal{A}$ on $\mathcal{C}' = \mathcal{C} \times_T T'$, flat over T', inducing rank one torsion-free sheaves on the fibers of $\mathcal{C}' \to T'$.*

Assume that the following is satisfied:

(i) *$\mathcal{C}|_{t_0} \cong C$ for some $t_0 \in T$, $\mathcal{C}|_t$ is linearly equivalent to C for all $t \in T$, and $\mathcal{C}|_t$ is smooth for $t \neq t_0$.*

(ii) *One irreducible component of T' is a finite cover of T, and the rest of the components of T' are mapped to $t_0 \in T$.*

(iii) *$\mathcal{A}|_{t'_0} \cong A$ for some $t'_0 \in T'$ mapping to $t_0 \in T$.*

(iv) $h^0(\mathcal{A}|_{t'}) \geq r+1$ *for all* $t' \in T'$.

Then the generalized Brill-Noether subvariety $\overline{W}^r_d(C)$ *of the compactified generalized Jacobian* $\overline{J}^d(C)$ *is connected.*

Proof. We will use the notation introduced in the previous section. The map $q : \overline{\mathcal{W}}^r_{d,|C|} \to U$ is a projective morphism. Recall that $\overline{W}^r_d(C)$ is the fiber of q over u_0, where u_0 is the point corresponding to C. By [F-L] the morphism q has connected fibers over V, thus a general fiber of q is connected, and we want to prove that the fiber over $u_0 \in U$ is also connected.

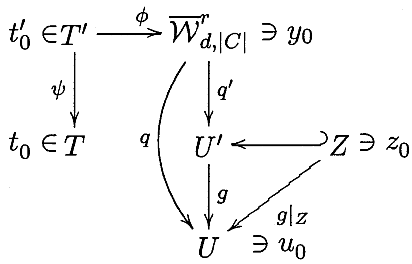
$$\begin{array}{ccc} \overline{W}^r_d(C) & \hookrightarrow & \overline{\mathcal{W}}^r_{d,|C|} \\ \downarrow & & \downarrow q \\ u_0 & \hookrightarrow & U \end{array}$$

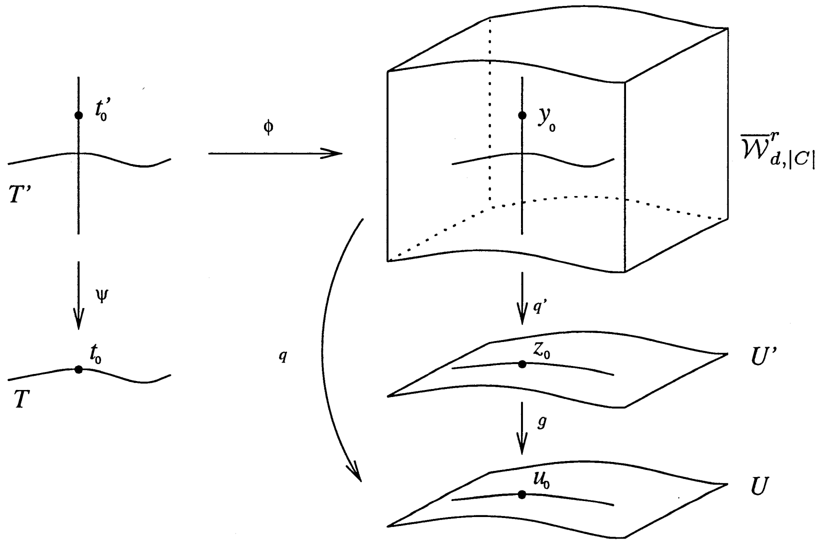
Let $\overline{\mathcal{W}}^r_{d,|C|} \xrightarrow{q'} U' \xrightarrow{g} U$ be the Stein factorization of q (see [H, III Corollary 11.5]), i.e., q' has connected fibers and g is a finite morphism. A general fiber of q is connected, and then U' has one irreducible component Z that maps to U birationally. The subset U is open in $|C|$ and hence normal, the restriction $g|_Z : Z \to U$ is finite and birational, Z and U are integral, thus by Zariski's main theorem (see [H, III Corollary 11.4]) each fiber of $g|_Z$ consists of just one point. Let z_0 be the point of Z in the fiber $g^{-1}(u_0)$.

Claim. Let y_0 be a point in the fiber $q^{-1}(u_0) = \overline{W}^r_d(C)$. Then y_0 is mapped by q' to z_0.

This claim implies that that $\overline{W}^r_d(C)$ is connected. Now we will prove the claim.

Let A be the sheaf on S corresponding to the point y_0. Let T', T, $t'_0 \in T'$, $t_0 \in T$, $\psi : T' \to T$ be the curves points and morphism given by the hypothesis of the lemma. Let $\phi : T' \to \overline{\mathcal{J}}^d_{|C|}$ be the morphism given by the universal property of the moduli space $\overline{\mathcal{J}}^d_{|C|}$. Item (iv) imply that the image of ϕ is in $\overline{\mathcal{W}}^r_{d,|C|}$.





The restriction of $q' \circ \phi$ to $T' \setminus \psi^{-1}(t_0)$ maps to Z, because for $t' \in T' \setminus \psi^{-1}(t_0)$ the sheaf $\mathcal{A}|_{t'}$ has smooth support by item (i). Items (c) and (i) imply that $g \circ q' \circ \phi(\psi^{-1}(t_0)) = u_0$. Thus $q' \circ \phi(\psi^{-1}(t_0))$ is a finite number of points (because it is in the fiber of g over u_0).

The facts that $q' \circ \phi(T' \setminus \psi^{-1}(t_0))$ is in Z and that $q' \circ \phi(\psi^{-1}(t_0))$ is a finite number of points imply that $q' \circ \phi(\psi^{-1}(t_0))$ is also in Z (because by item (b) the curve T' is connected and thus also its image under $q' \circ \phi$), and in fact $q' \circ \phi(\psi^{-1}(t_0)) = z_0$ because $q' \circ \phi(\psi^{-1}(t_0))$ is in the fiber of g over u_0.

By item (ii), $t'_0 \in \psi^{-1}(t_0)$. Then $q' \circ \phi(t'_0) = z_0$, and by item (iii) we have $y_0 = \phi(t'_0)$, then $q'(y_0) = q'(\phi(t'_0)) = z_0$ and the claim is proved. $\square$

In Section 2 we will construct this family under some assumptions on A (Proposition 2.5), and in Section 3 we will show how to use that to construct a family for any A. Note that because of Remark 1 we can assume $h^1(A) > 0$.

2. A particular case.

Given a rank one torsion-free sheaf A on an integral curve lying on a surface S, we define another sheaf A^* that is going to be some sort of dual. Let j be the inclusion of the curve C in the surface S. We define A^* as follows:

$$A^* = Ext^1(j_*A, \omega_S).$$

The operation $A \to A^*$ is a contravariant functor. Note that the support of A^* is C. It will be clear from the context when we are referring to A^* as a torsion sheaf on S or as a sheaf on C. In the case in which A is a line bundle, then $A^* = A^\vee \otimes \omega_C$. Now we prove some properties of this "dual".

Lemma 2.1. *Let A be a rank one torsion-free sheaf on an integral curve lying on a surface. Then $A^{**} = A$.*

Proof. First observe that if L is a line bundle on C, then $(A \otimes L)^* \cong A^* \otimes L^\vee$. To see this, take an injective resolution of ω_S

$$0 \to \omega_S \to \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots$$

Now we use this resolution to calculate the Ext sheaf.

$$Ext^1(A \otimes L, \omega_S) = h^1(Hom(A \otimes L, \mathcal{I}_\bullet)) = h^1(L^\vee \otimes Hom(A, \mathcal{I}_\bullet)) =$$

$$= L^\vee \otimes h^1(Hom(A, \mathcal{I}_\bullet)) = L^\vee \otimes Ext^1(A, \omega_S).$$

The third equality follows from the fact that $Hom(A, \mathcal{I}_\bullet)$ is supported on the curve and $L^\vee$ is locally free

It follows that $(L \otimes A)^{**} \cong L \otimes A^{**}$, and then proving the lemma for A is equivalent to proving it for $L \otimes A$. Multiplying with an appropriate very ample line bundle, we can assume that A is generated by global sections. Then we have an exact sequence

$$0 \to E \to V \otimes \mathcal{O}_S \to A \to 0, \tag{2.1}$$

where $V = H^0(A)$. The following lemma proves that E is locally free.

Lemma 2.2. *Let M be a torsion-free sheaf on an integral curve C that lies on a smooth surface S. Let $j : C \to S$ be the inclusion. Let F be a locally free sheaf on the surface. Let $f : F \to j_*M$ be a surjection. Then the elementary transformation F' of F, defined as the kernel of f*

$$0 \to F' \to F \xrightarrow{f} j_*M \to 0, \tag{2.2}$$

is a locally free sheaf.

Proof. M is torsion-free sheaf on C, and then j_*M has depth at least one, and because S is smooth of dimension 2, this implies that the projective dimension of j_*M is at most one ($Ext^i(j_*M, \mathcal{O}_S) = 0$ for $i \geq 2$). Now $Ext^i(F, \mathcal{O}_S) = 0$ for $i \geq 1$ because F is locally free, and then from the exact sequence (2.2), we get

$$0 \to Ext^i(F', \mathcal{O}_S) \to Ext^{i+1}(j_*M, \mathcal{O}_S) \to 0, \quad i \geq 1,$$

and then $Ext^i(F', \mathcal{O}_S) = 0$ for $i \geq 1$, and this implies that F' is locally free.
□

In particular, $E^{\vee\vee} = E$. Applying the functor $Hom(\cdot, \omega_S)$ twice to the sequence (2.1), we get

$$0 \to E \to V \otimes \mathcal{O}_S \to A^{**} \to 0.$$

Comparing with (2.1) we get the result (because the map on the left is the same for both sequences).

□

Lemma 2.3. $\mathrm{Ext}^1(A, \omega_S) \cong H^0(A^*)$*, and this is dual to* $H^1(A)$*.*

Proof. The local to global spectral sequence for Ext gives the following exact sequence

$$0 \to H^1(Hom(A, \omega_S)) \to \mathrm{Ext}^1(A, \omega_S) \to H^0(A^*) \to H^2(Hom(A, \omega_S))$$

But $Hom(A, \omega_S) = 0$ because A is supported in C and then the first and last terms in the sequence are zero and we have the desired isomorphism.

□

Now we will prove a lemma that we will need. The proof can also be found in [O], but for convenience we reproduce it here.

Lemma 2.4. *Let E and F be two vector bundles of rank e and f over a smooth variety X. Assume that $E^\vee \otimes F$ is generated by global sections. If $\phi : E \to F$ is a sheaf morphism, we define $D_k(\phi)$ to be the subset of X where $\mathrm{rk}(\phi_x) \leq k$ (there is an obvious determinantal description of $D_k(\phi)$ that gives a scheme structure). Let d_k be the expected dimension of $D_k(\phi)$*

$$d_k = \dim(X) - (e-k)(f-k).$$

Then there is a Zariski dense set U of $\mathrm{Hom}(E,F)$ such that if $\phi \in U$, then we have that $D_k(\phi) \setminus D_{k-1}(\phi)$ is smooth of the expected dimension (if $d_k < 0$ then it will be empty).

Proof. Let M_k be the set of matrices of dimension $e \times f$ and of rank at most k (there is an obvious determinantal description that gives a scheme structure to this subvariety). It is well known that the codimension of M_k in the space of all matrices is $(e-k)(f-k)$, and that the singular locus of M_k is M_{k-1}.

Now, because $E^\vee \otimes F$ is generated by global sections, we have a surjective morphism

$$H^0(E^\vee \otimes F) \otimes \mathcal{O}_X \to E^\vee \otimes F$$

that gives a morphism of maximal rank between the varieties defined as the total space of the previous vector bundles

$$p : X \times H^0(E^\vee \otimes F) \to \mathbb{V}(E^\vee \otimes F).$$

Define $\Sigma_k \subset \mathrm{V}(E^\vee \otimes F)$ as the set such that $\mathrm{rk}(\phi_x) \leq k$. The fiber of Σ_k over any point in X is obviously M_k. Define Z_k to be $p^{-1}(\Sigma_k)$. The fact that p has maximal rank implies that Z_k has codimension $(e-k)(f-k)$ in $X \times H^0(E^\vee \otimes F)$ and that the singular locus of Z_k is Z_{k-1}.

Now observe that the restriction of the projection

$$q|_{Z_k \setminus Z_{k-1}} : Z_k \setminus Z_{k-1} \to H^0(E^\vee \otimes F)$$

has fiber $q|_{Z_k \setminus Z_{k-1}}{}^{-1}(\phi) \cong D_k(\phi) \setminus D_{k-1}(\phi)$. Finally, by generic smoothness, for a general $\phi \in H^0(E^\vee \otimes F)$ this is smooth of the expected dimension (or empty). □

Now we will construct the deformation of A that we described in the section 1 in the particular case in which both A and A^* are generated by global sections.

Proposition 2.5. *Let A be a rank one torsion-free sheaf on an integral curve C lying on a surface S with $h^1(\mathcal{O}_S)=0$ and $-K_S$ generated by global sections. Denote $j : C \hookrightarrow S$. If A and A^* are both generated by global sections, then there exists a (not necessarily complete) smooth irreducible curve T and a sheaf $\mathcal{A}$ on $S \times T$ flat over T, such that*

(a) *the sheaf induced on the fiber of $S \times T \to T$ over some $t_0 \in T$ is j_*A*

(b) *the sheaf A_t induced on the fiber over any $t \in T$ with $t \neq t_0$ is supported on a smooth curve C_t and it is a rank one torsion-free sheaf when considered as a sheaf on C_t*

(c) *$h^0(A_t) \geq h^0(A)$ for every $t \in T$.*

Note that these are the hypothesis of Lemma 1.1 for the particular case in which both A and A^* are generated by global sections. We will lift this condition in the next section.

Proof. The fact that A is generated by global sections implies that there is an exact sequence

$$0 \to E \overset{f_0}{\to} V \otimes \mathcal{O}_S \to A \to 0 \qquad V = H^0(A),$$

with E locally free (by Proposition 2.2). Taking global sections in this sequence we see that $H^0(E)=0$, because

$$0 \to H^0(E) \to V \overset{\cong}{\to} H^0(A).$$

Consider a curve T mapping to $\operatorname{Hom}(E, V \otimes \mathcal{O}_S)$ with $t_0 \in T$ mapping to f_0 (so that item (a) is satisfied). Denote by f_t the morphism given for $t \in T$ by this map. After shrinking T we can assume that f_t is still injective. Let π_1 be the projection of $S \times T$ onto the first factor and let $\mathcal{E} = \pi_1^* E$. Using the universal sheaf and morphism on $\operatorname{Hom}(E, V \otimes \mathcal{O}_S)$ we can construct (by pulling back to $S \times T$) an exact sequence on $S \times T$

$$0 \to \mathcal{E} \overset{f}{\to} V \otimes \mathcal{O}_{S \times T} \to \mathcal{A} \to 0$$

that restricts for each t to an exact sequence

$$0 \to E \overset{f_t}{\to} V \otimes \mathcal{O}_S \to A_t \to 0, \tag{2.3}$$

where A_t is a sheaf supported in the degeneracy locus of f_t. It is clear that $\deg(A) = \deg(A_t)$.

Now we are going to prove that if the curve T and the mapping to $\mathrm{Hom}(E, V \otimes \mathcal{O}_S)$ are chosen generically, the quotient of the map gives the desired deformation.

The flatness of $\mathcal{A}$ over T follows from the fact that it has a short resolution and from the local criterion of flatness (we can apply [H, III Lemma 10.3.A]).

The condition on $h^0(A_t)$ follows because $H^0(E) = 0$ and we have a sequence

$$0 \to H^0(E) = 0 \to V \to H^0(A_t),$$

and then $h^0(A) \leq h^0(A_t)$. This proves item (c).

Using the long exact sequence obtained by applying $Hom(\cdot, \mathcal{O}_S)$ to (2.3), and the fact that E is locally free, we obtain that $Ext^i(A_t, \mathcal{O}_S)$ vanishes for $i \geq 2$, and so the projective dimension of A_t is 1, and this implies that A_t, when considered as a sheaf on its support C_t, is torsion-free.

We have to prove that we can choose the curve T and the map to $\mathrm{Hom}(E, V \otimes \mathcal{O}_S)$ such that C_t is smooth for $t \neq t_0$ (here we will use that A^* is generated by global sections).

First note that $Ext^1(A, \mathcal{O}_S)$ is generated by global sections, because $Ext^1(A, \mathcal{O}_S) = A^* \otimes \omega_S^{-1}$, and both A^* and ω_S^{-1} are generated by global sections. Now we see that $E^\vee$ is generated by global sections, because we have

$$0 \to V^\vee \otimes \mathcal{O}_S \to E^\vee \to Ext^1(A, \mathcal{O}_S) \to 0,$$

$Ext^1(A, \mathcal{O}_S)$ is generated by global sections and $H^1(V^\vee \otimes \mathcal{O}_S) = 0$. Then $E^\vee \otimes (V \otimes \mathcal{O}_S)$ is generated by global sections.

Now apply Lemma 2.4 with $F = V \otimes \mathcal{O}_S$. Then $n = m = r + 1$, $k = r$ and the expected dimension is 1. And the lemma gives that for ϕ in a Zariski open subset of $\mathrm{Hom}(E, V \otimes \mathcal{O}_S)$, the degeneracy locus $D_r(\phi)$ of ϕ is smooth away from the locus $D_{r-1}(\phi)$ where ϕ has rank $r - 1$, but again by Lemma 2.4 the locus $D_{r-1}(\phi)$ is empty. This proves item (b). □

3. General case.

Now we don't assume that A satisfies the properties of the particular case (i.e., A and A^* now might not be generated by global sections). We will find a new sheaf that satisfies those conditions. We know how to deform this new sheaf, and we will show how we can use this deformation to construct a deformation of the original A.

We start with a rank one torsion-free sheaf A with $h^0(A)$, $h^1(A) > 0$ on an integral curve C lying on a surface. First we define A' as the base point free part of A, i.e., A' is the image of the evaluation map

$$H^0(A) \otimes \mathcal{O}_C \to A.$$

We have assumed that $h^0(A) > 0$, and then A' is a (nonzero) rank one torsion-free sheaf. Obviously, $H^0(A) = H^0(A')$. We have a short exact sequence

$$0 \to A' \to A \to Q \to 0,$$

where Q has support of dimension 0. Now consider A'^*, and define B to be its base point free part. We have $h^0(A'^*) = h^1(A') = h^1(A) + h^0(Q) \geq h^1(A) > 0$. The first equality by lemma 2.3, and the last inequality by assumption. Then B is a (nonzero) rank one torsion-free sheaf. Finally define A'' to be equal to B^*.

Lemma 3.1. *Both A'' and A''^* are generated by global sections.*

Proof. Since B is the base point free part of A'^*, we have a sequence

$$0 \to B \to A'^* \to R \to 0$$

where R has support of dimension zero. Applying $Hom(\cdot, \omega_S)$ we get

$$0 \to A' \to B^* = A'' \to \widetilde{R} \to 0 \qquad \widetilde{R} = Ext^2(R, \omega_S),$$

whose associated cohomology long exact sequence gives

$$0 \to H^0(A') \to H^0(B^*) \to H^0(\widetilde{R}) \to H^1(A') \to H^1(B^*) \to 0.$$

To see that A'' is generated by global sections, it is enough to prove that the last map is an isomorphism, because then the first three terms make a short exact sequence, and the fact that A' and $\widetilde{R}$ are generated by global sections (the first by definition, the second because its support has dimension zero) will imply that B^* (that is equal to A'' by definition) is generated by global sections.

To prove that the last map is an isomorphism, we only need to show that $h^1(A') = h^1(B^*)$, and this is true because

$$h^1(A') = h^0(A'^*) = h^0(B) = h^0(B^{**}) = h^1(B^*).$$

The first equality is by Lemma 2.3, the second because B is the base point free part of A'^*, the third by Lemma 2.1, and the last again by Lemma 2.3.

To see that A''^* is generated by global sections, note that by definition $A''^* = B^{**} = B$, and this is generated by global sections.

□

We started with a rank one torsion-free sheaf A with $h^0(A)$ and $h^1(A) > 0$, and we have constructed new sheaves A' and A'' with (nontrivial) maps $A' \to A$ and $A' \to A''$. They give rise to exact sequences

$$0 \to A' \to A \to Q \to 0 \tag{3.1a}$$

$$0 \to A' \to A'' \to \widetilde{Q} \to 0. \tag{3.1b}$$

Lemma 3.2. *With the previous definitions we have $h^0(A') = h^0(A)$ and $h^1(A'') = h^1(A')$.*

Proof. By construction $h^0(A') = h^0(A)$ and $h^0(A''^*) = h^0(A'^*)$. By lemma 2.3 this last equality is equivalent to $h^1(A'') = h^1(A')$.

□

As A'' and A''^* are generated by global sections, then by Proposition 2.5 the sheaf A'' can be deformed in a family A''_t in such a way that the support of a general member of the deformation is smooth. The idea now is to find (flat) deformations of A' and A, so that for every t we still have maps like (3.1a) and (3.1b). From the existence of these maps we will be able to obtain the condition that $h^0(A_t) \geq h^0(A)$, then we will be able to apply Lemma 1.1 and then Theorem I will be proved. The details are in Section 4. We will start by showing how the condition on $h^0(A_t)$ is obtained, and then how we can find the deformations of A' and A.

Proposition 3.3. *Let A, A', A'' be rank one torsion-free sheaves on an integral curve C. Assume that they fit into exact sequences like (3.1) and that $h^0(A') = h^0(A)$ and $h^1(A') = h^1(A'')$. Let P be a curve (not necessarily complete), and let $\mathcal{A}$, $\mathcal{A}'$, and $\mathcal{A}''$ be sheaves on $S \times P$, flat over P, inducing for each $p \in P$ rank one torsion-free sheaves A_p, A'_p, A''_p, supported on a curve C_p of S, where $A_{p_0} = A$, $A'_{p_0} = A'$, and $A''_{p_0} = A''$ for some $p_0 \in P$. Assume that $h^0(A''_p) \geq h^0(A''_{p_0})$ for all $p \in P$ and that we have short exact sequences*

$$0 \to \mathcal{A}' \to \mathcal{A} \to \mathcal{Q} \to 0$$

$$0 \to \mathcal{A}' \to \mathcal{A}'' \to \widetilde{\mathcal{Q}} \to 0$$

with Q and $\widetilde{Q}$ flat over P (i.e., the induced sheaves Q_p, $\widetilde{Q}_p$ have constant length, equal to $l(Q)$ and $l(\widetilde{Q})$ respectively).

Then we have $h^0(A_p) \geq h^0(A_{p_0})$ for all $p \in P$.

Proof. For each $p \in P$ we have sequences

$$0 \to A'_p \to A_p \to Q_p \to 0$$

$$0 \to A'_p \to A''_p \to \widetilde{Q}_p \to 0.$$

The maps on the left are injective because they are nonzero and the sheaves have rank one and are torsion-free. Using the associated long exact sequences and the hypothesis we have

$$h^0(A_p) \geq h^0(A'_p) \geq h^0(A''_p) - l(\widetilde{Q}_p) \geq h^0(A'') - l(\widetilde{Q}) = h^0(A') = h^0(A).$$

□

It only remains to prove that those sheaves can be "deformed along", and that those deformations are flat, i.e. that given A, A' and A'' we can construct $\mathcal{A}'$ and $\mathcal{A}''$. This is proved in the following propositions.

Proposition 3.4. *Let L and M be rank one torsion-free sheaves on an integral curve C that lies on a surface S. Assume we have a short exact sequence*

$$0 \to L \to M \to Q \to 0. \tag{3.2}$$

Assume furthermore that we are given a sheaf $\mathcal{M}$ on $S \times P$ (where P is a connected but not necessarily irreducible curve) that is a deformation of M, flat over P. I.e., $\mathcal{M}|_{p_0} \cong M$ for some $p_0 \in P$, and for all $p \in P$ we have that $M_p = \mathcal{M}|_p$ are torsion-free sheaves on C_p, where C_p is a curve on S.

Then, there is a connected curve P' with a map $f : P' \to P$ and a sheaf $\mathcal{L}'$ over $S \times P'$ with the following properties:

One irreducible component of P' is a finite cover of P and the rest of the components map to $p_0 \in P$. The sheaf $\mathcal{L}'$ is a deformation of L, in the sense that $\mathcal{L}'|_{p'_0} \cong L$ for some $p'_0 \in P'$ mapping to $p_0 \in P$, the sheaf $\mathcal{L}'$ is flat over P' and induces rank one torsion-free sheaves on the fibers over P'. And if we define $\mathcal{M}'$ to be the pullback of $\mathcal{M}$ to $S \times P'$, there exists an exact sequence

$$0 \to \mathcal{L}' \to \mathcal{M}' \to \mathcal{Q}' \to 0,$$

inducing short exact sequences

$$0 \to L'_{p'} \to M'_{p'} \to Q'_{p'} \to 0$$

for every $p' \in P'$.

Proof. If the support of Q were in the smooth part of the curve, we would have $M \cong L \otimes \mathcal{O}_C(D)$, with D an effective divisor of degree $l(Q)$. Then, if we are given a deformation M_p of M, we only need to find a deformation D_p of the effective divisor D, with the only condition that D_p is an effective divisor on C_p, with degree $l(Q)$. This can easily be done if we are in the analytic category. In general we might need to do a base change of the parametrizing curve P and we will obtain a finite cover P' of P (What we are doing is moving a dimension zero and length $l(Q)$ subscheme of S, with the only restriction that for each p the corresponding scheme is in C_p). Then we only need to define $L_{p'} = M_{p'} \otimes \mathcal{O}_{C_{p'}}(-D_{p'})$ and the proposition would be proved (with P' a finite cover of P).

To be able to apply this, we will have to make first a deformation of L, keeping M fixed, until we get Q to be supported in the smooth part of C (the curve C also remains fixed in this deformation). This is the reason for the need of the curve P' with some irreducible components mapping to p_0.

We will prove this by induction on the length of the intersection of the support of Q and the singular part of C.

Lemma 3.5. *Let L and M be rank one torsion-free sheaves on an integral curve C that lies on a surface S. Assume we have a short exact sequence*

$$0 \to L \to M \to Q \to 0.$$

Assume that $Q = R \oplus Q'$ where Q' has length $l(Q) - 1$ and it is supported in the smooth part of C, and R has length one as it is supported in a singular point of C ("the length of the intersection of the support of Q and the singular part of C is one").

Then there is a flat deformation L_y of L parametrized by a connected curve Y (it might not be irreducible) such that $L_{y_0} = L$ for some $y_0 \in Y$ and for every $y \in Y$ there is an exact sequence

$$0 \to L_y \to M \to Q_y \to 0$$

and there is some $y_1 \in Y$ such that the support of Q_{y_1} is in the smooth part of C.

Proof. In this situation, the exact sequence (3.2) gives rise to another exact sequence

$$0 \to L \otimes I_Z^\vee \to M \to R \to 0$$

where the map on the right is the composition of $M \to Q$ and the projection $Q \to R$, and we denote by I_Z the ideal sheaf of the support Z of Q'. Because Z is in the smooth part of C, I_Z is an invertible sheaf. Note that Q' is the quotient of $\mathcal{O}_C$ by this ideal sheaf. Define $\widehat{L}$ to be $L \otimes I_Z^\vee$. If we know how to make a flat deformation $\widehat{L}_y$ of $\widehat{L}$ so that the quotient R_y is supported in the smooth part of C for some $y_1 \in Y$, then we can construct a deformation L_y of L defined as

$$L_y = \widehat{L}_y \otimes I_Z.$$

Note that this deformation is also flat. The cokernel Q_y of $L_y \to M$ is supported in the smooth part of C for the points $y \in Y$ for which R_y is supported in the smooth part of C.

This shows that to prove the lemma we can assume that Q has length one and its support is a singular point of C, i.e. $Q = \mathcal{O}_x$, where x is a singular point of C.

Consider the scheme $\mathrm{Quot}^1(M)$ representing the functor of quotients of M of length 1. If the support x of the quotient Q is in the smooth part of C, then there is only one surjective map (up to scalar) because $\dim \mathrm{Hom}(M, Q) = 1$, whose kernel is $M \otimes \mathcal{O}_C(-x)$.

If x is in the singular part, then in general $\dim \mathrm{Hom}(M, Q) > 1$, and the quotients are parametrized by $\mathbb{P}\mathrm{Hom}(M, Q)$ (the universal bundle is flat over $\mathbb{P}\mathrm{Hom}(M, Q)$). We want to show that $\mathrm{Quot}^1(M)$ is connected by constructing a flat family of quotients $M \to \widetilde{Q}_{\tilde{c}}$ (the family $\widetilde{Q}_{\tilde{c}}$ will be parametrized by an open set of the normalization $\widetilde{C}$ of C) such that for a general $\tilde{c}$ the support of $\widetilde{Q}_{\tilde{c}}$ is in the smooth part of C, and for some point $\tilde{c}_0$ the support of $\widetilde{Q}_{\tilde{c}_0}$ is a singular point of C.

Consider the normalization $\widetilde{C}$ of C, and let F be an open set of $\widetilde{C}$

$$\begin{array}{ccc} F & \xrightarrow{\ j\ } & C \times F \\ & & \big\downarrow{\scriptstyle \pi_1} \\ & & C \end{array} \tag{3.3}$$

Where π_1 is the projection to the first factor and $j = (\nu, i)$, the morphism $\nu : F \hookrightarrow \widetilde{C} \to C$ being the restriction to F of the normalization map and i the identity map. Note that j is a closed immersion, and its image is just $C \times_C F \cong F$.

Let $\tilde{c}_0$ be a point of $\widetilde{C}$ in $\nu^{-1}(x)$ (the family is going to be parametrized by an open neighborhood F of $\tilde{c}_0$). We have to construct a surjection of $\widetilde{\mathcal{M}} = \pi_1^* M$ onto $\widetilde{\mathcal{Q}} = j_* \mathcal{O}_F$. Note that $\widetilde{\mathcal{Q}}|_{C\times\tilde{c}} = \widetilde{Q}_{\tilde{c}} \cong \mathcal{O}_{\nu(\tilde{c})}$ and that $\widetilde{\mathcal{Q}}$ is flat over F.

Now, to define that quotient, it is enough to define it in the restriction to the image of j (because this is exactly the support of $\widetilde{\mathcal{Q}}$). So the map we have to define is

$$j^*\widetilde{\mathcal{M}} \to \mathcal{O}_F.$$

But $j^*\widetilde{\mathcal{M}} = \nu^* M$ is a rank one sheaf on the smooth curve F, so it is the direct sum of a line bundle and a torsion part T. Shrinking F if necessary, the line bundle part is isomorphic to $\mathcal{O}_F$, and we have

$$j^*\widetilde{\mathcal{M}} \cong T \oplus \mathcal{O}_F,$$

and then to define the quotient we just take an isomorphism in the torsion-free part. This finishes the proof of the lemma.

□

Now we go to the general case: the intersection of the support of Q with the singular part of C has length n. We are going to see how this can be reduced to the case $n = 1$.

Take a surjection from Q to a sheaf Q' of length $n-1$, such that Q is isomorphic to Q' at the smooth points. The kernel R of this surjection will have length 1, and will be supported in a singular point of C. It is isomorphic to $\mathcal{O}_x$, for some singular point x. We have a diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & L' & \longrightarrow & R & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & Q & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & Q' & = & Q' & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Observe that L, L' and R satisfy the hypothesis of Lemma 3.5, so we can find deformations L_y, R_y (parametrized by some curve Y and with $L_{y_0} = L$ and $R_{y_0} = R$ for some $y_0 \in Y$) such that for some $y_1 \in Y$ we have that the support of the corresponding sheaf R_{y_1} is a smooth point of C. All the maps of the previous diagram can be deformed along. To do this, we change L by L_y, R will be deformed to R_y and L' is kept constant. Then Q is deformed to a family Q_y defined as M/L_y. The cokernel of $R_y \to Q_y$ will be $Q_y/R_y = M/L' = Q'$, and hence we keep it constant. Then for each y we still have a commutative diagram, and furthermore it is easy to see that all deformations are flat (note that R_y is a flat deformation and Q' is kept constant, and then Q_y is a flat deformation). An important point is that M remains fixed, and the injection $L \to M$ is deformed to $L_y \to M$.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_y & \longrightarrow & L' & \longrightarrow & R_y & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_y & \longrightarrow & M & \longrightarrow & Q_y & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Q' & = & Q' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

For y_1 we have that the length of the intersection of the support of Q_{y_1} with the singular part of C is $n-1$. We repeat the process (starting now with L_{y_1}, M and Q_{y_1}), until all the points of the support of Q are moved to the smooth part of C. This finishes the proof of the proposition.

□

The following proposition is similar to Proposition 3.4, but now the roles of L and M are changed: we are given a deformation of L and we have to deform M along.

Proposition 3.6. *Let L and M be rank one torsion-free sheaves on an integral curve C that lies on a surface S. Assume we have a short exact sequence*

$$0 \to L \to M \to Q \to 0. \tag{3.4}$$

Assume furthermore that we are given a sheaf $\mathcal{L}$ on $S \times P$ (where P is a connected but not necessarily irreducible curve) that is a deformation of L, flat over P , i.e., $\mathcal{L}|_{p_0} \cong L$ for some $p_0 \in P$, and for all $p \in P$, we have that $L_p = \mathcal{L}|_p$ are torsion-free sheaves on C_p, where C_p is a curve on S.

Then, there is a connected curve P' with a map $f : P' \to P$ and a sheaf $\mathcal{M}'$ over $S \times P'$ with the following properties:

One irreducible component of P' is a finite cover of P and the rest of the components map to $p_0 \in P$. The sheaf $\mathcal{M}'$ is a deformation of M, in the sense that $\mathcal{M}'|_{p'_0} \cong M$ for some $p'_0 \in P'$ mapping to $p_0 \in P$, the sheaf $\mathcal{M}'$ is flat over P' and induces rank one torsion-free sheaves on the fibers over P'. And if we define $\mathcal{L}'$ to be the pullback of $\mathcal{L}$ to $S \times P'$, there exists an exact sequence

$$0 \to \mathcal{L}' \to \mathcal{M}' \to \mathcal{Q}' \to 0,$$

inducing short exact sequences

$$0 \to L'_{p'} \to M'_{p'} \to Q'_{p'} \to 0$$

for every $p' \in P'$.

Proof. The proof is very similar to the proof of Proposition 3.4. Again we start by observing that if the support of Q were in the smooth part of the curve, we would have $M \cong L \otimes \mathcal{O}_C(D)$, with D an effective divisor. Then if we are given a flat deformation L_p of L, we find a deformation D_p of D as in the first part, and the proposition would be proved. So again we need a lemma that deforms Q so that its support is in the smooth part of C.

Lemma 3.7. *Let L and M be rank one torsion-free sheaves on an integral curve C that lies on a surface S. Assume we have a short exact sequence*

$$0 \to L \to M \to Q \to 0.$$

Assume that the part of Q with support in the smooth part of C has length $l(Q) - 1$, i.e., $Q = R \oplus Q'$, where R has length one and is supported in a singular point of C and Q' has length $l(Q) - 1$ and is supported in the smooth part of C. Then there is a flat deformation M_y of M parametrized by a curve Y, such that for every $y \in Y$ there is an exact sequence

$$0 \to L \to M_y \to Q_y \to 0$$

with M_y a torsion-free sheaf, and there is some $y_1 \in Y$ such that the support of Q_{y_1} is in the smooth part of C.

Proof. Arguing as in the proof of Lemma 3.5, we see that it is enough to prove the case $l(Q) = 1$, and $Q = \mathcal{O}_x$ for x a singular point of C, then we can assume that the extension of the hypothesis of the lemma is

$$0 \to L \to M \to \mathcal{O}_x \to 0. \tag{3.5}$$

Now we will consider all extensions of $\mathcal{O}_x$ (for x any point in C) by L. If x is a smooth point, then there is only one extension that is not trivial (up to equivalence)

$$0 \to L \to M \cong L \otimes \mathcal{O}_C(x) \to \mathcal{O}_x \to 0.$$

All these extensions are then parametrized by the smooth part of C.

But if x is a singular point, we could have more extensions, because in general $s = \dim \operatorname{Ext}^1(\mathcal{O}_x, L) > 1$. They will be parametrized by a projective space $\mathbb{P}^{s-1}$. We call this space E_x. Note that there is a universal extension on $C \times E_x$ that is flat over E_x. We denote by e_1 the point in E_x corresponding to the extension (3.5).

Assume that $\widetilde{Q}_{\tilde{c}}$ is a family of torsion sheaves on C with length 1, parametrized by a curve F such that for a general point $\tilde{c} \in F$ of the parametrizing curve the support of $\widetilde{Q}_{\tilde{c}}$ in C is a smooth point, and for a special point $\tilde{c}_0 \in F$ the support of $\widetilde{Q}_{\tilde{c}_0}$ is a singular point. Now assume that we can construct a flat family (parametrized by F) of nontrivial extensions of $\widetilde{Q}_{\tilde{c}}$ by L. The extension corresponding to $\tilde{c}_0$ gives a point e_2 in E_x. The space E_x is a projective space, thus connected, and then there is a curve containing e_1 and e_2. Using this curve (together with the universal extension for E_x) and the curve F (together with the family of extensions that it parametrizes) we construct the curve Y that proves the lemma.

Now we need to construct F. As in the proof of Lemma 3.5, the parametrizing curve F will be an affine neighborhood of $\tilde{c}_0$ in the normalization $\widetilde{C}$ of C, where $\tilde{c}_0$ is a point that maps to the singular point x of C. Consider again the diagram (3.3) of the proof of Lemma 3.5. The family will be given by an extension of $\widetilde{\mathcal{Q}} = j_*\mathcal{O}_F$ by $\widetilde{\mathcal{L}} = \pi_1^* L$ on $C \times F$. These extensions are parametrized by the group $\operatorname{Ext}^1(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{L}})$. The following lemma gives information about this group and relates this extension with the extensions that we get after restriction for each slice $C \times \tilde{c}$. We will call $\widetilde{Q}_{\tilde{c}}$ and $\widetilde{L}_{\tilde{c}}$ the restrictions of $\widetilde{\mathcal{Q}}$ and $\widetilde{\mathcal{L}}$ to the slice $C \times \tilde{c}$. Note that the restriction $\widetilde{L}_{\tilde{c}}$ is isomorphic to L.

Lemma 3.8. *With the previous notation, we have*

1) $\operatorname{Ext}^1(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{L}}) \cong H^0(\mathcal{E}xt^1(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{L}}))$

2) *$Ext^1(\widetilde{\mathcal{Q}},\widetilde{\mathcal{L}})$ has rank zero outside of the support of $\widetilde{\mathcal{Q}}$, and rank 1 on the smooth points of the support of $\widetilde{\mathcal{Q}}$*

3) *Let I be the ideal sheaf corresponding to a slice $C\times\tilde{c}$. Then the natural map*
$$Ext^1_{\mathcal{O}_{C\times F}}(\widetilde{\mathcal{Q}},\widetilde{\mathcal{L}})\otimes\mathcal{O}_{C\times F}/I\to Ext^1_{\mathcal{O}_{C\times\tilde{c}}}(\widetilde{Q}_{\tilde{c}},\widetilde{L}_{\tilde{c}})$$
is injective.

Proof. Item 1 follows from the fact that $Hom(\widetilde{\mathcal{Q}},\widetilde{\mathcal{L}})=0$ and the exact sequence

$$0\to H^1(Hom(\widetilde{\mathcal{Q}},\widetilde{\mathcal{L}}))\to \mathrm{Ext}^1(\widetilde{\mathcal{Q}},\widetilde{\mathcal{L}})\to H^0(Ext^1(\widetilde{\mathcal{Q}},\widetilde{\mathcal{L}}))\to H^2(Hom(\widetilde{\mathcal{Q}},\widetilde{\mathcal{L}})).$$

To prove item 2 note that the stalk of $Ext^1(\widetilde{\mathcal{Q}},\widetilde{\mathcal{L}})$ at a point p is isomorphic to $\mathrm{Ext}^1(R/I,R)$, where R is the local ring at the point p, and I is the ideal defining the support or $\widetilde{\mathcal{Q}}$. The ideal I is principal if the point p is smooth, then R/I has a free resolution

$$0\to I\to R\to R/I\to 0$$

and it follows that $\mathrm{Ext}^1(R/I,R)\cong R/I$.

For item 3, consider the exact sequence

$$0\to\widetilde{\mathcal{Q}}\xrightarrow{\cdot f}\widetilde{\mathcal{Q}}\to\widetilde{Q}_{\tilde{c}}\to 0$$

where the first map is multiplication by the local equation f of the slice $C\times\tilde{c}$. Applying $Hom(\cdot,\widetilde{L}_{\tilde{c}})$ we get

$$Hom(\widetilde{\mathcal{Q}},\widetilde{L}_{\tilde{c}})=0\to Ext^1(\widetilde{Q}_{\tilde{c}},\widetilde{L}_{\tilde{c}})\to Ext^1(\widetilde{\mathcal{Q}},\widetilde{L}_{\tilde{c}})\to Ext^1(\widetilde{\mathcal{Q}},\widetilde{L}_{\tilde{c}}),$$

but the last map is zero. To see this, take a locally free resolution of $\widetilde{\mathcal{Q}}$. The map induced on the resolution by the multiplication with the equation f is just multiplication by the same f on each term

$$\begin{array}{ccccc}\mathcal{F}^\bullet & \longrightarrow & \widetilde{\mathcal{Q}} & \longrightarrow & 0\\ {\scriptstyle\cdot f}\downarrow & & {\scriptstyle\cdot f}\downarrow & & \\ \mathcal{F}^\bullet & \longrightarrow & \widetilde{\mathcal{Q}} & \longrightarrow & 0\end{array}$$

A local section of the sheaf $Ext^i(\widetilde{\mathcal{Q}},\widetilde{L}_{\tilde{c}})$ is represented by some local section $\varphi(\cdot)$ of $Hom(\mathcal{F}^i,\widetilde{L}_{\tilde{c}})$, and the endomorphism induced by multiplication by

f on $Ext^i(\widetilde{\mathcal{Q}}, \widetilde{L}_{\tilde{c}})$ is given by precomposition with multiplication $\varphi(f\cdot)$, but φ is a morphism of sheaves of modules and then this is equal to $f\varphi(\cdot)$, and this is equal to zero because $f\widetilde{L}_{\tilde{c}} = 0$. Then we have that

$$Ext^1(\widetilde{Q}_{\tilde{c}}, \widetilde{L}_{\tilde{c}}) \cong Ext^1(\widetilde{\mathcal{Q}}, \widetilde{L}_{\tilde{c}}). \tag{3.6}$$

Taking the exact sequence

$$0 \to \widetilde{\mathcal{L}} \xrightarrow{\cdot f} \widetilde{\mathcal{L}} \to \widetilde{L}_{\tilde{c}} \to 0$$

and applying $Hom(\widetilde{\mathcal{Q}}, \cdot)$ we get

$$Ext^1(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{L}}) \xrightarrow{\cdot f} Ext^1(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{L}}) \to Ext^1(\widetilde{\mathcal{Q}}, \widetilde{L}_{\tilde{c}})$$

and using this and the isomorphism (3.6) we have an injection

$$Ext^1(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{L}}) \otimes \mathcal{O}_{C\times F}/I \cong Ext^1(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{L}})/(f \cdot Ext^1(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{L}})) \hookrightarrow Ext^1(\widetilde{Q}_{\tilde{c}}, \widetilde{L}_{\tilde{c}}).$$

□

Now we are going to construct the family of extensions. By item 2 of the lemma the sheaf $\mathcal{E} = Ext^1(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{L}})$ is isomorphic to $\mathcal{O}_X \oplus T(\mathcal{E})$ (shrinking F if necessary) where X is the support of $\widetilde{\mathcal{Q}}$ and $T(\mathcal{E})$ is the torsion part. Take a nonvanishing section of the torsion-free part, and by item 1 this gives a nonzero element ψ of $\mathrm{Ext}^1(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{L}})$. This element gives a nontrivial extension

$$0 \to \widetilde{\mathcal{L}} \to \widetilde{\mathcal{M}} \to \widetilde{\mathcal{Q}} \to 0.$$

Observe that $\widetilde{\mathcal{M}}$ is flat over the base, because both $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{Q}}$ are flat.

By items 3 and 1 we have that the image of ψ under the restriction map

$$\mathrm{Ext}^1(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{L}}) \to \mathrm{Ext}^1(\widetilde{Q}_{\tilde{c}}, L)$$

is nonzero for any $\tilde{c}$ (recall that $\widetilde{L}_{\tilde{c}} = L$ for all $\tilde{c}$), and this means that the extensions that we obtain after restriction to the corresponding slices

$$0 \to L \to \widetilde{M_{\tilde{c}}} \to \widetilde{Q}_{\tilde{c}} \to 0 \tag{3.7}$$

are non trivial. Furthermore $\widetilde{M_{\tilde{c}}}$ is torsion-free. To prove this claim, let $T(\widetilde{M_{\tilde{c}}})$ be the torsion part of $\widetilde{M_{\tilde{c}}}$. The map $L \to T(\widetilde{M_{\tilde{c}}})$ coming from (3.7) is zero, because L is torsion-free, i.e. $T(\widetilde{M_{\tilde{c}}})$ injects in $\widetilde{Q}_{\tilde{c}}$. Then we have

$$\widetilde{Q}_{\tilde{c}} \cong \frac{\widetilde{M_{\tilde{c}}}}{L} \cong \frac{\widetilde{M_{\tilde{c}}}/T(\widetilde{M_{\tilde{c}}}) \oplus T(\widetilde{M_{\tilde{c}}})}{L} \cong \frac{\widetilde{M_{\tilde{c}}}/T(\widetilde{M_{\tilde{c}}})}{L} \oplus T(\widetilde{M_{\tilde{c}}}).$$

$\widetilde{Q}_{\bar{c}}$ doesn't decompose as the direct sum of two sheaves, and then one of these summands must be zero. The first summand cannot be zero, because this would imply that $L \cong \widetilde{M_{\bar{c}}}/T(\widetilde{M_{\bar{c}}})$ and then $\widetilde{M_{\bar{c}}} \cong L \oplus \widetilde{Q}_{\bar{c}}$, contradicting the hypothesis that the extension is not trivial. Then we must have $T(\widetilde{M_{\bar{c}}}) = 0$, and the claim is proved.

□

Now we are going to consider the general case, in which the part of Q supported in singular points has length n. We are going to see that this can be reduced to the case $n = 1$, in a similar way to proposition 3.4.

Let $R = \mathcal{O}_x$, where x is a singular point in the support of Q, and take a surjection from Q to R. We have a diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & L'/L & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & L' & \longrightarrow & M & \longrightarrow & R & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

Note that L', M and R satisfy the hypothesis of Lemma 3.7, then we can find (flat) deformations M_y and R_y parametrized by a curve Y such that for some $y_1 \in Y$ we have that the support of the corresponding sheaf R_{y_1} is a smooth point of C. All sheaves and maps can be deformed along. To do this we define $Q_y = M_y/L$ (we have $L \hookrightarrow L' \hookrightarrow M_y$, thus this quotient is well defined). The kernel of $Q_y \to R_y$ is L'/L. Then Q_y is a flat deformation (being the extension of a flat deformation R_y by a constant and hence flat deformation L'/L). Then for each y we have a commutative diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & L'/L & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & M_y & \longrightarrow & Q_y & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & L' & \longrightarrow & M_y & \longrightarrow & R_y & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

Observe that the length of the part of Q_{y_1} supported in singular points is $n-1$, so repeating this process we can deform Q until its support lies in the smooth part of C. This finishes the proof of the proposition.

□

4. Proof of Theorem I.

In this section we will prove Theorem I:

Proof. Nonemptiness follows from the fact that the Brill-Nother loci for smooth curves is nonempty, and by upper semicontinuity of $h^0(\cdot)$. By Remark 1 we can assume $r > d - p_a$. We will prove Theorem I by applying Lemma 1.1.

We start with a rank one torsion-free sheaf A corresponding to a point in $\overline{W^r_d}$, $d > 0$, $r \geq 0$, with $\rho(r,d) > 0$ (recall that we are assuming $r > d - p_a$). We have $h^0(A)$, $h^1(A) > 0$. As we explained at the beginning of the section 3, we call A' its base point free part. Then we take B to be the base point free part of A'^*, and finally define A'' to be B^*.

By Lemma 3.1, A'' and A''^* are rank one locally free sheaves on C generated by global sections. Then by Proposition 2.5 we find a deformation $\mathcal{A}''$ of A'' parametrized by a some smooth irreducible curve T.

The support of $\mathcal{A}''$ defines a family of curves $\mathcal{C}$ parametrized by the irreducible curve T. Note that $\mathcal{C}|_t$ is smooth for $t \neq 0$.

By the definition of A' and A'' we have exact sequences

$$0 \to A' \to A \to Q \to 0 \tag{4.1a}$$

$$0 \to A' \to A'' \to \widetilde{Q} \to 0 \tag{4.1b}$$

with $h^0(A') = h^0(A)$ and $h^1(A'') = h^1(A')$ (Lemma 3.2). If we look at (4.1b) we see that we are in the situation of Proposition 3.4, with $L = A'$, $M = A''$, $\mathcal{M} = \mathcal{A}''$, $P = T$. Then we get a family $\mathcal{A}'$ (parametrized by some connected but in general not irreducible curve). Now we use this family $\mathcal{A}'$ and the sequence (4.1a) to apply 3.6 with $L = A'$, $M = A$ and $\mathcal{L} = \mathcal{A}'$. We get a new family $\mathcal{A}$. We denote by T' the curve parametrizing the family $\mathcal{A}$.

This family satisfies all the hypothesis of Lemma 1.1 (item (iv) is given by Proposition 3.3), and then Theorem I is proved.

□

5. Moduli space of torsion-free sheaves on K3 surfaces.

In this section we will apply Theorem I to prove Theorem II. The proof is similar to an argument in [G-H]. Recall that S is a K3 complex surface. Given a line bundle L on S and an integer c_2, for any class ζ in the Neron-Severi group $NS(S)$ of S satisfying $L^2 - 4c_2 \leq \zeta^2 < 0$ and $\zeta \equiv L \bmod 2$, we define the associated wall $W^\zeta = \{M : \zeta \cdot M = 0$ and M $\in$ ample cone $(\subset NS(S) \otimes \mathbb{R})$ $\}$. We say that W^ζ is a wall of type (L, c_2) (see [Q]). We say that a polarization is (L, c_2)-generic if it doesn't lie on any wall. In this case, semistability implies stability (i.e., there are no strictly semistable sheaves), the moduli space is smooth of dimension

$$\dim \mathfrak{M}_H(L, c_2) = 4c_2 - L^2 - 6,$$

and then irreducibility is equivalent to connectivity (see [H-L2]).

First we prove that we can reduce the proof of Theorem II to a special case

Proposition 5.1. *Assume that* $\mathfrak{M}_H(L, c_2)$ *is irreducible under the additional hypothesis that* $\operatorname{Pic}(S) = \mathbb{Z}$. *Then* $\mathfrak{M}_H(L, c_2)$ *is also irreducible under the conditions of Theorem* II.

Proof. Let S be a surface with an (L, c_2)-generic polarization H. By [G-H, 2.1.1], there is a connected family of surfaces $\mathcal{S}$ parametrized by a curve T

and a line bundle $\mathcal{L}$ on $\mathcal{S}$ such that $(\mathcal{S}_0, \mathcal{L}_0) = (S, L)$ and $\mathrm{Pic}(\mathcal{S}_t) = \mathcal{L}_t \cdot \mathbb{Z}$ for $t \neq 0$. By [G-H, prop. 2.2], there is a smooth proper family $\mathcal{Z} \to T$ such that $\mathcal{Z}_0 \cong \mathfrak{M}_H(S, \mathcal{L}_0, c_2)$ (note that the polarization is H and not $\mathcal{L}_0$) and $\mathcal{Z}_t \cong \mathfrak{M}_{\mathcal{L}_t}(\mathcal{S}_t, \mathcal{L}_t, c_2)$ for $t \neq 0$.

By hypothesis we know that $\mathcal{Z}_t$ is irreducible for $t \neq 0$, and then by [H, III Ex 11.4], we obtain that $\mathcal{Z}_0$ is connected, but $\mathcal{Z}_0$ is smooth (because H is generic), and then this implies that $\mathcal{Z}_0$ is irreducible.

□

Then from now on we will assume that $\mathrm{Pic}(S) = \mathbb{Z}$ and hence $H = L$ is the ample generator.

Proposition 5.2. *Let V be a torsion-free stable rank two sheaf with $c_1 = L$, $c_2 \leq \frac{1}{2}L^2 + 3$.*

(a) *Then V fits in an exact sequence*

$$0 \to \mathcal{O}_S \to V \to L \otimes I_Z \to 0. \tag{5.1}$$

(b) *Conversely, every nonsplit extension of $L \otimes I_Z$ by $\mathcal{O}_S$ is a torsion-free stable sheaf.*

Proof. Take V stable. Using the Riemann-Roch theorem,

$$h^0(V) + h^2(V) \geq \frac{L^2}{2} - c_2 + 4 \geq 1.$$

If $h^2(V)$ were different from zero, by Serre duality we would have $\mathrm{Hom}(V, \mathcal{O}) \neq 0$, contradicting stability because this would give a nonzero morphism $V \to \mathcal{O}_S$.

Then $h^0(V) \neq 0$. Take a section of V. By stability, the quotient of the section is torsion-free, and we have an extension like (5.1). The extension is not split because V is stable. This proves (a).

To prove part (b), assume $L^{\otimes m} \otimes I_W$ is a destabilizing subsheaf. Then $m > 0$. By standard arguments we can assume that the quotient is torsion-free. The composition $L^{\otimes m} \otimes I_W \to V \to L \otimes I_Z$ is nonzero, because otherwise it would factor through $\mathcal{O}_S$ and this is impossible because $m > 0$. Then $m = 1$ and we have $I_W \hookrightarrow I_Z$. Furthermore, $l(W) > l(Z)$ because if $W = Z$, the sequence would split. Then we have a sequence

$$0 \to L \otimes I_W \to V \to I_{W'} \to 0,$$

but we reach a contradiction because $c_2 = l(W) + l(W') > l(Z) + l(W') = c_2 + l(W')$. Then there is no destabilizing subsheaf, and V is stable.

$\square$

Consider the moduli space $\overline{M}$ of stable framed modules (E, α) where E is a rank two torsion-free sheaf with $c_1(E) = L$ and second Chern class equal to c_2, and $\alpha : E \to L$ is a nontrivial homomorphism (see [H-L1]). The stability condition depends on a degree 1 polynomial $\delta(n) = \delta_1 n + \delta_0$. If $0 < \delta_1 < L^2$, then (E, α) is stable iff E is stable as a vector bundle (see [G-H, Lemma 1.1]). Let $N \subset \overline{M}$ be the subset corresponding to framed modules (E, α) such that $\ker(\alpha) = \mathcal{O}_S$. In [G-H, lemma 1.3], it is proved that N is a closed subset. Note that N can also be constructed as a moduli space of coherent systems [LP]. We have a diagram

(5.2)
$$\mathfrak{M}_L(L, c_2) \xleftarrow{\ \varphi\ } N \xrightarrow{\ \psi\ } Hilb^{c_2}(S)$$

By Proposition 5.2(a) φ is surjective, and by Proposition 5.2(b) the image of ψ is

$$X = \{Z \in \mathrm{Hilb}^{c_2}(S) : \dim \mathrm{Ext}^1(L \otimes I_Z, \mathcal{O}_S) \geq 1\}$$

Proposition 5.3. *The set X is connected.*

Proof. By Serre duality and looking at the sequence

$$0 \to H^0(L \otimes I_Z) \to H^0(L) \to H^0(\mathcal{O}_Z) \to H^1(L \otimes I_Z) \to 0,$$

we have $\dim \mathrm{Ext}^1(L \otimes I_Z, \mathcal{O}_S) \geq 1 \iff h^0(L \otimes I_Z) \geq \frac{1}{2}L^2 + 3 - c_2$. Now consider the following commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_S & \longrightarrow & L \otimes I_Z & \longrightarrow & j_*(\omega_C \otimes I_Z) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_S & \longrightarrow & L & \longrightarrow & L|_C = j_*\omega_C & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathcal{O}_Z & = & \mathcal{O}_Z & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

where $C \in \mathbb{P}(H^0(L \otimes I_Z))$ (the curve C might be singular, but we know it is irreducible and reduced because $\mathrm{Pic}(S) = \mathbb{Z}$ and L primitive), $j : C \hookrightarrow S$ is the inclusion, and $\omega_C = L|_C$ is the dualizing sheaf on C.

Using the top row we get $h^0(L \otimes I_Z) \geq \frac{1}{2}L^2 + 3 - c_2 \iff h^0(\omega_C \otimes I_Z) \geq \frac{1}{2}L^2 + 2 - c_2$. This condition can be restated in terms of Brill-Noether sets W^r_d:

$$\omega_C \otimes I_Z \in W^r_d$$

where $r = \frac{1}{2}L^2 + 1 - c_2$, and $d = L^2 - c_2$. We have

$$\rho(r, d) = 2c_2 - \frac{L^2}{2} - 3 = \frac{\dim \mathfrak{M}(L, c_2)}{2} > 0,$$

(recall that for $\dim \mathfrak{M}(L, c_2) = 0$ the irreducibility of the moduli space is known by the work of Mukai [M]) and we can apply Theorem I. Now consider the variety

$$N = \{(Z, C) : Z \subset C, \dim \mathrm{Ext}^1(L \otimes I_Z, \mathcal{O}_S) \geq 1\} \subset \mathrm{Hilb}^{c_2}(S) \times \mathbb{P}(H^0(L))$$

and the projections

$$\begin{array}{ccc} N & \xrightarrow{p_2} & \mathbb{P}(H^0(L)) \\ {\scriptstyle p_1}\downarrow & & \\ \mathrm{Hilb}^{c_2}(S) & & \end{array}$$

By Theorem I, p_2 is surjective with connected fibers. Then N is connected, and also the image of p_1, that is equal to X.

□

Finally we can prove Theorem II:

Proof. The fiber of ψ (see diagram (5.2)) over a point corresponding to Z is $\mathbb{P}(\mathrm{Ext}^1(L\otimes I_Z, \mathcal{O}_S))$. In particular it is connected. By the previous proposition X is connected, and then N is also connected. Finally the surjectivity of φ (Proposition 5.2(a)) proves that $\mathfrak{M}_H(H, c_2)$ is connected. This shows that the moduli space is irreducible under the additional hypothesis that $\mathrm{Pic}(S) = \mathbb{Z}$, and applying proposition 5.1 we obtain Theorem II.

□

Remark 2. Using similar techniques one can prove the irreducibility of the moduli space for any value of c_2. The proof is longer, due to the fact that we don't have a nice characterization of stable sheaves as in Proposition 5.2. The details are in [Go].

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