COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 9, Number 4, 683-723, 2001

Minimal Surfaces in a Wedge of a Slab

FRANCISCO J. LÓPEZ [1] AND FRANCISCO MARTÍN [1]

In this paper we construct a new family of minimal surfaces in a wedge of a slab, and study its geometrical properties in depth. These surfaces arise as a solution to Plateau's problem for some polygonal non compact boundaries.
By using these examples as barriers, we prove that any properly immersed minimal surface in a wedge of angle θ of a slab, where $\theta \in [0, \pi[$, satisfies the convex hull property. Moreover, we obtain some non existence results for properly immersed minimal surfaces with planar boundary.

1. Introduction.

A classical problem considered by Schwarz, Weierstrass and Riemann was to determine minimal surfaces bounded by straight lines. These authors obtained existence results for minimal surfaces with boundary a given polygon, where the sides of the polygon could be of finite or infinite length.

A comprehensive presentation of the Schwarz-Riemann-Weierstrass approach to the solution of Plateau's problem for polygonal boundaries can be found in Darboux's treatise [2, Vol. 1 and 3].

Very recently, López and Wei [9] have obtained an existence and uniqueness theorem for properly immersed minimal discs whose boundaries consist of two disjoint straight lines and a segment which meets the lines orthogonally.

In 1966, Jenkins and Serrin in [4] proved an existence and uniqueness theorem for minimal graphs bounded by straight lines. They obtained simple, necessary and sufficient conditions to solve the Dirichlet problem in a compact convex domain bounded by a polygon assuming values $+\infty$, $-\infty$ and continuous data on different straight segments in the boundary.

[1]Research partially supported by DGICYT grant number PB94-0796.

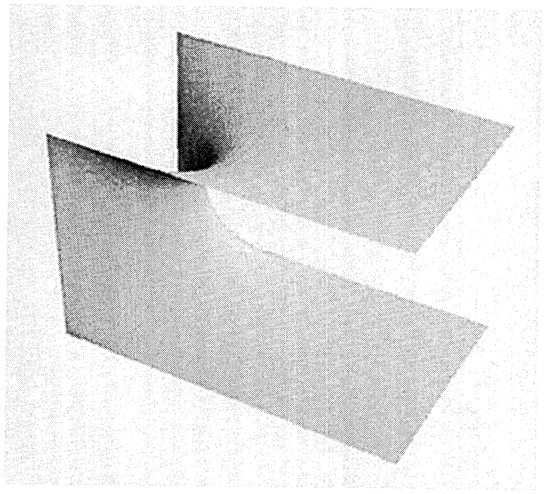

Figure 1: A Jenkins-Serrin graph.

In Figure 1, we illustrate a particular Jenkins-Serrin graph. In this case, the polygon is a rectangle and the data on the four edges are $+\infty$, 0, $+\infty$ and 0, respectively.

In this paper, we construct a deformation of these particular Jenkins-Serrin graphs by properly embedded minimal discs bounded by straight lines and contained in a wedge of a slab. Essentially, the deformation modifies the angle formed by the two planes containing the connected components of the boundary. In [7], the authors have obtained a natural uniqueness theorem for the surfaces in this new family. Furthermore, we use these new surfaces as barriers for maximum principle application. So, we have extended the family of minimal surfaces satisfying the convex hull property, and we have also proved some non existence results for minimal surfaces with planar boundaries.

First, we deal with the existence of properly embedded minimal surfaces whose boundary $\Gamma_{\theta\, d}$ consists of the following configuration of straight lines:

Fix $\theta \in [0, \pi]$ and $d \geq 0$, and consider two half-lines r_1^+ and r_1^- in $\mathbb{R}^3$, meeting at an angle of θ. If $\theta = 0$ this means that the straight lines are parallel and distinct. Let q_1^+ and q_1^- be two points in r_1^+ and r_1^-, respectively, such that they are symmetric with respect the inner bisector of this half-lines. If and only if $\theta \neq 0$, we allow $q_1^+ = q_1^-$. We choose q_1^+ and q_1^- in such a way that either $q_1^+ = q_1^-$ or the half-lines ℓ_1^+ and ℓ_1^- on r_1^+ and r_1^- starting at q_1^+ and q_1^-, respectively, do not intersect. Write $d = \text{dist}(q_1^+, q_1^-)$.

Let π_1 be the plane determined by ℓ_1^+ and ℓ_1^- and let π_2 denote a plane parallel and distinct to π_1. Let ℓ_2^+ and ℓ_2^- be the orthogonal projections to π_2 of ℓ_1^+ and ℓ_1^-, respectively. Denote q_2^+ (resp. q_2^-) as the orthogonal projection to π_2 of q_1^+ (resp. q_1^-), and label ℓ_0^+ (resp. ℓ_0^-) as the segment $[q_1^+, q_2^+]$ (resp. $[q_1^-, q_2^-]$). Finally, we write

$$\Gamma_{\theta\, d}^+ = \bigcup_{i=0}^{2} (\ell_i^+), \quad \Gamma_{\theta\, d}^- = \bigcup_{i=0}^{2} (\ell_i^-)$$

and define

$$\Gamma_{\theta\, d} = \Gamma_{\theta\, d}^+ \cup \Gamma_{\theta\, d}^-.$$

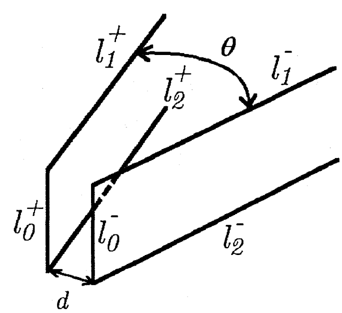

Figure 2: The curve $\Gamma_{\theta\, d}$.

Then, we study the following generalized Plateau's problem:

Problem.

Determine a properly immersed minimal surface $X : M \to \mathbb{R}^3$ satisfying:

(1) *M is homeomorphic to the closed unit disc $\overline{\mathbb{D}}$ minus two boundary points E_1 and E_2, that we call the ends of M.*

(2) *$X(\partial(M)) = \Gamma_{\theta\, d}$.*

(3) *If $d > 0$, X is an* embedding.

(4) *In the limit case $\ell_0^+ = \ell_0^-$ (i.e., $d = 0$), the maps $X|_{M-\gamma^+}$ and $X|_{M-\gamma^-}$ are injective, where γ^+ and γ^- are the two connected components of $\partial(M)$.*

(5) *$X(M)$ lies in the convex hull, $\mathcal{E}(\Gamma_{\theta\, d})$, of $\Gamma_{\theta\, d}$, $\theta \in [0, \pi[$.*

(6) *If* $\theta = \pi$*,* $X(M)$ *lies in one of the two half slab determined by* π_1*,* π_2 *and the plane containing* ℓ_i^+*,* ℓ_i^-*,* $i = 1, 2$*.*

In the limit case $\theta = 0$, it is known that $0 < d < \|\vec{v}\|$, where $\vec{v}$ is the vector with origin q_1^+ and end q_2^+, and the example is uniquely determined as one of the above mentioned Jenkins-Serrin graphs. See Remark 2, and Karcher's work [5] for a good survey.

Concerning to the general case $\theta > 0$, we have proved the following:

Theorem A. *There exists* $d_\theta > 0$ *such that, for any* $d \in [0, d_\theta]$*, the above Plateau's problem can be solved.*

The family of surfaces $\mathcal{M}$ arising from the Theorem A is analytical and depends on two parameters: the angle $\theta \in]0, \pi[$ and another one r which concerns to the complex structure induced by the immersion on the disc with piecewise analytical boundary. So, if we fix θ, there exist $r_\theta \in]-1, 1[$ and a subfamily $\mathcal{M}_\theta \subset \mathcal{M}$ of proper, conformal, minimal immersions

$$\mathcal{M}_\theta = \{X_{\theta\, r} : M_{\theta\, r} \to \mathbb{R}^3 \;/\; r \in]-1, r_\theta]\} \subset \mathcal{M}$$

such that $X_{\theta\, r}(\partial(M_{\theta\, r})) = \Gamma_{\theta\, d(r)}$ and $\mathcal{M} = \bigcup_{\theta \in]0,\pi[} \mathcal{M}_\theta$.

The *opening* function $d :]-1, r_\theta] \to [0, +\infty[$ has the following properties:

- d is analytical;
- d is positive in $]-1, r_\theta[$;
- $d(r_\theta) = \lim_{r \to -1} d(r) = 0$. In particular d is bounded;
- d has only one critical point $r'_\theta \in]-1, r_\theta[$, which is a *maximum*. Moreover, $d(r'_\theta) = d_\theta$.

In particular, if $d \in]0, d_\theta[$, the Plateau's problem has two solutions in $\mathcal{M}_\theta$. If $d = 0$ or $d = d_\theta$, the problem has a unique solution in $\mathcal{M}_\theta$. Finally, if $d > d_\theta$, there are no solutions in $\mathcal{M}_\theta$.

Next, we are going to make some remarks about the limits of the family of surfaces $\mathcal{M}$.

As we have mentioned above, the case $\theta = 0$ leads to Jenkins-Serrin graphs. See Remark 2, and [5]. When $\theta = \pi$, the resulting family of surfaces $\mathcal{M}_\pi = \{X_{\pi\, r} : M_{\pi\, r} \to \mathbb{R}^3 \;/\; r \in]-1, r_\pi]\}$ plays a special role in the Lorentz-Minkowski three dimensional space (see [6] for details). The surface

$X_{\pi\, r_\pi}(M_{\pi\, r_\pi})$ appeared first in [8] and motivated the discovery of the family of minimal twisted discs studied in [9].

In case $r = r_\theta$, the complete orientable minimal surface without boundary $\widetilde{X}_{\theta\, r_\theta} : \widetilde{M}_{\theta\, r_\theta} \to \mathbb{R}^3$ obtained from $X_{\theta\, r_\theta}(M_{\theta\, r_\theta})$ by successive Schwarz reflections about straight lines is singly periodic. If in addition $\frac{\theta}{\pi} \in \mathbb{Q}$, then the induced immersion $Y_\theta : \widetilde{M}_{\theta\, r_\theta}/\langle T\rangle \to \mathbb{R}^3/\langle T\rangle$ has four ends and finite total curvature. Here T is the translation by vector $2\vec{v}$, where as above $\vec{v}$ joins the points q_1^+ and q_2^+.

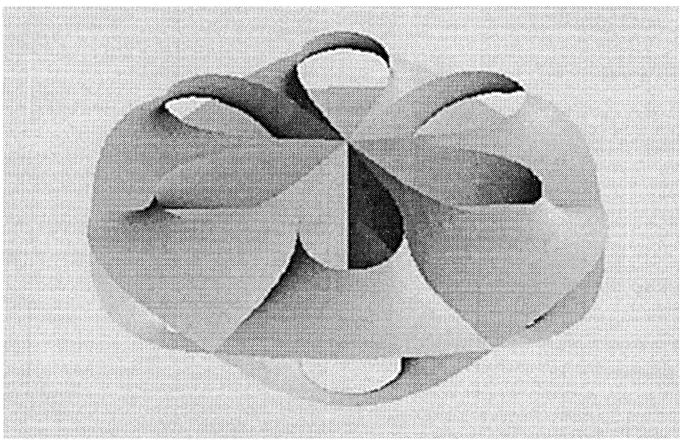

Figure 3: A fundamental piece of the surface $\widetilde{X}_{\theta r_\theta}(M_{\theta\, r_\theta})$ in case $\theta = \pi/3$.

Finally, as $r \to -1$, the surfaces $X_{\theta\, r}(M_{\theta\, r})$ converge to two parallel convex planar sectors in planes π_j, $j = 1, 2$, connected by the segment that joins the two vertices of the sectors. See Proposition 3.7 for details.

Hence, if we fix $\theta \in]0, \pi]$, the one parameter family of surfaces $X_{\theta\, r}(M_{\theta\, r})$, $r \in]-1, r_\theta]$, starts at a degenerate and *stable* surface determined by two parallel planar sectors, and ends in the *unstable* example $X_{\theta\, r_\theta}(M_{\theta\, r_\theta})$. Moreover, the spherical image of the Gauss map of the immersion $X_{\theta\, r}$ grows as a function of r. Then the geometrical and physical intuitions suggest that the immersions $X_{\theta\, r}$, $r \in]-1, r'_\theta]$, are stable, and the immersions $X_{\theta\, r}$, $r \in]r'_\theta, r_\theta]$ are unstable. Hence, the example $X_{\theta\, r'_\theta}$ would correspond to an almost-stable surface.

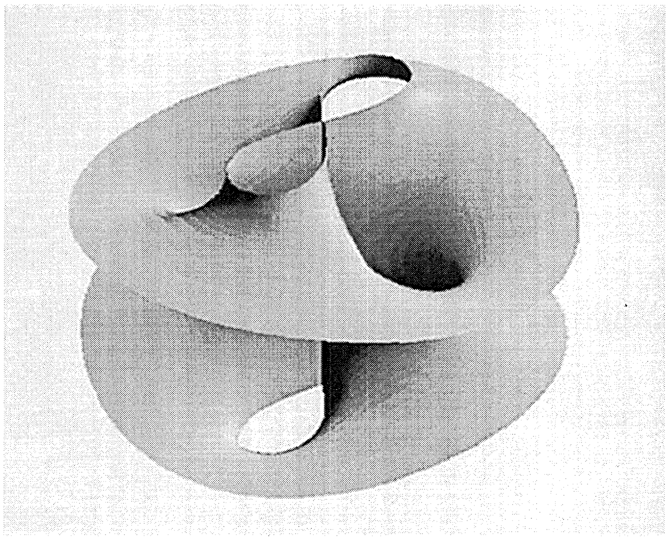

Figure 4: A fundamental piece of the surface $\widetilde{X}_{\theta\, r_\theta}(M_{\theta\, r_\theta})$ in case $\theta = \pi/2$.

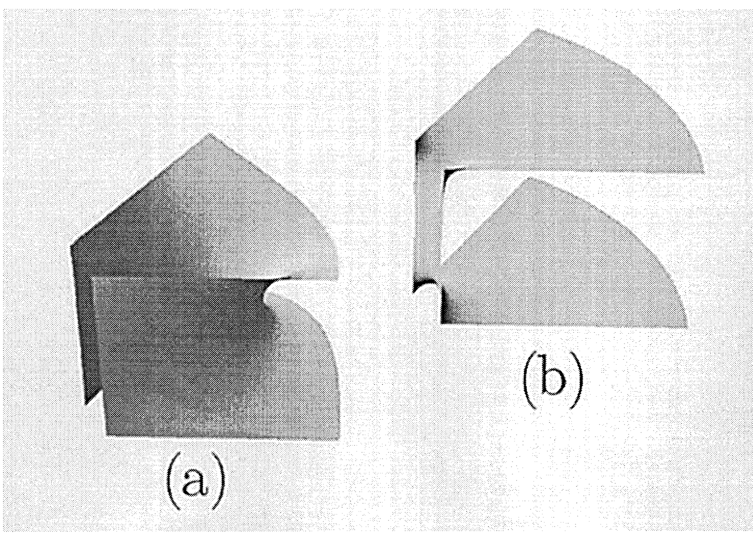

Figure 5: The two solutions in case $\theta = \pi/3$, $d = 0.28\,\|\vec{v}\|$. Figure **(a)** corresponds to the *unstable* example, and Figure **(b)** corresponds to the *stable* one.

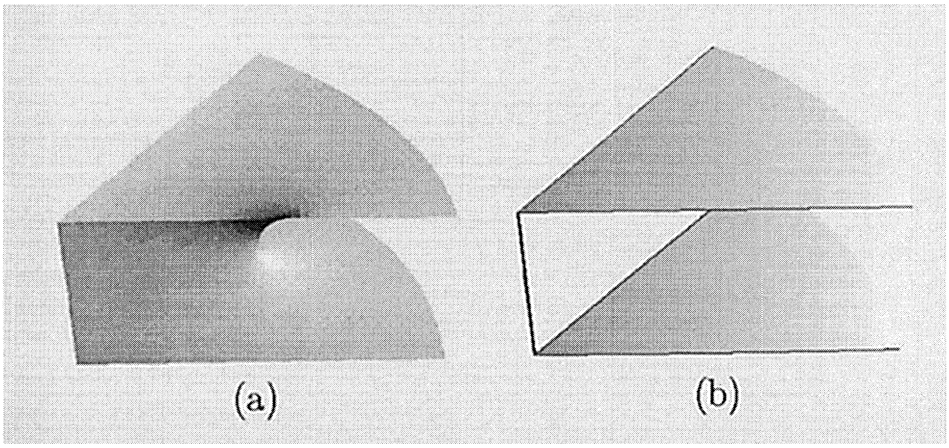

Figure 6: **(a)** The *unstable* surface $X_{\theta r_\theta}(M_{\theta r_\theta})$, for $\theta = \pi/3$. **(b)** The degenerate and *stable* surface determined by two parallel planar sectors, for $\theta = \pi/3$ and $r \to -1$.

The second part of the paper is devoted to using the new examples as barriers for the maximum principle application. This kind of technique was used by Schoen [15], Hoffman-Meeks [3] and Meeks-Rosenberg [11], among others.

It is well-known that any compact minimal surface in $\mathbb{R}^3$ verifies the convex hull property (see for instance [13]). Recall that a surface satisfies the convex hull property if and only if it lies in the convex hull of its boundary. In this paper we have proved that this property is also satisfied by any properly immersed minimal surface (compact or not) included in a *wedge of a slab*.

Label W_θ as the solid region of $\mathbb{R}^3$ determined by the intersection of:

- the convex region determined by two halfplanes P_1 and P_2, meeting at an angle θ along a straight line R, and
- a slab W, which is orthogonal to R.

Write $F_i = W \cap P_i$, $i = 1, 2$, and $L = W \cap R$. Moreover, denote W_0 as the intersection of two orthogonal slabs. To be more precise, we have proved:

Theorem B. *Let M be a properly immersed minimal surface, $M \subset W_\theta$, where $\theta \in [0,\pi[$. Then, M is contained in the convex hull of its boundary.*

Observe that we are not assuming in Theorem B that $\partial(M) \subset \partial(W_\theta)$. The hypothesis $M \subset W_\theta$ can not be substituted for the weaker one $\partial(M) \subset W_\theta$. In Figure 7 we illustrate two examples whose boundary lies in a wedge of a slab, but none of them is contained in the convex-hull of its boundary.

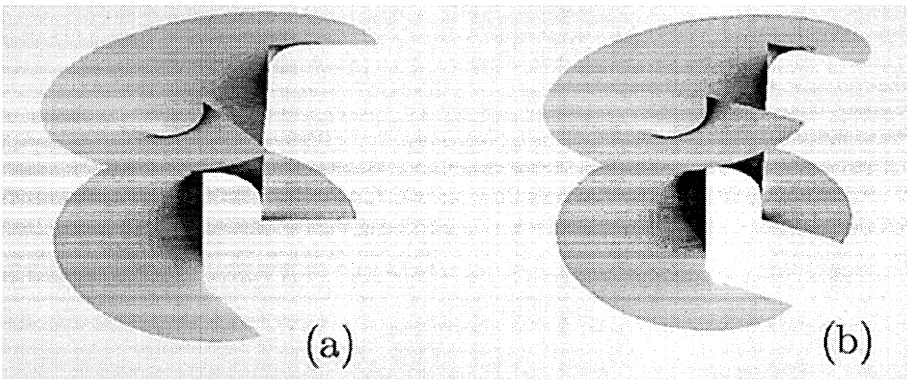

Figure 7: **(a)** A minimal disc bounded by $\Gamma_{\theta\, d}$, for $\theta = \pi/3$ and $d = 0.56\|\vec{v}\|$. **(b)** A minimal disc bounded by $\Gamma_{\theta\, d}$, for $\theta = 0$ and $d = 0.40\|\vec{v}\|$.

Theorem B does not hold for the wedges W_θ, $\theta \geq \pi$, as a suitable piece of the helicoid or the surfaces in $\mathcal{M}_\pi$ shows (see Figure 8). In this sense it is sharp. However, it holds for surfaces M lying in a wedge W_{θ_1}, $\theta_1 \in]0, 2\pi[$, whose boundary satisfies $\partial(M) \subset W_\theta \subset W_{\theta_1}$, $\theta \in]0,\pi[$ (see Corollary 4).

Finally, we have derived some non existence theorems for properly immersed minimal surfaces with planar boundary. Next theorem gives geometrical meaning to the number d_θ. Let h denote the thickness of the slab W containing the wedge W_θ.

Theorem C. *Let M be a properly immersed non flat minimal surface in a slab wedge W_θ, $\theta \in]0,\pi]$. Suppose that $\partial(M) \subset F_1 \cup F_2$. Then,*

$$\operatorname{dist}(L, \partial(M)) \leq \frac{d_\theta}{2\sin\left(\frac{\theta}{2}\right)}\, h.$$

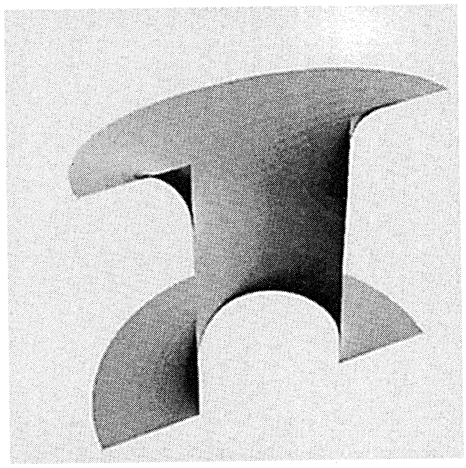

Figure 8: A surface $X_{\pi r}(M_{\pi r})$.

The surface $X_{\theta r'_\theta}(M_{\theta r'_\theta})$ corresponding to the maximum d_θ of the openingfunction is critical for Theorem C, and so the inequality is sharp.

Write $\{C_1, C_2\}$ and $\{D_1, D_2\}$ as the two pairs of opposite faces of W_0.

Theorem D. *Let M be a properly immersed non flat minimal surface in W_0 such that $\partial(M) \subset C_1 \cup C_2$. Assume that $\partial(M)$ lies in a halfspace orthogonal to W_0. Then*

$$dist(C_1, C_2) \leq dist(D_1, D_2).$$

Theorem D is a generalization of a classical result by Nitsche (see [12] and [14]).

This paper is laid out as follows. In Section 2, we give several results we need in this paper. In Section 3 we present the new family $\mathcal{M}$ of minimal surfaces and study its geometrical properties. In Section 4 we establish Theorems A, B, C and D.

Acknowledgements. We would like to thank A. Ros for helpful conversations. We would also like to thank Professor H. Rosenberg for suggesting that we use the new family of examples as barriers for maximum principle application.

2. Background and Notation.

The aim of this section is to fix the principal notation used in this paper, and to summarize some results about complete minimal surfaces.

Let $X : M \to \mathbb{R}^3$ a proper conformal minimal immersion, where M is a Riemann surface with piecewise analytic boundary homeomorphic to the closed unit disc $\mathbb{D}$ minus two boundary points E_1 and E_2, that we call the ends of M.

Remark 1. We say that M is a Riemann surface with piecewise analytic boundary if and only if M is a subset of an open Riemann surface M', the conformal structure of $M - \partial(M)$ is that induced by M' and $\partial(M)$ consists of a set of piecewise analytic curves. Meromorphic (resp. holomorphic) functions and 1-forms on M are, by definition, the restriction of meromorphic (resp. holomorphic) functions and 1-forms on M'.

The Weierstrass representation of X is denoted by (g, η). Recall that g is a meromorphic function and η a holomorphic 1-form on M. Both of them determine the minimal immersion X as follows:

$$X(P) = \mathrm{Re}\left(\int^P (\phi_1, \phi_2, \phi_3)\right)$$

where

$$\phi_1 = \frac{1}{2}(1-g^2)\eta,\ \phi_2 = \frac{i}{2}(1+g^2)\eta,\ \phi_3 = g\eta \tag{1}$$

are holomorphic 1-forms on M satisfying:

$$\sum_{j=1}^{3} |\phi_j|^2 \neq 0. \tag{2}$$

Furthermore, g is the stereographic projection of the Gauss map $N : M \to \mathbb{S}^2$.

Minimal surfaces containing straight lines have special properties. Among them, we emphasize Schwarz's reflection principle (see, for instance, [13]).

A particular case of minimal surfaces bounded by straight lines was studied by Jenkins and Serrin [4]. They prove the following interesting theorem:

Theorem 1 (Jenkins, Serrin). *Let D be a bounded covex domain whose boundary contains two sets of open straight segments $A_1, \ldots, A_k$ and $B_1, \ldots, B_l$, with the property that no two segments A_i and no two segments B_i has a common endpoint. The remaining portion of the boundary consists of endpoints of the segments A_i and B_i, and open arcs $C_1, \ldots, C_m$. Consider the Dirichlet problem:*

> *Determine a minimal graph in D which assumes the value $+\infty$ on each A_i, $-\infty$ on each B_i and assigned continuous data on each of the open arcs C_i.*

Let $\mathcal{P}$ denote a simple closed polygon whose vertices are chosen from among the ends points of the segments A_i and B_j. Let α, β be, respectively, the total lenght of the segments A_i and B_j which are part of $\mathcal{P}$. Finally, let γ denote the perimeter of $\mathcal{P}$.

Then, if the family of arcs $\{C_i\}$ is not empty, the Dirichlet problem stated above is solvable if and only if

$$2\alpha < \gamma \quad \textit{and} \quad 2\beta < \gamma.$$

Furthermore, the solution is unique if it exists.

We state two theorems which summarize the versions of the maximum principle which we require in the last section of this paper.

Theorem 2 (Interior maximum principle). *Suppose M_1, M_2 are minimal surfaces in $\mathbb{R}^3$. Suppose p is an interior point of both M_1 and M_2, and suppose $T_pM_1 = T_pM_2$. Assume that $T_pM_1 = \{x_3 = 0\}$ so that both M_1, M_2 are given near p as the graphs of two real analytic functions u_1 and u_2, respectively. If $u_1 \geq u_2$ in a neighbourhood of p, then $M_1 = M_2$.*

A elementary consequence of this result is the nonexistence of compact nonplanar minimal surfaces with boundary contained in a plane Π. Meeks and Rosenberg proved that this result remains true in the non compact case if in addition the surface lies in the slab determined by Π and a plane Π' parallel to Π. In fact, they obtained the following:

Theorem 3 (Meeks, Rosenberg). *Suppose $M \subset \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \geq 0\}$ is a properly immersed minimal surface with nonempty, possibly noncompact, boundary $\partial(M)$. If $x_3(\partial(M)) \geq \delta$, then $x_3(M) \geq \delta$.*

See [11, Lemma 2.1] for details.

3. The existence results.

In this section we construct the family of minimal surfaces described in the introduction. Moreover, we present in successive subsections a complete study of the main geometrical properties of these examples.

3.1. The Weierstrass representation and symmetry of the new family of examples.

Take $n \in [1,2]$ and $r \in]-1,1[$. Put $r = -cos(x_0)$, $x_0 \in]0,\pi[$. Consider the Riemann surface

$$\mathcal{N} = \{(z,w) \in \mathbb{C}^* \times \mathbb{C} \;/\; w^2 = (z - e^{ix_0/n})(z - e^{-ix_0/n})\},$$

and define in the z-plane:

$$\begin{aligned} s_1^+ &= \{\lambda e^{-\frac{\pi i}{n}} \;:\; \lambda \in]0,1]\}, \quad & s_1^- &= \{\lambda e^{\frac{\pi i}{n}} \;:\; \lambda \in]0,1]\},\\ s_2^+ &= \{\lambda e^{-\frac{\pi i}{n}} \;:\; \lambda \in [1,+\infty[\}, \quad & s_2^- &= \{\lambda e^{\frac{\pi i}{n}} \;:\; \lambda \in [1,+\infty[\},\\ s_0^+ &= \{e^{-it} \;:\; t \in [x_0/n, \pi/n]\}, \quad & s_0^- &= \{e^{it} \;:\; t \in [x_0/n, \pi/n]\}. \end{aligned}$$

Then, label $C \subset \mathcal{N}$ as the connected component of $z^{-1}\big(\mathbb{C} - \big(\cup_{i=0}^2 (s_i^+ \cup s_i^-)\big)\big)$ containing the point

$$P_0 = (1, +\sqrt{2(1-\cos(x_0/n))}).$$

Define

$$M = \overline{C} \tag{3}$$

where $\overline{C}$ means the closure of C in $\mathcal{N}$.

Finally, label $\gamma_i^+ = z^{-1}(s_i^+)$, $\gamma_i^- = z^{-1}(s_i^-)$, $i = 0,1,2$. Denote

$$\gamma^+ = \bigcup_{i=0}^2 \gamma_i^+, \quad \gamma^- = \bigcup_{i=0}^2 \gamma_i^-.$$

It is clear that $\partial(M) = \gamma^+ \cup \gamma^-$. Furthermore, note that $z|_{\gamma_i^+}$ and $z|_{\gamma_i^-}$ are bijective maps onto s_i^+ and s_i^-, respectively, $i = 1,2$. However, γ_0^+ and γ_0^- consist of two copies of s_0^+ and s_0^-, respectively. See Figure 9 for more details.

Since $z(M)$ is simply connected and $0 \notin z(M)$, then the function $z^n + z^{-n} + 2r$ is well defined on M. We choose the branch of z^n satisfying $1^n = 1$.

This choice of the branch of z^n implies that the function $z^n + z^{-n} + 2r$ has neither zeroes nor poles on $M - \partial(M)$.

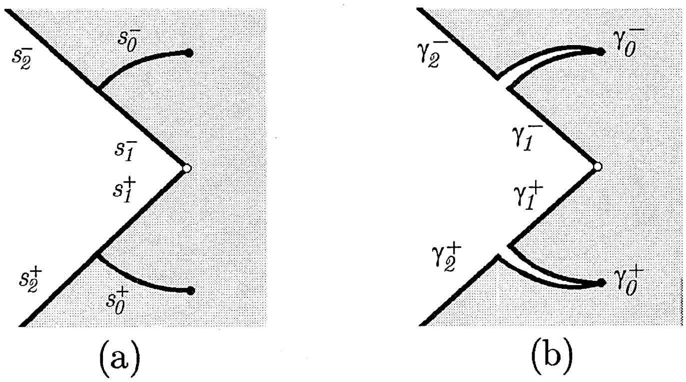

Figure 9: **(a)** The domain $z(M)$. **(b)** The surface M.

Hence, the function $\varphi((z,w)) = \sqrt{z^n + z^{-n} + 2r}$ has a well defined branch on the (simply connected) domain $M - \partial(M)$, that can be extended continuously to M. For convenience, we choose the branch verifying $\varphi(P_0) < 0$. Moreover, note that given $z_0 \in (s_0^+ \cup s_0^-)$ and denoting $\{P^+, P^-\} = z^{-1}(z_0)$, we have $\varphi(P^+) = -\varphi(P^-)$.

We consider the meromorphic data on M

$$g = iz, \quad \phi_3 = B\frac{dz}{z\varphi}, \tag{4}$$

where $B > 0$.

For simplicity, we write

$$\tau_r = \frac{dz}{z\varphi},$$

and as usual, we denote

$$(\phi_1, \phi_2, \phi_3) = \frac{B}{2}\left(-i(1/z + z)\tau_r, (1/z - z)\tau_r, \tau_r\right).$$

As M is homeomorphic to a closed disc minus two boundary points, then

$$X : M \longrightarrow \mathbb{R}^3$$

$$X(P) = \mathrm{Re}\int_{P_0}^{P} (\phi_1, \phi_2, \phi_3) \tag{5}$$

is a well defined conformal minimal immersion of M in $\mathbb{R}^3$.

Remark 2. If $n = 2$ and $r \in]-1, 1[$, the immersions associated to the Weierstrass data (4) are known and correspond to some Jenkins-Serrin graphs (see Figure 1). Following the notation of [4] and Theorem 2, these minimal graphs are the only with boundary values $+\infty$, 0, $+\infty$, 0 on a rectangle.

Extension by Schwarz reflections of these surfaces gives embedded doubly-periodic examples with two orthogonal planes of symmetry between adjacent saddle towers. As the quotient of such a surface under its group of translations has finite topology, then this quotient has finite total curvature (see Meeks-Rosenberg work [10]).

Karcher [5] proved that the arising Weierstrass data are those described in (4).

For the reasons explained in last remark, we are going to restrict our attention to the case $n < 2$, and in what follows we suppose $n \in [1, 2[$.

Concerning to its symmetries, let S_h, S_v denote the antiholomorphic transformations on M

$$S_h((z, w)) = (1/\overline{z}, \overline{w}/\overline{z}), \quad S_v((z, w)) = (\overline{z}, \overline{w}).$$

Notice that $S_h(P_0) = S_v(P_0)$, and

$$\begin{aligned} g \circ S_h &= 1/\overline{g},\ g \circ S_v = -\overline{g}, \\ S_h^*(\phi_3) &= -\overline{\phi_3},\ S_v^*(\phi_3) = \overline{\phi_3}; \end{aligned} \tag{6}$$

so elementary arguments imply that S_h (resp. S_v) induces on $X(M)$ a symmetry with respect to the plane $x_3 = 0$ (resp. $x_1 = 0$).

3.2. The boundary behaviour.

The following step is to study the behaviour of X along $\partial(M)$.

First, observe that

$$S_v(\gamma_i^+) = \gamma_i^-,\ i = 0, 1, 2 \tag{7}$$

$$S_h(\gamma_1^+) = \gamma_2^+,\ S_h(\gamma_1^-) = \gamma_2^-,\ S_h(\gamma_0^+) = \gamma_0^+,\ S_h(\gamma_0^-) = \gamma_0^-. \tag{8}$$

Introduce the following notation. Let $\ell_i^+ = X(\gamma_i^+)$ and $\ell_i^- = X(\gamma_i^-)$, $i = 0, 1, 2$, and label

$$\Gamma = \bigcup_{i=0}^{2} (\ell_i^+ \cup \ell_i^-). \tag{9}$$

We can prove the following:

Lemma 1. *The maps $X|_{\gamma^+}$, $X|_{\gamma^-}$ are injective, and:*

1. *The curves ℓ_1^+ and ℓ_1^- are half-lines contained in a plane $x_3 = k$, $k > 0$. Furthermore, they are symmetric with respect to plane $x_1 = 0$, and the straight lines containing ℓ_1^+ and ℓ_1^- meet at an angle $\theta = (2-n)\pi/n$.*
2. *The curve ℓ_2^+ (resp. ℓ_2^-) is the image of ℓ_1^+ (resp. ℓ_1^-) under the symmetry with respect to plane $x_3 = 0$.*
3. *The functions $x_1|_{\ell_i^+}$ and $x_1|_{\ell_i^-}$ diverge to $+\infty$ and $-\infty$, respectively, $i = 1, 2$.*

   If $n \in]1, 2[$, then $x_2|_{\ell_i^+}$, $x_2|_{\ell_i^-}$ diverge to $+\infty$, $i = 1, 2$. In case $n = 1$, $x_2|_{\ell_i^+}$ and $x_2|_{\ell_i^-}$ are constant.
4. *The curve ℓ_0^+ (resp. ℓ_0^-) is the vertical segment joining the end points of ℓ_1^+ and ℓ_2^+ (resp. ℓ_1^- and ℓ_2^-).*

Proof. On γ_1^-, put $z = t\, e^{\pi i/n}$, $t \in]0, 1]$. Taking into account that $\varphi(P_0) < 0$, an analytic continuation argument gives:

$$\phi_3(t) = -iB\frac{dt}{|t\sqrt{t^n + t^{-n} - 2r}|}$$
$$\phi_1(t) = -\frac{i}{2}\left((1/t + t)\cos(\pi/n) + i(t - 1/t)\sin(\pi/n)\right)\phi_3(t)$$
$$\phi_2(t) = \frac{1}{2}\left((1/t - t)\cos(\pi/n) - i(t + 1/t)\sin(\pi/n)\right)\phi_3(t).$$

These equations imply that ℓ_1^- is a half-line contained in a straight line $x_3 = k$, $x_2 - \tan(\pi/n)x_1 = k'$, for suitable $k, k' \in \mathbb{R}$. Note that this straight line meets the straight line $x_3 = k$, $x_1 = 0$ at an angle of $\frac{(2-n)\pi}{2n}$. Notice also that $\mathrm{Re}(\phi_1(t)/dt) > 0$, and so $X|_{\gamma_1^-}$ is injective.

Moreover, it is clear that $x_1|_{\ell_1^-}$ diverges to $-\infty$. If $n > 1$, then $x_2|_{\ell_1^-}$ diverges to $+\infty$, and $n = 1$ implies that $x_2|_{\ell_1^-}$ is constant.

The curve γ_0^- consists of two copies, δ_1 and δ_2, of s_0^-. We can assume that $\delta_1(t)$ and $\delta_2(t)$ are the two lifts to M of the curve $e^{it/n}$, $t \in [x_0, \pi]$, in the z-plane, satisfying $\delta_1(\pi) \in \gamma_1^-$ and $\delta_2(\pi) \in \gamma_2^-$, respectively.

Let $\delta(t)$ be the lift to M of the curve $e^{it/n}$, $t \in [0, x_0]$, in the z-plane. Observe $\delta(0) = P_0$ and $\delta(x_0) = \delta_1(x_0) = \delta_2(x_0)$.

Taking our choice of branches into account, we have

$$\varphi(\delta(t)) < 0, \quad i\varphi(\delta_1(t)) \in \mathbb{R}^-, \quad i\varphi(\delta_2(t)) \in \mathbb{R}^+.$$

Then, it is straightforward to check that:

$$\operatorname{Re}\left(\phi_i\left(\frac{d\delta_j}{dt}\right)\right) = 0,\ i = 1,2,\ j = 1,2,$$

and so ℓ_0^- is a vertical segment.

Furthermore, note that:

$$\operatorname{Re}\left(\phi_3\left(\frac{d\delta}{dt}\right)\right) = 0,\ \phi_3\left(\frac{d\delta_1}{dt}\right) \in \mathbb{R}^+,\ \phi_3\left(\frac{d\delta_2}{dt}\right) \in \mathbb{R}^-,$$

and this implies that $k > 0$ and $X|_{\gamma_0^-}$ is injective.

Taking into account the symmetries induced by S_h and S_v, (7) and (8), it is not hard to conclude the proof. □

Remark 3. Note that the angle function $\theta : [1, 2[\to]0, \pi]$, $\theta(n) = (2-n)\pi/n$, is an analytical diffeomorphism. Hence, we can use (θ, r) instead of (n, r) to parametrize our family of surfaces.

In the following, we label the plane $x_3 = k$ as π_1 and the plane $x_3 = -k$ as π_2.

Moreover, and in the remainder of this section, we fix $\theta \in]0, \pi]$ (or $n \in [1, 2[$).

Consider δ_1, δ_2 and δ as in the proof of Lemma 3.2.

If we label $h :]-1, 1[\longrightarrow \mathbb{R}$ as the *height* function, i.e.,

$$h(r) = 2\operatorname{Re}\left(\int_{\delta_1} \tau_r\right),$$

then a straightforward computation gives

$$h(r) = \frac{\sqrt{2}}{n}\int_{\arccos(-r)}^{\pi} \frac{dt}{\sqrt{-r-\cos(t)}} > 0,$$

where we are using the branch of arccos which maps $]-1, 1[$ into $]0, \pi[$.

In what follows, and up to homotheties, we suppose

$$B = 1/h(r). \tag{10}$$

In other words, and from Lemma 3.2, we are normalizing the immersion X in such a way that the distance between the planes π_1 and π_2 is 1 (i.e., $k = 1/2$).

On the other hand and taking into account Lemma 3.2 and the symmetries of $X(M)$, the *oriented* distance $d(r)$ between ℓ_0^+ and ℓ_0^- is given by:

$$d(r) = -2\mathrm{Re}\left(\int_\delta \phi_1\right). \tag{11}$$

This means that:

- $|d|$ is the distance between ℓ_0^+ and ℓ_0^-.
- $d > 0$ if and only if ℓ_i^+ and ℓ_i^- do not intersect, $i = 1, 2$. See Figure 2.
- $d = 0$ if and only if $\ell_0^+ = \ell_0^-$ (i.e., the end points of ℓ_i^+ and ℓ_i^- coincide, $i = 1, 2$).
- $d < 0$ if and only if the half-lines ℓ_i^+ and ℓ_i^- intersect at an interior point, $i = 1, 2$.

See Figure 10.

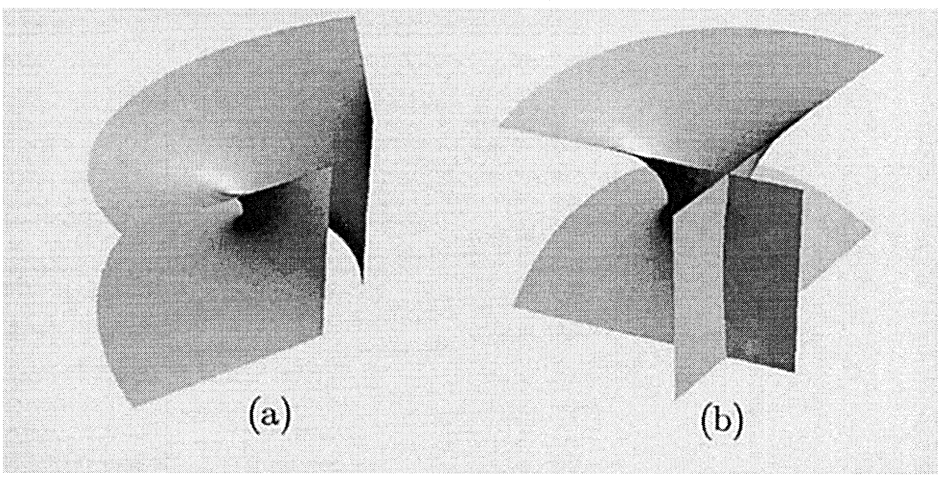

Figure 10: **(a)** A surface $X(M)$ for $\theta = 2\pi/3$ and $d > 0$. **(b)** A surface $X(M)$ for $\theta = 2\pi/3$ and $d < 0$.

3.3. The properness and convex hull property.

Next lemma is devoted to study some topological and algebraic properties of X.

Lemma 2. *The minimal immersion $X : M \to \mathbb{R}^3$ verifies:*

1. *X is proper,*
2. *If $\theta \neq \pi$, $X(M)$ is contained in the convex hull, $\mathcal{E}(\Gamma)$, of Γ.*
3. *If $\theta = \pi$, $X(M)$ is contained in the intersection of the slab $-1/2 \leq x_3 \leq 1/2$ and one of the two halfspaces determined by the plane containing ℓ_i^+, ℓ_i^-, $i = 0, 1, 2$.*

Proof. Firstly, we are going to study the behaviour of X around $E_1 = (0,1) = z^{-1}(0)$.

Taking into account the choice of the branch of φ, one has that

$$\phi_3 = -Bz^{\frac{n}{2}-1}\,dz + A(z)\,dz,$$

where the branch of $z^{n/2}$ satisfies $1^{n/2} = 1$. So

$$\begin{aligned} -\frac{i}{z}\phi_3 &= iBz^{\frac{n}{2}-2}\,dz - i\frac{A(z)}{z}\,dz, \\ iz\phi_3 &= -iBz^{\frac{n}{2}}\,dz + izA(z)\,dz, \end{aligned}$$

where $A(z)$ is holomorphic around $E_1 = z^{-1}(0)$ in M, and satisfies

$$\lim_{z\to 0}\frac{A(z)}{z^{\frac{3n}{2}-1}} \neq 0, \infty.$$

We define

$$\begin{aligned} F(z) &= -\frac{i}{2}\int \frac{1}{z}\phi_3 = i\frac{B}{n-2}z^{\frac{n}{2}-1} + F_1(z), \\ G(z) &= \frac{i}{2}\int z\phi_3 = -i\frac{B}{n+2}\,z^{\frac{n}{2}+1} + G_1(z), \\ H(z) &= \int \phi_3 = -\frac{B}{n}z^{\frac{n}{2}} + H_1(z), \end{aligned}$$

where the limits

$$\lim_{z\to 0}\frac{F_1(z)}{z^{\frac{3n}{2}-1}},\ \lim_{z\to 0}\frac{G_1(z)}{z^{\frac{3n}{2}+1}},\ \lim_{z\to 0}\frac{H_1(z)}{z^{\frac{3n}{2}}}$$

exist and are non zero. Then, X can be expressed locally around E_1 as

$$(12)\qquad X(z) = (X_1(z), X_2(z), X_3(z)) = \left(\overline{F(z)} - G(z), \mathrm{Re}\,(H(z))\right)$$
$$= \left(-\frac{iB}{n-2} z^{\frac{n}{2}-1}, 0\right) + (O_1(z), O_2(z), O_3(z)),$$

where $O_i(z)/|z|$ is a bounded function in a neighbourhood of E_1, $i = 1, 2, 3$. Equation (12) implies that for any sequence $\{P_m\}$ in M converging E_1, the sequence $\{X(P_m)\}$ diverges in $\mathbb{R}^3$. Taking this and the symmetry S_h into account, the same occurs for sequences converging to $E_2 = (\infty, \infty) = z^{-1}(\infty)$. Hence, we obtain that X is proper.

On the other hand, (12) yields that the limits $\lim_{P \to E_1} X_3(P)$ and $\lim_{P \to E_2} X_3(P)$ exist. Using Lemma 3.2 we deduce

$$(13)\qquad \lim_{P \to E_1} X_3(P) = \frac{1}{2}, \quad \lim_{P \to E_2} X_3(P) = -\frac{1}{2}.$$

As Γ lies in the slab $-1/2 \leq x_3 \leq 1/2$, then we can use (13) and Theorem 2 to obtain that $X(M)$ is contained in this slab.

Label $\widehat{\sigma}^+$ (resp. $\widehat{\sigma}^-$) as the plane containing ℓ_1^+ and ℓ_2^+ (resp. ℓ_1^- and ℓ_2^-).

If $d < 0$, we denote σ^+ (resp. σ^-) as the plane parallel to $\widehat{\sigma}^+$ (resp. $\widehat{\sigma}^-$) containing ℓ_0^- (resp. ℓ_0^+).

In case $d \geq 0$, we put $\sigma^+ = \widehat{\sigma}^+$ and $\sigma^- = \widehat{\sigma}^-$.

Label also σ_0 to the plane parallel to $x_2 = 0$ containing ℓ_0^+ and ℓ_0^-.

Let $\mathcal{H}^+$ and $\mathcal{H}^-$ be the open half spaces determined by σ^+ and σ^-, respectively, and containing Γ. Let $\mathcal{H}_0$ denote the closed half space determined by σ_0 and containing Γ too. Label $\mathcal{S}$ as the slab $x_3^{-1}([-1/2, 1/2])$, and finally denote $\mathcal{E} = \mathcal{H}^+ \cap \mathcal{H}^- \cap \mathcal{H}_0 \cap \mathcal{S}$. Then, a straightforward argument gives that

- If $\theta \neq \pi$ and $d < 0$, then $\mathcal{E}(\Gamma) = \mathcal{E} \cup \ell_0^+ \cup \ell_0^-$.
- If $\theta \neq \pi$ and $d \geq 0$, then $\mathcal{E} = \overline{\mathcal{E}}$.

On the other hand, consider $\{P_m\}$ a sequence in M converging to E_1, and assume that $\{\arg((X_1(P_m), X_2(P_m))\}$ converges to $\theta_0 \in [0, 2\pi[$. Let us prove that $\theta_0 \in [\pi/2 - \theta/2, \pi/2 + \theta/2]$.

Indeed, from (12) and for m large enough, one has that

$$(X_1(P_m), X_2(P_m)) = \frac{iB}{2-n} z^{\frac{n}{2}-1}(P_m) + q_m,$$

where $\{q_m\}$ is bounded. As $\arg(z(P_m)) \in [-\pi/n, \pi/n]$, then

$$\arg\left(\frac{iB}{2-n}\, z^{\frac{n}{2}-1}(P_m)\right) \in \left[\frac{\pi}{2} - \frac{(2-n)\pi}{2n}, \frac{\pi}{2} + \frac{(2-n)\pi}{2n}\right]$$
$$= [\pi/2 - \theta/2, \pi/2 + \theta/2].$$

Since q_m is bounded and $(X_1(P_m), X_2(P_m))$ is divergent, we deduce that $\theta_0 \in [\pi/2 - \theta/2, \pi/2 + \theta/2]$. This combined with equation (13) gives

$$\lim_{m\to\infty} \operatorname{dist}(P_m, \mathcal{E}(\Gamma)) = 0, \tag{14}$$

and this equation holds for arbitrary sequences converging to E_1. By using the symmetry S_h, the equation (14) holds for sequences converging to E_2 too.

Let κ be a plane which is not parallel to σ^+ and σ^-, and not intersecting Γ. Let $\mathcal{K}$ denote the closed half space determined by κ not containing Γ. Taking into account that X is proper and (14), we have that $\mathcal{K} \cap X(M)$ is compact, and so, by an elementary consequence of Theorem 2, it is empty. This proves Assertion *2* in the Lemma.

By a continuity argument, it is not hard to deduce Assertion *3* and complete the proof. □

Remark 4. By using the same Weierstrass data, (4), for $n \in [2/3, 1[$ and suitable $r \in]-1, 1[$, we can construct minimal examples with the same boundaries, which are not contained in their convex-hull. See Figure 7.

3.4. Embeddedness.

The following Lemmas are devoted to study under what conditions the immersion X is an embedding.

Let $M^+ = \{(z, w) \in M \,/\, \mathrm{Im}(z) \geq 0\}$ and $M^- = \{(z, w) \in M \,/\, \mathrm{Im}(z) \leq 0\}$, and define ρ as the lift to M of the divergent curve $]0, +\infty[$ in $z(M)$. We parametrize ρ as follows:

$$\rho(t) = z^{-1}(t), \quad t \in]0, +\infty[.$$

Obviously, the surfaces M^+ and M^- are topologically a closed disk minus two boundary points. Furthermore,

$$M = M^+ \cup M^-, \quad M^+ \cap M^- = \rho$$

and

$$\partial(M^+) = \rho \cup \gamma_1^+ \cup \gamma_2^+ \cup \gamma_0^+, \quad \partial(M^-) = \rho \cup \gamma_1^- \cup \gamma_2^- \cup \gamma_0^-.$$

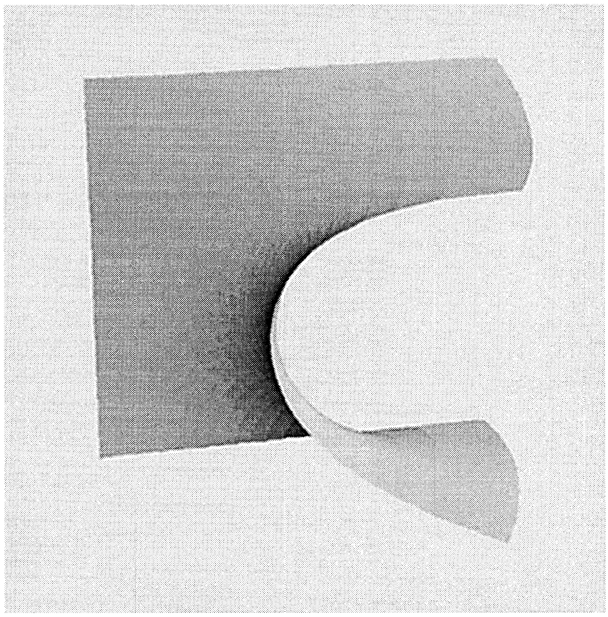

Figure 11: The surface $X(M^-)$ when $\theta = \pi/3$ and $r = 0.8$.

Lemma 3. *If $\theta \in]0, \pi[$, the surfaces $X(M^+)$ and $X(M^-)$ are graphs on the plane $x_1 = 0$. If $\theta = \pi$, the surfaces $X(M^+) - (\ell_1^+ \cup \ell_2^+)$ and $X(M^-) - (\ell_1^- \cup \ell_2^-)$ are graphs on the plane $x_1 = 0$.*

Proof. Suppose first that $\theta \neq \pi$. Note that $\phi_3/dt < 0$, $t \in]0, +\infty[$, and so $X_3|_\rho$ is injective. Moreover, from (10) and (13),

$$\lim_{t\to 0} X_3(\rho(t)) = 1/2, \quad \lim_{t\to +\infty} X_3(\rho(t)) = -1/2.$$

On the other hand, ρ is the fixed point set of S_v. Hence, $X(\rho)$ lies in the plane $x_1 = 0$.

By using the symmetry S_v, it suffices to prove that $X(M^+)$ is a graph on $x_1 = 0$.

Let $\mathfrak{p}_1$ denote the orthogonal projection on the plane $x_1 = 0$. Since $X(M) \subset \mathcal{E}(\Gamma)$ and $X(M) \cap \mathcal{E}(\Gamma) = \Gamma$ (see Theorem 2), then

$$\bigcup_{i=0}^{2} \mathfrak{p}_1(\ell_i^+) \subset \partial(\mathfrak{p}_1(M^+)),$$

and $\mathfrak{p}_1(M^+)$ is contained in the closed domain $W = \mathfrak{p}_1(\mathcal{E}(\Gamma))$ of the plane $x_1 = 0$, bounded by $\bigcup_{i=0}^{2} \mathfrak{p}_1(\ell_i^+)$. Let W_0 denote the closed domain in W bounded by $X(\rho) \cup (\bigcup_{i=0}^{2} \mathfrak{p}_1(\ell_i^+))$, and label $W_1 = W - W_0$. It is clear that the open domain W_1 is bounded by the curve $X(\rho)$.

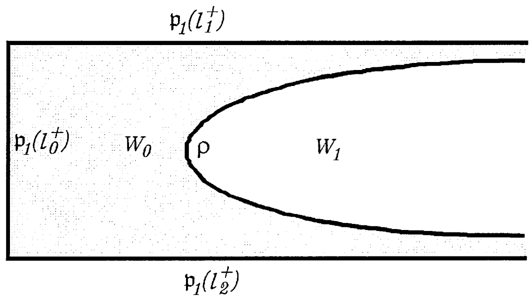

Figure 12: The domain $\mathfrak{p}_1(M^+)$.

As the normal vector $N(P)$ at any point $P \in M^+ - \rho$ does not lie in the plane $x_1 = 0$, then it is not hard to see that

$$\mathfrak{p}_1|_{X(M^+-\rho)} : M^+ - \rho \longrightarrow \mathfrak{p}_1(M^+)$$

is a local diffeomorphism.

In particular, $\mathfrak{p}_1(M^+) \cap W_1$ is an open subset of W_1. However, as X is proper (see Lemma 3.3), we have that $\mathfrak{p}_1(M^+) \cap W_1$ is also a closed subset of W_1. Since W_1 is connected, either $\mathfrak{p}_1(M^+) \cap W_1 = W_1$ or $\mathfrak{p}_1(M^+) \cap W_1 = \emptyset$. On the other hand, recall that X_3 extends continuously to the ends $\{E_1, E_2\}$ and $X_3(E_1) = 1/2$, $X_3(E_2) = -1/2$. This easily implies that only a compact subset of the x_2-axis is contained in $\mathfrak{p}_1(M^+)$, and so $\mathfrak{p}_1(M^+) \cap W_1 = \emptyset$.

A similar argument yields $\mathfrak{p}_1(M^+) \cap W_0 = W_0$, i.e., $\mathfrak{p}_1(M^+) = W_0$. Now it is easy to deduce that

$$\mathfrak{p}_1|_{X(M^+)} : M^+ \longrightarrow \mathfrak{p}_1(M^+)$$

is a submersion.

Since X is proper and $X(M) \subset \mathcal{E}(\Gamma)$, the map $\mathfrak{p}_1|_{X(M^+)} : M^+ \to \mathfrak{p}_1(M^+)$ is also proper, and so, it is a covering. Since $\mathfrak{p}_1(M^+) = W_0$ is simply-connected, $\mathfrak{p}_1|_{X(M^+)} : M^+ \to \mathfrak{p}_1(M^+)$ is a homeomorphism. This proves the Lemma for $\theta < \pi$.

If $\theta = \pi$, use a continuity argument. This concludes the proof. □

Lemma 4. *If $d(r) \geq 0$, then $X(M^+)$ (resp. $X(M^-)$) is contained in the half space $x_1 \geq 0$ (resp. $x_1 \leq 0$).*

In particular, $d(r) > 0$ implies that X is an embedding. If $d(r) = 0$, then $X|_{M-\gamma^+}$ and $X|_{M-\gamma^-}$ are injective.

Proof. In this proof we use similar ideas to those in Lemma 3.3.

By using the symmetry S_v, it suffices to prove the Lemma for $X(M^+)$.

First, note that $d(r) \geq 0$ and Lemma 3.2 imply that $X(\partial(M^+))$ lies in the half space $x_1 \geq 0$.

Let $\{Q_m\}$ be a sequence in M^+ converging to E_1, such that $\{\arg((X_1(Q_m), X_2(Q_m))\}$ converges to $\theta_0 \in [0, 2\pi[$. Let us see that $\theta_0 \in [0, \pi/2 + \theta/2]$.

Indeed, if m is large enough, then from (12) one has that

$$(X_1(Q_m), X_2(Q_m)) = \frac{iB}{2-n} z^{\frac{n}{2}-1}(Q_m) + p_m,$$

where $\{p_m\}$ is bounded. As $\arg(z(Q_m)) \in [-\pi/n, 0]$, then

$$\arg\left(\frac{iB}{2-n} z^{\frac{n}{2}-1}(Q_m)\right) \in \left[0, \frac{\pi}{2} + \frac{(2-n)\pi}{2n}\right] = [0, \pi/2 + \theta/2].$$

As p_m is bounded and $(X_1(Q_m), X_2(Q_m))$ diverges, we have $\theta_0 \in [0, \pi/2 + \theta/2]$. As a consequence, we obtain also that

$$\lim_{m\to\infty} \operatorname{dist}(Q_m, \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ / \ x_1 \geq 0\}) = 0. \tag{15}$$

By using S_h, then a symmetric equation to (15) holds for sequences converging to E_2.

Bcause of equation (15) and the properness of X, the same argument as at the end of the proof of Lemma 3.3 shows that $X(M^+)$ lies in the convex hull of its boundary. This concludes the proof. □

3.5. The analytical properties of the opening function d(r).

In order to understand the global behaviour of our family of surfaces, we are going to carry out a careful analysis of the function $d :]-1,1[\to \mathbb{R}$.

Define $\widehat{\delta}$ as the lift of the oriented curve $e^{ti/n}$, $t \in [-x_0, x_0]$, in the z-plane, and note that

$$d(r) = -\mathrm{Re}\left(\int_{\widehat{\delta}} \phi_1\right).$$

It is clear that $(S_h)_*(\widehat{\delta}) = \widehat{\delta}$ and $(S_v)_*(\widehat{\delta}) = -\widehat{\delta}$.

Therefore, from 6, we deduce that

$$\int_{\widehat{\delta}} \phi_1 = \overline{\int_{\widehat{\delta}} \phi_1} = -i \int_{\widehat{\delta}} z\phi_3.$$

If we define $f :]-1,1[\longrightarrow \mathbb{R}$

$$f(r) = i \int_{\widehat{\delta}} z\tau_r = 2\mathrm{Real}\left(i \int_{\delta} z\tau_r\right) = \frac{\sqrt{2}}{n} \int_0^{\arccos(-r)} \frac{\cos(t/n)}{|\sqrt{\cos(t)+r}|} dt, \tag{16}$$

then $d(r)$ is given by

$$d(r) = \frac{f(r)}{h(r)}. \tag{17}$$

To obtain the last equality in (16), we have taken into account the choice of the branch of φ.

Lemma 5. *The functions f, h, and d satisfy the following differential equations:*

1. $(1-r^2)f''(r) - 2rf'(r) + \dfrac{4-n^2}{4n^2} f(r) = 0.$
2. $(1-r^2)h''(r) - 2rh'(r) - \dfrac{1}{4}h(r) = 0.$
3. $-d(r)^2 + n^2\left(2 - 3n^2 - 2r^2 - n^2 r^2\right) d'(r)^2$
   $+ \left(4d(r) - 3n^2 d''(r)\right) n^2 \left(-1 + r^2\right) d''(r)$
   $+2n^2 d'(r) \left(2r\, d(r) + n^2 d^{(3)}(r) - 2n^2 r^2 d^{(3)}(r) + n^2 r^4 d^{(3)}(r)\right) = 0.$

Proof. Differentiating, one has

$$\frac{d^2}{dr^2}(\tau_r) - \frac{2r}{1-r^2}\frac{d}{dr}(\tau_r) - \frac{1}{4(1-r^2)}\tau_r = d(\widehat{h}), \tag{18}$$

$$\frac{d^2}{dr^2}(z\,\tau_r) - \frac{2r}{1-r^2}\frac{d}{dr}(z\,\tau_r) + \frac{4-n^2}{4n^2(1-r^2)}z\tau_r = d(\widehat{f}), \tag{19}$$

where $\widehat{h}$ and $\widehat{f}$ are the following meromorphic functions:

$$\widehat{h} = \frac{z^{2n}-1}{2n(1-r^2)z^n(z^n+z^{-n}+2r)^{3/2}},$$
$$\widehat{f} = z^{1-n}\frac{(n-2)-4rz^n-(n+2)z^{2n}}{2n^2(r^2-1)(z^n+z^{-n}+2r)^{3/2}}.$$

The function $y = z^n + z^{-n} + 2r$ is well defined on $z(M)$. Let M_φ denote the two-sheeted covering Riemann surface (with boundary) of $z(M)$ where $\varphi = \sqrt{y}$ is well defined, and label $p_\varphi : M_\varphi \to z(M)$ as the conformal covering map. Note that, in a natural way, M is biholomorphic to the closure in M_φ of $p_\varphi^{-1}(z(M - \partial(M)))$, and up this biholomorphism we can consider $M \subset M_\varphi$.

Let α be any simple closed curve in $z(M)$ winding once around $e^{ix_0/n}$ and $e^{-ix_0/n}$, and let $\widetilde{\alpha}$ be a lift of α to M_φ.

If ψ is a 1-form on M_φ, then up to a suitable choice of the orientation of α one has

$$\int_{\widetilde{\alpha}} \psi = 2\int_{\widehat{\delta}} \psi.$$

Applying the above equality to the one-forms in (19) and integrating by parts, we obtain *1.*

To obtain *2*, integrate by parts (18) along any curve in M homologous to $-\delta_2+\delta_1$, with the same end points that $-\delta_2+\delta_1$ and not passing through $(\pm e^{ix_0/n}, 0)$.

Moreover, using *1*, *2* and (17), it is not hard to obtain *3.* □

Lemma 6. *The function* $d :]-1,1[\to \mathbb{R}$ *satisfies:*

1. *It vanishes at only one point* $r_\theta \in]-1,1[$. *Furthermore* $d(r)$ *is positive in* $]-1, r_\theta[$ *and negative in* $]r_\theta, 1[$.

2. $\lim_{r\to -1} d(r) = 0$. *In particular,* d *is bounded in* $]-1, r_\theta[$.

3. *It has only a critical point $r'_\theta \in]-1, r_\theta[$ which is a maximum. In particular,* $\sharp\left[d^{-1}(\{x\})\right] = 2, \ \forall x \in]0, d(r'_\theta)[$.

Proof. From (17), it is obvious that $d(r_\theta) = 0$ if and only if $f(r_\theta) = 0$.

In order to study the function f, we consider $\widetilde{f} :]-1, 1[\to \mathbb{R}$ given by

$$\widetilde{f}(r) = \int_0^{+\infty} \sigma(r, z)\, dz,$$

where,

$$\sigma(r, z) = \frac{2mz^{\frac{2m-3}{2}}(z^m - r)}{(z^{2m} - 2rz^m + 1)^{3/2}}, \quad m = \frac{n}{2-n}.$$

Our purpose is to prove that $f(r) = \lambda \widetilde{f}$ for a suitable $\lambda > 0$.

To do this, observe that

(20)
$$\frac{d^2\sigma}{dr^2} - \frac{2r}{1-r^2}\frac{d\sigma}{dr} + \frac{4-n^2}{4n^2(1-r^2)}\sigma = \frac{d}{dz}\left(\frac{z^{-\frac{1}{2}+m} P(r, z)}{m\,(r^2-1)\,(1-2rz^m + z^{2m})^{\frac{5}{2}}}\right),$$

where:

$$P(r, z) = -r - 4mr + z^m + 6mz^m + 2r^2z^m + 2mr^2z^m - 3rz^{2m} - 4mrz^{2m} + z^{3m}.$$

Similar ideas to those used in the proof of Lemma 3.5 show that $\widetilde{f}$ satisfies Equation *1* in Lemma 3.5. Furthermore, it is not hard to check that

$$f(0) = \frac{2\sqrt{\pi}}{n\,\mathbf{\Gamma}\left(1 + \frac{2-n}{4n}\right)\mathbf{\Gamma}\left(\frac{3n-2}{4n}\right)}, \quad f'(0) = \frac{4\sqrt{\pi}}{n\,\mathbf{\Gamma}\left(\frac{2+n}{4n}\right)\mathbf{\Gamma}\left(\frac{n-2}{4n}\right)}$$

$$\widetilde{f}(0) = \frac{2\mathbf{\Gamma}\left(\frac{4m-1}{4m}\right)\mathbf{\Gamma}\left(\frac{2m+1}{4m}\right)}{\sqrt{\pi}}, \quad \widetilde{f}'(0) = -\frac{\mathbf{\Gamma}\left(\frac{2m-1}{4m}\right)\mathbf{\Gamma}\left(\frac{4m+1}{4m}\right)}{m\sqrt{\pi}},$$

where $\mathbf{\Gamma}$ is the classical Gamma function. Thus, it is straightforward to check that $\dfrac{f(0)}{f'(0)} = \dfrac{\widetilde{f}(0)}{\widetilde{f}'(0)}$.

As $f(0)$ and $\widetilde{f}(0)$ are positive, then there exists $\lambda > 0$ such that $f(r) = \lambda\widetilde{f}(r)$, which implies that f and $\widetilde{f}$ have got the same zeroes.

Then, it suffices to make a careful analysis of $\widetilde{f}$.

By deriving, we obtain

$$\frac{d\sigma}{dr} = -\frac{z^{m-3/2}}{(z^{2m} - 2rz^m + 1)^{3/2}} - 2\frac{d}{dz}\left(\frac{z^{m-1/2}}{(z^{2m} - 2rz^m + 1)^{3/2}}\right).$$

Hence, integrating by parts, we obtain

$$(21) \qquad \widetilde{f}'(r) = -\int_0^{+\infty} \frac{z^{m-3/2}}{(z^{2m} - 2rz^m + 1)^{3/2}} dz < 0, \ \forall r \in]-1,1[.$$

This implies that $\widetilde{f}$ has at most one zero.

From its definition, it is clear that $\widetilde{f}(r) > 0$, $\forall r \in]-1,0]$.

In order to compute the limit of $\widetilde{f}$ at 1, we use the Frobenius Method for the study of ordinary differential equations with singularities (see [1, §4.8]). Taking into account that $\widetilde{f}$ is a solution of Equation *1* in Lemma 3.5 and the above mentioned method, we deduce that

$$(22) \qquad \widetilde{f}(r) = a \log(1-r)\varphi_1(r) + b\varphi_2(r)$$

where $a, b \in \mathbb{R}$, $\varphi_1(1) \neq 0$ and φ_i, $i = 1, 2$, are analytic at $r = 1$.

As $\lim_{r\to 1} \widetilde{f}'(r) = -\infty$, we deduce $a \neq 0$, and so $\lim_{r\to 1} \widetilde{f}(r) = -\infty$. An intermediate value argument gives the existence of a unique zero r_θ of $\widetilde{f}$.

As $f(0) > 0$, then $d(r) > 0$, $\forall r \in]-1, r_\theta[$, and so $d(r) < 0$, $\forall r \in]r_\theta, 1[$. This concludes the proof of Statement *1*.

In order to prove *2*, observe that

$$(23) \qquad \lim_{r\to -1} \widetilde{f}(r) = \int_0^{+\infty} \frac{2mz^{\frac{2m-3}{2}}(z^m+1)}{(z^{2m} + 2z^m + 1)^{3/2}} dz \in \mathbb{R}^+,$$

Moreover, it is clear that $0 \leq \sqrt{-r - \cos(t)} \leq \sqrt{1 - \cos(t)}$, $t \in [\arccos(-r), \pi]$, then

$$h(r) \geq \frac{\sqrt{2}}{n} \int_{\arccos(-r)}^{\pi} \frac{dt}{\sqrt{1-\cos(t)}} = -\frac{2}{n} \log(\tan(\arccos(-r)/4)),$$

and so

$$(24) \qquad \lim_{r\to -1} h(r) \geq \lim_{r\to -1} \left(-\frac{2}{n} \log\left[\tan\left(\frac{\arccos(-r)}{4}\right)\right]\right) = +\infty.$$

Both (23) and (24) give Assertion *2*.

To obtain *3*, note that *3* in Lemma 3.5 yields that $d''(r'_\theta) < 0$ for each critical point $r'_\theta \in]-1, r_\theta[$ of d. Consequently, there exists only one critical point of d in $]-1, r_\theta[$ and it is a *maximum*. Obviously, $d(r'_\theta) = \text{Maximum}\{d(r) : r \in]-1, r_\theta[\}$.

Hence, it is clear that $\sharp\left[d^{-1}(\{x\})\right] \geq 2$, $\forall x \in]0, d(r'_\theta)[$. If $\sharp\left[d^{-1}(\{x\})\right] > 2$, for some $x \in]0, d(r'_\theta)[$, then it implies the existence of a local minimum of d in $]-1, r_\theta[$, which is absurd. This concludes the proof. $\square$

Definition 1. For each $\theta \in]0, \pi]$, we denote d_θ as the maximum of the opening function $d(r)$, $r \in]-1, r_\theta]$.

Remark 5. Since the function $d(n, r)$ is differentiable in $[1, 2]\times]-1, 1[$, then the function $\theta \longmapsto d_\theta$ is continuous in $]0, \pi]$.

In case $n = 2$ (i.e., $\theta = 0$), Theorem 2 implies that $0 < d(r) < 1$, $\forall r \in]-1, 1[$, and $Supremum\{d(r),\ r \in]-1, 1[\} = 1$, (see Remark 2). Hence, the function $\theta \longmapsto d_\theta$ extends continuously to $[0, \pi]$, defining $d_0 = 1$.

For $n \in [1, 2[$, it is straightforward to check from (17) that $\frac{\partial d}{\partial n} > 0$, i.e.,

$$\frac{\partial d}{\partial \theta} < 0.$$

This implies that the function $\theta \longmapsto d_\theta$ is decreasing in $]0, \pi]$, and in particular,

$$0 < d_\pi < d_\theta < 1, \quad \theta \in]0, \pi[.$$

3.6. Main Theorem.

We summarize the information in the preceding subsections (Lemmas 3.2, 3.3, 3.4, 3.4 and 3.5) in the the following:

Theorem 4. *Let $\theta \in]0, \pi]$ and $r \in]-1, 1[$.*

Consider the Weierstrass data given by (3) *and* (4)*, where $n = \frac{2\pi}{\theta+\pi}$ and B is given in* (10)*. Define the immersion X as in* (5)*. Then X satisfies:*

(i) *$X(M)$ is a properly immersed minimal disk with two boundary ends in $\mathbb{R}^3$.*

(ii) *$X(\partial(M)) = \Gamma = \bigcup_{i=0}^{2}(\ell_i^+ \cup \ell_i^-)$, where*

1. *The curves ℓ_1^+ and ℓ_1^- are half-lines contained in a plane $x_3 = 1/2$, they are symmetric with respect to the plane $x_1 = 0$, and the straight lines containing them meet at an angle θ.*
2. *The curve ℓ_2^+ (resp. ℓ_2^-) is the image under the symmetry with respect to the plane $x_3 = 0$ of ℓ_1^+ (resp. ℓ_1^-).*
3. *The curve ℓ_0^+ (resp. ℓ_0^-) is the vertical segment joining the end points of ℓ_1^+ and ℓ_2^+ (resp. ℓ_1^- and ℓ_2^-).*

(iii) *If $\theta \in]0, \pi[$, $X(M)$ is contained in the convex-hull $\mathcal{E}(\Gamma)$ of Γ. If $\theta = \pi$, $X(M)$ is contained in the intersection of the slab $-1/2 \leq x_3 \leq 1/2$ and one of the two halfspaces determined by the plane containing ℓ_i^+, ℓ_i^-, $i = 0, 1, 2$.*

(iv) *$X(M)$ is invariant under the symmetries with respect to the planes $x_1 = 0$ and $x_3 = 0$.*

(v) *If d is given as in (11), then this function of r and θ measures the* oriented *distance between ℓ_0^+ and ℓ_0^-. This means that*

1. $|d| = \text{dist}\,(\ell_0^+, \ell_0^-)$.
2. $d \leq 0$ *if and only if* $\ell_i^+ \cap \ell_i^- \neq \emptyset$, $i = 1, 2$, *and* $d > 0$ *otherwise.*
3. $d = 0$ *if and only if* $\ell_0^+ = \ell_0^-$.

(vi) *For each $\theta \in]0, \pi]$, there exists an only $r_\theta \in]0, 1[$ such that $d(r_\theta) = 0$. The function $d(r)$ is negative in $]r_\theta, 1[$ and positive in $] - 1, r_\theta[$. Furthermore, $\lim_{r \to -1} d(r) = 0$ and $d(r)$ has an unique critical point $r'_\theta \in] - 1, r_\theta[$. The number $d_\theta = d(r'_\theta)$ is the maximum of the opening function $d(r)$ on $] - 1, r_\theta]$.*

(vii) *X is an embedding if and only if $d > 0$. If $d = 0$ (i.e., $r = r_\theta$), then $X|_{M-\gamma^+}$ and $X|_{M-\gamma^-}$ are injective, where γ^+ and γ^- are the connected components of $\partial(M)$.*

The numbers $r \in] - 1, 1[$ and $\theta \in]0, \pi]$ are analytical parameters of our family of surfaces. From now on we will refer to $X_{\theta\, r}$ as the immersion arising in above theorem for the values r and θ. Analogously we indicated by $M_{\theta\, r}$ the disc with the corresponding complex structure.

Then we can describe our family of surfaces $\mathcal{M}$ as follows:

$$\mathcal{M} = \{X_{\theta\, r} : M_{\theta\, r} \to \mathbb{R}^3 \,/\, r \in] - 1, 1[,\ \theta \in]0, \pi]\}. \tag{25}$$

Note that θ has a clear geometrical meaning: the angle that ℓ_i^+ makes with ℓ_i^-, $i = 1, 2$. The meaning of parameter r concerns to the underlying complex structure of the surface.

Remark 6. Elementary geometrical arguments give that the complete orientable minimal surface without boundary

$$\widetilde{X}_{\theta\, r} : \widetilde{M}_{\theta\, r} \longrightarrow \mathbb{R}^3$$

obtained from $X_{\theta\, r}(M_{\theta\, r})$ by successive Schwarz reflections about straight lines is invariant under the vertical translation T by vector $(0, 0, 2)$.

The case $\frac{\theta}{\pi} \in \mathbb{Q}$ and $r = r_\theta$ is specially interesting. The immersion $\widetilde{X}_\theta = \widetilde{X}_{\theta\, r_\theta}$ is singly periodic and the induced immersion

$$Y_\theta : \widetilde{M}_{\theta\, r_\theta} / \langle T \rangle \longrightarrow \mathbb{R}^3 / \langle T \rangle$$

has four ends and finite total curvature. If we write $\frac{\pi}{\theta} = p/q$, p, $q \in \mathbb{N}$, $gcd(p,q) = 1$, then it is not hard to check that:

- If p is even the surface $\widetilde{M_{\theta}}_{r_\theta}$ is the two sheeted orientable covering of a nonorientable minimal surface properly immersed in $\mathbb{R}^3$. Moreover, Y_θ has four ends, its total curvature is $-8\pi(p+q)$ and $\widetilde{M_{\theta}}_{r_\theta}/\langle T\rangle$ has genus $2p-1$. A fundamental piece bounded by straight lines of the surface $\widetilde{X}_\theta(\widetilde{M_{\theta}}_{r_\theta})$, $\theta = \pi/2$, is illustrated in Figure 13.

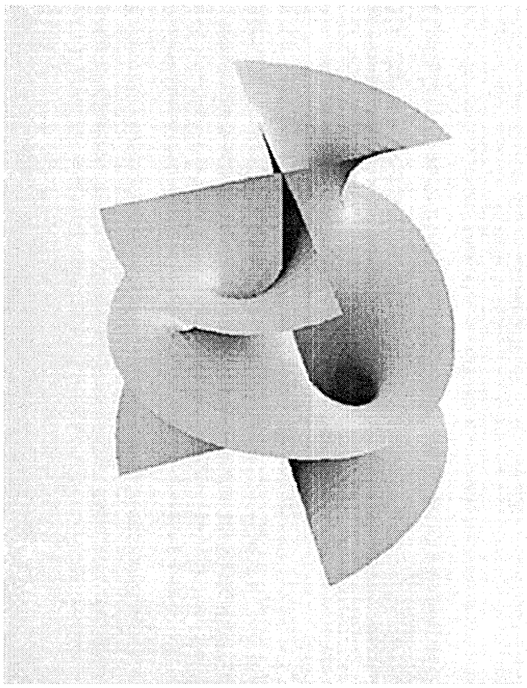

Figure 13: A fundamental piece of the surface $\widetilde{X}_\theta(\widetilde{M_{\theta}}_{r_\theta})$, for $\theta = \pi/2$, contained in the slab $-1/2 \leq x_3 \leq 3/2$.

- If p is odd the surface $X_{\theta\, r_\theta}(M_{\theta\, r_\theta})$ is invariant under a translation T' by vector $(0,0,1)$, and the induced immersion

$$Y'_\theta : \widetilde{M_{\theta}}_{r\theta}/\langle T'\rangle \longrightarrow \mathbb{R}^3/\langle T'\rangle$$

has two ends. Moreover, if q is even (resp. q is odd), Y'_θ has total curvature $-8\pi(p+q)$ (resp. $-4\pi(p+q)$) and $\widetilde{M_{\theta}}_{r_\theta}/\langle T'\rangle$ has genus

$2p$ (resp. p). Figure 14 shows a fundamental piece of the surface $\widetilde{X}_\theta(\widetilde{M}_{\theta\, r_\theta})$, $\theta = \pi/3$.

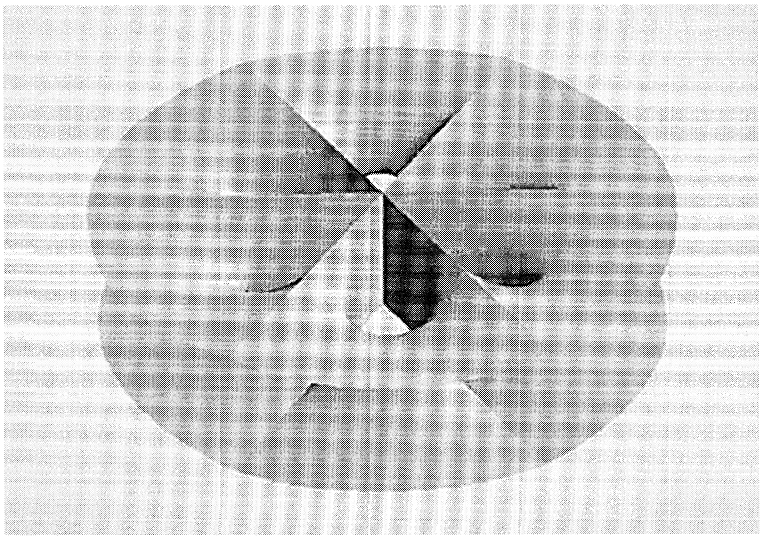

Figure 14: A fundamental piece of the surface $\widetilde{X}_\theta(\widetilde{M}_{\theta\, r_\theta})$, for $\theta = \pi/3$, contained in the slab $-1/2 \leq x_3 \leq 1/2$.

- The ends of $Y_{\theta\, r_\theta}$ are embedded if and only if π/θ belongs to $\mathbb{N}$.

3.7. Stability and limits.

Finally, a few words about the stability of the surfaces $X_{\theta\, r}(M_{\theta\, r})$, $r \in]-1, r_\theta]$.

From (16), it is straightforward to check that $f(\frac{n\pi}{2}) > 0$. Since $\lim_{r\to 1} f(r) = -\infty$ (see the proof of Lemma 3.5), then $r_\theta \in]\frac{n\pi}{2}, 1[$. Therefore, for $r = r_\theta$, the spherical image of the Gauss map of X contains a hemisphere, and so this immersion is *unstable.*

However, and in the sense explained in the following proposition, the limit surface obtained as $r \to -1$ is *stable.*

Proposition 1. *As set of points in $\mathbb{R}^3$, the surfaces $X_{\theta\, r}(M_{\theta\, r})$ converge on compact subsets of $\mathbb{R}^3$, as $r \to -1$, to the union of the segment*

$[(0,0,-1/2),(0,0,1/2)]$ *and the two parallel planar sectors in* π_1 *and* π_2 *determined by the planes* $\cos(\frac{\theta}{2})x_1 - \sin(\frac{\theta}{2})x_2 = 0$ *and* $\cos(\frac{\theta}{2})x_1 + \sin(\frac{\theta}{2})x_2 = 0$ *and contained in the half space* $x_2 \geq 0$.

Furthermore, for each $\epsilon \in]0,1/2[$, *the surfaces* $X_{\theta\, r}(M_{\theta\, r}) \cap \{|x_3| \leq 1/2-\epsilon\}$ *converge on compact subsets of* $\mathbb{R}^3$, *as* $r \to -1$, *to the segment* $[(0,0,-1/2+\epsilon),(0,0,1/2-\epsilon)]$.

An example of these limit surfaces has been illustrated in Figure 6.

Proof. For convenience, we write

$$g_r = iz, \quad \eta_r = B(r)\frac{dz}{z^2\sqrt{z^n + z^{-n} + 2r}},$$

and observe

$$2X_{\theta\, r}(P) = \mathrm{Re}\left(\int_{P_0}^{P} ((1-g_r^2)\eta_r, i(1+g_r^2)\eta_r, 2g_r\eta_r)\right),$$

for $r \in]-1,1[$. Recall that $z(P_0) = 1$.

Note that $z(M_{\theta\, r}) = z(M_{\theta\, r'})$, r, $r' \in]-1, r_\theta]$, and label $\Omega = z(M_{\theta\, r})$, $r \in]-1, r_\theta]$.

If we write $A = \mathbb{D} \cap \Omega$, then $z|_{z^{-1}(A)}$ is injective, and so we identify the sets $z^{-1}(A)$ and A.

We deal with the convergence of the family of minimal surfaces $\{X_{\theta\, r}(\overline{A})\}$, as $r \to -1$.

Introduce the change of variable $z = B(r)^{\frac{2}{2-n}}\, x$. In the x-plane, $\overline{A}$ corresponds to the set $B(r)^{\frac{2}{n-2}} \cdot \overline{A}$, the Weierstrass data become

$$g_r = iB(r)^{\frac{2}{2-n}}\, x, \quad \eta_r = -i\frac{dx}{x^{2-n/2}\sqrt{B(r)^{\frac{4n}{2-n}}x^{2n} + 2rB(r)^{\frac{2n}{2-n}}x^n + 1}},$$

and the initial condition is $P_0 \equiv x(P_0) = B(r)^{\frac{2}{n-2}}$.

Taking into account that $\lim_{r\to -1} B(r) = 0$ (see the proof of Lemma 3.5), then, as $r \to -1$, the set $B(r)^{\frac{2}{n-2}}\, \overline{A}$ converges to Ω, and the Weierstrass data to

$$g_{-1} = 0, \quad \eta_{-1} = -i\frac{dx}{x^{2-n/2}}.$$

Moreover, note that $\lim_{r\to -1} x(P_0) = \infty$.

Hence, if r_j is a sequence converging to -1 as $j \to \infty$, and $P_j \in B(r_j)^{\frac{2}{n-2}} \cdot \overline{A}$, then the sequence $X_{\theta\, r_j}(P_j)$ converges in $\mathbb{R}^3$ if and only if the sequence P_j converges in $\Omega \cup \{\infty\}$.

If P_j converges to a finite point (in the x-plane), then $z(P_j) = B(r_j)^{\frac{2}{2-n}} P_j$ converges to 0 in the z-plane. Thus, it is obvious that $X_{r_j}(P_j)$ converges to a point in π_1 (lying in the sector described in the statement of the lemma).

If P_j converges to ∞ (in the x-plane), an analogous argument gives that $X_{\theta\, r_j}(P_j)$ converges to a point of the x_3-axis whose third coordinate lies in $[0, 1/2]$. Furthermore, as $d(r) \to 0$ as $r \to -1$ (see Lemma 3.5), then, for any $q \in [(0,0,0),(0,0,1/2)]$, we can find a sequence P_j as above converging to q.

By using the symmetry S_h, we conclude the proof of the first part of the proposition.

For the second part, take $\epsilon \in]0, 1/2[$. The third coordinate function of $X_{\theta r}$ is continuous at $(0,-1)$ and $(\infty, -1)$, as function of (z, r). Then, we can find $\delta_\epsilon > 0$ and a compact set $K_\epsilon \subset \Omega$, such that

$$z\left(X_{\theta\, r}^{-1}\left(\{|x_3| \leq 1/2 - \epsilon\}\right)\right) \subset K_\epsilon, \quad \forall r \in]-1, -1+\delta_\epsilon[.$$

Hence, if r_j is a sequence converging to -1 as $j \to \infty$, and in the x-plane, $P_j \in B(r_j)^{\frac{2}{n-2}} \cdot (\overline{A} \cap K_\epsilon)$, then P_j converges to ∞, and so, as we have mentioned above, the sequence $X_{\theta\, r_j}(P_j)$ converges to a point of $[(0,0,0),(0,0,1/2)]$.

Taking the symmetry S_h into account once again, we conclude the proof. □

4. Properly immersed minimal surfaces in a wedge of a slab.

By the strong halfspace theorem [3], a properly immersed minimal surface in a wedge of a slab has non empty boundary. In this section we prove that such a surface satisfies the *convex hull property.* Furthermore, we obtain some non-existence theorems for properly immersed minimal surfaces with planar boundaries.

First, we introduce some notation.

Let L be the segment $\{(0,0,t) \;:\; t \in]-1/2, 1/2[\}$. Label

$$W = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \;:\; -\frac{1}{2} \leq x_3 \leq \frac{1}{2}\right\}.$$

For $\theta \in]0, 2\pi[$, we write

$$W_\theta = \{(x_1, x_2, x_3) \in W - L \;:\; \arg((x_1, x_2)) \in [0, \theta]\} \cup L.$$

Define also

$$\Sigma_\theta = \{(x_1, x_2, x_3) \in W - L \ : \ \arg((x_1, x_2)) = \theta\} \cup L,$$

where $\theta \in [0, 2\pi[$.

Recall that the surfaces $X_{\theta\, r}(M_{\theta\, r})$, $r \in]-1, r_\theta[$, constructed in Section 3 are contained in a wedge of angle $\theta \in]0, \pi]$ of a horizontal slab. Furthermore the thickness of this slab is 1. Thus, after a rigid motion (unique if $\theta \neq \pi$), we can assume that $\partial(X_{\theta\, r}(M_{\theta\, r})) \subset \partial(W_\theta)$. In the case of $\theta = \pi$ we choose the rigid motion in such a way that $X_{\pi\, r}(M_{\pi\, r})$ is symmetric with respect to the plane $x_1 = 0$.

For the sake of simplicity, we label $S_{\theta\, r} = X_{\theta\, r}(M_{\theta\, r})$. Moreover, we write S_θ instead of $S_{\theta\, r_\theta}$.

We start with the following lemma:

Lemma 7. *Let M be a connected properly immersed minimal surface in the wedge W_{θ_1}, $\theta_1 < 2\pi$, and suppose that $\partial(M) \subset \Sigma_0$. Then M is a planar region in Σ_0.*

Proof. Up to a homothety and a translation in the direction of the x_1-axis, we can assume that:

- The distance from $\partial(M)$ to the planes $x_3 = 1/2$ and $x_3 = -1/2$ is positive. As the immersion is proper, the distance from M to the planes $x_3 = 1/2$ and $x_3 = -1/2$ is also positive (see Theorem 2).
- There exists $\delta > 0$ such that $x_1 \geq \delta$, $\forall (x_1, x_2, x_3) \in \partial(M)$.

First suppose that $\theta_1 \leq \pi$.

An application of Theorem 2 gives that $\Sigma_\pi \cap M = \emptyset$. Then, consider

$$\theta_0 = \text{Infimum}\,\{\theta \in [0, \pi] \ : \ \Sigma_\theta \cap M = \emptyset\}\,.$$

The theorem holds if and only if $\theta_0 = 0$.

We proceed by contradiction, and suppose $\theta_0 > 0$. Note that Theorem 2 gives $M \subset W_{\theta_0}$.

From Definition 1, it is clear that

$$\text{dist}(L, \partial(S_{\theta_0\, r})) \leq \frac{d_{\theta_0}}{2 \sin\left(\frac{\theta_0}{2}\right)}, \quad r \in]-1, r_{\theta_0}[.$$

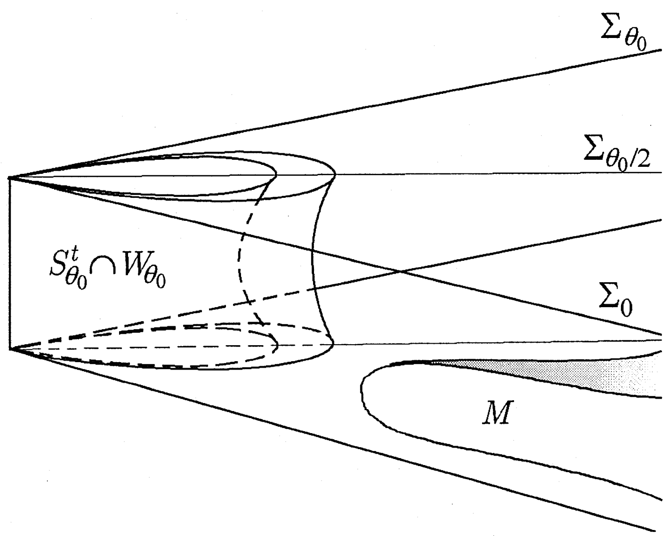

Figure 15: The surfaces $S^t_{\theta_0} \cap W_{\theta_0}$ and M.

Up to a translation in the direction of the x_1-axis, we can assume

$$\delta > \frac{d_{\theta_0}}{2\sin\left(\frac{\theta_0}{2}\right)}, \tag{26}$$

and so $\partial(S_{\theta_0 r}) \cap M = \emptyset$, $\forall r \in]-1, r_{\theta_0}]$. In the following we prove that, in fact, $S_{\theta_0 r} \cap M = \emptyset$, $\forall r \in]-1, r_{\theta_0}]$. Otherwise, we can find $r' \in]-1, r_{\theta_0}]$ such that $S_{\theta_0 r'} \cap M \neq \emptyset$. On the other hand, taking into account that the surfaces are properly immersed, we use Proposition 3.7 to infer the existence of $s \in]-1, r_\theta]$ such that $S_{\theta_0 r} \cap M = \emptyset$, $\forall r \in]-1, s]$. Let $r'' = \text{Infimum}\{r \in]-1, r_{\theta_0}] : S_{\theta_0 r} \cap M \neq \emptyset\}$. Since $S_{\theta_0 r}$ and M are properly immersed and do not have any contact at infinity, $\forall r \in]-1, r_{\theta_0}]$, then $S_{\theta_0 r''} \cap M \neq \emptyset$ and, obviously, $S_{\theta_0 r} \cap M = \emptyset$, $\forall r \in]-1, r''[$. Moreover, the above arguments imply that:

$$(S_{\theta_0 r''} \cap M) \cap (\partial(S_{\theta_0 r''}) \cup \partial(M)) = \emptyset.$$

Hence, Theorem 2 leads to a contradiction.

In particular, $S_{\theta_0} \cap M = \emptyset$.

Denote by $S^t_{\theta_0}$ the homothetic shrinking of S_{θ_0} by t, $t \geq 1$. It is clear that $S^t_{\theta_0}$ and M do not contact at infinity. Taking into account that S_{θ_0} lies in W_{θ_0} and $x_3\left((\partial(S_{\theta_0}) - L) \cap (\Sigma_0 \cup \Sigma_\pi)\right) = \{-1/2, 1/2\}$, it follows that $\partial(S^t_{\theta_0} \cap \overline{W_{\theta_0}}) \subset \{|x_3| = 1/2\} \cup L$, and so $\left(S^t_{\theta_0} \cap M\right) \cap \left(\partial(S^t_{\theta_0}) \cup \partial(M)\right) = \emptyset$, $t \geq 1$. Reasoning as above, Theorem 2 implies that $S^t_{\theta_0} \cap M = \emptyset$, $t \geq 1$.

For any $t > 1$, the surface $S^t_{\theta_0} \cap W_{\theta_0}$ splits W_{θ_0} into two connected components, one of them bounded: $\mathrm{Int}(S^t_{\theta_0})$, and the other one unbounded: $\mathrm{Ext}(S^t_{\theta_0})$. For $t = 1$, the surface S_{θ_0} also divides W_{θ_0} into two connected components, that we continue calling $\mathrm{Int}(S_{\theta_0})$ and $\mathrm{Ext}(S_{\theta_0})$.

Taking into account that M is connected, $\partial(M) \subset \mathrm{Ext}(S^t_{\theta_0})$ and $S^t_{\theta_0} \cap M = \emptyset$, then it is easy to check that $M \subset \mathrm{Ext}(S^t_{\theta_0})$, $\forall t \in [1, +\infty[$. Therefore, $M \subset \Omega = \bigcap_{t \geq 1} \mathrm{Ext}(S^t_{\theta_0})$. Taking into account that $(S_{\theta_0} \cap \Sigma_{\theta_0/2}) - L$ is a connected arc diverging to both ends of S_{θ_0}, it is not hard to deduce that $\Sigma_{\theta_0/2} \cap \Omega = \emptyset$. Hence,

$$M \cap \Sigma_{\theta_0/2} = \emptyset.$$

Since M is connected, this fact contradicts the choice of θ_0.

Finally, suppose that $\theta_1 > \pi$. Let us see that in fact $M \subset W_\pi$. Otherwise, $M_1 = M \cap \overline{W_{\theta_1} - W_\pi}$ is a surface contained in a wegde of angle $\theta_1 - \pi < \pi$ whose boundary lies in Σ_π. So, by using the preceding reasoning, M_1 is a piece of a plane, which is absurd. This proves the lemma. □

To state the next algebraic lemma, we will need the following notation.

For any $\vec{a} \in \mathbb{S}^2$ and $y \in \mathbb{R}^3$, define

$$H^y_{\vec{a}} = \left\{ x \in \mathbb{R}^3 \; : \; \langle \vec{a}, (x - y) \rangle \geq 0 \right\}.$$

As usual, we identify $\mathbb{R}^2 = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \; : \; x_3 = 0 \right\}$ and $\mathbb{S}^1 = \mathbb{S}^2 \cap \mathbb{R}^2$.

Lemma 8. *Given $\vec{a} \in \mathbb{S}^2 - \{(0, 0, \pm 1)\}$ and $y \in \mathbb{R}^3$, there exists $\vec{a}_0 \in \mathbb{S}^1$ and $z \in \mathbb{R}^2$ such that:*

- $\left(H^z_{\vec{a}_0} \cap W \right) \subset \left(H^y_{\vec{a}} \cap W \right)$.
- *The map $x \longmapsto \langle \vec{a}, (x - y) \rangle$ is bounded in $\left(H^y_{\vec{a}} - H^z_{\vec{a}_0} \right) \cap W$.*

The proof is merely an exercise. We will omit it.

We can now prove the main result of this section.

Theorem 5. *Any connected properly immersed minimal surface in a slab wedge W_θ, $\theta \in]0, \pi[$, lies in the convex hull of its boundary.*

Proof. Let M be a minimal surface satisfying the hypotheses of the theorem.

If M is a piece of a plane the result obviously holds. Suppose that M is not flat.

Consider $\vec{a} \in \mathbb{S}^2$ and $y \in \mathbb{R}^3$ such that $\partial(M)$ lies in $H^y_{-\vec{a}}$. We have to prove that $M \subset H^y_{-\vec{a}}$ too.

We proceed by contradiction, and suppose that $M \cap (H^y_{\vec{a}} - \partial(H^y_{\vec{a}})) \neq \emptyset$. Let M' be a connected component of $M \cap H^y_{\vec{a}}$. If $\vec{a} = (0,0,\pm 1)$, and taking into account that $\partial(M') \subset \partial(H^y_{\vec{a}})$, it is easy to deduce from Theorem 2 that M' (and so M) is flat, which is absurd. In what follows, we assume that $\vec{a}$ is not vertical. Consider $\vec{a}_0 \in \mathbb{S}^1$ and $z \in \mathbb{R}^2$ given by Lemma 4 when applied to $(\vec{a}, y)$. If $M' \subset H^y_{\vec{a}} - H^z_{\vec{a}_0}$, then, from Theorem 2 we deduce, as above, that M' is a planar region, which is absurd. Hence, $M' \cap H^z_{\vec{a}_0} \neq \emptyset$.

Let M'' be a connected component of $M' \cap H^z_{\vec{a}_0}$, and take $\vec{b}_0 \in \mathbb{S}^1$ orthogonal to $\vec{a}_0$. Since $\partial(M'') \subset \left(\partial(H^z_{\vec{a}_0}) \cap W_\theta\right)$ and $\theta < \pi$, then the set $\{\langle x, \vec{b}_0\rangle : x \in \partial(M'')\}$ is bounded either from above or below.

Up to a rigid motion, we can suppose that $\vec{a}_0 = (0,1,0)$, $z = (0,0,0)$ and $\partial(M'') \subset \Sigma_0$. Thus, we can apply Lemma 4 to deduce that M'' is a planar region. This is contrary to our assumptions, and concludes the proof. □

Corollary 1. *Let M be a connected properly immersed minimal surface in the halfslab W_π. Assume $\partial(M) \subset W_\theta$, $\theta \in]0,\pi[$. Then M lies in the convex hull of $\partial(M)$.*

Proof. It suffices to prove that $M \subset W_\theta$. Indeed, if $M \not\subset W_\theta$, take a connected component M'' of $M - W_\theta$. Since $\partial(M'') \subset \Sigma_\theta$, then, up to a rigid motion, we can apply Lemma 4 to infer that M'' is a planar region in Σ_θ, which is absurd. □

Corollary 2. *Let M be a connected properly immersed minimal surface in the halfslab W_{θ_1}, $\theta_1 \in]0,2\pi[$. Assume $\partial(M) \subset W_\theta$, $\theta \in]0,\pi[$. Then M lies in the convex hull of $\partial(M)$.*

Proof. It suffices to prove that $M \subset W_\pi$, and use Corollary 4. Indeed, if $M \not\subset W_\pi$, take a connected component M'' of $M - W_\pi$. Since $\partial(M'') \subset \Sigma_\pi$ and M'' lies in a wedge of angle $\theta_1 - \pi < \pi$, then, up to a rigid motion, we can apply Lemma 4 to infer that M'' is a planar region in Σ_π, which is absurd. □

Theorem 4 can be extended to the case of $\theta = 0$. For any $d \in]0,+\infty[$, define

$$C_d = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \; : \; 0 \leq x_2 \leq d, \; -\frac{1}{2} \leq x_3 \leq \frac{1}{2} \right\}.$$

Theorem 6. *Any connected properly immersed minimal surface in C_d, $d \in]0, +\infty[$, lies in the convex hull of its boundary.*

The proof is as in Theorem 4.

Remark 7. Theorem 4 and Corollary 4 fail in the case of $\theta = \pi$. A simple counterexample is the surface $S_{\pi\, r}$, for any $r \in] -1, r_\pi[$. A suitable piece of a helicoid is another counterexample.

The following theorem represents another interesting application of the above techniques.

Theorem 7. *Let M be a connected properly immersed non flat minimal surface in a wedge W_θ, $\theta \in]0, \pi]$. Suppose that $\partial(M) \subset \Sigma_0 \cup \Sigma_\theta$. Then,*

$$\operatorname{dist}(L, \partial(M)) \leq \frac{d_\theta}{2\sin\left(\frac{\theta}{2}\right)},$$

where d_θ is given in Definition 1.

Proof. First observe that the sets $\partial(M) \cap \Sigma_0$ and $\partial(M) \cap \Sigma_\theta$ are not empty. Otherwise, Lemma 4 says that M is flat, which is absurd.

Suppose that $\operatorname{dist}(L, \partial(M)) > \frac{d_\theta}{2\sin(\frac{\theta}{2})}$. After a suitable homothetic shrinking of M, we can suppose that $\operatorname{dist}(L, \partial(M)) > \frac{d_\theta}{2\sin(\frac{\theta}{2})}$ and $\operatorname{dist}(\partial(M), \{|x_3| = 1/2\}) > 0$. ¿From Definition 1, it follows that $\operatorname{dist}(L, > \partial(S_{\theta\, r})) \leq \frac{d_\theta}{2\sin(>\frac{\theta}{2})}$, and so we can use the surfaces $S_{\theta\, r}$, $r \in] -1, r_\theta[$, as barriers for the maximum principle application. So, reasoning as in the proof of Lemma 4, we obtain $M \cap S_\theta = \emptyset$.

Consider the homothetic shrinking S_θ^t of S_θ by t, $t \geq 1$. Following the proof of Lemma 4, we use these surfaces as barriers to deduce that $M \cap \Sigma_{\theta/2} = \emptyset$, which contradicts the fact that M is connected. □

The surface $S_{\theta\, r'_\theta}$, corresponding to the maximum d_θ of the opening function, reaches the equality in Theorem 4. Hence, this result is sharp.

Finally, we prove a non existence theorem for properly immersed non flat minimal surfaces with planar boundary. As stated in the introduction, this theorem is a generalization of a well-known result of Nitsche [12] in the non compact case (see also [14]).

For $d \in]0, +\infty[$, label

$$F_d^0 = \left\{ (x_1, x_2, x_3) \in C_d \, : \, x_3 = -\frac{1}{2} \right\},$$
$$F_d^1 = \left\{ (x_1, x_2, x_3) \in C_d \, : \, x_3 = \frac{1}{2} \right\}.$$

Moreover, for any $d \in]0, 1[$, and $t \in \mathbb{R}$, we denote S_d^t as the Jenkins-Serrin graph over the rectangle $\{t\} \times [0, d] \times [-\frac{1}{2}, \frac{1}{2}]$ with boundary values:

- $-\infty$ on $\{t\} \times]0, d[\times \{-\frac{1}{2}, \frac{1}{2}\}$,
- 0 on $\{t\} \times \{0, d\} \times [-\frac{1}{2}, \frac{1}{2}]$.

See Remarks 2 and 5.

Theorem 8. *Let M be a connected properly immersed minimal surface in $\mathbb{R}^3$ satisfying:*

1. $M \subset C_d$ *and* $\partial(M) \subset (F_d^0 \cup F_d^1)$, *where* $0 < d < 1$,
2. $x_1 > 0$, $\forall (x_1, x_2, x_3) \in \partial(M)$.

Then M is a planar region in $F_d^0 \cup F_d^1$.

Proof.

Suppose M is not flat.

From Theorem 4 we get $x_1 > 0$, $\forall (x_1, x_2, x_3) \in M$. Thus, $M \cap S_d^t = \emptyset$, $t \leq 0$. If it is necessary, we take d', $1 > d' > d$, and translate M in the direction of the x_2-axis, in such a way that the distance from M to the planes $x_2 = 0$ and $x_2 = d'$ is positive. In particular, $M \cap S_{d'}^t$ does not meet $\partial(M) \cup \partial(S_{d'}^t)$, $t > 0$.

On the other hand, as the set $C_{d'} - \left(\bigcup_{t \in \mathbb{R}} S_{d'}^t \right)$ is included in $\{x_3 = -\frac{1}{2}\} \cup \{x_3 = \frac{1}{2}\}$ and M is not flat, then there is $t' > 0$ large enough such that $M \cap S_{d'}^{t'} \neq \emptyset$.

Label $t_0 = \text{Infimum}\{t \in \mathbb{R} \, : \, M \cap S_{d'}^t \neq \emptyset\}$. From the above arguments, $t_0 \in [0, +\infty[$. As M and $S_{d'}^t$, $t \in \mathbb{R}$, are properly immersed, then $S_{d'}^{t_0} \cap M \neq \emptyset$. Therefore, $S_{d'}^{t_0}$ and M have an interior contact point, and so Theorem 2 leads to a contradiction. □

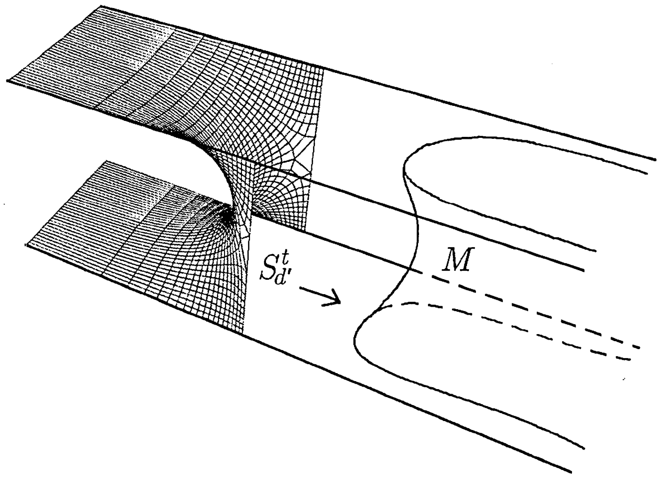

Figure 16: The surfaces $S_{d'}^t$, $t \in \mathbb{R}$, and M.

References.

[1] E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, Mc Graw-Hill Book Company, New York, 1955.

[2] G. Darboux, *Tèorie gènèrale des surfaces*, **1**, Chelsea Publishing Co., New York, 1972.

[3] D. Hoffman and W.H. Meeks, III, *The strong halfspace theorem for minimal surfaces*, Invet. Math., **101** (1990), 373–377.

[4] H. Jenkins and J. Serrin, *Variational problems of minimal surface type. II. Boundary value problems for the minimal surface equation*, Arch. Rat. Mech. Anal., **21** (1966), 321–342.

[5] H. Karcher, *Construction of minimal surfaces*, Surveys in Geometry 1989/90, University of Tokyo, 1989. Also: Vorlesungsreihe Nr. 1, SFB 256, Bonn, 1989.

[6] F.J. Lopez, R. Lopez and R. Souam, *Maximal surfaces of Riemann type in Lorentz-Minkowski space* $\mathbb{L}^3$, Michigan Math. J., **47**(**3**) (2000), 469–497.

[7] F.J. Lopez and F. Martin, *Uniqueness of properly embedded minimal surfaces bounded by straight lines*, J. Austral. Math. Soc. Ser. A, **69(3)** (2000), 362–402.

[8] F.J. Lopez, M. Ritore and F. Wei, *A characterization of Riemann's minimal surfaces*, J. Differential Geom., **47(2)** (1997), 376–397.

[9] F.J. Lopez and F. Wei, *Properly immersed minimal discs bounded by straight lines*, Math. Ann., **318(4)** (2000), 667–706.

[10] W.H. Meeks, III and H. Rosenberg, *The global theory of doubly periodic minimal surfaces*, Invent. Math., **97** (1989), 351–379.

[11] W. H. Meeks III and H. Rosenberg, *The geometry and conformal structure of properly embedded minimal surfaces of finite topology in* $\mathbb{R}^3$, Invent. Math., **114** (1993), 625–639.

[12] J.C.C. Nitsche, *A supplement to the condition of J. Douglas*, Rend. Circ. Matem. Palermo, Serie II, Tomo XIII, (1964).

[13] R. Osserman, *A survey of minimal surfaces*, Dover Publications, New York, second edition, 1986.

[14] W. Rossman, *Minimal surfaces with planar boundary curves*, Kyushu J. Math., **52** (1998), 209–225.

[15] R. Schoen, *Uniqueness, symmetry and embeddedness of minimal surfaces*, J. Differential Geom., **18** (1983), 791–809.

Departamento de Geometría y Topología
Universidad de Granada
18071 Granada, Spain
E-mail address: fjlopez@goliat.ugr.es

Departamento de Geometría y Topología
Universidad de Granada
18071 Granada, Spain
E-mail address: fmartin@goliat.ugr.es

Received December 1, 1998.