COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 9, Number 4, 657-682, 2001

Energy Minimizing Maps to Piecewise Uniformly Regular Lipschitz Manifolds

CHANGYOU WANG

We prove the optimal partial regularity of energy-minimizing maps into polyhedral and certain other approximately polyhedral manifolds. We also estimate the size of preimages of points in the $(k-2)$ skeleton of a polyhedral manifold.

1. Introduction.

For a bounded smooth $\Omega \subset R^m$ and a closed $X \subset R^K$, we define

$$H^1(\Omega, X) = \{u \in H^1(\Omega, R^K) | u(x) \in X \text{ for a.e. } x \in \Omega\}$$

of maps $u : \Omega \to X$ with energy $E(u) = \int_\Omega |Du|^2 \, dx < \infty$. A map $u \in H^1(\Omega, X)$ is *energy minimizing* if

$$E(u) \leq E(v), \tag{1.1}$$

for every $v \in H^1(\Omega, X)$, with $v|_{\partial\Omega} = u|_{\partial\Omega}$ in the sense of trace. Whenever a given Dirichlet boundary data $g : \partial\Omega \to X$ admits an extension in $H^1(\Omega, X)$, there exists an energy minimizing extension. It is a very interesting question to ask whether such a minimizing map is regular or at least regular off a small closed set. When X is a smooth compact Riemannian manifold without boundary, the problem has been well studied by many people. It was first proven by Schoen-Uhlenbeck ([SU], [SU1]) (see also Giaquinta-Giusti [GG]) that minimizing maps are smooth in Ω except a closed subset whose Hausdorff codimension is at least 3. Later, their results were generalized to p-energy minimizing maps by Hardt-Lin [HL], Fuchs [Fm], and Luckhaus [Ls] for $1 < p < \infty$. When X is an Alexander space, which has nonpositive curvature, it was first proven by Gromov-Schoen [GS] and then by Korevaar-Schoen [KS], [KS1], Jost [J1], [J2], and Serbinowski [St1] that any minimizing map is Lipschitz continuous in Ω and continuous up to the boundary $\partial\Omega$ if the boundary data are Lipschitz continuous. In the thesis [St2], Serbinowski also showed a small energy regularity theorem in case that X has curvature bounded above. Our main theorem, stated below, allows the possibility of X having infinite curvature. When X is a round cone in R^K, it has been treated by [Lf], [HL1]. As a matter of fact, the theory developed by Jost [J1] also allows the domain to be certain singular spaces (cf. also [C], [Lf1] for related results).

In this note, we will consider a special class of Lipschitz submanifolds $X \subset R^K$, which is called *piecewise uniformly regular Lipschitz manifolds* which include both C^1 and polyhedral submanifolds (but not varieties with cusps) and which will be defined below. Our main result is the following:

Theorem I. *Let $X \subset R^K$ be a compact k-dimensional piecewise uniformly regular Lipschitz manifold and $u \in H^1(\Omega, X)$ be an energy minimizing map with $u|_{\partial\Omega} = g|_{\partial\Omega}$ where $g : \Omega \to X$ is a given Lipschitz continuous map. Then*

I1. *There exists a closed subset $\Sigma \subset \Omega$, with $\dim_H \Sigma \leq m - 3$, such that $u \in C^\alpha(\bar{\Omega} \setminus \Sigma, N)$ for some $0 < \alpha < 1$. Here $\dim_H$ denotes the Hausdorff dimension of a set in R^m, and $\bar{\Omega} = \Omega \cup \partial\Omega$. For $m = 3$, Σ is discrete.*

I2. *If, in addition, $X \subset R^{k+1}$ is a k-dimensional polyhedron. Then for any $p \in X_{k-2}$, either $u \equiv p$ on Ω or $\dim_H(u^{-1}(p)) \leq m - 1$. Here $u^{-1}(p) \equiv \Sigma \cup (u|_{\Omega\setminus\Sigma})^{-1}(p)$.*

Remark II. In fact, it follows from the proof of I2 (see §4 below) that if $p \in X_{k-2}$ has the property that there is no minimizing geodesics in X passing through p, then either $u \equiv p$ or $\dim_H(u^{-1}(p)) \leq m-2$. In particular, if $X \subset R^3$ is a 2-dimensional convex polyhedron such that X has infinite positive curvature at each $a \in X_0$ (i.e., the enclosed angle of X at a is less than 2π), then we actually obtain that either $u \equiv a$ or $\dim_H(u^{-1}(a)) \leq m - 2$.

Now we define a piecewise uniformly regular Lipschitz manifold. We will denote the <u>cone</u> over a $Y \subset S^{K-1}$ by

$$C(Y) = \{\lambda x | \lambda > 0, x \in Y\}.$$

The *tangent cone* of a subset $X \subset R^K$ at a point a is

$$T_a X = C\left(\cap_{\epsilon>0} \text{Clos}\left\{\frac{x-a}{|x-a|} : 0 < |x-a| < \epsilon, x \in X\right\}\right).$$

A k-dimensional piecewise uniformly regular Lipschitz manifold $X \subset R^K$ is roughly a C^1 triangulated, uniformly asymptotically conical, Lipschitz submanifold. For $k = 1$, X is a piecewise C^1 Jordan curve. For $k \geq 2$, X being a k-dimensional Lipschitz submanifold of R^K means that X is a closed set which is locally the graph of a R^{K-k} valued Lipschitz function defined on a domain in R^k. Second the triangulation of X is assumed to be a bilipschitz map from X to the support of a simplicial complex so that, for the induced skeleta, $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots X_{k-1} \subset X_k = X$, each difference $X_l \setminus X_{l-1}$, for $l = 0, \cdots, k$, is an open l-dimensional C^1 submanifold. Third we assume that, for each $l \in \{0, \cdots, k\}$ and $a \in X_l \setminus X_{l-1}$, there exists $(k-l-1)$-dimensional piecewise uniformly regular Lipschitz submanifold Y_a of $S^{K-1} \cap (T_a X_l)^\perp$ (inductively defined) such that the tangent cone

$$T_a X = T_a(X_l \setminus X_{l-1}) \times C(Y_a) (\equiv R^l \times C(Y_a)). \tag{1.2}$$

Moreover, for each $a \in X$, there exist neighbourhoods U_a ($\subset X$) of a, V_a ($\subset R^K$) of 0, and a Lipschitz map $\Psi_a : T_a X \cap V_a \to X \cap U_a$ such that $\Psi_a(0) = a$ and for some $\theta_a \in (0,1)$,

$$\lim_{r \downarrow 0} \sup_{b \in B(a,r) \cap (X_i \setminus X_{i-1})} \mathrm{Lip}(\Psi_b|_{T_b X \cap B(0, r_{b,a})}) = 1, \tag{1.3}$$

for all $i \in \{0, \cdots, k\}$, where $r_{b,a} = \theta_a \min\{r, \mathrm{dist}(b, X_{i-1})\}$. Finally, if $\{b_n\} \subset X$ converges to $a \in X$, then there exists a l-dimensional piecewise uniformly regular Lipschitz submanifold Z_a of S^{K-1} with $-1 \leq l \leq k-1$, which may also depend on $\{b_n\}$, and bilipschitz maps $T_{b_n,a} : T_{b_n} X \to R^{k-l-1} \times C(Z_a)$ such that $T_{b_n,a}(0) = 0$ and

$$\lim_{b_n \to a} \max\{1 + \|T_{b_n,a} - Id\|, \mathrm{Lip}(T_{b_n,a}), \mathrm{Lip}(T_{b_n,a}^{-1})\} = 1. \tag{1.4}$$

Here $\|T_{b_n,a} - Id\| = \sup_{0 \neq v \in T_{b_n} X} \frac{|T_{b_n,a}(v) - v|}{|v|}$.

It is easy to verify that any C^1 manifold $M \subset R^K$ is a piecewise uniformly regular Lipschitz manifold. An example of a k-dimensional piecewise uniformly regular Lipschitz manifold with singularity is a k-dimensional polyhedron $P = \partial U \subset R^{k+1}$, here $U \subset R^{k+1}$ is a simply connected bounded polyhedral domain.

Notice that in the terminology of [GS], a k-dimensional piecewise uniformly regular Lipschitz manifold X may have infinite curvature at $p \in X_{k-2}$. Hence it doesn't seem possible to apply the analytic method developed by [GS] directly. In fact, an energy minimizing map here may have discontinuity. Moreover, since X has singularity in general, it seems impossible to have the usual Euler-Lagrange equations for minimizing maps. Our idea is follows. First we prove the Hölder continuity for energy minimizing maps into any tangent cone of a compact piecewise uniformly regular Lipschitz manifold. Then we combine the usual blowup arguments of the domain and target and the extension Lemma due to Luckhaus [Ls] (a generalization of that of [SU]) to prove that, at zero energy density points, suitable rescalings of an energy minimizing map into a compact piecewise uniformly regular Lipschitz manifold X converge strongly in H^1 to an energy minimizing map from the unit ball $B \subset R^m$ into a tangent cone of X. Both steps involve an induction on k (the dimension of X), the structures of tangent cones (cf.(1.2)), and local approximation property to a piecewise uniformly regular manifold by its tangent cones (cf.(1.3)–(1.4)). In the process of proving the boundary regularity, we give a simple proof of the boundary monotonicity inequality (cf. [SU], [HL], [Fm] for smooth X), which covers the case that $X \subset R^K$ is any closed subset. To prove I2 of Theorem I, we generalize the dimension reduction argument by [Lf] and [GS].

The paper is written as follows. In Section 2, we prove the continuity for minimizing maps into tangent cones of a compact piecewise uniformly regular Lipschitz manifold and the interior partial regularity for minimizing maps into a compact piecewise uniformly regular Lipschitz manifold. In Section 3, we prove the boundary monotonicity inequality and boundary regularity. In Section 4, we prove the Hausdorff dimension estimation for preimages.

Acknowledgement. The author wishes to thank Professor Hardt for suggesting this problem. The author is partially supported by NSF DMS 9970549.

2. Interior Partial Regularity.

In this section, we first prove Hölder continuity for minimizing maps into tangent cones of a compact piecewise uniformly regular Lipschitz manifold and then show the small energy regularity for minimizing maps into a compact piecewise uniformly regular Lipschitz manifold.

Let us first recall both the energy monotonicity inequality and monotonicity of order functions for minimizing maps into cones.

Lemma 2.1. *Assume that $X \subset R^K$ is a cone and $u \in H^1(\Omega, X)$ is energy minimizing. Then*

(1) *For any* $x \in \Omega$ *and* $0 < t \leq s < \operatorname{dist}(x, \partial\Omega)$,

$$(2.1)\qquad t^{2-m}\int_{B_t(x)} |Du|^2 + 2\int_{B_s(x)\setminus B_t(x)} |y-x|^{2-m}\left|\frac{\partial u}{\partial r}\right|^2 = s^{2-m}\int_{B_s(x)} |Du|^2,$$

(2)

$$(2.2)\qquad \Delta u \cdot u = 0,\ \ \Delta |u|^2 = 2|Du|^2, \quad in\ \Omega,$$

in the sense of distribution.

(3) *For* $a \in \Omega$ *and* $0 < r < \operatorname{dist}(a, \partial\Omega)$. *Either* $u \equiv 0$ *on* $B(a,r)$ *or the order function* $N(a,s) = \frac{s\int_{B_s(a)} |Du|^2}{\int_{\partial B_s(a)} |u|^2}$ *is monotonically nondecreasing for* $s \in [r, \operatorname{dist}(a, \partial\Omega))$.

Proof. Since the interior monotonicity equality for minimizing maps can be proven by only using the variations of domain, (1) follows exactly from that of [HL] (cf. also [GS]). Let $\phi \in C_0^1(\Omega, R)$ be given. Since X is a cone, $u_t(x) = (1 + t\phi(x))u(x) : \Omega \to X$ for $|t|$ small is a comparision map to u. By minimality of u, we have

$$0 = \frac{d}{dt}|_{t=0}\int_B |Du_t|^2 = 2\int_B Du\cdot D(u\phi),$$

which clearly implies both equations of (2.2). To prove (3), we first notice that (2.2) implies $|u|^2$ is a (nonnegative) subharmonic function. Hence if $\int_{\partial B_r(a)} |u|^2 = 0$, then the mean-value inequality for $|u|^2$ yields $u \equiv 0$ on $B_r(a)$. Otherwise $\int_{\partial B_s(a)} |u|^2 > 0$ for all $s \in [r, \operatorname{dist}(a, \partial\Omega))$ and $N(a,s)$ is absolutely continuous for $s \in [r, \operatorname{dist}(a, \partial\Omega))$ so that it is differentiable for a.e. s. In fact, for a.e. $s \in [r, \operatorname{dist}(a, \partial\Omega))$,

$$\frac{d}{ds}N(a,s) = \frac{\left(\int_{B_s(a)} |Du|^2\,dx + s\int_{\partial B_s(a)} |Du|^2\right)}{\int_{\partial B_s(a)} |u|^2} - \frac{s\int_{B_s(a)} |Du|^2\left(\int_{\partial B_s(a)} |u|^2\right)_s}{\left(\int_{\partial B_s(a)} |u|^2\right)^2},$$

and

$$(2.3)\qquad \left(\int_{\partial B_s(a)} |u|^2\right)_s = \frac{m-1}{s}\int_{\partial B_s(a)} |u|^2 + \int_{\partial B_s(a)} \frac{\partial}{\partial s}|u|^2.$$

On the other hand, approximating the characteristic function of $B_s(a)$ by suitable test functions φ, (2.2) implies

$$(2.4)\qquad 2\int_{B_s(a)} |Du|^2 = \int_{\partial B_s(a)} \frac{\partial}{\partial s}|u|^2, \text{ a.e } s \in [r, \operatorname{dist}(a, \partial\Omega)).$$

Combining (2.3) with (2.4), we have

$$\frac{d}{ds}N(a,s) = 2s\frac{\int_{\partial B_s(a)} |\frac{\partial u}{\partial s}|^2 \int_{\partial B_s(a)} |u|^2 - \left(\int_{B_s(a)} |Du|^2\right)^2}{\left(\int_{\partial B_s(a)} |u|^2\right)^2}.$$

Observe that, by the Cauchy inequality, (2.4) implies

$$\int_{B_s(a)} |Du|^2 \leq \left(\int_{\partial B_s(a)} |u|^2\right)^{\frac{1}{2}} \left(\int_{\partial B_s(a)} |\frac{\partial u}{\partial s}|^2\right)^{\frac{1}{2}}.$$

Therefore

$$\frac{d}{ds}N(a,s) \geq 0, \text{ a.e. } s \in [r, \operatorname{dist}(a, \partial\Omega)).$$

The proof is complete. □

Corollary 2.2. *Let $X \subset R^K$ be a cone and $u \in H^1(\Omega, X)$ be an energy minimizing map. Then*

$$(2.5)\qquad r^{2-m}\int_{B_r(a)} |Du|^2 \leq \left(\log\left(\frac{s}{r}\right)\right)^{-1} s^{1-m}\int_{\partial B_s(a)} |u|^2,$$

for $0 < r < s < \operatorname{dist}(a, \partial\Omega)$.

Proof. First we notice that (2.3) and (2.4) imply

$$(2.6)\qquad \frac{d}{dr}\left(\frac{1}{r^{m-1}}\int_{\partial B_r(a)} |u|^2\right) = \frac{2}{r^{m-1}}\int_{B_r(a)} |Du|^2, \forall r \in (0, \operatorname{dist}(a, \partial\Omega)).$$

Integrating (2.6) from r to s, we get

$$2\int_r^s \left(\sigma^{2-m}\int_{B_\sigma(a)}|Du|^2\right)\frac{d\sigma}{\sigma} \le s^{1-m}\int_{\partial B_s(a)}|u|^2 d\sigma.$$

This, combined with (2.1), clearly implies (2.5). □

Now we are ready to prove the interior partial regularity for energy minimizing maps into a compact piecewise uniformly regular Lipschitz manifold. It is well known that iterations of the following energy improvement Lemma and the Morrey's decay Lemma (cf. [Mc]) yield the interior partial regularity (cf. [SU], [HL]).

Lemma 2.3. *Assume that $X \subset R^K$ is a k-dimensional compact piecewise uniformly regular Lipschitz manifold. There exist $\epsilon_0 = \epsilon_0(m, X) > 0$ and $\theta_0 = \theta_0(m, X) \in (0, \frac{1}{2})$ such that if $u \in H^1(\Omega, X)$ is energy minimizing and satisfies, on $B_r(x) \subset \Omega$, $r^{2-m}\int_{B_r(x)}|Du|^2 \le \epsilon_0^2$, then*

$$(\theta_0 r)^{2-m}\int_{B_{\theta_0 r}(x)}|Du|^2 \le \frac{1}{2}r^{2-m}\int_{B_r(x)}|Du|^2. \tag{2.7}$$

Proof. First notice that if we define $u_{x,r}(y) = u(x+ry) : B_1 \to X$ then $u_{x,r} \in H^1(B_1, X)$ is also energy minimizing. Hence we may assume that $x = 0$, $r = 1$. Suppose that the Lemma were false. Then, for any $\theta \in (0, \frac{1}{2})$, there exist minimizing maps $\{u_n\} \subset H^1(B_1, X)$ such that $\int_{B_1}|Du_n|^2 = \epsilon_n^2 \downarrow 0$ but

$$\theta^{2-m}\int_{B_\theta}|Du_n|^2 > \frac{1}{2}\epsilon_n^2. \tag{2.8}$$

Let $a_n = \frac{1}{|B_1|}\int_{B_1} u_n$. Then the Poincaré inequality implies that

$$\operatorname{dist}^2(a_n, X) \le C\int_{B_1}|u_n - a_n|^2 \le C\int_{B_1}|Du_n|^2 \le C\epsilon_n^2. \tag{2.9}$$

Hence there exist $\{b_n\} \subset X$, with

$$|a_n - b_n| \le C\epsilon_n. \tag{2.10}$$

Passing to subsequences, we may assume that there exists $a \in X$ such that $b_n \to a$. Denote $R_n = |b_n - a|$. Then we proceed as follows.

Case 1. $R = \lim_{n\to\infty} \frac{R_n}{\epsilon_n} < \infty$: we know

$$\int_{B_1} \left|\frac{u_n - a}{\epsilon_n}\right|^2 \leq 2\left(\int_{B_1}\left|\frac{u_n - a_n}{\epsilon_n}\right|^2 + \frac{R_n^2 + |b_n - a_n|^2}{\epsilon_n^2}\right) \\ \leq C$$

and

$$\int_{B_1}\left|D\frac{u_n - a}{\epsilon_n}\right|^2 = 1.$$

Hence we may assume that $v_n = \frac{u_n - a}{\epsilon_n} \to v$ weakly in H^1 and it is readily seen that $v(B_1) \subset T_aX$.

We may always, after passing to subsequences, that there exists $i_0 \in \{0, 1, \cdots, k\}$ such that $\{b_n\} \subset X_{i_0} \setminus X_{i_0 - 1}$.

Case 2. $R = \infty$: we divide it into two cases.

Case 2(a). $a \in X_{i_0} \setminus X_{i_0-1}$: it then follows from the definition of X that there exists a l-dimensional piecewise uniformly regular Lipschitz submanifold $Z_a \subset S^{K-1}$, with $-1 \leq l \leq k-1$, and bilipschitz maps $T_{b_n,a} : T_{b_n}X \to R^{k-l-1} \times C(Z_a)$ such that $T_{b_n,a}(0) = 0$ and satisfy (1.4). Since $v_n = \frac{u_n - b_n}{\epsilon_n}$ is bounded in $H^1(B_1, R^K)$, we may assume that $v_n \to v$ weakly in H^1. We need to show that Im $(v) \subset R^{k-l-1} \times C(Z_a)$. By the Egroff's theorem, we can assume that for any $\delta > 0$ there exists $E_\delta \subset B_1$, with $|E_\delta| \leq \delta$, such that $|v_n|$ and $|v|$ are bounded on $B_1 \setminus E_\delta$ and v_n converges to v uniformly on $B_1 \setminus E_\delta$. By the defintion of tangent cones, we know that there exists $w_n : B_1 \setminus E_\delta \to T_{b_n}X$ such that

$$|v_n - w_n| \leq \eta_n, \tag{2.11}$$

for some η_n, with $\lim_{n\to\infty} \eta_n = 0$. Let $\bar{w}_n = T_{b_n,a}(w_n) : B_1 \setminus E_\delta \to R^{k-l-1} \times C(Z_a)$. Then, by (1.4),

$$\begin{aligned} |\bar{w}_n - v| &\leq |v - v_n| + |v_n - w_n| + |w_n - \bar{w}_n| \\ &\leq |v - v_n| + |v_n - w_n| + \|T_{b_n,a} - Id\||w_n| \to 0. \end{aligned} \tag{2.12}$$

Hence, v maps $B_1 \setminus E_\delta$ to $R^{k-l-1} \times C(Z_a)$. Since δ is arbitrary, we conclude that v maps B_1 to $R^{k-l-1} \times C(Z_a)$. Moreover, we see that

$$\lim_{n\to\infty} \frac{\operatorname{dist}(b_n, X_{i_0-1})}{\epsilon_n} = \infty. \tag{2.13}$$

Case 2(b). $a \in X_{i_0-1}$. We may assume that there exist $\{\bar{b}_n\} \subset X_{i_0-1}$ such that $|b_n - \bar{b}_n| = \text{dist}(b_n, X_{i_0-1})$. If

$$\frac{|\bar{b}_n - b_n|}{\epsilon_n} \to \infty, \tag{2.14}$$

we still let $v_n = \frac{u_n - b_n}{\epsilon_n}$. Similar to the discussion of Case 2(a), we can show that $v_n \to v$ weakly in H^1 and Im $(v) \subset R^{k-l-1} \times C(Z_a)$, where $Z_a \subset S^{K-1}$ is given by Case 1. Otherwise, $\frac{|\bar{b}_n - b_n|}{\epsilon_n} \to C < \infty$. Hence one can see that $\frac{|\bar{b}_n - a|}{\epsilon_n} \to \infty$. Now we repeat Case 2 with b_n replaced by $\bar{b}_n$ and $v_n = \frac{u_n - \bar{b}_n}{\epsilon_n}$ so that, after repeating finitely many times, there exists $\{\tilde{b}_n\} \subset X_{m_0}$ for some $m_0 \in \{0, \cdots, i_0 - 1\}$ such that

$$\frac{\text{dist}(\tilde{b}_n, X_{m_0-1})}{\epsilon_n} \to \infty, \tag{2.15}$$

and there also exists a p-dimensional piecewise uniformly regular Lipschitz submanifold $W_a \subset S^{K-1}$, with $-1 \leq p \leq k-1$, such that $v_n = \frac{u_n - \tilde{b}_n}{\epsilon_n} \to v$ weakly in H^1, and Im $(v) \subset R^{k-p-1} \times C(W_a)$.

Now we need to show that $v_n \to v$ strongly in $H^1(B_{\frac{3}{4}})$ and $v : B_{\frac{3}{4}} \to T_a X($ or $R^{k-l-1} \times C(Z_a)$, or $R^{k-p-1} \times C(W_a))$ is energy minimizing. In order to do this, we need to use (1.2)–(1.4) and the extension Lemma due to Luckhaus [Ls] (cf. also [SU]).

Lemma 2.4. *For a given closed subset $X \subset R^K$. Let $v, w \in H^1(S^{m-1}, X)$, $0 < \lambda < \frac{1}{2}$, $\epsilon \in (0,1)$. Suppose*

$$\int_{S^{m-1}} |Dv|^2 + |Dw|^2 + \frac{|v-w|^2}{\epsilon^2} = K^2.$$

There exist $C_1 = C_1(m), C_2 = C_2(m)$ and a map $\phi \in H^1(B_1 \setminus B_{1-\lambda}, R^K)$ such that

$$\begin{aligned} \phi(z) &= v(z), \quad \forall |z| = 1, \\ &= w\left(\frac{z}{1-\lambda}\right), \quad \forall |z| = 1-\lambda, \end{aligned}$$

$$\int_{B_1 \setminus B_{1-\lambda}} |D\phi|^2 \leq C_1 K^2 \left(1 + \left(\frac{\epsilon}{\lambda}\right)^2\right) \lambda,$$

$$\phi(B_1 \setminus B_{1-\lambda}) \subset \{y \in R^K |\ \text{dist}\,(y, X) \leq r\}$$

with $r = C_2 K \epsilon^{\frac{1}{4}} \lambda^{\frac{2-m}{2}}$.

Now we can proceed as follows.

Case a. Im $(v) \subset T_aX$: Take any comparision map $\tilde{v} \in H^1(B_1, T_aX)$ coinciding with v in $B_1 \setminus B_{1-\lambda_0}$, where $0 < \lambda_0 < 1$ is sufficiently small. By the Fatou's Lemma and the Fubini's theorem, there exists $\rho_0 \in (1-\lambda_0, 1)$ such that

$$(2.16) \qquad \int_{\partial B_{\rho_0}} |v_n - \tilde{v}|^2 \to 0, \quad \int_{\partial B_{\rho_0}} |Dv_n|^2 + |D\tilde{v}|^2 \le C.$$

Choose $R_n \to \infty$ such that $\epsilon_n R_n \to 0$. Define

$$(2.17) \qquad \tilde{v}_n = \frac{R_n \tilde{v}}{\max(R_n, |\tilde{v}|)}, \qquad \tilde{u}_n = \Psi_a(\epsilon_n(\tilde{v}_n)).$$

Here Ψ_a is given by (1.3) in the definition of X so that

$$(2.18) \qquad \lim_{n\to\infty} \mathrm{Lip}(\Psi_a|_{T_aX \cap B(0, R_n\epsilon_n)}) = 1.$$

Case b. Im $v \subset R^{k-l-1} \times C(Z_a)$ for some l-dimensional piecewise uniformly regular Lipschitz $Z_a \subset S^{K-1}$, with $-1 \le l \le k-1$. Here we consider the case 2(a) above only, since the other cases in case 2 can be handled in the same way. Taking any comparision map $\tilde{v} \in H^1(B_1, R^{k-l-1} \times C(Z_a))$ coinciding with v in $B_1 \setminus B_{1-\lambda_0}$, where $0 < \lambda_0 < 1$ is sufficiently small. Hence, by the Fatou's Lemma and the Fubini's theorem, (2.16) holds too. From (2.14), we can choose $R_n \to \infty$ such that $\epsilon_n R_n \to 0$ and $R_n \le \theta_a \frac{\mathrm{dist}(b_n, X_{i-1})}{\epsilon_n}$, where $\theta_a \in (0,1)$ is given by the definition of X. Define

$$(2.19) \qquad \tilde{v}_n = \frac{R_n \tilde{v}}{\max(R_n, |\tilde{v}|)}, \qquad \tilde{u}_n = \Psi_{b_n}(\epsilon_n T_{a,b_n}(\tilde{v}_n)).$$

Here Ψ_{b_n} and $T_{a,b_n} : R^{k-l-1} \times C(Z_a) \to T_{b_n}X$ is given by the definition of X. Therefore, (1.3) and (1.4) imply that

$$(2.20) \qquad \lim_{n\to\infty} \max\{\mathrm{Lip}(T_{a,b_n}),\ \mathrm{Lip}(\Psi_{b_n}|_{T_{b_n}X \cap B(0,R_n\epsilon_n)})\} = 1.$$

Applying Lemma 2.4 to $\tilde{u}_n$ and u_n, there exists a map $\tilde{\tilde{u}}_n \in H^1(B_1, R^K)$ such that

$$(2.21) \qquad \tilde{\tilde{u}}_n(z) = \tilde{u}_n\left(\frac{z}{1-\lambda_n}\right), \quad \forall |z| < \rho_0(1-\lambda_n),$$

$$(2.22) \qquad \tilde{\tilde{u}}_n = u_n, \ \forall |z| \ge \rho_0,$$

$$(2.23) \qquad \int_{B_{\rho_0} \setminus B_{\rho_0(1-\lambda_n)}} |D\tilde{\tilde{u}}_n|^2 \le C\lambda_n \epsilon_n^2,$$

$\lambda_n \to 0$, and $\operatorname{dist}(\tilde{\tilde{u}}_n, X) \to 0$ uniformly in $B_{\rho_0} \setminus B_{\rho_0(1-\lambda_n)}$. Notice that $\tilde{\tilde{u}}_n$ has its image out of X only in $B_{\rho_0} \setminus B_{\rho_0(1-\lambda_n)}$ but with uniformly small distance to X. On the other hand, since X is a compact piecewise uniformly regular Lipschitz manifold, there exist a $\eta_0 > 0$ and a Lipschitz retraction map $F_{\eta_0} : X_{\eta_0} \to X$ (i.e. $F(y) = y$ for $y \in X$) such that $\operatorname{Lip}(F_{\eta_0}) \le C_0$, here $X_{\eta_0} = \{x \in R^K | \operatorname{dist}(x, X) \le \eta_0\}$. Therefore if we define $w_n : B_{\rho_0} \to X$ by

$$
\begin{aligned}
&w_n(z) = \tilde{\tilde{u}}_n(z), \ \forall |z| \le \rho_0(1-\lambda_n) \\
&w_n(z) = F_{\eta_0}(\tilde{\tilde{u}}_n(z)), \quad \forall |z| \in (\rho_0(1-\lambda_n), \rho_0).
\end{aligned}
$$

Then w_n is a comparision map to u_n. Now we calculate the energy as follows. For simplicity, we only do the calculation in the case b.

$$
\begin{aligned}
\int_{B_{\rho_0}} |Dv|^2 &\le \lim_{n\to\infty} \int_{B_{\rho_0}} |Dv_n|^2 \\
&= \lim_{n\to\infty} \epsilon_n{}^{-2} \int_{B_{\rho_0}} |Du_n|^2 \\
&\le \lim_{n\to\infty} \epsilon_n{}^{-2} \int_{B_{\rho_0}} |Dw_n|^2 \\
&= \lim_{n\to\infty} \epsilon_n^{-2} \left(\int_{B_{\rho_0(1-\lambda_n)}} |D\tilde{\tilde{u}}_n|^2 + \int_{B_{\rho_0} \setminus B_{\rho_0(1-\lambda_n)}} |D(F_{\eta_0}(\tilde{\tilde{u}}_n))|^2 \right) \\
&\le \lim_{n\to\infty} \epsilon_n^{-2} \Bigg(\int_{B_{\rho_0(1-\lambda_n)}} \left| D\left(\Psi_{b_n} \left(\epsilon_n T_{a,b_n} \tilde{v}_n \left(\frac{\cdot}{1-\lambda_n} \right) \right) \right) \right|^2 \\
&\quad + \int_{B_{\rho_0} \setminus B_{\rho_0(1-\lambda_n)}} |D(F_{\eta_0}(\tilde{\tilde{u}}_n))|^2| \Bigg) \\
&\le \lim_{n\to\infty} \epsilon_n^{-2} \Bigg(\epsilon_n^2 \operatorname{Lip}^2(\Psi_{b_n}|_{T_{b_n} X \cap B(0, R_n \epsilon_n)}) \operatorname{Lip}^2(T_{a,b_n}) \\
&\quad \cdot (1-\lambda_n)^{m-2} \int_{B_{\rho_0}} |D\tilde{v}_n|^2 + \operatorname{Lip}^2(F_{\eta_0}) \int_{B_{\rho_0} \setminus B_{\rho_0(1-\lambda_n)}} |D\tilde{\tilde{u}}_n|^2 \Bigg) \\
&\le \lim_{n\to\infty} \left(\left(1 + 0\left(\frac{1}{n}\right) \right) (1-\lambda_n)^{m-2} \int_{B_{\rho_0(1-\lambda_n)}} |Dv|^2 + C\lambda_n \right) \\
&= \int_{B_{\rho_0}} |Dv|^2.
\end{aligned}
$$

Since the limit cone $R^{k-l-1} \times C(Z_a)$ appearing in case b above is also a tangent cone of a k-dimensional picewise uniformly regular Lipschitz submanifold $Y \subset R^K$, the conclusion of Lemma 2.3 follows if we can prove

Lemma 2.5. *Assume that X is a k-dimensional compact piecewise uniformly regular Lipschitz manifold. Then for any $a \in X$ there exists $\theta_0 = \theta_0(m, a, X) \in (0, \frac{1}{2})$ such that if $u \in H^1(B_1, T_a X)$ is an energy minimizing map then*

$$\theta_0^{2-m} \int_{B_{\theta_0}} |Du|^2 \leq \frac{1}{2} \int_{B_1} |Du|^2. \tag{2.24}$$

Proof. It is done by an induction on k.

(1) $k = 1$: Since X is a piecewise C^1 Jordan curve. For $a \in X_1$, we know that $T_a X = R^1$ and a minimizing $u \in H^1(B_1, R^1)$ is a harmonic function so that (2.24) holds trivially. For $a \in X_0$, we have $T_a X = \overline{OA_1} \cup \overline{OA_2}$ and the angle between $\overline{OA_1}$ and $\overline{OA_2}$ is positive. Here $\overline{OA_i}$ for $i = 1, 2$ is a ray in R^2 emmitting from the origin of R^2. Observe that there exists an isometric map $F : \overline{OA_1} \cup \overline{OA_2} \to R^1$ so that $F(u) : B_1 \to R^1$ is a harmonic function, hence u is Lipschitz continuous and (2.24) holds again.

(2) $k \geq 2$: Suppose that the Lemma is true for any l-dimensional piecewise uniformly regular Lipschitz manifold for all $1 \leq l \leq k-1$. We need to show that the Lemma remains to be true for a k-dimensional piecewise uniformly regular Lipschitz manifold X. To do it, we proceed as follows. For $a \in X_k \setminus X_{k-1}$, since $T_a X = R^k$ we know that a minimizing $u \in H^1(B_1, R^k)$ is a vector valued harmonic function so that (2.24) holds trivially. For $a \in X_l \setminus X_{l-1}$ for some $0 \leq l \leq k-1$, we know that $T_a X = R^l \times C(Y_a)$ with Y_a being a $(k-l-1)$-dimensional piecewise uniformly regular Lipschitz manifold in S^{K-1}. Therefore the minimality of $u = (u_1, u_2) : B_1 \to R^l \times C(Y_a)$ implies that $u_1 : B_1 \to R^l$ is a harmonic function and $u_2 : B_1 \to C(Y_a)$ is energy minimizing. Therefore, our proof is complete if we can prove (2.24) for any minimizing map $w : B_1 \to C(Y_a)$. To do it, we first observe that we can assume that $\int_{\partial B_1} |w|^2 > 0$ (otherwise Lemma 2.1 implies that $w \equiv 0$ on B_1 so that (2.24) holds trivially). Also notice that, since $C(Y_a)$ is a cone, $w \in H^1(B_1, C(Y_a))$ is energy minimizing implies that $\lambda w \in H^1(B_1, C(Y_a))$ is also minimizing for any $\lambda > 0$. Therefore, to prove (2.24) for w is equivalent to prove (2.24) for λw, for some $\lambda > 0$. By choosing $\lambda = (\int_{\partial B_1} |w|^2)^{-\frac{1}{2}} > 0$, we may assume that $w \in H^1(B_1, C(Y_a))$ satisfies $\int_{\partial B_1} |w|^2 = 1$. It follows

from Corollary 2.2 that

$$\theta^{2-m}\int_{B_\theta}|Dw|^2\le\left(\log\frac{1}{\theta}\right)^{-1}. \tag{2.25}$$

Hence for any fixed number $\epsilon_0>0$ and minimizer $w\in H^1(B_1,C(Y_a))$ with $\int_{\partial B_1}|w|^2=1$ and $\int_{B_1}|Dw|^2>\epsilon_0^2$ we have

$$\theta^{2-m}\int_{B_\theta}|Dw|^2\le\frac{1}{2}\int_{B_1}|Dw|^2, \tag{2.26}$$

provided that we choose $\theta\le e^{-\frac{2}{\epsilon_0^2}}$.

Claim. Assume that Y_a is given as above. There exist $\epsilon_0=\epsilon_0(m,Y_a)>0$, $\theta_1=\theta_1(m,Y_a)\in(0,\frac{1}{2})$ such that if $w\in H^1(B_1,C(Y_a))$ is energy minimizing satisfying $\int_{B_1}|Dw|^2\le\epsilon_0^2$ and $\int_{\partial B_1}|w|^2=1$ then (2.24) holds.

Proof of Claim. We use induction on the dimension of Y_a. It is easy to see that (2.24) is true when the dimension of Y_a is 0. Suppose that (2.24) is true for any l-dimensional piecewise uniformly regular Lipschitz submanifold $Z\subset S^{K-1}$ with $l<\dim(Y_a)$. We want to show that (2.24) is also true for Y_a itself. Suppose that it were false. Then for any $\theta\in(0,\frac{1}{2})$ there exist minimizing maps $\{w_n\}\subset H^1(B_1,C(Y_a))$ such that

$$\int_{B_1}|Dw_n|^2=\epsilon_n^2\downarrow 0,\quad\int_{\partial B_1}|w_n|^2=1, \tag{2.27}$$

but (2.24) fails. Denote $a_n=\frac{1}{|\partial B_1|}\int_{\partial B_1}w_n$. Then

$$|a_n|\le C(m)\left(\int_{\partial B_1}|w_n|^2\right)^{\frac{1}{2}}\le C.$$

The Poincaré inequality implies,

$$\int_{\partial B_1}|w_n-a_n|^2\le C(m)\int_{B_1}|Dw_n|^2\le C\epsilon_n^2, \tag{2.28}$$

and

$$\operatorname{dist}^2(a_n,C(Y_a))\le\frac{1}{|\partial B_1|}\int_{\partial B_1}|w_n-a_n|^2\le C(m)\epsilon_n^2. \tag{2.29}$$

Therefore there exist $\{b_n\} \subset C(Y_a)$ with $|b_n - a_n| \leq C\epsilon_n$. Passing to subsequence, b_n converges to $b \in C(Y_a)$. Note that $|b| = |\partial B_1|^{-\frac{1}{2}}$ because

$$|b|^2 - |\partial B_1|^{-1} = |\partial B_1|^{-1} \left(\int_{\partial B_1} |b|^2 - |w_n|^2 \right) \to 0, \text{ as } n \to \infty.$$

Since $T_{b_n}(C(Y_a)) = R^1 \times T_{\frac{b_n}{|b_n|}}(Y_a)$ and $p = \dim(Y_a) \leq k-1$, it follows from the definition of Y_a that there exists a piecewise uniformly regular manifold $Z_b \subset S^{K-1}$, with $q = \dim(Z_b) \leq p-1$, and bilipschitz maps $T_{\frac{b_n}{|b_n|}, b} : T_{\frac{b_n}{|b_n|}}(Y_a) \to R^{p-q-1} \times C(Z_b)$ which satisfy (1.4). Now we can repeat the argument similar to that of Lemma 2.3 to show that $v_n = \frac{w_n - b_n}{\epsilon_n} \to v$ strongly in H^1, where $v \in H^1(B_{\frac{3}{4}}, R^{p-q} \times C(Z_b))$ is an energy minimizing map. Denote $v = (v_1, v_2) : B_{\frac{3}{4}} \to R^{p-q} \times C(Z_b))$. Then we have that $v_1 : B_{\frac{3}{4}} \to R^{p-q}$ is a harmonic function and $v_2 : B_{\frac{3}{4}} \to C(Z_b)$ is a mimimizing map. Since Z_b has dimension less than the dimension of Y_a. It follows from the induction hypothesis that (2.24) holds for v_2 for some small θ_1 so does (2.24) hold for v. This contradicts with the choices of u_n. This finishes proof of the claim. Hence Lemma 2.5 follows by letting $\theta_0 = \min\{e^{-\frac{2}{\epsilon_0^2}}, \theta_1\}$. Therefore the proof of Lemma 2.3 is also complete. □

Completion of Proof of Interior Partial Regularity..

Lemma 2.6. *Assume that $X \subset R^K$ is a k-dimensional compact piecewise uniformly regular Lipschitz manifold. Suppose that $u \in H^1(\Omega, X)$ is energy minimizing. Then there exists a closed subset $\Sigma \subset \Omega$, with $\dim_H(\Sigma) \leq m-3$, such that $u \in C^\alpha(\Omega \setminus \Sigma, X)$ for some $\alpha \in (0,1)$.*

Proof. Define $\Sigma = \{x \in \Omega | \lim_{r\to 0} r^{2-m} \int_{B_r(x)} |Du|^2 \geq 2^{2-m}\epsilon_0^2\}$, where ϵ_0 is given by Lemma 2.3. Then it follows from (2.1) and a standard covering argument (cf. [SU]) that Σ is closed with $H^{m-2}(\Sigma) = 0$. On the other hand, for any $x_0 \in \Omega \setminus \Sigma$, there exists $r_0 > 0$ such that

$$r_0^{2-m} \int_{B_{r_0}(x_0)} |Du|^2 \leq 2^{2-m}\epsilon_0^2. \tag{2.30}$$

It follows from (2.1) that

$$r^{2-m} \int_{B_r(x)} |Du|^2 \leq 2^{m-2} \int_{B_{r_0}(x_0)} |Du|^2 \leq \epsilon_0^2, \tag{2.31}$$

for any $x \in B_{\frac{r_0}{2}}(x_0)$ and $0 < r \le \frac{r_0}{2}$. Applying Lemma 2.3 repeatedly, we know that there exists $\theta_0 = \theta_0(m, X) \in (0, \frac{1}{2})$ such that for any $k \ge 1$

$$(\theta_0^k r)^{2-m} \int_{B_{\theta_0^k r}(x)} |Du|^2 \le 2^{-k}\epsilon_0^2, \tag{2.32}$$

for any $x \in B_{\frac{r_0}{2}}(x_0)$ and $0 < r \le \frac{r_0}{2}$. Hence there exists $\alpha_0 = \alpha_0(m, X) \in (0,1)$ so that

$$r^{2-m} \int_{B_r(x)} |Du|^2 \le C(\epsilon_0, m, X) r^{2\alpha_0}, \tag{2.33}$$

for any $x \in B_{\frac{r_0}{2}}(x_0)$ and $0 < r \le \frac{r_0}{2}$. Therefore, Morrey's decay Lemma (cf. [Mc]) implies that $u \in C^{\alpha_0}(B_{\frac{r_0}{4}}(x_0))$.

One can follow the dimension reduction argument of [SU] to show that Σ has Hausdorff dimension at most $m-3$. The key is to show that the set of minimizing maps into X is compact.

Lemma 2.7. *Assume that $X \subset R^K$ is a Lipschitz neighbourhood retraction. Suppose that $\{u_n\} \subset H^1(B_1, X)$ is a sequence of minimizing maps and $u_n \to u$ weakly in $H^1(B_1, X)$. Then $u_n \to u$ strongly in $H^1(B_{\frac{3}{4}}, X)$ and $u : B_{\frac{3}{4}} \to X$ is energy minimizing.*

Proof. Take any comparision map $w \in H^1(B_1, X)$ coinciding with u in $B_1 \setminus B_{1-\lambda_0}$ for some small $\lambda_0 \in (0, \frac{1}{4})$. By the Fatou's Lemma and the Fubini's theorem, there exists $\rho_0 \in (1-\lambda_0, 1)$ such that

$$\int_{\partial B_{\rho_0}} |u_n - w|^2 \to 0, \quad \int_{\partial B_{\rho_0}} |Du_n|^2 + |Dw|^2 \le C < \infty. \tag{2.34}$$

Applying Lemma 2.4 to u_n and w, we have that there exists $\tilde{u}_n \in H^1(B_{\rho_0}, R^K)$ such that

$$\begin{aligned} \tilde{u}_n(x) &= w\left(\frac{x}{1-\lambda_n}\right), \quad |x| \le \rho_0(1-\lambda_n) \\ &= u_n(x), \ |x| = \rho_0. \end{aligned}$$

$$\int_{B_{\rho_0} \setminus B_{\rho_0(1-\lambda_n)}} |D\tilde{u}_n|^2 \le C\lambda_n. \tag{2.35}$$

$\operatorname{dist}(\tilde{u}_n, X) \to 0$, as $\lambda_n \to 0$, uniformly in $B_{\rho_0} \setminus B_{\rho_0(1-\lambda_n)}$. Let $F : X_{\delta_0} \to X$ be a Lipschitz retraction map. Here X_{δ_0} is the δ_0 neighbourhood of X in R^K. Define

$$\begin{aligned} w_n(x) &= w\left(\frac{x}{1-\lambda_n}\right), \quad |x| \le \rho_0(1-\lambda_n) \\ &= F(\tilde{u}_n(x)), \quad \rho_0(1-\lambda_n) \le |x| \le \rho_0. \end{aligned}$$

Then w_n is a comparision map to u_n, we have

$$\begin{aligned} \int_{B_{\rho_0}} |Du|^2 &\le \lim_{n\to\infty} \int_{B_{\rho_0}} |Du_n|^2 \\ &\le \lim_{n\to\infty} \int_{B_{\rho_0}} |Dw_n|^2 \\ &\le \lim_{n\to\infty} \left[\int_{B_{\rho_0(1-\lambda_n)}} \left| Dw\left(\frac{\cdot}{1-\lambda_n}\right)\right|^2 + \int_{B_{\rho_0}\setminus B_{\rho_0(1-\lambda_n)}} |D\tilde{u}_n|^2 \right] \\ &\le \lim_{n\to\infty} \left[(1-\lambda_n)^{m-2} \int_{B_{\rho_0}} |Dw|^2 + C \operatorname{Lip}^2(F)\lambda_n \right] \\ &\le \int_{B_{\rho_0}} |Dw|^2. \end{aligned}$$

This clearly implies both the minimality of u and the strong convergence of u_n to u. □

3. Boundary Regularity.

In this section, we prove that any minimizing map from Ω into a k-dimensional compact piecewise uniformly regular Lipschitz manifold X is Hölder continuous near $\partial\Omega$, provided that $g : \partial\Omega \to X$ is Lipschitz continuous. The argument is a generalization of [SU1], [HL]. Three key points are: small energy boundary regularity, boundary monotonicity inequality, and nonexistence of boundary minimizing tangent maps.

Here we only sketch the proof for $\Omega = B_1^+ \equiv \{x = (x_1, x_m) \in B_1 : x_m \ge 0\}$. One can refer to [HL] for the modification to a general Ω. Denote $T_r^+ = \{x = (x_1, x_m) \in B_r : x_m \ge 0\}$ for $0 < r \le 1$. We first give a new proof of boundary monotonicity inequality (cf. [SU1], [HL] for smooth X), which doesn't rely on the nearest point projection from neighbourhoods of X to X.

Lemma 3.1. *Assume that $X \subset R^K$ is a closed subset. Let $u \in H^1(B_1^+, X)$ be energy minimizing with $u|_{T_1} = g$, where $g : B_1^+ \to X$ is a given Lipschitz map. Then there exist $\delta_0 = \delta_0(m, g, X) \in (0, \frac{1}{2})$ and $C_0 = C_0(m, g, X) > 0$ such that, for $0 < r \leq s \leq \delta_0$,*

$$(3.1)\qquad \begin{aligned} &r^{2-m} \int_{B_r^+} |Du|^2 + \int_{B_s^+ \setminus B_r^+} |x|^{2-m} \left| \frac{\partial u}{\partial |x|} \right|^2 \\ &\leq e^{C_0(s-r)} s^{2-m} \int_{B_s^+} |Du|^2 + C_0(s-r). \end{aligned}$$

Proof. We shall consider the energy of a comparision map on B_ρ^+ obtained by homogeneous extension from $(0, \ldots, 0, \rho^2)$. We use the polar coordinates (r, θ, ω), center at $(0, \ldots, 0, \rho^2)$, and denote the polar angle functions center at 0 as $(\phi, \omega) \in [0, \frac{\pi}{2}] \times S^{m-2}$. Then it follows from [HL] p. 578 that

$$\theta = \phi + \sin^{-1}(\rho \sin \theta).$$

Now we define

$$\begin{aligned} v(r, \theta, \omega) &= u(\rho, \phi, \omega) : 0 \leq \theta \leq \Theta(\rho) \\ &= g(\rho^2 \tan(\pi - \theta), \omega) : \Theta(\rho) \leq \theta \leq \pi. \end{aligned}$$

Here $\Theta(\rho) = \pi - \sin^{-1}(1 + \rho^2)^{-\frac{1}{2}}$. Then v is a comparison map of u, we have

$$\begin{aligned} &\int_{B_\rho^+} |Du|^2 \\ &\leq \int_{B_\rho^+} |Dv|^2 \\ &= \int_0^{\Theta(\rho)} d\theta \int_0^{R(\rho,\phi)} r^{m-3}\, dr \int_{S^{m-2}} \left(\left| \frac{\partial v}{\partial \omega} \right|^2 \sin^{-2}\theta + \left| \frac{\partial v}{\partial \theta} \right|^2 \right) \sin^{m-2}\theta \, d\omega \\ &\quad + \int_0^{\rho^2} dt \int_{B^{m-1}_{\frac{\rho^2 - t}{\rho}}} \left| D_{x,t} g \left(\frac{\rho^2}{\rho^2 - t} x \right) \right|^2 dx \\ &\leq (m-2)^{-1} R(\rho, \phi)^{m-2} \int_0^{\Theta(\rho)} \int_{S^{m-2}} \left(\left| \frac{\partial v}{\partial \omega} \right|^2 \sin^{-2}\theta + \left| \frac{\partial v}{\partial \theta} \right|^2 \right) \sin^{m-2}\theta \, d\omega \, d\theta \\ &\quad + C \int_0^{\rho^2} \left(\frac{\rho^2}{\rho^2 - t} \right)^{3-m} \int_{B_\rho} |Dg|^2 \\ &= I + II. \end{aligned}$$

Here $R(\rho,\phi)=\rho\sqrt{1+\rho^2-2\rho\cos\phi}$. It is easy to see that

$$|II|\le C\rho^2\int_{B_\rho^{m-1}}|Dg|^2\le C\mathrm{Lip}^2(g)\rho^{m+1},$$

To estimate I, we use the change of coordinates: $(\theta,\omega)\to(\phi,\omega)$, and observe that there exists $\delta_0=\delta_0(g,m,X)\in(0,\frac{1}{2})$ such that for any $\rho\in(0,\delta_0)$

$$\left|\frac{\partial\phi}{\partial\theta}\right|^2=\left|1-\frac{\rho\cos\theta}{\sqrt{1-\rho^2\sin^2\theta}}\right|^2\le 1+C\rho,$$

$$\left|\frac{\sin\theta}{\sin\phi}\right|\le 1+C\rho,\quad \sin^{m-2}\theta\,d\theta\,d\omega\le(1+C\rho)\sin^{m-2}\phi\,d\phi\,d\omega,$$

and

$$R(\rho,\phi)\le\rho(1+C\rho).$$

Hence

$$I\le(1+C\rho)\frac{\rho}{m-2}\int_{\partial B_\rho^+}|D_Tu|^2.$$

Here $\partial B_\rho^+=\{x=(x_1,x_m)\in\partial B_\rho|x_m>0\}$ and D_T denotes the tangential derivative. Therefore,

$$\int_{B_\rho^+}|Du|^2\le(1+C\rho)\frac{\rho}{m-2}\int_{\partial B_\rho^+}|D_Tu|^2+C\mathrm{Lip}^2(g)\rho^{m+1}.$$

This clearly implies (3.1). □

Now we prove boundary energy improvement Lemma for minimizing maps into a compact piecewise uniformly regular Lipschitz manifold, under the small energy hypothesis.

Lemma 3.2. *Assume that $X\subset R^K$ is a k-dimensional piecewise uniformly regular Lipschitz manifold. There exist $\epsilon_0=\epsilon_0(m,X)>0$, $\theta_0=\theta_0(m,X)\in(0,\frac{1}{2})$, and $C_0=C_0(m,X)>0$ such that if $u\in H^1(B_1^+,X)$ is energy minimizing with $u|_{T_1}=g$, here $g:B_1^+\to X$ is a given Lipschitz map, and $\int_{B_1^+}|Du|^2\le\epsilon_0^2$, then*

$$\theta_0^{2-m}\int_{B_{\theta_0}^+}|Du|^2\le\frac{1}{2}\max\left\{\int_{B_1^+}|Du|^2,C_0Lip^2g\right\}. \tag{3.2}$$

Proof. If the Lemma were false. Then, for any $\theta \in (0, \frac{1}{2})$, there would exist a sequence of minimizing maps $u_n \in H^1(B_1^+, X)$ such that $u_n|_{T_1} = g_n$ with $g_n : B_1^+ \to X$ given Lipschitz maps and

$$\int_{B_1^+} |Du_n|^2 = \epsilon_n^2 \to 0, \tag{3.3}$$

and

$$\frac{\mathrm{Lip}(g_n)}{\epsilon_n} \to 0, \tag{3.4}$$

but

$$\theta^{2-m} \int_{B_\theta^+} |Du_n|^2 > \frac{1}{2}\epsilon_n^2. \tag{3.5}$$

Assume that $g_n(0) \to a \in X$. For simplicity, we assume that

$$\lim_{n\to\infty} \frac{|g_n(0) - a|}{\epsilon_n} < \infty.$$

One can refer to the discussion of section 2 above for all other cases. Hence there exists a $q \in T_aX$ such that $\frac{g_n(0)-a}{\epsilon_n} \to q$. Define $v_n = \frac{u_n - a}{\epsilon_n} : B_1^+ \to X$. Then we know that $\int_{B_1^+} |Dv_n|^2 = 1$, and by the Poincaré inequality,

$$\int_{B_1^+} |v_n|^2 \le C\epsilon_n^{-2} \left(\int_{B_1^+} |u_n - g_n|^2 + \mathrm{Lip}^2(g_n) + |g_n(0) - a|^2 \right) \le C.$$

Hence we may assume that $v_n \to v$ weakly in $H^1(B_1^+, R^K)$. Since $v_n|_{T_1}(x) = \frac{g_n(x)-a}{\epsilon_n} \to q$ uniformly (by (3.4)), we know that $v|_{T_1} = q$. Next we show that

$$v : B_1^+ \to T_aX, \text{ and is energy minimizing from } B_{\frac{3}{4}}^+ \text{ into } T_aX, \tag{3.6}$$

$$v_n \to v \text{ strongly in } H^1(B_{\frac{3}{4}}^+, R^K). \tag{3.7}$$

The proof of the first part of (3.6) is as same as that of Lemma 2.3. To prove the minimality of v and the strong convergence of v_n to v, we apply Lemma 2.4 in the following way. Take any comparision map $w \in H^1(B_1^+, T_aX)$ coinciding with v in $B_1^+ \setminus B_{1-\lambda_0}^+$ and $w|_{T_1} = q$. Here $\lambda_0 \in (0, \frac{1}{4})$ is sufficiently small. By the Fatou's Lemma and the Fubini's theorem, there exists $\rho_0 \in (1-\lambda_0, 1)$ such that

$$\int_{\partial B^+_{\rho_0}} |v_n - w|^2 \to 0, \int_{\partial B^+_{\rho_0}} |Dv_n|^2 + Dw|^2 \le C. \tag{3.8}$$

As in Lemma 2.3, we choose $R_n \to \infty$ such that $R_n\epsilon_n \to 0$. Define $w_n = \frac{R_n w}{\max\{R_n, |w|\}} : B^+_1 \to T_a X$. Then we know that $w_n|_{T_1} = q$. For $\lambda_n \to 0$, denote $\Omega_n = \{x = (x_1, x_m) \in B_{\rho_0(1-\lambda_n)} : x_m \ge \rho_0\lambda_n\} (\subset B^+_{\rho_0})$. Notice that $\partial\Omega_n = A^1_n \cup A^2_n$, where $A^1_n = \{x = (x_1, x_m) \in \partial\Omega_n : x_m > \rho_0\lambda_n\}$ and $A^2_n = \{x = (x_1, x_m) \in \partial\Omega_n : x_m = \rho_0\lambda_n\}$. Then it is easy to see that there exists a bilipschitz map $F_n : \Omega_n \to B^+_{\rho_0}$ such that $F(A^1_n) = \partial B^+_{\rho_0}$, $F_n(A^2_n) = T_{\rho_0}$ and

$$\lim_{n\to\infty} \mathrm{Lip}(F_n) = 1. \tag{3.9}$$

Define $\tilde{u}_n : \Omega_n \to X$ by

$$\tilde{u}_n(x) = \Psi_a(\epsilon_n w_n(F_n(x))).$$

Here Ψ_a is given by the definition of X. One can see from (3.8) that

$$\int_{\partial\Omega_n} |\tilde{u}_n - u_n(F_n)|^2 \to 0, \int_{\partial\Omega_n} |D\tilde{u}_n|^2 + |D(u_n(F_n))|^2 \le C. \tag{3.10}$$

Hence, we can apply Lemma 2.4 to $\tilde{u}_n$ and u_n on $B^+_\rho \setminus \Omega_n$ to conclude that there exist maps $\tilde{\tilde{u}}_n \in H^1(B^+_{\rho_0} \setminus \Omega_n, R^K)$ such that

$$\begin{aligned} \tilde{\tilde{u}}_n(x) &= \tilde{u}_n(x), \quad \forall x \in \partial\Omega_n \\ &= u_n(x), \quad \forall x \in \partial B^+_{\rho_0} \cup T_{\rho_0}. \end{aligned} \tag{3.11}$$

$$\int_{B^+_{\rho_0}\setminus\Omega_n} |D\tilde{\tilde{u}}_n|^2 \le C\lambda_n\epsilon_n^2, \tag{3.12}$$

and dist $(\tilde{\tilde{u}}_n, X) \to 0$ uniformly in $B^+_{\rho_0} \setminus \Omega_n$. Then, similar to the discussion of Lemma 2.3, we have a comparision map $p_n : B^+_{\rho_0} \to X$ to u_n, which is given by

$$\begin{aligned} p_n(x) &= \tilde{u}_n(x), \quad \forall x \in \Omega_n \\ &= F_{\eta_0}(\tilde{\tilde{u}}_n(x)), \quad \forall x \in B^+_{\rho_0} \setminus \Omega_n. \end{aligned}$$

Here F_{η_0} is the same Lipschitz retraction map as in Lemma 2.3. Then, as in the proof of Lemma 2.3, we have

$$\begin{aligned}
\int_{B^+_{\rho_0}} |Dv|^2 &\le \lim_{n\to\infty} \int_{B^+_{\rho_0}} |Dv_n|^2 \\
&= \lim_{n\to\infty} \epsilon_n^{-2} \int_{B^+_{\rho_0}} |Du_n|^2 \\
&\le \lim_{n\to\infty} \epsilon_n^{-2} \int_{B^+_{\rho_0}} |Dp_n|^2 \\
&= \lim_{n\to\infty} \epsilon_n^{-2} \left(\int_{\Omega_n} |D\tilde{u}_n|^2 + \int_{B^+_{\rho_0}\setminus\Omega_n} |DF_{\eta_0}(\tilde{\tilde{u}}_n)|^2 \right) \\
&\le \lim_{n\to\infty} \epsilon_n^{-2} \Big(\mathrm{Lip}^{2-m}(F_n)\mathrm{Lip}^2(\Psi_a|_{T_aX\cap B(0,R_n\epsilon_n)}) \\
&\qquad \cdot \int_{B^+_{\rho_0}} |D\epsilon_n w_n|^2 + C\mathrm{Lip}^2(F_{\eta_0}) \int_{B^+_{\rho_0}\setminus\Omega_n} |D\tilde{\tilde{u}}_n|^2 \Big) \\
&\le \lim_{n\to\infty} \left(\left(1 + 0\left(\frac{1}{n}\right)\right) \int_{B^+_{\rho_0}} |Dv|^2 + C\lambda_n \right) \\
&= \int_{B^+_{\rho_0}} |Dv|^2 .
\end{aligned}$$

This clearly implies both (3.6) and (3.7). Therefore, we reach the desired contradiction, if we assume the following Lemma.

Lemma 3.3. *Assume that $X \subset R^K$ is a k-dimensional compact piecewise uniformly regular Lipschitz manifold. Then, for any $a \in X$, if $u \in H^1(B_1^+, T_aX)$ is an energy minimizing map with $u|_{T_1} = 0$, then there exists $\theta_0 = \theta_0(m, a, X) \in (0, \frac{1}{2})$ such that*

$$\theta_0^{2-m} \int_{B^+_{\theta_0}} |Du|^2 \le \frac{1}{2} \int_{B_1^+} |Du|^2. \tag{3.13}$$

Proof. The proof is based on an induction of k. Here we would like to point out that the proof of Lemma 2.1 implies

$$\Delta u \cdot u = 0, \quad \Delta |u|^2 = 2|Du|^2, \ \text{ in } B_1^+, \tag{3.14}$$

in the sense of distribution. Hence, similar to (2.5) (cf. Corollary 2.2), we have

$$\theta^{2-m}\int_{B_\theta^+}|Du|^2 \le \left(\log\left(\frac{1}{\theta}\right)\right)^{-1}\int_{\partial B_1^+}|u|^2, \quad \forall\theta\in(0,1). \tag{3.15}$$

The rest of the proof can be carried by the same way as Lemma 2.5 and is omitted here. □

To obtain the full boundary regularity, we also need

Lemma 3.4. *Assume that $X \subset R^K$ is a closed subset. Suppose that $\phi \in H^1(B_1^+, X)$ is energy minimizing with $\phi|_{T_1}$ = constant and $\phi(x) = \phi(\frac{x}{|x|})$. Then $\phi \equiv$ constant.*

Proof. It follows exactly from [HL] §5. □

Lemma 3.5. *Assume that $X \subset R^K$ is a k-dimensional compact piecewise uniformly regular Lipschitz manifold. Let $u \in H^1(B_1^+, X)$ is an energy minimizing map with $u|_{T_1} = g$ for a given Lipschitz map $g : B_1^+ \to X$. Then there exist $\delta_0 = \delta_0(g, m, X) \in (0,1)$ and $\alpha_0 \in (0,1)$ so that $u \in C^{\alpha_0}(B_{1-\delta_0}^+ \cap \{x = (x_1, x_m) : x_m \le \delta_0\}, X)$.*

Proof. First we notice that iterations of Lemma 3.2 and Lemma 2.3 imply that there exist $\delta_0 = \delta_0(m, g, X) \in (0,1)$ and $\alpha_0 \in (0,1)$ and a closed subset $\Sigma \subset T_1$ with $H^{m-2}(\Sigma) = 0$ such that $u \in C^{\alpha_0}(B_{1-\delta_0}^+ \cap \{x = (x_1, x_m) : x_m \le \delta_0\}\setminus\Sigma, X)$ (cf. also [SU1], [HL]). Now we need to show that $\Sigma = \emptyset$. Suppose $\Sigma \ne \emptyset$, then, for any $x_0 \in \Sigma$ and $r_i \downarrow 0$, $u(x_0 + r_i\cdot) : B_1^+ \to X$ converges strongly in $H^1(B_1^+, X)$ to a nonconstant map $v : B_1^+ \to X$, which is an energy minimizing map such that $v|_{T_1}$ = constant and $v(x) = v(\frac{x}{|x|})$, which is impossible by Lemma 3.4. Here we have used (3.1) and a compactness result similar to Lemma 2.7. □

4. Hausdorff Dimension Estimation for Preimages.

In this section, we prove I2 of Theorem I. So we now assume $X \subset R^{k+1}$ is a k-dimensional polyhedron and $u \in H^1(\Omega, X)$ is energy minimizing. For $p \in X_{k-2}$, denote $S_p = \{x \in \Omega \setminus \Sigma | u(x) = p\}$. When X is a round cone in R^4 and p is its vertex, Lin [Lf] proved that S_p has Hausdorff dimension at most $m-1$, provided that $u \not\equiv p$. Here we generalize his argument, which is based on Federer dimension reduction principle [Fh]. Observe that it suffices to show that $\dim_H S_{p,\delta} \leq m-1$ for any small $\delta > 0$. Here $S_{p,\delta} = \{x \in S_p : \mathrm{dist}(x, \partial\Omega \cup \Sigma) \geq 2\delta\}$. The key is the following Lemma.

Lemma 4.1. *Suppose that $\{x_n\}, \{x_0\} \subset S_{p,\delta}$ satisfy $x_n \to x_0$. Then there exists a nonzero energy minimizing map $\phi : R^m \to T_pX$ of homogeneous degree α for some $\alpha \geq 0$ (i.e., $\phi(x) = |x|^\alpha \phi(\frac{x}{|x|})$) such that $\phi(0) = \phi(y_0) = 0$ for some $y_0 \in \partial B_1$. Moreover, there exists a nonzero energy minimizing map $\psi : R^{m-1} \to T_pX$ of homogeneous degree α_1 for some $\alpha_1 \geq 0$ such that $\psi(0) = 0$.*

Proof. We may assume that p is the origin of R^K. Since u is continuous near S_p, there exists $\delta_0 > 0$ such that $u(S_{p,\delta_0}) \subset V_p$, where V_p is a neighbourhood of X at p such that $\lambda V_p \subset X$ for $0 < \lambda \leq 2$. Hence, we can apply Lemma 2.1 to u (with $\Omega = B(x_0, \delta_0)$) to conclude that $N(x,r) = \frac{r\int_{B_r(x)} |Du|^2}{\int_{\partial B_r(x)} |u|^2}$ is monotonically nondecreasing with respect to $0 < r < \delta_0$ for all $x \in S_{p,\delta_0}$. Therefore, $N(x,0) = \lim_{r\downarrow 0} N(x,r)$ exists for all $x \in S_{p,\delta_0}$ and is upper semicontinuous. Define $v_n(y) = \frac{u(x_0+r_n y)}{\lambda_n} : r_n^{-1}(B(x_0,\delta_0)\backslash\{x_0\}) \to \lambda_n^{-1} V_p$, where $r_n = |x_n - x_0|$ and $\lambda_n = (r_n^{1-m} \int_{\partial B_{2r_n}(x_0)} |u|^2)^{\frac{1}{2}}$. For n sufficiently large, we have $N(x_0, 2r_n) \leq 2N(x_0, 0)$. Notice that v_n is a sequence of minimizing maps into $\lambda_n^{-1} V_p \subset T_pX$ and satisfies

$$\int_{\partial B_2} |v_n|^2 = 1, \quad \int_{B_2} |Dv_n|^2 \leq 2N(x,0), \ \forall n \gg 1. \tag{4.1}$$

Hence $\{v_n\} \subset H^1(B_2, T_pX)$ is bounded. Applying Lemma 2.6 and 2.8, we can assume that $v_n \to \phi$ in $H^1 \cap C^0(B_2, T_pX)$ locally so that $\phi : B_2 \to T_pX$ is energy minimizing, and $\phi(0) = \phi(y_0) = 0$ for some $y_0 \in \partial B_1(0)$. Moreover, $\phi \not\equiv 0$ and

$$\frac{r\int_{B_r} |D\phi|^2}{\int_{\partial B_r} |\phi|^2} = N(x_0, 0), \quad \forall 0 < r \leq 2. \tag{4.2}$$

To see these, we may assume that $\int_{\partial B_r} |u_n|^2 \le 1$ for all $r \in (\frac{3}{2}, 2)$ and observe that

$$
\begin{aligned}
1 - \int_{\partial B_r} |u_n|^2 &= \int_r^2 \frac{d}{dt} \int_{\partial B_t} |u_n|^2 \\
&\le \frac{\epsilon}{2} \int_{B_2} |Du_n|^2 + C(\epsilon) \int_{B_2 \setminus B_r} |u_n|^2 \\
&\le \epsilon N(x_0, 0) + C(\epsilon)(2 - r).
\end{aligned}
$$

Hence, for sufficiently small ϵ and r_0 sufficiently close to 2, we have

$$\int_{\partial B_{r_0}} |u_n|^2 \ge \frac{1}{2}.$$

In particular, ϕ is nonzero. (4.2) follows from

$$\frac{r \int_{B_r} |D\phi|^2 \, dx}{\int_{\partial B_r} |\phi|^2} = \lim_{n \to \infty} \frac{r_n r \int_{B_{r_n r}(x_0)} |Du|^2}{\int_{\partial B_{r_n r}(x_0)} |u|^2} = N(x_0, 0).$$

By (4.2) and the proof of Lemma 2.1, there exists $h : [0, 2] \to R$ so that $\frac{d}{dr}\phi(r, \theta) = h(r)\phi(r, \theta)$ for all $\theta \in S^{m-1}$ and $r \in (0, 2)$. It is easy to see that $h(r) = \frac{N(x_0,0)}{r}$ so that $\phi(x) = |x|^{N(x_0,0)}\phi(\frac{x}{|x|})$. Since $\phi(y_0) = 0$ for some $y_0 \in \partial B_1$, we can repeat the same argument with center at y_0 to conclude that there exists a nonzero energy minimizing map $\psi : R^m \to T_pX$ with homogeneous degree α_1 for some $\alpha_1 \ge 0$, which is independent of one direction and $\psi(0) = 0$. □

Completion of Proof of I2 of Theorem I.

Following [Lf] or [GS], we can show that if $\dim_H S_{p,\delta} > m-1$ then there exists a nontrival minimizing geodesic $\psi : R^1 \to T_pX$ such that $\psi(0) = \psi(1) = 0$, which is clearly impossible. □

References.

[C] J.Y. Chen, *On energy minimizing mappings between and into singular spaces*, Duke Math. J., **79(1)** (1995), 77–99.

[Fh] H. Federer, *The singular set of area minimizing rectifiable currents with codimmension one and of area minimizing flat chains modulo two with arbitary codimmension*, Bull. Amer. Math. Soc., **76** (1970), 767–771.

[Fm] M. Fuchs, *p-harmonic maps obstacle problems, Part* 1, *Partial regularity Theorey*, Anali. Mat. Pura Appl., **156(4)** (1990), 127–158.

[HL] R. Hardt and F.H. Lin, *Mappings minimizing the L^p norm of the gradient*, CPAM, **40** (1987), 555–588.

[HL1] R. Hardt and F.H. Lin, *Harmonic maps into round cones and singularities of nematic liquid srystals*, Math. Z., **213** (1993), 575–593.

[GG] M. Giaquinta and E. Giusti, *The singular set of the minima of certain quadratic functionals*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **11(1)** (1984), 45–55.

[GS] M. Gromov and R. Schoen, *Harmonic maps into singular spaces and p-adic superrigidity for lattices in group of rank* 1, Pub. Math. Inst. IHES, **76** (1992), 165–246.

[J1] J. Jost, *Equilibrium Maps between Metric Spaces*, Calc. Var. Partial Differential Equations, **2**(**2**) (1994), 173–204.

[J2] J. Jost, *Convex functionals and generalized harmonic maps into spaces of non-positive curvature*, Comm. Math. Helv., **70(4)** (1995), 659–673.

[KS] J. Korevaar and R. Schoen, *Sobolev spaces and harmonic maps for metric space targets*, Comm. Anal. Geom., **1** (1993), 561–659.

[KS1] J. Korevaar and R. Schoen, *Global Existence Theorems for Harmonic Maps to Non-locally Compact Spaces*, Comm. Anal. Geom., **5**(**2**) (1997), 213–266.

[Lf] F.H. Lin, *On nematic liquid crystals with variable degree of orientation*, Comm. Pure Appl. Math., **44(4)** (1991), 453–468.

[Lf1] F.H. Lin, *Analysis on Singular Spaces*, in 'Collection of papers on Geomtry analysis and mathematical physics', 114–126, World Sci. Publishing, River Edge, NJ, 1997.

[Ls] S. Luckhaus, *Partial Hölder continuity for Minima of certain Energy among maps into a Riemanian Manifold*, Indiana Journal of Maths., **37**(**2**) (1988), 349–367.

[Mc] C.B. Morrey, Jr., *Multiple integrals in the Calculus of Variations*, Springer-Verlag, 1966.

[SU] R. Schoen and K. Uhlenbeck, *A regularity theorey for harmonic mappins*, J. Diff. Geom., **17** (1982), 307–335.

[SU1] R. Schoen and K. Uhlenbeck, *Boundary regularity and the Dirchlet problems of harmonic mappings*, J. Diff. Geom., **18** (1983), 253–268.

[St1] T. Serbinowski, *Boundary Regularity of Harmonic Maps to Nonpositively Curved Metric Spaces*, Comm. Anal. Geom., **2**(**1**) (1994), 139–153.

[St2] T. Serbinowski, *Harmonic Maps into Metric Spaces with Curvature Bounded Above*, Ph. D. Thesis, University of Utah, 1995.

Department of Mathematics
University of Kentucky
Lexington, KY 40506
E-mail address: `cywang@ms.uky.edu`

Received February 4, 1998 and revised November 6, 2000.