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Potential theory on Lipschitz domains in Riemannian manifolds: L^P Hardy, and Hölder space results

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1. Introduction.

This is a continuation of our paper "Boundary layer methods for Lipschitz domains in Riemannian manifolds," [MT]. In that paper we have initiated a program aimed at extending the layer potential theory for the flat-space Laplacian on Lipschitz domains in the Euclidean space to the setting of variable coefficients, and more generally to the context of Lipschitz domains in Riemannian manifolds.

We recall the general setting, which will also be in effect in this paper. Let M be a smooth, compact Riemannian manifold, of real dimension $\dim M = n$, with a Riemannian metric tensor, which we assume is Lipschitz. That is, M is covered by local coordinate charts in which the components g_{jk} of the metric tensor are Lipschitz functions. (Actually, in [MT] it was assumed that the metric tensor was of class C^1; we will extend the results of [MT] to the Lipschitz case in §2 of this paper.) Then the Laplace-Beltrami operator on M is given in local coordinates by

$$\Delta u := g^{-1/2}\,\partial_j(g^{jk}g^{1/2}\,\partial_k u), \tag{1.1}$$

where we use the summation convention, take (g^{jk}) to be the matrix inverse to (g_{jk}), and set $g := \det(g_{jk})$. For $V \in L^\infty(M)$ we introduce the second order, elliptic differential operator

$$L := \Delta - V. \tag{1.2}$$

We assume $V \geq 0$ on M and also $V > 0$ on a set of positive measure in each connected component of $M\setminus\overline{\Omega}$. Amongst other things, this guarantees that

$$L : H^{1,p}(M) \longrightarrow H^{-1,p}(M) \tag{1.3}$$

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is an isomorphism, for each $p \in (1,\infty)$, where $H^{s,p}(M)$ denotes the class of L^p-Sobolev spaces on M. Let $\Omega \subset M$ be a connected open set that is a Lipschitz domain; i.e., $\partial\Omega$ is locally representable as the graph of a Lipschitz function.

Consider the Dirichlet boundary problem

$$Lu = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f, \tag{1.4}$$

and the Neumann boundary problem

$$Lu = 0 \text{ in } \Omega, \quad \partial_\nu u\big|_{\partial\Omega} = g, \tag{1.5}$$

where $\partial_\nu = \partial/\partial\nu$ is the normal derivative on $\partial\Omega$. Hereafter, all boundary traces are taken in the nontangential limit sense. More specifically, given a function u defined and continuous on Ω, set

$$u\big|_{\partial\Omega}(x) := \lim_{\substack{y\to x\\ y\in\gamma(x)}} u(y), \qquad x \in \partial\Omega, \tag{1.6}$$

when this limit exists. In (1.6), $\gamma(x) \subset \Omega$ is a nontangential approach region with "vertex" at x; cf. [MT] for more details. Furthermore, $\partial_\nu u\big|_{\partial\Omega} := \langle \nu, du\big|_{\partial\Omega}\rangle$. Natural estimates (involving the nontangential maximal function; cf. below) also accompany (1.4)–(1.5).

When Ω is a Lipschitz domain in the Euclidean space and $L = \partial_1^2 + \cdots + \partial_n^2$ is the flat-space Laplacian, these were first treated in [Dah], [JK] by means of harmonic measure estimates and, shortly thereafter, in [Ver], [DK] using layer potential techniques. The latter papers, building on [FJR] where the case of C^1 domains was treated, made use of boundedness properties of Cauchy integrals on Lipschitz surfaces due to [CMM] (following the pioneering work of [C]).

In [MT] we extended these operator norm estimates on Cauchy integrals to a variable coefficient setting, allowing for an analysis of single and double layer potentials in the manifold setting described above. To be explicit, denote by $E(x,y)$ the integral kernel of L^{-1}, so

$$L^{-1}u(x) = \int_M E(x,y)u(y)\, d\mathrm{Vol}(y) \qquad x \in M, \tag{1.7}$$

where $d\mathrm{Vol}$ is the volume element on M determined by its Riemannian metric. For a function $f : \partial\Omega \to \mathbb{R}$, define the single layer potential

$$\mathcal{S}f(x) := \int_{\partial\Omega} E(x,y)f(y)\, d\sigma(y), \qquad x \notin \partial\Omega, \tag{1.8}$$

where $d\sigma$ is the natural area element on $\partial\Omega$, and define the double layer potential by

$$\mathcal{D}f(x) := \int_{\partial\Omega} \frac{\partial E}{\partial \nu_y}(x,y)\, f(y)\, d\sigma(y), \qquad x \notin \partial\Omega. \tag{1.9}$$

The following results on the behavior of these potentials were demonstrated in [MT], extending previously known results for the flat Euclidean case.

Define $\Omega_+ := \Omega$ and $\Omega_- := M \setminus \overline{\Omega}$; note that $\Omega_\pm$ are Lipschitz domains. Given $f \in L^p(\partial\Omega)$, $1 < p < \infty$, we have, for a.e. $x \in \partial\Omega$,

$$\mathcal{S}f\big|_{\partial\Omega_+}(x) = \mathcal{S}f\big|_{\partial\Omega_-}(x) = Sf(x), \tag{1.10}$$

and

$$\mathcal{D}f\big|_{\partial\Omega_\pm}(x) = \left(\pm\frac{1}{2}I + K\right) f(x), \tag{1.11}$$

where, for a.e. $x \in \partial\Omega$,

$$\begin{aligned} Sf(x) &:= \int_{\partial\Omega} E(x,y) f(y)\, d\sigma(y), \\ Kf(x) &:= \text{P.V.} \int_{\partial\Omega} \frac{\partial E}{\partial \nu_y}(x,y)\, f(y)\, d\sigma(y). \end{aligned} \tag{1.12}$$

Here P.V. $\int_{\partial\Omega}$ indicates that the integral is taken in the principal value sense. More concretely, fix a smooth background metric which, in turn, induces a distance function on M. In particular, we can talk about balls and P.V. $\int_{\partial\Omega}$ is defined in the sense of removing such geodesic balls.

Furthermore, for a.e. $x \in \partial\Omega$,

$$\partial_\nu \mathcal{S}f\big|_{\partial\Omega_\pm}(x) = \left(\mp\frac{1}{2}I + K^*\right) f(x), \tag{1.13}$$

where K^* is the formal transpose of K. Moreover, the operators

$$K, K^* : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \quad 1 < p < \infty, \tag{1.14}$$

and

$$S : L^p(\partial\Omega) \longrightarrow H^{1,p}(\partial\Omega), \quad 1 < p < \infty, \tag{1.15}$$

are bounded and we have nontangential maximal function estimates

$$\|(\nabla \mathcal{S}f)^*\|_{L^p(\partial\Omega)} \leq C_p\|f\|_{L^p(\partial\Omega)}, \quad \|(\mathcal{D}f)^*\|_{L^p(\partial\Omega)} \leq C_p\|f\|_{\dot{L}^p(\partial\Omega)}, \tag{1.16}$$

for $1 < p < \infty$. Here and in the sequel, if u is defined in Ω then u^* will denote the nontangential maximal function of u, defined at boundary points by

$$u^*(x) := \sup\{|u(y)| : y \in \gamma(x)\}, \qquad x \in \partial\Omega. \tag{1.17}$$

Extending results produced in the Euclidean case by [Ver], we showed in [MT] that the operators

$$\pm\frac{1}{2}I + K,\ \pm\frac{1}{2}I + K^* : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega) \tag{1.18}$$

are Fredholm, of index zero. In particular, if $V > 0$ on a set of positive measure in each connected component of $M \setminus \overline{\Omega}$, then the operators

$$\frac{1}{2}I + K,\ \frac{1}{2}I + K^* : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega) \tag{1.19}$$

are invertible. Also, if $V > 0$ on a set of positive measure in Ω, then

$$-\frac{1}{2}I + K,\ -\frac{1}{2}I + K^* : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega) \tag{1.20}$$

are isomorphisms, while, if $V = 0$ on $\bar{\Omega}$, then

$$-\frac{1}{2}I + K^* : L^2_0(\partial\Omega) \longrightarrow L^2_0(\partial\Omega) \tag{1.21}$$

is an isomorphism, where $L^2_0(\partial\Omega)$ consists of elements of $L^2(\partial\Omega)$ integrating to zero.

Given these results, we can produce a solution to the Dirichlet problem (1.4) with $f \in L^2(\partial\Omega)$, in the form

$$u = \mathcal{D}\left(\left(\frac{1}{2}I + K\right)^{-1} f\right), \tag{1.22}$$

and this, in turn, satisfies

$$\|u^*\|_{L^2(\partial\Omega)} \leq C\|f\|_{L^2(\partial\Omega)}. \tag{1.23}$$

A solution to the Neumann problem (1.5) with $g \in L^2(\partial\Omega)$ is given by

$$u = \mathcal{S}\left(\left(-\frac{1}{2}I + K^*\right)^{-1} g\right), \tag{1.24}$$

and this satisfies

$$\|(\nabla u)^*\|_{L^2(\partial\Omega)} \leq C\|g\|_{L^2(\partial\Omega)}. \tag{1.25}$$

If $V = 0$ on $\partial\Omega$, we require $\int_{\partial\Omega} g\, d\sigma = 0$. Uniqueness of solutions to (1.4) and (1.5), satisfying (1.22) and (1.25) respectively, was also established.

Also [MT] treated the regularity problem. Given f in the Sobolev space $H^{1,2}(\partial\Omega)$, the Dirichlet problem (1.4) has a solution satisfying

$$\|(\nabla u)^*\|_{L^2(\partial\Omega)} \leq C\|f\|_{H^{1,2}(\partial\Omega)}. \tag{1.26}$$

Regarding results with L^p-data, it was noted in [MT] that interpolation of (1.23) with the classical result for bounded data yields a solution to the Dirichlet problem (1.4), for $f \in L^p(\partial\Omega)$, satisfying

$$\|u^*\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}, \quad 2 \leq p \leq \infty. \tag{1.27}$$

Also it was noted in [MT] that, in view of results of [Sn], one has invertibility of (1.19), etc., on $L^p(\partial\Omega)$, for $|p-2| < \varepsilon$, for some $\varepsilon = \varepsilon(\Omega) > 0$. Hence the L^p-solvability for the Dirichlet and Neumann problem, and the L^p-regularity for the Dirichlet problem, are established for $|p-2| < \varepsilon(\Omega)$ in [MT].

One of our primary goals in this paper is to establish the unique solvability of the Neumann problem (1.5), with data $g \in L^p(\partial\Omega)$, satisfying

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C_p\|g\|_{L^p(\partial\Omega)}, \quad 1 < p \leq 2, \tag{1.28}$$

and the unique solvability of the Dirichlet problem (1.4), with data $f \in H^{1,p}(\partial\Omega)$, satisfying

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C_p\|f\|_{H^{1,p}(\partial\Omega)}, \quad 1 < p \leq 2. \tag{1.29}$$

For $\Omega \subset \mathbb{R}^n$ and $L = \Delta_0$, the flat-space Laplacian, (1.29) was first established in [Ver], whereas a unified approach to both (1.28) and (1.29) was developed in [DK]. The setting of Lipschitz domains in Riemannian manifolds introduces significant additional difficulties which require new techniques and ideas to overcome. For example, at small scales, one has to understand the structure of the singularity in the kernel $E(x,y)$ of L^{-1} (cf. (1.7)) along the diagonal. Also, at large scales, one has to find appropriate substitutes for techniques in the flat, Euclidean setting which utilize the asymptotic behavior at infinity for solutions of elliptic PDE's; cf. [DK].

In fact, we will solve the Neumann problem (1.5) with data

$$g \in \mathfrak{h}^1(\partial\Omega), \tag{1.30}$$

the local Hardy space, which we define below. We will show that there is a unique solution (modulo an additive constant, and assuming $\int_{\partial\Omega} g\, d\sigma = 0$, if $V = 0$ on Ω), satisfying

$$\|(\nabla u)^*\|_{L^1(\partial\Omega)} \le C\|g\|_{\mathfrak{h}^1(\partial\Omega)}. \tag{1.31}$$

Also, we will solve the Dirichlet problem (1.4), with f satisfying

$$\nabla_T f \in \mathfrak{h}^1(\partial\Omega), \tag{1.32}$$

where ∇_T stands for the tangential gradient, obtaining a unique solution satisfying

$$\|(\nabla u)^*\|_{L^1(\partial\Omega)} \le C\|\nabla_T f\|_{\mathfrak{h}^1(\partial\Omega)} + C\|f\|_{L^1(\partial\Omega)}. \tag{1.33}$$

The L^p results mentioned above will follow from these Hardy space results and the previously described L^2 results, by interpolation.

By $\mathfrak{h}^1(\partial\Omega)$ we mean the localization of the atomic Hardy space $\mathfrak{H}^1_{\rm at}(\partial\Omega)$. We recall some definitions. We say a function $f \in L^\infty(\partial\Omega)$ is an *atom* if

$$\operatorname{supp} f \subseteq B_r(x_0) \cap \partial\Omega \tag{1.34}$$

for some $x_0 \in \partial\Omega$, $r \in (0, \operatorname{diam} \Omega]$, and

$$\|f\|_{L^\infty(\partial\Omega)} \le \frac{1}{r^{n-1}}, \quad \int_{\partial\Omega} f\, d\sigma = 0. \tag{1.35}$$

Then $g \in L^1(\partial\Omega)$ is said to belong to $\mathfrak{H}^1_{\rm at}(\partial\Omega)$ provided it can be written in the form

$$g = \sum_{\nu\ge 1} a_\nu f_\nu, \quad f_\nu \text{ an atom}, \quad \sum_{\nu \ge 1} |a_\nu| < \infty, \tag{1.36}$$

and there is the norm

$$\|g\|_{\mathfrak{H}^1_{\rm at}(\partial\Omega)} := \inf \left\{ \sum |a_\nu| : g = \sum a_\nu f_\nu,\ f_\nu \text{ atom} \right\}. \tag{1.37}$$

This corresponds to the approach in [CW] considering $\partial\Omega$ equipped with the measure $d\sigma$ and the geodesic distance as a space of homogeneous type. Then we can set

$$\mathfrak{h}^1(\partial\Omega) := \mathfrak{H}^1_{\text{at}}(\partial\Omega) + \mathcal{C}, \tag{1.38}$$

where $\mathcal{C}$ consists of functions on $\partial\Omega$ that are constant on each connected component of $\partial\Omega$. Equivalently,

$$\mathfrak{h}^1(\partial\Omega) = \mathfrak{H}^1_{\text{at}}(\partial\Omega) + L^q(\partial\Omega), \quad \forall\, q \in (1, \infty]. \tag{1.39}$$

The space $\mathfrak{h}^1(\partial\Omega)$ is "local" in the sense that, under $f \mapsto \varphi f$, it is a module over $C^r(\partial\Omega)$, for any $r > 0$.

There is also the space $\text{bmo}(\partial\Omega)$ of functions of bounded mean oscillation, localized to be a module over $C^r(\partial\Omega)$ for any $r > 0$. It is a deep result due to [FS] (with complements by [Sar] and by [CW] to apply to the current setting) that

$$\mathfrak{h}^1(\partial\Omega)^* = \text{bmo}(\partial\Omega), \quad \text{vmo}(\partial\Omega)^* = \mathfrak{h}^1(\partial\Omega), \tag{1.40}$$

where $\text{vmo}(\partial\Omega)$ is the closure of $C(\partial\Omega)$ in $\text{bmo}(\partial\Omega)$.

Analogously to (1.34)–(1.35), we can also define p-atoms on $\partial\Omega$ for $(n-1)/n < p \leq 1$. The only difference is that the upper bound in the normalization condition is, this time, $r^{-(n-1)/p}$. We denote the ℓ^p span of p-atoms by $\mathfrak{H}^p_{\text{at}}(\partial\Omega)$ and set

$$\|g\|_{\mathfrak{H}^p_{\text{at}}(\partial\Omega)} := \inf\left\{\left(\sum |a_\nu|^p\right)^{1/p} : g = \sum a_\nu f_\nu,\ f_\nu \ p\text{-atom}\right\}. \tag{1.41}$$

Also, introduce $\mathfrak{h}^p(\partial\Omega) := \mathfrak{H}^p_{\text{at}}(\partial\Omega) + \mathcal{C}$ and endow it with the natural "norm". The analogue of (1.39) remains valid in this setting too. Now, the precise sense in which $\mathfrak{h}^p(\partial\Omega)$ is "local" is that, under $f \mapsto \varphi f$, this space is a module over $C^r(\partial\Omega)$, for any $r > (n-1)(p^{-1} - 1)$. Let us also point out that, if $(n-1)/n < p < 1$, then $\mathfrak{h}^p(\partial\Omega)$ is only a quasi-Banach space and that its dual is

$$(\mathfrak{h}^p(\partial\Omega))^* = C^\alpha(\partial\Omega), \qquad \alpha := (n-1)(p^{-1} - 1), \tag{1.42}$$

i.e., the space of Hölder continuous functions on $\partial\Omega$.

We outline the structure of the rest of this paper. In §2 we indicate the modifications which are necessary in order to extend the L^2 theory of [MT] from the context of C^1 metric tensors to Lipschitz continuous metric tensors. Section 3 contains results regarding the pointwise boundary behavior of null solutions of L in Lipschitz domains. In §§4–5 we tackle the Neumann problem (1.5). Combining techniques developed in [KP] and [MT], we estimate solutions in the case that g is an atom, in order to establish (1.31). In §6 we treat the regularity problem for (1.4). We estimate the solution when $\nabla_T f$ is a vector atom and use this to establish (1.33).

Our work in §§3–6 makes use of L^2 results in [MT] which, in turn, is based on the L^2-invertibility results of the operators in (1.19) and (1.20). It is also of interest to extend these invertibility results to $L^p(\partial\Omega)$ for optimal ranges of p's, as well as to other function spaces such as Hardy, Hölder, $\mathrm{bmo}(\partial\Omega)$ and Besov spaces. We do this in §7 via an approach which emphasizes the inherent links between the functional analytic properties of the operators (1.19)–(1.20) on such spaces.

For the purpose of this introduction, let us illustrate this point by indicating the main steps in the proof of the invertibility of $\frac{1}{2}I+K$ on the Hölder space $C^\alpha(\partial\Omega)$ for $\alpha \in (0, \alpha_0)$ with $\alpha_0 = \alpha_0(\partial\Omega) > 0$ small. (This extends results for the ordinary Laplacian in Euclidean star-like Lipschitz domains given in [Br], where a structurally different approach was presented.) First, the atomic theory implies that $\frac{1}{2}I + K^*$ is invertible at the level of Hardy spaces $\mathfrak{h}^p(\partial\Omega)$ when $p = 1$. In turn, with operator bounds on K^* in hand, this automatically entails the same property for $p \in (1-\varepsilon, 1)$, for some small $\varepsilon > 0$. This latter fact is a consequence of rather general stability results on complex interpolation scales of quasi-Banach spaces established in [KM]. Then the desired result follows by duality.

In §8 we show that the Helmholtz projection is bounded on $L^p(\Omega)$ for $3/2-\varepsilon < p < 3+\varepsilon$, for some $\varepsilon = \varepsilon(\Omega) > 0$. This extends to the Riemannian manifold setting earlier work done in the Euclidean setting in [FMM] and [MMP]. Finally, in Appendices A and B we collect several useful results about $\mathfrak{h}^p(\partial\Omega)$ and Cauchy type operators on such spaces.

2. Layer potentials for Lipschitz metric tensors.

As stated in §1, [MT] studied the single layer potential (1.8) and double layer potential (1.9) when Ω was a Lipschitz domain in a compact manifold M with C^1 metric tensor. Here we extend this study to the case of a Lipschitz metric tensor. We retain the other assumptions on Ω, V, etc., made in §1.

A good bit of the necessary work was already done in [MT]. In §3 of that paper it was shown that, if the metric tensor is Lipschitz, then the integral kernel $E(x,y)$ of $(\Delta - V)^{-1}$ has the following properties. For one,

$$E(\cdot, y) \in C^s_{\mathrm{loc}}(M \setminus \{y\}), \quad \forall\, s < 2, \tag{2.1}$$

and in fact, if K and $\widetilde{K}$ are disjoint compact subsets of M,

$$y \mapsto E(\cdot, y)\big|_{\widetilde{K}} \text{ is continuous from } K \text{ to } C^s(\widetilde{K}), \quad \forall\, s < 2. \tag{2.2}$$

Furthermore, in local coordinates, in which the metric tensor is given by (g_{jk}), we can set

$$e_0(x-y, y) = C\Big(\sum g_{jk}(y)(x_j - y_j)(x_k - y_k)\Big)^{-(n-2)/2}, \tag{2.3}$$

for appropriate $C = C_n$, and then define the remainder $e_1(x, y)$ so that

$$E(x, y)\sqrt{g(y)} = e_0(x-y, y) + e_1(x, y). \tag{2.4}$$

This remainder satisfies, for each $\varepsilon \in (0, 1)$,

$$|e_1(x, y)| \le C_\varepsilon |x-y|^{-(n-3+\varepsilon)}, \quad |\nabla_x e_1(x, y)| \le C_\varepsilon |x-y|^{-(n-2+\varepsilon)}, \tag{2.5}$$

and, from (2.2) and (2.4),
(2.6)
$$e_1(x, y),\ \nabla_x e_1(x, y) \text{ are well defined in } C^0(\partial\Omega), \quad \text{for each } x \in M \setminus \partial\Omega.$$

Note that $e_0(z, y)$ is smooth in $z \in \mathbb{R}^n \setminus 0$ and homogeneous of degree $-(n-2)$ in z, but only Lipschitz in y. (If the metric tensor is C^1 then $e_0(z, y)$ is C^1 in y.) In particular, in (1.8), $\mathcal{S}f(x)$ and $\nabla\mathcal{S}f(x)$ are both well defined for all $x \in M\setminus\partial\Omega$, for any $f \in L^1(\partial\Omega)$, and the limiting result (1.10) clearly holds. Now we can still bring to bear Propositions 1.5–1.6 of [MT] to deduce that the estimate on $(\nabla\mathcal{S}f)^*$ in (1.16) and the jump relation (1.13) for $\partial_\nu\mathcal{S}f$ continue to hold, even for a Lipschitz metric tensor, and we also have $K^* : L^p(\partial\Omega) \to L^p(\partial\Omega)$ for $1 < p < \infty$. Since K is simply the formal transpose of K^*, we also have the boundedness of K, so (1.14) continues to hold for Lipschitz metric tensors.

Turning attention to (1.9), let us first observe that, since $E(x, y) = E(y, x)$, it follows that $E(x, y)$ has more smoothness in y than a separate analysis of the terms in (2.4) would indicate. In particular, we have

$$\nabla_y E(x, y) \text{ is well defined in } C^0(\partial\Omega), \quad \text{for each } x \in M \setminus \partial\Omega, \tag{2.7}$$

despite the fact that $\nabla_y e_0(x-y, y)$ is defined only a.e. on M, and is perhaps not well defined on $\partial\Omega$, if the metric tensor is only Lipschitz. Hence, $\mathcal{D}f(x)$ is well defined for any x in $M \setminus \partial\Omega$.

It remains to estimate $(\mathcal{D}f)^*$ and extend the jump relation (1.11) to the present setting. This requires understanding the nature of the singularity of

$\partial_{\nu_y} E(x, y)$ on $\partial\Omega$. Note that even though, by a variant of (2.7), $\partial_{\nu_y} E(x, y)$ makes (pointwise) sense if $x \in \partial\Omega \setminus \{y\}$ for a.e. $y \in \partial\Omega$, the formula (2.4) is not suitable for the task at hand since it involves $\nabla_y e_0(x - y, y)$. Thus, we need a different asymptotic expansion for $E(x, y)$, which avoids this difficulty.

To do this, use the symmetry $E(x, y) = E(y, x)$ to write

$$E(x, y)\sqrt{g(x)} = e_0(x - y, x) + e_1(y, x), \tag{2.8}$$

so that

$$\frac{\partial E}{\partial \nu_y}(x, y) = \frac{1}{\sqrt{g(x)}} \left\{ \frac{\partial}{\partial \nu_y} e_0(x - y, x) + \frac{\partial}{\partial \nu_y} e_1(y, x) \right\}. \tag{2.9}$$

The point is that, this time, $\nabla_y e_0(x - y, x)$ no longer exhibits the problem we encountered earlier and, moreover, we are differentiating e_1 with respect to its first set of arguments, so (2.5)–(2.6) apply.

With these identities and estimates in hand, we are again in a position to apply Propositions 1.5–1.6 of [MT] to justify (1.11) and the rest of (1.16) in the setting of Lipschitz metric tensors. Now all the results of [MT] work with the hypothesis that the metric tensor on M is C^1 relaxed to just being Lipschitz.

We mention parenthetically that some of the material of this section can be pushed to more singular metric tensors, using some techniques developed in [T1] and in Chapter 13 of [T2]. We pursue this in [MT3].

3. Existence and behavior of boundary values.

We begin with a Fatou-type result for null-solutions (and their gradients) of the operator $L = \Delta - V$. We retain the hypotheses on M, Ω, L and V made in §1.

Proposition 3.1. *Let $u \in C^1_{loc}(\Omega)$ satisfy*

$$Lu = 0 \ \text{ in } \Omega, \quad (\nabla u)^*(x) < +\infty \ \text{ for a.e. } x \in \partial\Omega. \tag{3.1}$$

Then the pointwise nontangential boundary trace $\nabla u\big|_{\partial\Omega}$ exists at almost every point of $\partial\Omega$. Similarly, if $u \in C^0_{loc}(\Omega)$ satisfies

$$Lu = 0 \ \text{ in } \Omega, \quad u^*(x) < +\infty \ \text{ for a.e. } x \in \partial\Omega, \tag{3.2}$$

then $u\big|_{\partial\Omega}$ exists a.e. on $\partial\Omega$.

Proof. For each $k \in \mathbb{Z}^+$, set

$$\Lambda_k := \{x \in \partial\Omega : (\nabla u)^*(x) \leq k\}, \quad \Omega_k := \bigcup_{x \in \Lambda_k} \gamma(x). \tag{3.3}$$

Recall that $\gamma(x) \subset \Omega$ stands for the nontangential approach region with vertex at $x \in \partial\Omega$ (introduced in connection with (1.6)). Note that $\Omega_k \subset \Omega$ and $\Lambda_k \subset \partial\Omega_k$. Since $(\nabla u)^*(x) < +\infty$ at a.e. $x \in \partial\Omega$, we infer that the set

$$A := \partial\Omega \setminus \bigcup_{k \geq 1} \Lambda_k \tag{3.4}$$

has zero surface measure. Also, Ω_k is a Lipschitz domain and

$$\|(\nabla u)^*\|_{L^\infty(\partial\Omega_k)} < \infty$$

for each k. Now it follows from the results on the L^2-regularity problem in §8 of [MT] that there exists $B_k \subset \partial\Omega_k$ of zero surface measure such that $(\nabla u)(x)$ exists in the nontangential sense at each $x \in \Lambda_k \setminus B_k$.

Thus the pointwise nontangential boundary trace $(\nabla u)\big|_{\partial\Omega}$ exists at every point of $\partial\Omega$ except perhaps those in $A \cup (\cup_k B_k)$. Since this latter set has zero surface measure, the first part of Proposition 3.1 is proven. The second part is proven similarly, based on the solution of the L^2-Dirichlet problem in [MT]. □

We next obtain conditions guaranteeing that various boundary data belong to the Hardy space $\mathfrak{h}^1(\partial\Omega)$. Let ν denote the unit conormal to Ω. Also, let d, δ stand, respectively, for the usual exterior derivative operator and its adjoint, and denote by $\wedge$ the exterior product of forms.

Proposition 3.2. *Suppose $u \in C^1_{loc}(\Omega)$ and $(\nabla u)^* \in L^1(\partial\Omega)$. Then $\nu \wedge du\big|_{\partial\Omega} \in \mathfrak{h}^1(\partial\Omega)$ and*

$$\|\nu \wedge du\|_{\mathfrak{h}^1(\partial\Omega)} \leq C\|(\nabla u)^*\|_{L^1(\partial\Omega)}. \tag{3.5}$$

Furthermore, if $\Delta u = 0$ in Ω, then also $\partial_\nu u \in \mathfrak{h}^1(\partial\Omega)$ and

$$\|\partial_\nu u\|_{\mathfrak{h}^1(\partial\Omega)} \leq C\|(\nabla u)^*\|_{L^1(\partial\Omega)}. \tag{3.6}$$

Proof. As in [DK], our proof uses two ingredients: the duality result $\mathfrak{h}^1(\partial\Omega) = \text{vmo}(\partial\Omega)^*$, and Varopoulos's extension theorem ([Var]), to the effect that $f \in \mathrm{Lip}(\partial\Omega)$ has an extension $F \in \mathrm{Lip}(\overline{\Omega})$ such that $|\nabla F|\, d\mathrm{Vol}$ is a Carleson measure in Ω with norm $\leq C\|f\|_{\mathrm{bmo}(\partial\Omega)}$. Recall that a positive measure μ on Ω is called Carleson if

$$\mu(\Omega \cap B_r(p)) \leq C r^{n-1}, \qquad \forall\, p \in \partial\Omega,\ r > 0. \tag{3.7}$$

The least constant in (3.7) is called the Carleson norm of μ. Now Varopoulos's extension F also satisfies

$$\|F\|_{L^p(\Omega)} \leq C_p \|f\|_{\mathrm{bmo}(\partial\Omega)}, \quad \forall\ p < \infty. \tag{3.8}$$

In particular, with $p = n$, this implies that $|F|\, d\mathrm{Vol}$ is also a Carleson measure in Ω with norm controlled in terms of $\|f\|_{\mathrm{bmo}(\partial\Omega)}$.

Then, with a slight change in notation, i.e., considering f to be a 2-form, we have

$$\begin{aligned} \left| \int_{\partial\Omega} \langle f, \nu \wedge du\rangle \, d\sigma \right| &= \left| \int_{\Omega} \langle \delta F, du\rangle \, d\mathrm{Vol} \right| \\ &\leq C\|f\|_{\mathrm{bmo}(\partial\Omega)} \|(\nabla u)^*\|_{L^1(\partial\Omega)}, \end{aligned} \tag{3.9}$$

integrating by parts, utilizing $|\delta F| \leq C|\nabla F| + C|F|$ and invoking the basic Carleson measure estimate (cf., e.g., [St]). This proves (3.5).

The proof of (3.6) is similar. In this case, take f to be scalar and use the identity

$$\int_{\Omega} (\Delta u) F \, d\mathrm{Vol} + \int_{\Omega} \langle \nabla u, \nabla F\rangle \, d\mathrm{Vol} = \int_{\partial\Omega} (\partial_\nu u) f \, d\sigma \tag{3.10}$$

to obtain

$$\left| \int_{\partial\Omega} (\partial_\nu u) f \, d\sigma \right| \leq C\|f\|_{\mathrm{bmo}(\partial\Omega)} \|(\nabla u)^*\|_{L^1(\partial\Omega)}, \tag{3.11}$$

provided $\Delta u = 0$. This finishes the proof of the proposition. □

If we replace the hypothesis $\Delta u = 0$ by $Lu = h$, then (3.11) is replaced by

$$\left| \int_{\partial\Omega} (\partial_\nu u) f \, d\sigma \right| \leq C\|f\|_{\mathrm{bmo}(\partial\Omega)} \|(\nabla u)^*\|_{L^1(\partial\Omega)} + \int_{\Omega} |h + Vu| \cdot |F| \, d\mathrm{Vol}. \tag{3.12}$$

We have

$$\|u\|_{L^{n/(n-1)}(\Omega)} \le C\|(\nabla u)^*\|_{L^1(\partial\Omega)} + C\|u\|_{L^1(\Omega)}, \tag{3.13}$$

so, since $V \in L^\infty(\Omega)$, (3.7) and (3.12) give

$$\left| \int_{\partial\Omega} (\partial_\nu u) f \, d\sigma \right| \le C\|f\|_{\mathrm{bmo}(\partial\Omega)} \left\{ \|(\nabla u)^*\|_{L^1(\partial\Omega)} + \|u\|_{L^1(\Omega)} + C_q\|Lu\|_{L^q(\Omega)} \right\}, \tag{3.14}$$

for each $q > 1$. Thus we can prove:

Proposition 3.3. *If $u \in C^1_{loc}(\Omega)$, $(\nabla u)^* \in L^1(\partial\Omega)$, and $Lu = h \in L^q(\Omega)$ for some $q > 1$, then $\partial_\nu u \in \mathfrak{h}^1(\partial\Omega)$, and we have*

$$\|\partial_\nu u\|_{\mathfrak{h}^1(\partial\Omega)} \le C\|(\nabla u)^*\|_{L^1(\partial\Omega)} + C_q\|Lu\|_{L^q(\Omega)} + C\left| \int_\Omega u \, d\,Vol \right|. \tag{3.15}$$

Proof. It only remains to note that

$$\|u\|_{L^1(\Omega)} \le \left| \int_\Omega u \, d\mathrm{Vol} \right| + C\|(\nabla u)^*\|_{L^1(\partial\Omega)}. \tag{3.16}$$

□

4. Estimates on the Neumann kernel.

In this section we define the Neumann kernel and establish some estimates that will be useful for results on the Neumann problem in §5. As in §1, we assume M is a compact Riemannian manifold with a Lipschitz metric tensor, having a Laplace-Beltrami operator Δ. Also, $V \in L^\infty(M)$ satisfies $V \ge 0$ on M and $V > 0$ on some set of positive measure in each connected component of $M \setminus \overline{\Omega}$. Assume $\dim M > 2$ and let Ω be a connected Lipschitz domain in M.

We first treat the case when $V > 0$ on a set of positive measure in Ω, so

$$L = \Delta - V : H^{1,2}(\Omega) \longrightarrow H^{1,2}(\Omega)^* \tag{4.1}$$

is invertible. Then the unique solution $u \in H^{1,2}(\Omega)$ to

$$(\Delta - V)u = f \quad \text{in } \Omega, \quad \partial_\nu u\big|_{\partial\Omega} = 0 \tag{4.2}$$

is given by

$$u(x) = \int_\Omega N(x,y) f(y)\, d\mathrm{Vol}(y), \qquad x \in \Omega. \tag{4.3}$$

In this section we produce some estimates on the Neumann kernel $N(x,y)$. It is easy to see that, for each $y \in \overline{\Omega}$, $N(\cdot,y) \in C^1_{\mathrm{loc}}(\Omega \setminus \{y\})$. Further local regularity follows from material in §3 of [MT]. Here we seek further control of $N(x,y)$ as $x \to \partial\Omega$ and as $x \to y$. We begin with the following simple but useful estimate on solutions to (4.2).

Proposition 4.1. *Given $f \in L^2(\Omega)$, the solution $u \in H^{1,2}(\Omega)$ to (4.2) satisfies*

$$\|u\|_{L^{2n/(n-2)}(\Omega)} \le C\, \|f\|_{L^{2n/(n+2)}(\Omega)}. \tag{4.4}$$

Proof. Sobolev's inequality gives

$$\|u\|_{L^{2n/(n-2)}(\Omega)} \le C\, \|u\|_{H^{1,2}(\Omega)}. \tag{4.5}$$

The invertibility of (4.1) gives

$$\|u\|^2_{H^{1,2}(\Omega)} \le C(V) \int_\Omega \{|\nabla u|^2 + V|u|^2\}\, d\mathrm{Vol}, \tag{4.6}$$

and the variational characterization of u as a solution to (4.2) shows the right side of (4.6) is equal to $C(V)\int_\Omega u\bar{f}\, d\mathrm{Vol}$, so we have

$$\|u\|^2_{L^{2n/(n-2)}(\Omega)} \le C \int_\Omega u\bar{f}\, d\mathrm{Vol}, \tag{4.7}$$

from which (4.4) follows. □

Now we obtain a pointwise estimate on solutions to (4.2).

Proposition 4.2. *Suppose $f \in L^2(\Omega)$ and $u \in H^{1,2}(\Omega)$ solves (4.2). Assume $K \subset \overline{\Omega}$ is compact and*

$$\mathit{supp}\, f \subset K. \tag{4.8}$$

Then, for $x \in \overline{\Omega} \setminus K$,

$$|u(x)| \le C\, \|f\|_{L^{2n/(n+2)}(\Omega)}\, \mathit{dist}(x,K)^{-(n-2)/2}. \tag{4.9}$$

Proof. As in [DK], [KP], we can extend u by "reflection" to $\tilde{u}$ on a neighborhood $\tilde{\Omega}$ of $\overline{\Omega}$ such that $\tilde{u} \in H^{1,2}_{\text{loc}}(\tilde{\Omega})$ satisfies

$$L\tilde{u} = \tilde{f}, \tag{4.10}$$

where $\tilde{f}$ is obtained from f by such a reflection, and hence is supported on $\widetilde{K}$, the union of K and its image under this reflection. Here L is a uniformly elliptic, divergence-form operator with L^∞ coefficients, and we have from Proposition 4.1 the estimate

$$\|\tilde{u}\|_{L^{2n/(n-2)}(\tilde{\Omega})} \leq C\,\|f\|_{L^{2n/(n+2)}(\Omega)}. \tag{4.11}$$

Assume dist$\,(x, K) = r_0,\ x \in \overline{\Omega}$. Then dist$\,(x, \widetilde{K}) \geq r_0/A$ for some $A \in (1, \infty)$. If $r_0 \leq \inf\{\text{dist}\,(y, M \setminus \tilde{\Omega}) : y \in \overline{\Omega}\}$, then

$$L\tilde{u} = 0 \quad \text{in } B_{r_0/A}(x). \tag{4.12}$$

If we dilate $B_{r_0/A}(x)$ out to a ball of unit radius, we have the $L^{2n/(n-2)}$-norm of the dilated solution bounded by

$$C\|f\|_{L^{2n/(n+2)}(\Omega)} \left(r_0^{-n}\right)^{(n-2)/2n}. \tag{4.13}$$

The DeGiorgi-Nash-Moser estimates (cf. [Mor], [GT]) then imply (4.9). □

We can now use (4.3) to estimate $N(x, y)$.

Proposition 4.3. *We have*

$$|N(x,y)| \leq C\ \mathit{dist}\,(x,y)^{-(n-2)}. \tag{4.14}$$

Proof. Let $r := \text{dist}\,(x, y) > 0$ and let $K := \overline{\Omega} \cap B_r(y)$. Applying Proposition 4.2 to f supported in K, we have by duality

$$\left\{ \int_{\Omega\cap B_r(y)} |N(x,y)|^{2n/(n-2)}\, d\text{Vol}(y) \right\}^{(n-2)/2n} \leq C\, r^{-(n-2)/2}. \tag{4.15}$$

Now $N(x, y) = N(y, x)$, so $N(x, y)$ also solves a uniformly elliptic, divergence-form PDE with L^∞ coefficients as a function of y, for $y \in B_r(x)$. Hence another application of the DeGiorgi-Nash-Moser theory gives (4.14). □

The DeGiorgi-Nash-Moser theory also gives Hölder estimates, with an exponent depending on the ellipticity constant of L. Keeping in mind that these uniform Hölder estimates apply to the dilates of functions on B_r, dilated out to a ball of unit radius, we have:

Proposition 4.4. *There exists* $s \in (0,1)$ *such that*

$$|N(x,y) - N(x',y)| \le C\frac{dist\,(x,x')^s}{dist\,(x,y)^{n-2+s}}, \tag{4.16}$$

for $dist\,(x,x') \le \frac{1}{2} dist\,(x,y)$, *and*

$$|N(x,y) - N(x,y')| \le C\frac{dist\,(y,y')^s}{dist\,(x,y)^{n-2+s}}, \tag{4.17}$$

for $dist\,(y,y') \le \frac{1}{2} dist\,(x,y)$.

Proof. The estimate (4.16) follows from the discussion above, and then (4.17) follows from the symmetry

$$N(x,y) = N(y,x). \tag{4.18}$$

□

We now describe the modifications for the case when $V = 0$ on Ω. In that case, given $f \in H^{1,2}(\Omega)^*$, the problem

$$\Delta u = f \text{ in } \Omega, \quad \partial_\nu u\big|_{\partial\Omega} = 0, \tag{4.19}$$

has a solution $u \in H^{1,2}(\Omega)$ if and only if $\int_\Omega f\,d\mathrm{Vol} = 0$, and the solution is unique modulo an additive constant. We define

$$T : H^{1,2}(\Omega)^* \longrightarrow H^{1,2}(\Omega) \tag{4.20}$$

by $T1 = 0$, and, if $\int_\Omega f\,d\mathrm{Vol} = 0$, Tf is the unique solution to (4.19) satisfying $\int_\Omega u\,d\mathrm{Vol} = 0$. Then $N(x,y)$ is the integral kernel of this operator:

$$Tf(x) = \int_\Omega N(x,y) f(y)\,d\mathrm{Vol}(y), \qquad x \in \Omega. \tag{4.21}$$

As before, one readily verifies that, for each $y \in \overline{\Omega}$, $N(\cdot,y) \in C^1_{\mathrm{loc}}(\Omega \setminus \{y\})$. Also $N(x,y)$ still has the symmetry property (4.18). As in Proposition 4.1, we have

$$\|Tf\|_{L^{2n/(n-2)}(\Omega)} \le C\|f\|_{L^{2n/(n+2)}(\Omega)}. \tag{4.22}$$

The arguments used in Propositions 4.2 and 4.3 also extend, and we see that the estimates (4.16)–(4.17) also hold in this case.

5. Hardy space and L^p estimates for the Neumann problem.

We retain the hypotheses on M, Δ, V and Ω made in §4. Here we examine the Neumann problem

$$(5.1)\qquad (\Delta - V)u = 0 \quad \text{in } \Omega, \quad \partial_\nu u\big|_{\partial\Omega} = g.$$

In the setting of a Riemannian manifold with C^1 metric tensor, this problem was studied in [MT]. Furthermore, as explained in §2, the same results continue to hold when the metric tensor has only Lipschitz components. What we shall need here is the fact that, if $g \in L^2(\partial\Omega)$, then (5.1) has a unique solution $u \in C^{2-\varepsilon}_{\text{loc}}(\Omega)$, $\forall\, \varepsilon > 0$, satisfying

$$(5.2)\qquad \|(\nabla u)^*\|_{L^2(\partial\Omega)} \leq C\|g\|_{L^2(\partial\Omega)}.$$

The solution is given by

$$(5.3)\qquad u = \mathcal{S}\left(\left(-\frac{1}{2}I + K^*\right)^{-1} g\right),$$

where

$$(5.4)\qquad -\frac{1}{2}I + K^* : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega)$$

is shown to be invertible. All this holds if $V > 0$ on a set of positive measure in Ω. If $V = 0$ on Ω, we assume $\int_{\partial\Omega} g\, d\sigma = 0$ in (5.1), the solution u satisfying (5.2) is unique up to an additive constant, etc.

Here we aim to estimate the L^1-norm of $(\nabla u)^*$ when g belongs to the Hardy space $\mathfrak{h}^1(\partial\Omega)$. This can be done once we estimate $(\nabla u)^*$ when g is an atom. Recall that an atom on $\partial\Omega$ is a function $g \in L^\infty(\partial\Omega)$ satisfying

$$(5.5)\qquad \text{supp}\ g \subset B_r(x_0) \cap \partial\Omega,$$

for some $x_0 \in \partial\Omega$, $0 < r \leq \text{diam}\, M$, and

$$(5.6)\qquad \|g\|_{L^\infty(\partial\Omega)} \leq \frac{1}{r^{n-1}}, \quad \int_{\partial\Omega} g\, d\sigma = 0.$$

The following is a key estimate. A result of this sort was first given in [KP]; our proof is closer in spirit to an argument in [Sh]. For now, assume $V > 0$ on a set of positive measure in Ω.

Proposition 5.1. *If $g \in L^\infty(\partial\Omega)$ is an atom, then the solution u to (5.1) satisfies*

$$\|(\nabla u)^*\|_{L^1(\partial\Omega)} \leq C. \tag{5.7}$$

In the proof of this proposition, the following pointwise estimate is useful. Recall that we are assuming $V > 0$ on a set of positive measure in Ω.

Lemma 5.2. *If u is as in Proposition 5.1, we have*

$$|u(x)| \leq C \frac{r^s}{dist\,(x, x_0)^{n-2+s}}, \tag{5.8}$$

for $x \in \overline{\Omega}$, $dist\,(x, x_0) \geq 4r$, where $s \in (0,1)$ is as in (4.17).

Proof. The solution u to the Neumann problem (5.1) can be written

$$u(x) = \int_{\partial\Omega} N(x,y) g(y)\, d\sigma(y), \qquad x \in \Omega, \tag{5.9}$$

where $N(x,y)$ is the Neumann kernel, studied in §4. Since $\int_{\partial\Omega} g\, d\sigma = 0$, we can write

$$u(x) = \int_{\partial\Omega} \{N(x,y) - N(x,x_0)\}\, g(y)\, d\sigma(y), \qquad x \in \Omega. \tag{5.10}$$

Then the estimate (5.8) follows from (4.17). □

We now present the

Proof of Proposition 5.1. Given (5.5)–(5.6), let $S_1 := B_{4r}(x_0) \cap \partial\Omega$, and for $\ell \geq 2$ (and $2^\ell r \leq \operatorname{diam}\Omega$), set

$$B_\ell := B_{2^{\ell+1}r}(x_0) \setminus B_{2^\ell r}(x_0), \quad S_\ell := B_\ell \cap \partial\Omega. \tag{5.11}$$

We will estimate $(\nabla u)^*$ on each set S_ℓ. First

$$\begin{aligned} \int_{S_1} (\nabla u)^*\, d\sigma &\leq C\, r^{(n-1)/2} \left\{ \int_{S_1} |(\nabla u)^*|^2\, d\sigma \right\}^{1/2} \\ &\leq C\, r^{(n-1)/2} \left\{ \int_{\partial\Omega} |(\nabla u)^*|^2\, d\sigma \right\}^{1/2} \\ &\leq C\, r^{(n-1)/2}\, \|g\|_{L^2(\partial\Omega)}, \end{aligned} \tag{5.12}$$

where the last inequality follows by (5.2). By (5.6), $\|g\|_{L^2(\partial\Omega)} \le Cr^{-(n-1)/2}$ so the contribution coming from S_1 has the proper control.

Estimates on S_ℓ for $\ell \ge 2$ will involve several ingredients, including (5.8) and Caccioppoli estimates. To proceed, we introduce

$$I_\ell := \int_{S_\ell} (\nabla u)^* \, d\sigma, \tag{5.13}$$

for $\ell \ge 2$, and set

$$\Omega_{\ell,t} := \Omega \setminus B_{2^\ell rt}(x_0), \quad \frac{1}{2} \le t \le 1. \tag{5.14}$$

Notice that all domains (5.14) have a uniformly bounded Lipschitz constant (at least if $2^\ell r$ is not large; say $2^\ell r \le A$). Also, pick A so that there is a set Q of positive measure in Ω, disjoint from all the sets B_ℓ with $2^\ell r \le A$, such that $V > 0$ on Q.

Now, we have

$$\begin{aligned} I_\ell &\le C(2^\ell r)^{(n-1)/2} \left\{ \int_{S_\ell} |(\nabla u)^*|^2 \, d\sigma \right\}^{1/2} \\ &\le C(2^\ell r)^{(n-1)/2} \left\{ \int_{\partial\Omega_{\ell,t}} |(\nabla u)^*|^2 \, d\sigma \right\}^{1/2} \\ &\le C(2^\ell r)^{(n-1)/2} \left\{ \int_{\partial\Omega_{\ell,t}} |\partial_\nu u|^2 \, d\sigma \right\}^{1/2}. \end{aligned} \tag{5.15}$$

The last inequality holds by the analogue of (5.2) for this family of Lipschitz domains.

Note that, in the last integral in (5.15), $\partial_\nu u$ is supported on $\partial B_{2^\ell rt}(x_0) \cap \overline{\Omega}$. Integrating over $t \in [1/2, 1]$ gives

$$I_\ell \le C(2^\ell r)^{(n-2)/2} \left\{ \int_{\Omega\cap B_{\ell-1}} |\nabla u|^2 \, d\mathrm{Vol} \right\}^{1/2}. \tag{5.16}$$

Using the fact that $-\Delta u + Vu = 0$ on $B_{\ell-1} \cap \Omega$ and keeping in mind the reflection argument mentioned in §4, we now apply Caccioppoli's inequality (cf., e.g., Lemma 1.1.5 in [Ke]) to this last integral; we arrive at

$$I_\ell \leq C(2^\ell r)^{(n-4)/2} \left\{ \int_{\Omega \cap \mathcal{B}_\ell} |u|^2 \, d\mathrm{Vol} \right\}^{1/2}, \tag{5.17}$$

with $\mathcal{B}_\ell := B_{\ell-2} \cup B_{\ell-1} \cup B_\ell \cup B_{\ell+1}$. Now we apply the estimate (5.8) on u to deduce

$$I_\ell \leq C(2^\ell r)^{(n-4)/2} (2^\ell r)^{n/2} \frac{r^s}{(2^\ell r)^{n-2+s}} = C\, 2^{-s\ell}, \tag{5.18}$$

for some $s \in (0,1)$. Note that all the r's cancel in (5.18).

We have from (5.18) that

$$\sum_{\ell=1}^{N} \|(\nabla u)^*\|_{L^1(S_\ell)} \leq C,$$

where N is chosen so that $2^N r \approx A$. The estimate of $(\nabla u)^*$ on the remainder of $\partial\Omega$ follows from the same analysis as that for $(\nabla u)^*$ on S_N just done. Thus Proposition 5.1 is proven. □

Given the characterization (1.37)–(1.38) of the Hardy space $\mathfrak{h}^1(\partial\Omega)$, we have:

Proposition 5.3. *The solution operator $g \mapsto u$ for the Neumann boundary problem (5.1) has a unique continuous extension from $L^2(\partial\Omega)$ to $\mathfrak{h}^1(\partial\Omega)$, with*

$$\|(\nabla u)^*\|_{L^1(\partial\Omega)} \leq C\|g\|_{\mathfrak{h}^1(\partial\Omega)}. \tag{5.19}$$

The modifications needed for the case when $V = 0$ on Ω follow a well-worn path so, below, we no longer insist that $V > 0$ on a set of positive measure in Ω. We state the result for the Neumann problem.

Theorem 5.4. *Let $g \in \mathfrak{h}^1(\partial\Omega)$; if $V = 0$ on Ω assume also $\int_{\partial\Omega} g \, d\sigma = 0$. Then the Neumann problem (5.1) has a solution u satisfying (5.19), and also $u \in C^{2-\varepsilon}_{loc}(\Omega)$, $\forall\, \varepsilon > 0$. If $V > 0$ on a set of positive measure on Ω, then such u is unique. If $V = 0$ on Ω, then u is unique up to an additive constant.*

It remains to prove uniqueness, which we now do.

Proposition 5.5. *Let $u \in C^1_{loc}(\Omega)$ satisfy $(\nabla u)^* \in L^1(\partial\Omega)$. Then*

$$Lu = 0 \ \text{in} \ \Omega, \quad \partial_\nu u\big|_{\partial\Omega} = 0 \Longrightarrow u \ \text{constant in} \ \Omega. \tag{5.20}$$

The constant is zero if $V > 0$ on a set of positive measure in Ω.

Proof. Let Θ be a smooth vector field on M that is everywhere transverse to $\partial\Omega$, pointing into $\overline{\Omega}$. Denote by $\mathcal{F}_t$ the flow on M generated by Θ. Let $g_t = \mathcal{F}_t^* g$ denote the metric tensor on M that is the pull-back of the original metric g under $\mathcal{F}_t$. Also, let Δ_t denote the Laplace operator on M for the metric g_t and $d\sigma_t$ the surface measure on $\partial\Omega$ induced by this metric tensor. Set $v_t(x) = \mathcal{F}_t^* v(x) = v(\mathcal{F}_t x)$ and $L_t = -\Delta_t + V_t$. Finally, denote the Neumann kernel on Ω for L_t by $N^t(x,y)$.

By the hypotheses on u, for each $t > 0$, $L_t u_t = 0$ on Ω, and $u_t\big|_{\overline{\Omega}}$ is regular enough that we can write (modulo constants if $V = 0$ on Ω)

$$u_t(x) = \int_{\partial\Omega} N^t(x,y)\,\frac{\partial u_t}{\partial \nu_t}(y)\rho_t(y)\,d\sigma(y), \quad \text{for each } x \in \Omega, \tag{5.21}$$

where $\rho_t := d\sigma_t/d\sigma$ is the Radon-Nikodym derivative of $d\sigma_t$ with respect to the original surface measure $d\sigma$. In the light of the discussion in §3 and our hypotheses, we see that

$$\frac{\partial u_t}{\partial \nu_t}\rho_t \longrightarrow \frac{\partial u}{\partial \nu}, \quad \text{in } L^1(\partial\Omega) \quad \text{as } t \searrow 0. \tag{5.22}$$

Also, by the results in §4, we have

$$|N^t(x,y)| \le C(x), \quad \text{uniformly for } t \in (0,A] \text{ and } y \in \partial\Omega. \tag{5.23}$$

If $V > 0$ on a set of positive measure in Ω, pick A so this holds on $\mathcal{F}_t(\Omega)$, for $t \in [0, A]$.

The hypothesis that $\partial u/\partial\nu = 0$ on $\partial\Omega$ then yields that, as $t \searrow 0$, the integral on the right side of (5.21) vanishes, and the proposition is proven. □

We now have the following result on the Neumann problem with L^p data.

Theorem 5.6. *There is an $\varepsilon = \varepsilon(\Omega) > 0$ with the following property. Take $p \in (1, 2+\varepsilon)$. Let $g \in L^p(\partial\Omega)$. If $V = 0$ on Ω assume also $\int_{\partial\Omega} g\,d\sigma = 0$. Then the Neumann problem (5.1) has a solution satisfying*

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \le C_p \|g\|_{L^p(\partial\Omega)}. \tag{5.24}$$

If $V > 0$ on a set of positive measure in Ω, then such u is unique. If $V = 0$ on Ω, then u is unique up to an additive constant.

Proof. As discussed at the beginning of this section, the case $p = 2$ was treated in [MT]. The existence of a solution satisfying (5.24) for $p \in (1, 2]$ then follows from Theorem 5.4 by interpolation. The result for $|p - 2| < \varepsilon$ was also treated in [MT]. Finally, uniqueness follows from Proposition 5.5. □

6. The Dirichlet regularity problem.

We retain the hypotheses on M, Δ, V and Ω made in §1. Here we wish to examine the Dirichlet problem

$$(\Delta - V)u = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f. \tag{6.1}$$

In the setting of a Riemannian manifold with a C^1 metric tensor this was studied in [MT]. There it was shown that, if $\nabla_T f \in L^2(\partial\Omega)$, then (6.1) has a unique solution $u \in C^{2-\varepsilon}_{\text{loc}}(\Omega)$, $\forall\, \varepsilon > 0$, satisfying

$$\|u^*\|_{L^2(\partial\Omega)} + \|(\nabla u)^*\|_{L^2(\partial\Omega)} \le C\|\nabla_T f\|_{L^2(\partial\Omega)} + C\|f\|_{L^2(\partial\Omega)}. \tag{6.2}$$

The solution is given by

$$u = \mathcal{S}(S^{-1} f), \tag{6.3}$$

where

$$S : L^2(\partial\Omega) \longrightarrow H^{1,2}(\partial\Omega) \tag{6.4}$$

is shown to be invertible. Again we recall that the extension of this material to the setting of Lipschitz metric tensors follows from the material of §2.

Here we aim to estimate the L^1-norm of $(\nabla u)^*$ when $\nabla_T f$, the tangential gradient of f, belongs to the (vector) Hardy space $\mathfrak{h}^1(\partial\Omega)$. This can be done once we estimate $(\nabla u)^*$ when $\nabla_T f$ is a vector atom. That is, we assume $f \in \text{Lip}(\partial\Omega)$ satisfies

$$\operatorname{supp} f \subset B_r(x_0) \cap \partial\Omega \tag{6.5}$$

for some $x_0 \in \partial\Omega$, $0 < r \le \operatorname{diam} M$, and

$$\|\nabla_T f\|_{L^\infty(\partial\Omega)} \le \frac{1}{r^{n-1}}. \tag{6.6}$$

For now we merely note that (6.5)–(6.6) imply

$$\|f\|_{L^\infty(\partial\Omega)} \le \frac{C}{r^{n-2}}. \tag{6.7}$$

Proposition 6.1. *If $f \in Lip(\partial\Omega)$ satisfies (6.5)–(6.6), then the solution u to (6.1) satisfies*

$$\|(\nabla u)^*\|_{L^1(\partial\Omega)} \le C. \tag{6.8}$$

Proof. Define S_ℓ and B_ℓ as in the proof of Proposition 5.1. As in that case we will estimate $(\nabla u)^*$ on each set S_ℓ. First, parallel to (5.12), we have

$$\begin{aligned} \int_{S_1} (\nabla u)^*\, d\sigma &\le C\, r^{(n-1)/2} \left\{ \int_{\partial\Omega} |(\nabla u)^*|^2\, d\sigma \right\}^{1/2} \\ &\le C\, r^{(n-1)/2} \left\{ \|f\|_{L^2(\partial\Omega)} + \|\nabla_T f\|_{L^2(\partial\Omega)} \right\}, \end{aligned} \tag{6.9}$$

where the last inequality is a consequence of the estimate (6.2). Since, by (6.6), $\|f\|_{L^2(\partial\Omega)} + \|\nabla_T f\|_{L^2(\partial\Omega)} \le C\, r^{-(n-1)/2}$, we have the right bound for this piece.

As in the proof of Proposition 5.1, we will need a (pointwise) decay estimate on u. We claim that, under the current hypotheses,

$$|u(x)| \le C \frac{r^\alpha}{\operatorname{dist}(x, x_0)^{n-2+\alpha}}, \tag{6.10}$$

for $x \in \overline{\Omega}$, for some $\alpha \in (0,1)$ independent of the atomic data; we will establish this result below. Armed with (6.10) we now proceed to estimate

$$I_\ell = \int_{S_\ell} (\nabla u)^*\, d\sigma \tag{6.11}$$

for $\ell \ge 2$. Define $\Omega_{\ell,t}$ as in (5.9), again for $2^\ell r \le A$. Now, parallel to (5.15), we have

$$\begin{aligned} I_\ell &\le C(2^\ell r)^{(n-1)/2} \left\{ \int_{\partial\Omega_{\ell,t}} |(\nabla u)^*|^2\, d\sigma \right\}^{1/2} \\ &\le C(2^\ell r)^{(n-1)/2} \left\{ \int_{\partial\Omega_{\ell,t}} \left[|\nabla_T u|^2 + |u|^2 \right] d\sigma \right\}^{1/2}. \end{aligned} \tag{6.12}$$

The last inequality in (6.12) holds by the analogue of (6.2) for this family of Lipschitz domains. The integrand in the last integral in (6.12) is supported in $\partial B_{2^\ell rt}(x_0) \cap \overline{\Omega}$. Integrating over $t \in [1/2, 1]$ gives

$$I_\ell \leq C(2^\ell r)^{(n-2)/2} \left\{ \int_{\Omega \cap B_{\ell-1}} \left[|\nabla u|^2 + |u|^2 \right] d\mathrm{Vol} \right\}^{1/2}, \tag{6.13}$$

which is the analogue of (5.16) in the present context. Since the reflection argument from §4 is no longer available, this time we invoke a boundary Caccioppoli estimate in order to obtain

$$I_\ell \leq C(2^\ell r)^{(n-4)/2} \left\{ \int_{\Omega \cap \mathcal{B}_\ell} |u|^2 \, d\mathrm{Vol} \right\}^{1/2}, \tag{6.14}$$

with $\mathcal{B}_\ell := B_{\ell-2} \cup B_{\ell-1} \cup B_\ell \cup B_{\ell+1}$. That this works is guaranteed by the fact that $u|_{\partial\Omega \cap \mathcal{B}_\ell} = 0$; see Lemma 1.1.21 in [Ke] for details in similar circumstances. Now, as in (5.18), an application of the estimate (6.10) yields

$$I_\ell \leq C\, 2^{-\alpha \ell}, \tag{6.15}$$

and hence

$$\sum_{\ell=1}^{N} \|(\nabla u)^*\|_{L^1(S_\ell)} \leq C,$$

where N is chosen so that $2^N r \approx A$. The estimate of $(\nabla u)^*$ on the remainder of $\partial\Omega$ follows by the same sort of analysis as that for $(\nabla u)^*$ on S_N just done. Thus Proposition 6.1 is proven, modulo the task of establishing (6.10). □

By (6.7), the estimate (6.10) is an obvious consequence of the following more general result.

Proposition 6.2. *There exists $\alpha \in (0,1)$ and $C \in (0,\infty)$ with the following property. Given $p \in \partial\Omega$, $r \in (0, \mathrm{diam}\,\Omega)$, and $f \in L^\infty(\partial\Omega)$ with support in $B_r(p) \cap \partial\Omega$, let u solve the Dirichlet problem*

$$Lu = 0 \ \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f. \tag{6.16}$$

Then

$$|u(x)| \leq C\, \|f\|_{L^\infty(\partial\Omega)}\, r^{\alpha+n-2}\, \mathrm{dist}\,(x,p)^{2-n-\alpha}, \tag{6.17}$$

for all $x \in \overline{\Omega}$.

Proof. It suffices to show that, given $\operatorname{supp} f \subset B_r(p) \cap \partial\Omega$,

$$\|f\|_{L^\infty(\partial\Omega)} \le r^{2-n-\alpha} \Longrightarrow |u(x)| \le C \operatorname{dist}(x,p)^{2-n-\alpha}, \quad \forall x \in \overline{\Omega}. \tag{6.18}$$

To do this, we will show that there exist constants $C \in (0,\infty)$, $\alpha \in (0,1)$ and, for each $p \in \partial\Omega$, a function $\varphi_p \in C(\overline{\Omega} \setminus p)$ such that

$$\operatorname{dist}(x,p)^{2-n-\alpha} \le \varphi_p(x) \le C \operatorname{dist}(x,p)^{2-n-\alpha} \quad \text{on } \overline{\Omega}, \tag{6.19}$$

and

$$(\Delta - V)\varphi_p \le 0, \quad \text{in } \Omega. \tag{6.20}$$

Granted this, we establish (6.18) as follows. There is no loss of generality in assuming $f \ge 0$. Then the hypotheses on f imply

$$0 \le f \le \varphi_p \text{ on } \partial\Omega, \quad (\Delta - V)\varphi_p \le (\Delta - V)u \text{ in } \Omega. \tag{6.21}$$

Consequently, the maximum principle implies

$$0 \le u \le \varphi_p \text{ on } \overline{\Omega}, \tag{6.22}$$

which yields (6.18).

It remains to construct the functions $\varphi_p(x)$. We begin with the case where Ω is a Lipschitz domain in $\mathbb{R}^n$ and $\Delta = \Delta_0$ is the flat-space Laplacian, $\Delta_0 = \partial_1^2 + \cdots + \partial_n^2$. Given $p \in \partial\Omega$, let Γ_p be a truncated circular cone in $\mathbb{R}^n \setminus \Omega$, with vertex at p. Extend Γ_p to an infinite cone C_p with vertex at p. We will construct a function ψ_p on the complementary cone $K_p = \mathbb{R}^n \setminus C_p$, from which φ_p will in turn be constructed.

Translate p to the origin and use spherical polar coordinates (r,ω) on $\mathbb{R}^n$; $\omega \in S^{n-1}$, $r = |x| = |x-p|$. Say $S^{n-1} \cap K_p = \mathcal{O}_p$, so K_p is the cone over $\mathcal{O}_p$. We produce ψ_p in the form

$$\psi_p(r\omega) = r^{2-n-\alpha}\,\beta_p(\omega), \tag{6.23}$$

for some β_p to be specified shortly. In spherical polar coordinates we have

$$\Delta_0\psi_p = \frac{\partial^2\psi_p}{\partial r^2} + \frac{n-1}{r}\frac{\partial\psi_p}{\partial r} + \frac{1}{r^2}\Delta_S\psi_p, \tag{6.24}$$

where Δ_S is the Laplace-Beltrami operator on the unit sphere S^{n-1}. Hence (6.23) yields

$$\Delta_0\psi_p(r\omega) = \{\Delta_S\beta_p(\omega) + \alpha(n-2+\alpha)\beta_p(\omega)\}\, r^{-n-\alpha}. \tag{6.25}$$

We specify β_p as the solution to

$$\Delta_S\beta_p(\omega)+\alpha(n-2+\alpha)\beta_p(\omega)=-a_1 \text{ in } \mathcal{O}_p, \quad \beta_p\big|_{\partial\mathcal{O}_p}=b_1, \tag{6.26}$$

for certain positive constants a_1 and b_1 to be described below.

Note that if $\alpha\in(0,1)$ is small enough then $\alpha(n-2+\alpha)$ (which is positive) is smaller than the smallest eigenvalue of $-\Delta_S$ on $L^2(\mathcal{O}_p)$, with the Dirichlet boundary condition. The sets $\mathcal{O}_p$ belong to a family of circular caps in S^{n-1} with radii bounded away from 0 and π. Hence we can take $b_1>0$ large enough and $\alpha>0$, $a_1>0$ small enough to guarantee

$$1\le\beta_p(\omega)\le A, \quad |\nabla\beta_p(\omega)|\le A, \quad \forall\,\omega\in\mathcal{O}_p, \tag{6.27}$$

for some $A\in(0,\infty)$. Then we have

$$|x-p|^{2-n-\alpha}\le\psi_p(x)\le A\,|x-p|^{2-n-\alpha} \text{ in } K_p, \tag{6.28}$$

and

$$\Delta_0\psi_p(x)=-a_1\,|x-p|^{-n-\alpha} \text{ in } K_p. \tag{6.29}$$

Let us move back to $\Omega\subset M$, with a Lipschitz metric tensor. Given $p\in\partial\Omega$, pick a coordinate system centered at p such that $g_{jk}(p)=\delta_{jk}$. In this coordinate system

$$\Delta=\Delta_0+A_p(x,D)+B_p(x,D) \tag{6.30}$$

where $A_p(x,D)$ is a second-order differential operator with Lipschitz coefficients, vanishing at $x=p$, and $B_p(x,D)$ is a first-order differential operator with L^∞ coefficients. Then, with ψ_p as above, we have

$$(\Delta-V)\psi_p(x)\le-a_2\,\mathrm{dist}\,(x,p)^{-n-\alpha}, \quad \forall\,x\in B_\rho(p)\cap K_p, \tag{6.31}$$

for some $a_2>0$, $\rho>0$. Pick a cutoff function $\chi\in C_0^\infty(B_\rho(p))$ such that $\chi=1$ on $B_{\rho/2}(p)$, and set $\psi_p^\#=\chi\psi_p$. We have

$$(\Delta-V)\psi_p^\#(x)\le c_1-a_2\,\mathrm{dist}\,(x,p)^{-n-\alpha} \text{ on } \overline{\Omega}, \tag{6.32}$$

and (possibly increasing A)

$$A^{-1}\,\mathrm{dist}\,(x,p)^{2-n-\alpha}-c_2\le\psi_p^\#(x)\le A\,\mathrm{dist}\,(x,p)^{2-n-\alpha} \text{ on } \overline{\Omega}, \tag{6.33}$$

for some constants $c_1,c_2\in(0,\infty)$. Now, for some $\gamma>0$, take

$$\varphi_p:=A\psi_p^\#+\gamma\psi^b, \tag{6.34}$$

where ψ^b solves

$$(\Delta - V)\psi^b = -1 \text{ on } \overline{\Omega}, \quad \psi^b\big|_{\partial\Omega} = 1. \tag{6.35}$$

The L^∞ theory of the Dirichlet problem gives $\psi^b \in L^\infty(\Omega)$ and the maximum principle implies $\psi^b(x) \geq 1$ for all $x \in \Omega$. Taking $\gamma \in (0,\infty)$ sufficiently large, we have a function satisfying (6.19)–(6.20), and the proof of Proposition 6.2 is complete. □

Again using the atomic decomposition of $\mathfrak{h}^1(\partial\Omega)$, we have:

Theorem 6.3. *Given $f \in L^1(\partial\Omega)$ with $\nabla_T f \in \mathfrak{h}^1(\partial\Omega)$, the Dirichlet problem (6.1) has a unique solution u, satisfying $u \in C^{2-\varepsilon}_{loc}(\Omega)$, $\forall\, \varepsilon > 0$, and*

$$\|u^*\|_{L^1(\partial\Omega)} + \|(\nabla u)^*\|_{L^1(\partial\Omega)} \leq C\|\nabla_T f\|_{\mathfrak{h}^1(\partial\Omega)} + C\|f\|_{L^1(\partial\Omega)}, \tag{6.36}$$

for some $C > 0$ independent of f.

It remains only to establish uniqueness, which we now do.

Proposition 6.4. *Assume that $u \in C^1_{loc}(\Omega)$ satisfies $(\nabla u)^* \in L^1(\Omega)$. Then*

$$Lu = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = 0 \Longrightarrow u = 0 \text{ in } \Omega. \tag{6.37}$$

Proof. Take two sequences of approximating domains:

$$\Omega_j \nearrow \Omega, \quad \Omega'_k \searrow \Omega. \tag{6.38}$$

Let $\theta \in C^\infty(M)$ be given. For each k, let v_k denote the unique solution to the Dirichlet problem

$$Lv_k = 0 \text{ in } \Omega'_k, \quad v_k\big|_{\partial\Omega'_k} = \theta\big|_{\partial\Omega'_k}. \tag{6.39}$$

In particular, $v_k \in C^1(\overline{\Omega})$ for each k. Then, for each pair j, k, Green's formula in Ω_j gives

$$\int_{\partial\Omega_j} v_k \frac{\partial u}{\partial \nu_j}\, d\sigma_j = \int_{\partial\Omega_j} u \frac{\partial v_k}{\partial \nu_j}\, d\sigma_j. \tag{6.40}$$

For fixed k, as $j \to \infty$, the right side of (6.40) vanishes, while the left side converges to $\int_{\partial\Omega} v_k(\partial u/\partial\nu)\, d\sigma$. Letting then $k \to \infty$, we obtain

$\int_{\partial\Omega} \theta(\partial u/\partial\nu)\, d\sigma = 0$. Since θ is arbitrary, this gives $\partial u/\partial\nu = 0$. With this in hand, we obtain from Green's integral representation formula that $u = 0$ in Ω. □

We have the following L^p-regularity result.

Theorem 6.5. *There is an $\varepsilon = \varepsilon(\Omega) > 0$ with the following property. Take $p \in (1, 2+\varepsilon)$. Let $f \in L^p(\partial\Omega)$ and assume $\nabla_T f \in L^p(\partial\Omega)$. Then the Dirichlet problem (6.1) has a unique solution, satisfying*

$$\|u^*\|_{L^p(\partial\Omega)} + \|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C\|\nabla_T f\|_{L^p(\partial\Omega)} + C\|f\|_{L^p(\partial\Omega)}. \tag{6.41}$$

Proof. The existence follows for $1 < p \leq 2$ by interpolating the L^2 result (6.2) and the result of Theorem 6.3. The result for $|p-2| < \varepsilon$ was treated in [MT]. Finally, uniqueness follows from Proposition 6.4. □

7. Invertibility properties for layer potential operators.

Once again, retain the hypotheses on M, Δ, V and Ω which have been made in §1. In this section we study invertibility properties for layer potential operators on several function spaces of interest. We debut with:

Theorem 7.1. *Under our standard hypotheses,*

$$\frac{1}{2}I + K : L^q(\partial\Omega) \longrightarrow L^q(\partial\Omega) \text{ is invertible, } 2-\varepsilon < q < \infty. \tag{7.1}$$

If, in addition, $V = 0$ on $\overline{\Omega}$, then

$$-\frac{1}{2}I + K^* : L^p_0(\partial\Omega) \longrightarrow L^p_0(\partial\Omega) \text{ is invertible, } \quad 1 < p < 2+\varepsilon. \tag{7.2}$$

If, on the other hand, it is also assumed that $V > 0$ on a set of positive measure in Ω then

$$-\frac{1}{2}I + K^* : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \text{ is invertible, } \quad 1 < p < 2+\varepsilon. \tag{7.3}$$

Proof. The first step is to show that

$$\pm\frac{1}{2}I + K^* : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \text{ are Fredhlom with index zero} \tag{7.4}$$

for any $p \in (1, 2+\varepsilon)$. Once this is proved, the theorem follows from (7.4) and the fact that the operators in (7.1)–(7.3) are isomorphisms for $p = q = 2$ (a proof of which is contained in [MT]), via a general functional analytic argument.

To proceed in this direction, let $f \in L^p(\partial\Omega)$, $1 < p < 2 + \varepsilon$, and set $u_\pm := \mathcal{S}f$ in $\Omega_\pm$. Based on the estimates derived in §§5–6, we may write:

(7.5)

$$
\begin{aligned}
\|f\|_{L^p(\partial\Omega)} &\leq \left\|\left(\frac{1}{2}I + K^*\right) f\right\|_{L^p(\partial\Omega)} + \left\|\left(-\frac{1}{2}I + K^*\right) f\right\|_{L^p(\partial\Omega)} \\
&= \left\|\frac{\partial u_\mp}{\partial \nu}\right\|_{L^p(\partial\Omega)} + \left\|\left(\mp\frac{1}{2}I + K^*\right) f\right\|_{L^p(\partial\Omega)} \\
&\leq C\,\|(\nabla u_\mp)^*\|_{L^p(\partial\Omega)} + \left\|\left(\mp\frac{1}{2}I + K^*\right) f\right\|_{L^p(\partial\Omega)} \\
&\leq C\,\|\nabla_T u_\mp\|_{L^p(\partial\Omega)} + C\|\mathcal{S}f\|_{L^p(\partial\Omega)} + \left\|\left(\mp\frac{1}{2}I + K^*\right) f\right\|_{L^p(\partial\Omega)} \\
&= C\|\nabla_T u_\pm\|_{L^p(\partial\Omega)} + C\|\mathcal{S}f\|_{L^p(\partial\Omega)} + \left\|\left(\mp\frac{1}{2}I + K^*\right) f\right\|_{L^p(\partial\Omega)} \\
&\leq C\,\|(\nabla u_\pm)^*\|_{L^p(\partial\Omega)} + C\|\mathcal{S}f\|_{L^p(\partial\Omega)} + \left\|\left(\mp\frac{1}{2}I + K^*\right) f\right\|_{L^p(\partial\Omega)} \\
&\leq C\left\|\frac{\partial u_\pm}{\partial \nu}\right\|_{L^p(\partial\Omega)} + C\|\mathcal{S}f\|_{L^p(\partial\Omega)} + \left\|\left(\mp\frac{1}{2}I + K^*\right) f\right\|_{L^p(\partial\Omega)} \\
&= C\left\|\left(\mp\frac{1}{2}I + K^*\right) f\right\|_{L^p(\partial\Omega)} + C\|\mathcal{S}f\|_{L^p(\partial\Omega)}.
\end{aligned}
$$

That is, for each $p \in (1, 2+\varepsilon)$ there exists a constant $C = C(p, \Omega) > 0$ so that

$$
\|f\|_{L^p(\partial\Omega)} \leq C\left\|\left(\mp\frac{1}{2}I + K^*\right) f\right\|_{L^p(\partial\Omega)} + C\|\mathcal{S}f\|_{L^p(\partial\Omega)}, \tag{7.6}
$$

uniformly for $f \in L^p(\partial\Omega)$. In particular, $\mp\frac{1}{2}I + K^*$ are semi-Fredholm on $L^p(\partial\Omega)$ for each $p \in (1, 2+\varepsilon)$. The fact that (7.4) is valid for $1 < p < 2+\varepsilon$ now follows from this, the fact that (7.4) holds when $p = 2$ (cf. [MT]) and Lemma 7.2 below. □

Here is the lemma that finishes the proof of Theorem 7.1.

Lemma 7.2. *Assume that (E_0, E_1), (F_0, F_1) are two compatible couples of reflexive Banach spaces and fix $T \in \mathcal{L}(E_0, F_0) \cap \mathcal{L}(E_1, F_1)$. Let $E_\theta :=$*

$[E_0,E_1]_\theta$, $F_\theta := [F_0,F_1]_\theta$ be defined by complex interpolation, so $T \in \mathcal{L}(E_\theta,F_\theta)$ for $0 \le \theta \le 1$.

Then the set of points $\theta \in (0,1)$ at which $T : E_\theta \to F_\theta$ is semi-Fredholm is open and the index is locally constant on this set.

Proof. This is a version of Theorem 2.9 in [KM]. □

Theorem 7.3. *The operators*

$$S : L^p(\partial\Omega) \longrightarrow H^{1,p}(\partial\Omega), \quad 1 < p < 2+\varepsilon, \tag{7.7}$$

and

$$\frac{1}{2}I + K : H^{1,p}(\partial\Omega) \longrightarrow H^{1,p}(\partial\Omega), \quad 1 < p < 2+\varepsilon, \tag{7.8}$$

are well-defined, bounded and invertible.

Proof. Let $f \in L^p(\partial\Omega)$, $1 < p < 2+\varepsilon$, and again set $u_\pm := \mathcal{S}f$ in $\Omega_\pm$. We have

$$\begin{aligned}
\|f\|_{L^p(\partial\Omega)} &\le \left\| \left(\frac{1}{2}I + K^*\right) f \right\|_{L^p(\partial\Omega)} + \left\| \left(-\frac{1}{2}I + K^*\right) f \right\|_{L^p(\partial\Omega)} \\
&= \left\| \frac{\partial u_-}{\partial \nu} \right\|_{L^p(\partial\Omega)} + \left\| \frac{\partial u_+}{\partial \nu} \right\|_{L^p(\partial\Omega)} \\
&\le C\|(\nabla u_-)^*\|_{L^p(\partial\Omega)} + C\|(\nabla u_+)^*\|_{L^p(\partial\Omega)} \\
&\le C\|\nabla_T u_-\|_{L^p(\partial\Omega)} + C\|\nabla_T u_+\|_{L^p(\partial\Omega)} + C\|Sf\|_{L^p(\partial\Omega)} \\
&\le C\|Sf\|_{H^{1,p}(\partial\Omega)}.
\end{aligned} \tag{7.9}$$

With this in hand, the conclusion about the operator (7.7) follows as in the proof of Theorem 7.1. All the other points in the theorem also follow from this, Theorem 7.1 and the intertwining property $S(\pm\frac{1}{2}I+K^*) = (\pm\frac{1}{2}I+K)S$. □

An immediate consequence of Theorems 7.1–7.2 is that the solutions to the L^p Neumann, Dirichlet and Regularity problems have natural integral representation formulas. The specifics are contained in the next corollary.

Corollary 7.4. *The solution to the Neumann problem (5.1) with g in $L^p(\partial\Omega)$ (or $L^p_0(\partial\Omega)$ if $V \equiv 0$ on $\bar{\Omega}$), $1 < p < 2+\varepsilon$, is representable in the form (5.3).*

The solution to the Dirichlet problem (1.4) with $f \in L^p(\partial\Omega)$, $2-\varepsilon < p < \infty$, can be expressed in the form (1.22). Finally, the solution to the regularity problem (6.1) with $f \in H^{1,p}(\partial\Omega)$, $1 < p < 2+\varepsilon$, has the integral representation formula (6.3).

In order to state our next result, we recall that $B_s^p(\partial\Omega) = B_s^{pp}(\partial\Omega)$ is the usual scale of Besov spaces on $\partial\Omega$. Also, denote by $\tilde{B}_s^p(\partial\Omega)$ the subspace of $B_s^p(\partial\Omega)$ consisting of distributions that annihilate constants.

Proposition 7.5. *There exists $\varepsilon = \varepsilon(\Omega) > 0$ such that*
(7.10)
$$\frac{1}{2}I + K : B^p_{1-1/p}(\partial\Omega) \to B^p_{1-1/p}(\partial\Omega) \text{ is invertible}, \quad \forall\, p \in (3/2-\varepsilon, 3+\varepsilon).$$
Also, if $V = 0$ in Ω,
(7.11)
$$-\frac{1}{2}I + K^* : \tilde{B}^q_{-1/q}(\partial\Omega) \to \tilde{B}^q_{-1/q}(\partial\Omega) \text{ is invertible}, \quad \forall\, q \in (3/2-\varepsilon, 3+\varepsilon).$$

In the Euclidean case, such a result has been obtained in [FMM].

Proof. If we denote by T_q the operators in (7.1) and by $T_{1,p}$ the operators in (7.8), then obvious inclusions show that

$$\begin{aligned} &T_q^{-1} = T_2^{-1} \text{ on } L^{q\vee 2}(\partial\Omega), \quad \forall\, q \in (2-\varepsilon, \infty), \\ &T_2^{-1} = T_{1,2}^{-1} \text{ on } H^{1,2}(\partial\Omega), \\ &T_{1,p}^{-1} = T_{1,2}^{-1} \text{ on } H^{1,p\vee 2}(\partial\Omega), \quad \forall\, p \in (1, 2+\varepsilon). \end{aligned} \tag{7.12}$$

It follows that all these operators agree on $\mathrm{Lip}(\partial\Omega)$, and hence

$$T_q^{-1} = T_{1,p}^{-1} \text{ on } L^q(\partial\Omega) \cap H^{1,p}(\partial\Omega), \quad \forall\, p \in (1, 2+\varepsilon),\ q \in (2-\varepsilon, \infty). \tag{7.13}$$

Thus we can apply $\frac{1}{2}I + K$ and $(\frac{1}{2}I + K)^{-1}$ to various spaces obtained from those in (7.1) and (7.8) via real and complex interpolation. Complex interpolation yields

$$\frac{1}{2}I + K : H^{s,p}(\partial\Omega) \longrightarrow H^{s,p}(\partial\Omega) \text{ is invertible}, \quad \forall\, (s, 1/p) \in \mathcal{R}_\varepsilon, \tag{7.14}$$

where the region $\mathcal{R}_\varepsilon$ is the interior of the parallelogram with vertices at the points $(0,0)$, $(0, 1/(2-\varepsilon))$, $(1,1)$, and $(1/(2+\varepsilon), 1)$ in $\mathbb{R}^2$. Now we have the real interpolation result (cf. [BL])

$$(H^{s_0,p}, H^{s_1,p})_{\theta,p} = B^p_\sigma, \quad \sigma = (1-\theta)s_0 + \theta s_1, \text{ for } 0 < \theta < 1. \tag{7.15}$$

Hence we have

$$\frac{1}{2}I+K : B^p_s(\partial\Omega) \longrightarrow B^p_s(\partial\Omega) \text{ invertible, } \quad \forall\, (s,1/p) \in \mathcal{R}_\varepsilon. \tag{7.16}$$

Taking $(s,1/p) \in \mathcal{R}_\varepsilon$ for which $s + 1/p = 1$ yields (7.10). Finally, (7.11) follows via a similar argument, plus duality. □

We next establish invertibility results on $\mathfrak{h}^1(\partial\Omega)$ and $\mathrm{bmo}(\partial\Omega)$.

Theorem 7.6. *The operators*

$$\frac{1}{2}I+K^* : \mathfrak{h}^1(\partial\Omega) \longrightarrow \mathfrak{h}^1(\partial\Omega) \text{ and } \frac{1}{2}I+K : bmo(\partial\Omega) \longrightarrow bmo(\partial\Omega) \tag{7.17}$$

are invertible. If, in addition, $V = 0$ on $\overline{\Omega}$, then

$$-\frac{1}{2}I + K^* : \mathfrak{H}^1_{at}(\partial\Omega) \longrightarrow \mathfrak{H}^1_{at}(\partial\Omega) \tag{7.18}$$

is also invertible.

Proof. The boundedness of the operators under discussion follows from (1.40) and Proposition B.6. Next, paralleling the estimate (7.5) in the present context we arrive at

$$\|f\|_{\mathfrak{h}^1(\partial\Omega)} \le C\|(\pm\frac{1}{2}I + K^*)f\|_{\mathfrak{h}^1(\partial\Omega)} + C\|Sf\|_{L^1(\partial\Omega)} + C\|\mathcal{S}f\|_{L^1(\Omega)}.$$

It is elementary to check that S is compact on $L^1(\partial\Omega)$ (using, e.g., the compactness criterion in [Ed]). Furthermore, $\mathcal{S}$ maps $\mathfrak{h}^1(\partial\Omega)$ boundedly into $H^{1,1}(\Omega)$ and, hence, $\mathcal{S} : \mathfrak{h}^1(\partial\Omega) \to L^q(\Omega)$ is also compact for $1 \le q < n/(n-1)$.

In particular, the operators in (7.17)–(7.18) have closed ranges. Since $L^2(\partial\Omega)$, $L^2_0(\partial\Omega)$ embed densely in $\mathfrak{h}^1(\partial\Omega)$ and $\mathfrak{H}^1_{\mathrm{at}}(\partial\Omega)$, respectively, we conclude that the operators $\pm\frac{1}{2}I + K^*$ also have dense ranges and, hence, are onto. The fact that they are one-to-one follows from the uniqueness in the atomic Neumann and Dirichlet problems via a familiar reasoning based on jump relations. Note that Proposition B.5 is used here. This takes care of $\pm\frac{1}{2}I + K^*$. Finally, the claim about $\frac{1}{2}I + K$ follows by duality. □

We now discuss the issue of invertibility on local Hardy spaces with subunitary index and Hölder spaces with small exponent.

Theorem 7.7. *There exist* $\alpha_0 > 0$ *and* $\varepsilon > 0$ *so that*

$$\frac{1}{2}I + K : C^\alpha(\partial\Omega) \longrightarrow C^\alpha(\partial\Omega) \text{ is invertible } \forall \alpha \in (0, \alpha_0) \tag{7.19}$$

and

$$\frac{1}{2}I + K^* : \mathfrak{h}^p(\partial\Omega) \longrightarrow \mathfrak{h}^p(\partial\Omega) \text{ is invertible } \forall p \in (1-\varepsilon, 1). \tag{7.20}$$

Proof. We begin with (7.20). Indeed, the family $\{\mathfrak{h}^p(\partial\Omega)\}_{p\leq 1}$ continued with $\{L^p(\partial\Omega)\}_{p>1}$ is a complex interpolation scale (cf. the discussion in [KM]). Also, the operator $\frac{1}{2}I + K^*$ maps this scale boundedly into itself for $(n-1)/n < p < \infty$. This follows from [MT] and Proposition B.6. Returning to the original claim made regarding the operator (7.20), we have this result based on the fact that the first operator in (7.17) is an isomorphism and the stability results on complex interpolation scales of quasi-Banach spaces in [KM]. Dualizing (7.20) gives (7.19) for some small $\alpha_0 > 0$. The proof of Theorem 7.7 is therefore finished. □

Remark. Note that, occasionally, we are led to considering quasi-Banach spaces, such as $\mathfrak{h}^p(\partial\Omega)$ for $p < 1$. The same applies to $\mathfrak{h}^{1,p}_{\text{at}}(\partial\Omega)$ for $p < 1$, the inhomogeneous version of the ℓ^p-span of regular atoms (these are functions satisfying (6.5) and a version of (6.6) adapted to L^p). It is also implicit in our work so far that $\frac{1}{2}I + K$ is an isomorphism of $\mathfrak{h}^{1,p}_{\text{at}}(\partial\Omega)$ for $p < 1$, p close to 1.

Now, generally speaking, for a linear and bounded operator T on some quasi-Banach space X, the property of being an isomorphism is preserved by passing to the "minimal enlargement" of X to a Banach space $\hat{X}$, its so called Banach envelope. Quite recently, in [MM], the Banach envelopes of all Besov and Triebel-Lizorkin spaces have been identified. The relevance of this result in the present context is that atomic Hardy spaces fit in the scale of Triebel-Lizorkin spaces. In particular, it is proved in [MM] that $\widehat{\mathfrak{h}^p(\partial\Omega)} = B^{1,1}_{-s}(\partial\Omega)$ and $\widehat{\mathfrak{h}^{1,p}(\partial\Omega)} = B^{1,1}_{1-s}(\partial\Omega)$, where $s = (n-1)(\frac{1}{p} - 1)$. Thus, in particular, $\frac{1}{2}I + K$ continues to be an isomorphism of $B^{1,1}_{1-s}(\partial\Omega)$ for each $s \in (0, s_0)$ with $s_0 = s_0(\partial\Omega) > 0$ small. In turn, results as such can be used to solve the general Poisson problem (with Dirichlet and Neumann boundary conditions) for the Laplace-Beltrami operator in Lipschitz domains for optimal ranges of indices. We will address this topic in detail in a separate paper; cf. [MT2].

Continuing our discussion along the lines of Theorem 7.7 we note the following.

Corollary 7.8. *Assume that the metric tensor on M is of class C^{1+r} for some $r>0$. Then there exists $\alpha_0>0$ with the following property. For any $\alpha\in(0,\alpha_0)$ and $g\in C^\alpha(\partial\Omega)$, the L^2-solution of the Dirichlet problem*

$$(7.21)\qquad Lu=0 \ \text{ in } \Omega,\ \ u^*\in L^2(\partial\Omega),\ \ u\big|_{\partial\Omega}=g,$$

has the property that

$$(7.22)\qquad u\in C^\alpha(\overline{\Omega}) \ \text{ and } \ \|u\|_{C^\alpha(\overline{\Omega})}\leq C\|g\|_{C^\alpha(\partial\Omega)}.$$

Furthermore, $u=\mathcal{D}h$ in Ω for some $h\in C^\alpha(\partial\Omega)$ with $\|h\|_{C^\alpha(\partial\Omega)}\approx\|u\|_{C^\alpha(\overline{\Omega})}$.

Proof. In the light of Theorem 7.7, we only need to check that

$$(7.23)\qquad \mathcal{D}:C^\alpha(\partial\Omega)\longrightarrow C^\alpha(\overline{\Omega}) \ \text{ is bounded for any } \alpha\in(0,1).$$

As is well known, this will follow from the estimate

$$(7.24)\qquad \operatorname{dist}(x,\partial\Omega)^{1-\alpha}|\nabla\mathcal{D}f(x)|\leq C\|f\|_{C^\alpha(\partial\Omega)}, \ \text{ uniformly for } x\in\Omega.$$

To see this, fix a function $f\in C^\alpha(\partial\Omega)$, a point $x\in\Omega$ and select $p\in\partial\Omega$ so that $d:=\operatorname{dist}(x,\partial\Omega)=\operatorname{dist}(x,p)$. Since $\nabla\mathcal{D}1\in L^\infty(\Omega)$, there is no loss of generality in assuming that $f(p)=0$.

Next, for a large constant C, split the domain of integration in $\mathcal{D}f$ into $\{y\in\partial\Omega:\operatorname{dist}(y,p)\leq Cd\}$ and $\mathcal{D}f$ into $\{y\in\partial\Omega:\operatorname{dist}(y,p)>Cd\}$. In the first resulting integral majorize the kernel of $\nabla\mathcal{D}$ by Cd^{-n}, while in the second one by $C\operatorname{dist}(y,p)^{-n}$. That this works, is guaranteed by the expansion (2.4) and the estimates that e_0, e_1 satisfy in the present context, i.e., when the metric tensor is C^{1+r}, $r>0$. In this case, it has been proved in [MMT] that $e_1(x,y)$ satisfies (for any $\varepsilon>0$)

$$(7.25)\qquad |\nabla_x\nabla_y e_1(x,y)|\leq C_\varepsilon|x-y|^{-(n-1+\varepsilon)}.$$

Also, from (2.3), it is clear that

$$(7.26)\qquad \begin{aligned}&(\nabla_z^2 e_0)(z,y)=\mathcal{O}(|z|^{-n}) \quad \text{as } z\to 0 \text{ uniformly in } y,\\ &(\nabla_z\nabla_y e_0)(z,y)=\mathcal{O}(|z|^{-(n-1)}) \quad \text{as } z\to 0 \text{ uniformly in } y.\end{aligned}$$

These suffice to justify the aforementioned estimates for the kernel of $\nabla\mathcal{D}$. Finally, on account of $|f(y)|\leq C\operatorname{dist}(y,p)^\alpha$, the desired estimate, (7.24), follows. □

Next we discuss the Neumann problem with data in $\mathfrak{h}^p(\partial\Omega)$ for $1-p>0$, small. To set the stage, let Θ be a smooth vector field on M which is transversal to $\partial\Omega$ and points into Ω. Denote by $\mathcal{F}_t$ the flow generated by Θ on M and introduce g_t, $d\sigma_t$, ν_t, etc., as in the proof of Proposition 5.5. If $u\in C^1(\Omega)$, we say that $\partial_\nu u=g\in\mathfrak{h}^p(\partial\Omega)$ for some $(n-1)/n<p<1$ provided

$$\lim_{t\searrow 0}\int_{\partial\Omega}\frac{\partial u_t}{\partial\nu_t}\psi\,d\sigma_t=\int_{\partial\Omega}g\psi\,d\sigma,\qquad\forall\,\psi\in C^\alpha(\partial\Omega),\tag{7.27}$$

where $\alpha:=(n-1)(p^{-1}-1)>0$. An important observation is that the jump formula

$$\partial_\nu\mathcal{S}f=\Big(-\frac{1}{2}I+K^*\Big)f,\qquad\forall\,f\in\mathfrak{h}^p(\partial\Omega),\tag{7.28}$$

continues to hold in this context since, by Propositions B.5–B.6 both sides depend continously on $f\in\mathfrak{h}^p(\partial\Omega)$ and (7.28) is valid for p-atoms, in which case the L^2 theory applies.

Theorem 7.9. *There exists $\varepsilon>0$ with the following significance. Fix $p\in(1-\varepsilon,1)$. If $V>0$ on a set of positive measure in Ω, then the Neumann boundary problem*

$$u\in C^1(\Omega),\quad Lu=0\ \text{ in }\Omega,\quad \partial_\nu u=g\in\mathfrak{h}^p(\partial\Omega),\quad(\nabla u)^*\in L^p(\partial\Omega),\tag{7.29}$$

has a unique solution, which satisfies

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)}\leq C\|g\|_{\mathfrak{h}^p(\partial\Omega)}.\tag{7.30}$$

If, on the other hand, $V=0$ on $\bar\Omega$, then the same is true for data in $\mathfrak{H}^p_{at}(\partial\Omega)$, except that uniqueness is now valid only modulo an additive constant.

This extends results for star-like Lipschitz domains in the flat Euclidean setting from [Br].

Proof. Let us assume first that $V=0$ on $\bar\Omega$. Recall $\rho_t=d\sigma_t/d\sigma$. With a self explanatory piece of notation which emphasizes the dependence on the metric tensor, we claim that

$$\frac{\partial u_t}{\partial\nu_t}\in\mathfrak{h}^p(\partial\Omega,d\sigma_t),\ \text{ hence }\ \frac{\partial u_t}{\partial\nu_t}\rho_t\in\mathfrak{h}^p(\partial\Omega,d\sigma),\quad\forall\,t>0,\tag{7.31}$$

and

$$\left\| \frac{\partial u_t}{\partial \nu_t} \rho_t \right\|_{\mathfrak{h}^p(\partial\Omega, d\sigma)} \leq C \left\| \frac{\partial u_t}{\partial \nu_t} \right\|_{\mathfrak{h}^p(\partial\Omega, d\sigma_t)} \leq C\|(\mathrm{grad}_t u_t)^*\|_{L^p(\partial\Omega, d\sigma_t)} \leq C\|(\nabla u)^*\|_{L^p(\partial\Omega)}, \tag{7.32}$$

uniformly for $t > 0$. The first membership in (7.31) together with the second estimate in (7.32) follow from the constructive approach to atomic decompositions in $\mathbb{R}^n$ developed in [Wi] (building on some earlier work in [Co] and [La]). Wilson's approach is rather flexible and extends to the setting of Lipschitz domains on manifolds since it only employs the ordinary nontangential maximal operator and cancellations based on integrations by parts. A typical example of the latter phenomenon is $\int_{\partial D} \partial_{\nu_t} u_t \, d\sigma_t = 0$, for any subdomain D of Ω (here is where we need $V = 0$ in $\overline{\Omega}$).

Going further, the second membership in (7.31) together with the first inequality in (7.32) are discussed in Appendix A, whereas the third estimate in (7.32) is clear. Thus, (7.31)–(7.32) are taken care of. In the case when V is not necessarily zero in Ω we use the previous results for $\tilde{u} := u - \int_\Omega E(\cdot, y)V(y)u(y)\, d\mathrm{Vol}$ and we obtain similar conclusions. Let us point out that all arguments so far work for $(n-1)/n < p \leq 1$.

Turning attention to the Neumann problem (7.29), assume first that $V > 0$ on a set of positive measure in Ω. Existence and estimates follow from (7.28), the fact that

$$-\frac{1}{2}I + K^* : \mathfrak{h}^p(\partial\Omega) \longrightarrow \mathfrak{h}^p(\partial\Omega) \tag{7.33}$$

is invertible (which, in turn, is dealt with much as (7.21)) and Proposition B.5. As for uniqueness, let us first show that, for each $0 < \alpha < s$ with s as in Proposition 4.4,

$$N^t(x, \cdot) \longrightarrow N(x, \cdot), \quad \text{in } C^\alpha(\partial\Omega) \text{ as } t \searrow 0, \tag{7.34}$$

provided $x \in \Omega$ is fixed. Indeed, since $C^s(\partial\Omega) \hookrightarrow C^\alpha(\partial\Omega)$ compactly and since, by the results in §4, $\{N^t(x, \cdot)\}_{t>0}$ is bounded in $C^s(\partial\Omega)$, it follows that there exists $\tilde{N}(x, \cdot)$ so that $N^t(x, \cdot) \longrightarrow \tilde{N}(x, \cdot)$ in $C^\alpha(\partial\Omega)$ as $t \searrow 0$. Now, if $u \in C^1_{\mathrm{loc}}(\Omega)$ is such that $(\nabla u)^* \in L^2(\partial\Omega)$ and $Lu = 0$ on Ω, reasoning as in (5.21)–(5.22) and then letting $t \searrow 0$ gives

$$u(x) = \int_{\partial\Omega} \tilde{N}(x, y) \frac{\partial u}{\partial \nu}(y)\, d\sigma(y), \quad \text{for each } x \in \Omega. \tag{7.35}$$

This, the L^2 theory for the Neumann problem and (5.9) then yield that $\tilde{N}(x,y) = N(x,y)$. Thus, (7.34) is proved.

Returning to the task of proving uniqueness in (7.29), note that (5.21), (7.27), (7.34) and (7.32) easily give the desired conclusion when $V > 0$ on a set of positive measure in Ω. The case when $V = 0$ on $\overline{\Omega}$ is similar and we omit it. □

We conclude this section with a discussion of the Dirichlet problem (7.21) with boundary data in $\mathrm{bmo}(\partial\Omega)(\subset L^2(\partial\Omega))$.

Proposition 7.10. *Assume that the metric tensor on M is of class C^{1+r} for some $r > 0$. Then for any $g \in bmo(\partial\Omega)$, the L^2-solution of the Dirichlet problem (7.21) has the property that*

$$\begin{aligned} &\textit{dist}\,(\cdot,\partial\Omega)|\nabla u|^2\, d\,\textit{Vol is a Carleson measure} \\ &\qquad \textit{in } \Omega \textit{ with norm} \leq C\|g\|^2_{bmo(\partial\Omega)}. \end{aligned} \tag{7.36}$$

Proof. Since the second operator in (7.17) is an isomorphism, u has the form $\mathcal{D}h$ for some $h \in \mathrm{bmo}(\partial\Omega)$ with $\|h\|_{\mathrm{bmo}(\partial\Omega)} \approx \|g\|_{\mathrm{bmo}(\partial\Omega)}$. Therefore, it suffices to show that if $h \in \mathrm{bmo}(\partial\Omega)$ is arbitrary then $\mathrm{dist}\,(\cdot,\partial\Omega)|\nabla\mathcal{D}h|^2\, d\mathrm{Vol}$ is a Carleson measure in Ω whose norm is controlled by $\|h\|^2_{\mathrm{bmo}(\partial\Omega)}$.

Given the estimates on the kernel of $\mathcal{D}$ in [MT], [MMT], this follows with minor modifications from the corresponding flat, Euclidean result in [FK] as soon as the square-function estimate

$$\int_{\Omega} |u|^2\, d\mathrm{Vol} + \int_{\Omega} \mathrm{dist}\,(x,\partial\Omega)|\nabla u(x)|^2\, d\mathrm{Vol}(x) \leq C \int_{\partial\Omega} |u|^2\, d\sigma, \tag{7.37}$$

uniformly for u so that $Lu = 0$ in Ω, $u^* \in L^2(\partial\Omega)$, is available. In the present context, this estimate is a consequence of results in [MT] and §§1–2 of [MMT]. □

Remark. It is also possible to show that the estimate obtained by reversing the inequality in (7.37) remains valid. In the present context, a natural proof can be obtained by adapting the approach in [M]. See also, e.g., [DJK], [DKPV] for more on related topics. In turn, such an estimate allows one to show (much as in [FN]) that if u is so that $Lu = 0$ in Ω and $\mathrm{dist}(\cdot,\partial\Omega)|\nabla u|^2\, d\mathrm{Vol}$ is a Carleson measure in Ω, then $\exists\, u\big|_{\partial\Omega}$ and $u\big|_{\partial\Omega} \in \mathrm{bmo}(\partial\Omega)$. Also, a naturally accompanying estimate holds. However, we shall not develop this point here any further.

8. Helmholtz type decompositions.

Assume that Ω is a connected Lipschitz domain in a Riemannian manifold M equipped with

$$\text{(8.1)} \qquad \text{a metric with } H^{2,r} \text{ coefficients, } r > \max\{3, \dim M\}.$$

The goal is to establish L^p based Helmholtz type decompositions for vector fields (or, equivalently, 1-forms) in Ω. We shall do this in a constructive fashion and for an optimal range of p's. The main results are contained in Propositions 8.1 and 8.3.

To get started, we denote by $\Lambda^1 TM$ the first exterior power bundle of the manifold M and by d the usual exterior differential operator. The Riemannian metric naturally extends to the fibers of $\Lambda^1 TM$ and we let δ stand for the adjoint operator of d. Finally, let $\vee$ be the interior product of forms. As is well known, there is a canonical isomorphism between 1-forms and vector fields on M. Under this isometrical correspondence, d and δ become, respectively, the gradient and divergence operators, whereas the interior multiplication by ν (the outward conormal to $\partial\Omega$) becomes the scalar product with the unit normal to $\partial\Omega$.

Going further, if $u \in L^1(\Omega, \Lambda^1 TM)$ is so that $\delta u \in L^1(\Omega)$ then we can define the scalar distribution $\nu \vee u$ on M (actually supported on $\partial\Omega$) by

$$\text{(8.2)} \qquad \langle \nu \vee u, \varphi \rangle := \int_\Omega \varphi\, \delta u \, d\mathrm{Vol} + \int_\Omega \langle u, d\varphi \rangle \, d\mathrm{Vol}, \qquad \forall\, \varphi \in C^1(M).$$

Recall the scale $B^p_s(\partial\Omega) = B^{pp}_s(\partial\Omega)$ of Besov spaces on $\partial\Omega$. Then, (cf., e.g., [MMT]), the mapping

$$\text{(8.3)} \qquad \{u \in L^p(\Omega, \Lambda^1 TM) : \delta u \in L^p(\Omega)\} \ni u \mapsto \nu \vee u \in B^p_{-1/p}(\partial\Omega)$$

is well defined and bounded for $1 < p < \infty$, in the sense that

$$\text{(8.4)} \qquad \|\nu \vee u\|_{B^p_{-1/p}(\partial\Omega)} \le C(\Omega, p) \left(\|u\|_{L^p(\Omega)} + \|\delta u\|_{L^p(\Omega)} \right).$$

Let us point out that if $u \in L^p(\Omega, \Lambda^1 TM)$ has $\delta u = 0$ then $\nu \vee u$, regarded as a functional in $\left(B^q_{1-1/q}(\partial\Omega)\right)^*$, $\frac{1}{p} + \frac{1}{q} = 1$, annihilates constants, by Stokes' theorem. We denote the collection of all such functionals by $\tilde{B}^p_{-1/p}(\partial\Omega)$.

For $1 < p < \infty$, introduce the closed subspace of $L^p(\Omega, \Lambda^1 TM)$

$$\text{(8.5)} \qquad L^p_{\delta,0}(\Omega) := \{u \in L^p(\Omega, \Lambda^1 TM) : \delta u = 0 \text{ and } \nu \vee u = 0\}$$

and denote by P the orthogonal projection of $L^2(\Omega, \Lambda^1 TM)$ onto $L^2_{\delta,0}(\Omega)$. Analogously, $dH^{1,p}(\Omega)$ is a closed subspace of $L^p(\Omega, \Lambda^1 TM)$ for $1 < p < \infty$ and we denote by Q the orthogonal projection of $L^2(\Omega, \Lambda^1 TM)$ onto $dH^{1,2}(\Omega)$.

Proposition 8.1. *For each Lipschitz domain Ω in M, with arbitrary topology, there exists a positive number ε depending on Ω such that P and Q extend to bounded operators from $L^p(\Omega, \Lambda^1 TM)$ onto $L^p_{\delta,0}(\Omega)$ and $d\,H^{1,p}(\Omega)$, respectively, for each $p \in (\frac{3}{2} - \varepsilon, 3 + \varepsilon)$. Hence, in this range,*

$$L^p(\Omega, \Lambda^1 TM) = d\,H^{1,p}(\Omega) \oplus L^p_{\delta,0}(\Omega) \tag{8.6}$$

where the direct sum is topological.

In the class of Lipschitz domains, this result is sharp. If, however, $\partial\Omega \in C^1$ and $r = \infty$ then we may take $1 < p < \infty$.

Of course, we will assume without loss of generality that Ω is connected. To begin the proof, we need a preliminary result.

Lemma 8.2. *Let Ω be a Lipschitz domain in M. Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ so that the Neumann boundary problem*

$$\begin{cases} \Delta v = 0 \ \textit{in}\ \Omega, \\ \dfrac{\partial v}{\partial \nu} = g \in \tilde{B}^p_{-1/p}(\partial\Omega), \\ v \in H^{1,p}(\Omega), \end{cases} \tag{8.7}$$

has a unique (modulo constants) solution for each $p \in (\frac{3}{2} - \varepsilon, 3 + \varepsilon)$. Each solution satisfies

$$\|dv\|_{L^p(\Omega)} \leq C\|g\|_{B^p_{-1/p}(\partial\Omega)}. \tag{8.8}$$

Note that in (8.7), the normal derivative of v is to be interpreted as $\nu \vee dv$, in the sense of (8.2).

Proof. If $\varepsilon > 0$ is so that $-\frac{1}{2}I + K^*$ is an isomorphism of $\tilde{B}^p_{-1/p}(\partial\Omega)$ for $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$ then, so we claim, $v := \mathcal{S}((-\frac{1}{2}I + K^*)^{-1} g)$ solves (8.7) and satisfies (8.8). Indeed, due to Proposition 7.5, we only need to prove that the operator

$$\mathcal{S} : B^p_{-1/p}(\partial\Omega) \to H^{1,p}(\Omega) \tag{8.9}$$

is bounded for $r/(r-1) < p < \infty$.

To this end, let Π stand for the Newtonian potential in Ω at the level of 1-forms. That is, Π is the integral operator in Ω whose kernel is the Schwartz kernel of $(-d\delta - \delta d - V)^{-1} : H^{-1,2}(M, \Lambda^1 TM) \to H^{1,2}(M, \Lambda^1 TM)$ for some V as in §1 with $V \equiv 0$ on $\overline{\Omega}$. Given our assumption on the metric, $\Pi : L^q(\Omega) \to H^{2,q}(\Omega)$ for $1 < q < r$; see [MMT]. In particular, $\delta\Pi : L^q(\Omega) \to H^{1,q}(\Omega)$, $\forall\, q \in (1, r)$. Recall that $\mathrm{Tr} : H^{s,q}(\Omega) \to B^q_{s-1/q}(\partial\Omega)$, $1/q < s < 1 + 1/q$, $1 < q < \infty$, is the usual trace operator.

For any reasonable function f on $\partial\Omega$ and 1-form g in Ω, we have

$$\int_\Omega \langle dSf, g\rangle \, d\mathrm{Vol} = \int_{\partial\Omega} \langle \mathrm{Tr}(\delta\Pi g), f\rangle \, d\sigma + \int_\Omega \langle Rf, g\rangle \, d\mathrm{Vol}, \tag{8.10}$$

where R is an integral operator with a weakly singular kernel (cf. (6.17) of [MMT]). In particular, $R : H^{-1,p}(\partial\Omega) \to L^p(\Omega, \Lambda^1 TM)$ is bounded for $1 < p < \infty$. From this and (8.10) we deduce that

$$\left| \int_\Omega \langle dSf, g\rangle \, d\mathrm{Vol} \right| \le C \|g\|_{L^q(\Omega, \Lambda^1 TM)} \|f\|_{B^p_{-1/p}(\partial\Omega)}, \tag{8.11}$$

with $1/p + 1/q = 1$. Now, the fact that the operator (8.9) is bounded readily follows from this.

Finally, there remains the issue of uniqueness (modulo constants) for (8.7). This can be done in several ways. For instance, if v is a null-solution then, from Green's representation formula, $\mathrm{Tr}\, v \in B^p_{1-1/p}(\partial\Omega)$ satisfies $(-\frac{1}{2}I + K)(\mathrm{Tr}\, v) = 0$. Consequently, by the dual of (7.11), $\mathrm{Tr}\, v$ must be a constant on $\partial\Omega$ and, further, $v \equiv$ const. in Ω. □

We are now ready to present the

Proof of Proposition 8.1. We follow the approach in [FMM] with natural alterations. Specifically, recall the Newtonian potential Π and define $\tilde{P} : L^p(\Omega, \Lambda^1 TM) \to L^p_{\delta,0}(\Omega) \hookrightarrow L^p(\Omega, \Lambda^1 TM)$ by setting

$$\tilde{P}u := u - d\delta\Pi u - dv, \qquad \forall u \in L^p(\Omega, \Lambda^1 TM), \tag{8.12}$$

where v is the unique solution to the (scalar) Neumann boundary problem

$$\begin{cases} \Delta v = 0 \text{ in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = \nu \vee (u - d\delta\Pi u) \in \tilde{B}^p_{-1/p}(\partial\Omega), \\ v \in H^{1,p}(\Omega), \\ \displaystyle\int_\Omega v = 0. \end{cases} \tag{8.13}$$

Note that $u - d\delta\Pi u = \delta d\Pi u$ on Ω. As we have commented, $\nu \vee \delta d\Pi u$ integrates to zero over $\partial\Omega$. The range $p \in (\frac{3}{2} - \varepsilon, 3 + \varepsilon)$, with ε small ensures the solvability of the boundary problem (8.13). By construction, $\tilde{P}$ is well-defined, linear and bounded and, moreover, $I - \tilde{P}$ maps $L^p(\Omega)$ boundedly into $dH^{1,p}(\Omega)$.

Next, we aim at proving that $\tilde{P}$ is onto $L^p_{\delta,0}(\Omega)$. Indeed, so we claim,

$$\tilde{P}(u) = u \qquad \forall\, u \in L^p_{\delta,0}(\Omega). \tag{8.14}$$

To see this, note that if $u \in L^p_{\delta,0}(\Omega)$ then $(\Delta - V)\delta\Pi u = \delta(\Delta - V)\Pi u - [V, \delta]\Pi u = \delta u = 0$ on Ω. Hence, if v solves (8.13), then the function $v + \delta\Pi u$ is harmonic, belongs to $H^{1,p}(\Omega)$ and has vanishing normal derivative. Invoking uniqueness for the Neumann problem (cf. Lemma 8.2), it follows that $\tilde{P}(u) = u - d\delta\Pi u - dv = u$ as claimed.

The fact that on $L^2(\Omega) \cap L^p(\Omega)$ the operator $\tilde{P}$ acts as the orthogonal projection onto $L^2_{\delta,0}(\Omega)$ is easily seen from (8.12). Thus, P extends to a bounded mapping of $L^p(\Omega)$ onto $L^p_{\delta,0}(\Omega)$, as desired. From this, the statement about $Q = I - P$ follows as well.

That we may take $p \in (1, \infty)$ when $\partial\Omega \in C^1$ is due to the fact that, in this context, the problem (8.13) is uniquely solvable in this range. Finally, the optimality of the range $p \in (\frac{3}{2} - \varepsilon, 3 + \varepsilon)$ in the class of Lipschitz domains follows from the counterexamples in [FMM]. □

For each $1 < p < \infty$, let us now consider

$$L^p_\delta(\Omega) := \left\{ u \in L^p(\Omega, \Lambda^1 TM) : \delta u = 0 \right\}. \tag{8.15}$$

Proposition 8.3. *Let Ω be an arbitrary connected Lipschitz domain in M. Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that*

$$L^p(\Omega, \Lambda^1 TM) = d\, H^{1,p}_0(\Omega) \oplus L^p_\delta(\Omega), \quad \textit{for each } p \in \left(\frac{3}{2} - \varepsilon, 3 + \varepsilon\right), \tag{8.16}$$

where the direct sum is topological.

Once again, in the class of Lipschitz domains, this result is sharp. If, however, $\partial\Omega \in C^1$ *and* $r = \infty$ *then we may take* $1 < p < \infty$.

Again, we need a preliminary result.

Lemma 8.4. *Let* Ω *be a Lipschitz domain in* M. *Then there exists* $\varepsilon = \varepsilon(\Omega) > 0$ *so that the Dirichlet boundary problem*

$$\begin{cases} \Delta v = 0 \ \text{in}\ \Omega, \\ Tr v = f \in B^p_{1-1/p}(\partial\Omega), \\ v \in H^{1,p}(\Omega), \end{cases} \tag{8.17}$$

has a unique solution for each $p \in (\frac{3}{2} - \varepsilon, 3 + \varepsilon)$. *This solution satisfies*

$$\|v\|_{H^{1,p}(\Omega)} \leq C\|f\|_{B^p_{1-1/p}(\partial\Omega)}. \tag{8.18}$$

Proof. If ε is so that $\frac{1}{2}I + K$ is invertible on $B^p_{1-1/p}(\partial\Omega)$ for $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$ (cf. Proposition 7.5), then we may take $v := \mathcal{D}((\frac{1}{2}I + K)^{-1}f)$ in Ω. Accepting that

$$\mathcal{D} : B^p_{1-1/p}(\partial\Omega) \to H^{1,p}(\Omega) \tag{8.19}$$

is a bounded operator when $1 < p < r$, we deduce that v solves (8.17) and satisfies (8.18). In turn, the claim about (8.19) is readily seen from the identity

$$\mathcal{D}(\operatorname{Tr} w) = w + \delta\Pi(dw) + R(dw), \tag{8.20}$$

valid for any scalar $w \in H^{1,p}(\Omega)$, and Gagliardo's trace lemma. Here R is an integral operator with a weakly singular kernel; cf. [MMT]. For our purposes, we only need to know that R maps $L^p(\Omega, \Lambda^1 TM)$ boundedly into $H^{1,p}(\Omega)$. Based on these, (8.19) follows.

Thus, we are left with proving the uniqueness of the solution of (8.17). However, this can be done in a similar manner to the uniqueness part in Lemma 8.2, by taking advantage of the fact that $\frac{1}{2}I + K^*$ is invertible on $\tilde{B}^p_{-1/p}(\partial\Omega)$ for $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$. □

Finally, we are ready to present the

Proof of Proposition 8.3. Again, we follow closely [FMM]. Here the departure point is to consider the operator

$$L^p(\Omega, \Lambda^1 TM) \ni u \mapsto d(\delta \Pi u - v) \in dH_0^{1,p}(\Omega), \tag{8.21}$$

where v is the unique solution to the Dirichlet problem

$$\begin{cases} \Delta v = 0 \text{ in } \Omega, \\ \mathrm{Tr}\, v = \mathrm{Tr}\,(\delta \Pi u) \in B^p_{1-1/p}(\partial\Omega), \\ v \in H^{1,p}(\Omega). \end{cases} \tag{8.22}$$

Lemma 8.4 guarantees that this assignment is well defined, linear and bounded if $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$ for some $\varepsilon = \varepsilon(\Omega) > 0$. Using this and paralleling the argument in Proposition 8.1 yields the desired conclusion; we omit the details. The sharpness of the range of p's is proved in [FMM]. □

A. Remarks on $\mathfrak{h}^p(\partial\Omega)$.

In (1.38) we defined $\mathfrak{h}^1(\partial\Omega)$ in terms of $\mathfrak{H}^1_{\mathrm{at}}(\partial\Omega)$, which in turn was defined in terms of atoms. As explained in §1, the spaces $\mathfrak{h}^p(\partial\Omega)$ and $\mathfrak{H}^p_{\mathrm{at}}(\partial\Omega)$ for $(n-1)/n < p \leq 1$ are defined analogously. Here we characterize $\mathfrak{h}^p(\partial\Omega)$ for p in this range in terms of *ions*, defined as follows. Pick $\alpha \in (0,1)$. We say $f \in L^\infty(\partial\Omega)$ is an ion, or an (α, p)-ion, provided

$$\operatorname{supp} f \subseteq B_r(x_0) \cap \partial\Omega \tag{A.1}$$

for some $x_0 \in \partial\Omega$, $r \in (0, \operatorname{diam} \Omega]$, and

$$\|f\|_{L^\infty(\partial\Omega)} \leq r^{-(n-1)/p}, \qquad \left| \int_{\partial\Omega} f \, d\sigma \right| \leq r^\alpha. \tag{A.2}$$

Lemma A.1. *For* $(n-1)/n < p \leq 1$,

$$\mathfrak{h}^p(\partial\Omega) = \left\{ \sum a_\nu f_\nu : f_\nu \ (\alpha,p)\text{-}ion, \ \sum |a_\nu|^p < \infty \right\}. \tag{A.3}$$

Proof. Temporarily denote the right side of (A.3) by $\mathfrak{h}^p_{\mathrm{i}}(\partial\Omega)$. Clearly $\mathfrak{h}^p(\partial\Omega) \subset \mathfrak{h}^p_{\mathrm{i}}(\partial\Omega)$. To establish the reverse inclusion we only need to check that there exists $C > 0$ so that

$$f \ (\alpha, p)\text{-ion} \Longrightarrow f \in \mathfrak{h}^p(\partial\Omega) \text{ and } \|f\|_{\mathfrak{h}^p(\partial\Omega)} \leq C. \tag{A.4}$$

Then the inclusion $\mathfrak{h}^p_{\mathrm{i}}(\partial\Omega) \subset \mathfrak{h}^p(\partial\Omega)$ will follow from $\mathfrak{h}^p(\partial\Omega) = \mathfrak{H}^p_{\mathrm{at}}(\partial\Omega) + L^q(\partial\Omega)$ plus the fact that $\ell^p \hookrightarrow \ell^1$ if $(n-1)/n < p \leq 1$, and $q > 1$.

As for (A.4), if f is as in (A.1)–(A.2), we write $f = g + h$ with

$$\text{(A.5)} \qquad h := b\,r^{-(n-1)/p}\chi_{B_r(x_0)\cap\partial\Omega}, \quad b := \frac{r^{(n-1)/p}}{A(x_0,r)}\int_{\partial\Omega} f\,d\sigma,$$

where $A(x_0, r)$ is the area of $B_r(x_0) \cap \partial\Omega$. Thus g is a p-atom, up to a factor of $1 + |b|$. Note that, if $\kappa = \kappa(\Omega) > 0$ is a constant satisfying

$$\text{(A.6)} \qquad \frac{r^{n-1}}{\kappa} \leq A(x,r) \leq r^{n-1}\kappa, \quad \forall\, x \in \partial\Omega,\ \forall\, r \in (0, \operatorname{diam}\Omega],$$

then $|b| \leq C(\kappa)$. In particular, $\|g\|_{\mathfrak{H}^p_{\mathrm{at}}(\partial\Omega)} \leq C(\kappa)$. The remainder h is a function we call a *charge* (or an α-charge); it satisfies

$$\text{(A.7)} \qquad \operatorname{supp} h \subseteq B_r(x_0) \cap \partial\Omega, \quad \|h\|_{L^\infty(\partial\Omega)} \leq \kappa\, r^{\alpha-(n-1)}.$$

Upon noting that a charge satisfies

$$\text{(A.8)} \qquad \begin{aligned} \|h\|_{L^q(\partial\Omega)} &\leq \kappa\, r^{\alpha-(n-1)}\, A(x,r)^{1/q} \\ &\leq \kappa^{1+1/q}\, r^{\alpha-(n-1)(q-1)/q} = \kappa^{1+1/q}, \end{aligned}$$

provided $q := (n-1)/(n-1-\alpha) > 1$, the desired conclusion follows. □

The ionic characterization of $\mathfrak{h}^p(\partial\Omega)$ readily shows the following.

Lemma A.2. *Let $(n-1)/n < p \leq 1$. Then the space $\mathfrak{h}^p(\partial\Omega)$ is a module over $C^\alpha(\partial\Omega)$ for any $\alpha > (n-1)(p^{-1}-1)$.*

Proof. Suppose that $\|\varphi\|_{L^\infty(\partial\Omega)} \leq A$ and $|\varphi(x) - \varphi(y)| \leq B \operatorname{dist}(x,y)^\alpha$, for some $\alpha > (n-1)(p^{-1}-1)$. If f is a p-atom, supported on $B_r(x_0) \cap \partial\Omega$, then the decomposition

$$\text{(A.9)} \quad \varphi f = af + (\varphi - a)f, \quad a = \varphi(x_0), \quad \|(\varphi-a)f\|_{L^\infty} \leq B\, r^{\beta-(n-1)},$$

writes f as a linear combination of a p-atom and a β-charge, where $\beta := \alpha - (n-1)(p^{-1}-1) > 0$. In particular, $M_\varphi : \mathfrak{h}^p(\partial\Omega) \to \mathfrak{h}^p(\partial\Omega)$ is bounded. □

While independent of the (smooth) background metric used to define a distance on M, the space $\mathfrak{h}^p(\partial\Omega)$ does depend, generally speaking, on the Riemannian metric g which induces the surface element $d\sigma$ on $\partial\Omega$. Specifically, for further reference we record the following useful observation.

Lemma A.3. *Assume that g and g' are two metric tensors with continuous coefficients on M and denote by $d\sigma$, $d\sigma'$ the surface measures induced on $\partial\Omega$. Also, for $(n-1)/n < p \leq 1$, denote by $\mathfrak{h}^p(\partial\Omega, d\sigma)$, $\mathfrak{h}^p(\partial\Omega, d\sigma')$ the corresponding (homogeneous) Hardy spaces and set $\rho := d\sigma'/d\sigma \in L^\infty(\partial\Omega)$.*

Then M_ρ, the operator of multiplication by ρ is an isomorphism of $\mathfrak{h}^p(\partial\Omega, d\sigma)$ onto $\mathfrak{h}^p(\partial\Omega, d\sigma')$.

B. Cauchy integrals and layer potentials on $\mathfrak{h}^p(\partial\Omega)$, $(n-1)/n < p \leq 1$.

To begin, let Γ be a Lipschitz graph in $\mathbb{R}^n$, of the form $x_n = \varphi(x_1, \dots, x_{n-1})$ for some Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. The first result is perhaps part of the "folklore" of the subject, but we believe it deserves a written proof.

Proposition B.1. *There exists $N = N(n)$ such that, if $k \in C^N(\mathbb{R}^n \setminus 0)$ is odd and homogeneous of degree $-(n-1)$, then*

$$\mathcal{K}f(x) = \int_\Gamma k(x-y) f(y)\, d\sigma(y), \quad x \in \mathbb{R}^n \setminus \Gamma \tag{B.1}$$

satisfies the nontangential maximal function estimate

$$\|(\mathcal{K}f)^*\|_{L^p(\Gamma)} \leq C(p, \Gamma) \left\| k\big|_{S^{n-1}} \right\|_{C^N} \|f\|_{\mathfrak{H}^p_{at}(\Gamma)}, \tag{B.2}$$

for each $p \in ((n-1)/n, 1]$, where $C(p,\Gamma)$ depends only on p and $\|\nabla\varphi\|_{L^\infty}$.

Proof. It suffices to estimate $(\mathcal{K}f)^*$ when $f \in L^\infty(\Gamma)$ is a p-atom, and considering the transformations of our various objects under translations and dilations, it suffices to consider normalized atoms, i.e., $f \in L^\infty(\Gamma)$ satisfying

$$\operatorname{supp} f \subset B_1(0) \cap \Gamma, \quad \|f\|_{L^\infty(\Gamma)} \leq 1, \quad \int_\Gamma f\, d\sigma = 0, \tag{B.3}$$

assuming that $0 \in \Gamma$. The hypotheses clearly yield

$$|x| \geq 2 \Longrightarrow |\mathcal{K}f(x)| \leq C|x|^{-n}, \tag{B.4}$$

so

$$\begin{aligned} x \in \Gamma,\ |x| \geq 2 &\Longrightarrow (\mathcal{K}f)^*(x) \leq C|x|^{-n} \\ &\Longrightarrow \int_{\Gamma\setminus B_2(0)} [(\mathcal{K}f)^*]^p\, d\sigma \leq C_p, \end{aligned} \tag{B.5}$$

as long as $(n-1)/n < p \leq 1$. The L^2 theory due to [CMM] gives

$$\int_{B_2(0)} [(\mathcal{K}f)^*]^p \, d\sigma \leq C_p \|(\mathcal{K}f)^*\|^p_{L^2(\Gamma)} \leq C'_p < +\infty, \tag{B.6}$$

so we are done. □

Granted Proposition B.1, we can establish the following variable coefficient extension, by the same argument as used in [MT].

Proposition B.2. *There exists $M = M(n)$ such that the following holds. Let $b(x,z)$ be odd in z and homogeneous of degree $-(n-1)$ in z, and assume $D_z^\alpha b(x,z)$ is continuous and bounded on $\mathbb{R}^n \times S^{n-1}$, for $|\alpha| \leq M$. Then*

$$\mathcal{B}f(x) := \int_\Gamma b(x, x-y) f(y) \, d\sigma(y), \quad x \in \mathbb{R}^n \setminus \Gamma, \tag{B.7}$$

satisfies

$$\|(\mathcal{B}f)^*\|_{L^p(\Gamma)} \leq C(\Gamma) \sup_{|\alpha| \leq M} \|D_z^\alpha b(x,z)\|_{L^\infty(\mathbb{R}^n \times S^{n-1})} \|f\|_{\mathfrak{H}^p_{at}(\Gamma)}, \tag{B.8}$$

for each $p \in ((n-1)/n, 1]$.

Proof. This follows by expanding $b(x,\cdot)$ in spherical harmonics,

$$b(x,z) = \sum_{j\geq 0} b_j(x) \varphi_j(z/|z|) \, |z|^{-(n-1)}, \tag{B.9}$$

and then invoking Proposition B.1 for each term. Since $\|b_j\|_{L^\infty} \|\varphi_j\|_{C^N} \leq C_\kappa j^{-\kappa}$, we can use the p-homogeneity and the subadditivity of $\|\cdot\|^p_{L^p(\Gamma)}$ to prove (B.8) by taking κ large; compare the proof of Proposition 1.2 in [MT]. □

We are also interested in estimates on

$$\widetilde{\mathcal{B}}f(x) := \int_\Gamma b(y, x-y) f(y) \, d\sigma(y), \qquad x \in \mathbb{R}^n \setminus \Gamma. \tag{B.10}$$

In this case, $b_j(x)$ in the expansion (B.9) is replaced by $b_j(y)$, which acts as a multiplication operator on f. Now neither $\mathfrak{H}^p_{\rm at}(\Gamma)$ nor $\mathfrak{h}^p(\Gamma)$ is a module over the space of bounded continuous functions, but, for any compact $\Gamma_0 \subset \Gamma$, $\mathfrak{h}^p(\Gamma_0)$ is a module over $C^r(\Gamma_0)$, for $r > (n-1)(p^{-1}-1)$, so we have the following result.

Proposition B.3. *In Proposition B.2, assume in addition that* $D_z^\alpha b(x,z)$ *is Hölder continuous on* $\mathbb{R}^n \times S^{n-1}$*, of exponent* $r > (n-1)(p^{-1}-1)$*. Let* $\Gamma_0 \subset \Gamma$ *be compact. Then*

$$\text{(B.11)} \qquad \|(\widetilde{\mathcal{B}}f)^*\|_{L^p(\Gamma_0)} \le C \sup_{|\alpha|\le M} \sup_{|z|=1} \|D_z^\alpha b(x,z)\|_{C^r(\Gamma_0)} \|f\|_{\mathfrak{h}^p(\Gamma_0)},$$

for f *supported on* Γ_0*.*

The following is a simple consequence; compare Proposition 1.5 of [MT]. Denote by $C^r S^m_{\text{cl}}$ the classical m-symbols $p(x,\xi)$ which are C^r in x, for some $r \in [0,\infty)$, while still smooth in $\xi \in \mathbb{R}^n \setminus 0$. Also, let Γ_0 be a compact subset of Γ.

Proposition B.4. *If* $p(x,\xi) \in C^0 S^{-1}_{cl}$ *has principal symbol that is odd in* ξ*, then the integral kernel* $K(x,y)$ *of* $p(x,D)$ *has the property that*

$$\text{(B.12)} \qquad \mathcal{K}f(x) = \int_\Gamma K(x,y) f(y)\, d\sigma(y), \quad x \in \mathbb{R}^n \setminus \Gamma,$$

satisfies, for each $(n-1)/n < p \le 1$*,*

$$\text{(B.13)} \qquad \|(\mathcal{K}f)^*\|_{L^p(\Gamma_0)} \le C\|f\|_{\mathfrak{h}^p(\Gamma_0)}, \quad \operatorname{supp} f \subseteq \Gamma_0.$$

If $(n-1)/n < p \le 1$*,* $r > (n-1)(p^{-1}-1)$ *and* $q(\xi,x) \in C^r S^{-1}_{cl}$ *has principal symbol odd in* ξ*, then the integral kernel* $\widetilde{K}(x,y)$ *of* $q(D,x)$ *has the property that*

$$\text{(B.14)} \qquad \widetilde{\mathcal{K}}f(x) = \int_\Gamma \widetilde{K}(x,y) f(y)\, d\sigma(y), \quad x \in \mathbb{R}^n \setminus \Gamma,$$

satisfies

$$\text{(B.15)} \qquad \|(\widetilde{\mathcal{K}}f)^*\|_{L^p(\Gamma_0)} \le C\|f\|_{\mathfrak{h}^p(\Gamma_0)}, \quad \operatorname{supp} f \subseteq \Gamma_0.$$

Proposition B.4 extends to the setting of pseudodifferential operators on a compact manifold, containing a Lipschitz domain Ω, with $\partial\Omega$ replacing Γ. In the process, Lemma A.3 is used.

Consider now the single layer potential (1.8), under the hypotheses on M, Ω, L made in §1 and recall the decomposition (2.4). Now $\nabla_x e_0(x-y,y)$ satisfies the conditions of Proposition B.3 and $e_1(x,y)$ satisfies (2.5). Hence the contribution of $e_1(x,y)$ is easy to estimate (e.g., as in (B.5)–(B.6) but more elementary), and we have:

Proposition B.5. *For $(n-1)/n < p \leq 1$,*

$$\|(\nabla \mathcal{S} f)^*\|_{L^p(\partial\Omega)} \leq C \|f\|_{\mathfrak{h}^p(\partial\Omega)}, \tag{B.16}$$

uniformly for $f \in \mathfrak{h}^p(\partial\Omega)$.

Turning to boundary operators, recall first the notation introduced in connection with (7.27). Now, if $u \in C^1(\Omega)$ is a scalar function and g is a two-form with coefficients in $\mathfrak{h}^p(\partial\Omega)$ for some $(n-1)/n < p < 1$, we shall say that $\nu \wedge du = g$ provided

$$\lim_{t \searrow 0} \int_{\partial\Omega} \langle \nu_t \wedge du_t, \psi \rangle \, d\sigma_t = \int_{\partial\Omega} \langle g, \psi \rangle \, d\sigma, \tag{B.17}$$

for each two-form ψ with coefficients in $C^\alpha(\partial\Omega)$, where $\alpha := (n-1)(p^{-1} - 1) > 0$. Recall that we do not make any notational distinction between functions and forms with coefficients in $\mathfrak{h}^p(\partial\Omega)$.

Proposition B.6. *The operators*

$$K^* : \mathfrak{h}^p(\partial\Omega) \longrightarrow \mathfrak{h}^p(\partial\Omega) \tag{B.18}$$

and

$$\nu \wedge dS : \mathfrak{h}^p(\partial\Omega) \longrightarrow \mathfrak{h}^p(\partial\Omega) \tag{B.19}$$

are well defined and bounded for each $(n-1)/n < p \leq 1$.

Proof. In dealing with the first operator, there is no loss of generality in assuming that $V = 0$ on $\overline{\Omega}$, which we shall do. Consider next f in $L^2(\partial\Omega)$, say. Then

$$\partial_\nu \mathcal{S} f \in \mathfrak{h}^p(\partial\Omega) \quad \text{and} \quad \|\partial_\nu \mathcal{S} f\|_{\mathfrak{h}^p(\partial\Omega)} \leq C \|(\nabla \mathcal{S} f)^*\|_{L^p(\partial\Omega)}. \tag{B.20}$$

As explained in the course of the proof of Theorem 7.9, this follows by adapting the results in [Wi] to the present context. Now, (B.20) and Proposition B.5 imply that

$$\left\| \left(-\frac{1}{2} I + K^* \right) f \right\|_{\mathfrak{h}^p(\partial\Omega)} \leq C \|f\|_{\mathfrak{h}^p(\partial\Omega)} \tag{B.21}$$

and, as far as (B.18) is concerned, the desired conclusion follows by the usual density argument.

The arguments for (B.19) is similar and we omit it. This finishes the proof of the proposition. □

In closing, let us point out that by combining the result (1.13) with Proposition 3.3, we deduce from Proposition B.5 an alternative proof of the fact that K^* and $\nabla_T S$ are bounded operators on $\mathfrak{h}^1(\partial\Omega)$. Also, if M has a metric tensor of class C^{1+r} for some $r > 0$, then we can make use of the following lemma below to give an alternative proof of the fact that K^* and $\nabla_T S$ are bounded on $\mathfrak{h}^p(\partial\Omega)$ for $1-\varepsilon < p \leq 1$.

Lemma B.7 ([CW]). *Let (X, μ, d) be a homogeneous space in the sense of [CW], where d is the quasi-distance and μ is the doubling measure on X. Also, denote by $\mathfrak{H}^p$ the class of atomic Hardy spaces associated with (X, μ, d) and let m stand for the measure distance, i.e., $m(x,y) := inf\{d\mu(B) : x, y \in B,\ B$ ball$\}$. Consider*

$$\text{(B.22)} \qquad Tf(x) := \int_X k(x,y) f(y)\, d\mu(y), \qquad x \in X,$$

an integral operator on X. Assume that T is bounded on $L^2(X, d\mu)$ and that, for some $C_0, C_1, \theta > 0$, the kernel satisfies

$$\text{(B.23)} \qquad |k(x,y) - k(x,y_0)| \leq C_0 \left[\frac{m(y,y_0)}{m(x,y_0)}\right]^\theta \frac{1}{m(x,y_0)},$$

uniformly for $m(x,y_0) > C_1 m(y,y_0)$. Then there exists $\varepsilon > 0$ so that

$$\text{(B.24)} \qquad T : \mathfrak{H}^p \longrightarrow \mathfrak{H}^p \quad \text{is bounded}$$

for $1-\varepsilon < p < 1$. If, in addition, Tf integrates to zero for $f \in L^2(X, d\mu)$, then we may also take $p = 1$ in (B.24).

Let us check that Lemma B.7 applies to, for instance, the operator K^* on $\mathfrak{h}^p(\partial\Omega)$. Recall that its integral kernel is $\langle \nu(x), d_x E(x,y)\rangle$, where $E(x,y)$ has been introduced in (1.7). From §2 we know that $E(x,y)\sqrt{g(y)} = e_0(x-y, y) + e_1(x,y)$, where $e_0(z,y)$ is independent of V and satisfies (7.26) and, if the metric tensor is C^{1+r} for some $r > 0$ then $e_1(x,y)$ satisfies (7.25). Thus, in the case under discussion, (B.23) follows by estimating the contributions from $e_0(x-y,y)$ and $e_1(x,y)$ separately. Consequently, the operator K^* is bounded on $\mathfrak{h}^p(\partial\Omega)$ for $1-\varepsilon < p < 1$. There remains the situation when $p = 1$. To this end, assume for a moment that $V = 0$ on $\overline{\Omega}$. Since in this case, by the divergence theorem, $K^* f$ has a vanishing first moment,

$\forall f \in L^2(\partial\Omega)$, the desired conclusion follows directly from the last part in Lemma B.7. Now, a different choice of V will (by the fact that $e_0(x-y,y)$ is independent of V and (7.25)), affect K^* only by a bounded map from $L^1(\partial\Omega)$ into $L^{p_0}(\partial\Omega)$, for some $p_0 > 1$. By (1.39), this suffices to conclude that K^* remains bounded for $p = 1$ also. The argument for $\nabla_T S$ is similar.

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