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The Curvature of Minimal Surfaces in Singular Spaces

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1. Introduction.

Let D be an unit disk in $\mathbf{R}^2$ and (M, g) a smooth Riemannian manifold. If an immersed surface $u : D \to M$ is minimal, i.e. stationary with respect to the area functional

$$\int_D \sqrt{det(\tau_{ij})}dx$$

where $(\tau_{ij})=u^*g$ is the pull back metric on D, then $b_{11} + b_{22} = 0$ with b_{ij} the components of the vector valued second fundamental form. The Gauss equation then gives,

$$K_\Sigma = K_M + b_{11}b_{22} - b_{12}^2 \leq K_M$$

where K_Σ is the Gauss curvature of the surface and K_M is the sectional curvature of the tangent plane to the surface in the manifold. This shows that the curvature of a minimal surface is less than or equal to that of the ambient space. In this paper, we will show that this fundamental curvature property of minimal surfaces also holds in certain singular spaces.

When a smooth surface Σ has a conformal metric with conformal factor λ, it is well known that the Gaussian curvature K_Σ is given by the formula

$$K_\Sigma = -\frac{1}{2\lambda}\triangle \log \lambda.$$

Hence, the condition that the curvature be bounded from above by κ reduces to the inequality

$$\triangle \log \lambda \geq -2\kappa\lambda.$$

Our main theorem states that this same type of inequality holds when we replace the smooth Riemannian manifold with a complete metric space of curvature bounded from above by κ. We will call a map from a surface a minimal surface if it is conformal and locally energy minimizing. Recall that these conditions on a map are equivalent to minimality in the smooth setting. Because our target space can be quite singular, we can only expect the following weak inequality:

Main Theorem. *Let λ be a conformal factor of the pull back metric under a minimal surface $u : D \to X$ where (X, d) is a complete metric space of curvature bounded from above by κ. Then for all non-negative $\varphi \in C_c^\infty(D)$,*

$$\int_D \log \lambda \Delta\varphi \geq -2\kappa \int_D \varphi\lambda.$$

The complete metric spaces considered are length spaces, i.e. any two points can be joined by a distance realizing curve. Furthermore, we impose a curvature bound from above; here, the curvature bound is defined in terms of comparing geodesic triangles with comparison triangles in a constant curvature surface (see Section 3 for the precise definition.) Sometimes called Alexandrov spaces with curvature bounded from above, they were studied by A.D. Alexandrov [A] in the 1950's and advanced by him and the Russian school of mathematicians. They include smooth Riemannian manifolds with an upper bound on the sectional curvature but allow singularities of a very general type. In fact, no restriction is made on the singularities. If we consider a class of Riemannian manifolds with upper bound κ on sectional curvature and a lower bound on the injectivity radius, the completion by Gromov-Hausdorff metric turns out to be these metric spaces of curvature bounded from above by κ.

The motivation of this paper is twofold. One, we wish to extend the study of minimal surfaces in Riemannian manifolds to spaces with singularities. Second, we wish to introduce analytical tools in the study of singular spaces.

Recently, there has been much interest in the study of harmonic map theory for spaces with singularities. A general existence and regularity theories of harmonic maps into Riemannian simplicial complexes of non-positive curvature was developed in [GS] to solve certain rigidity problems. This theory was further generalized for maps into complete metric spaces with non-positive curvature by [KS] and independently by [J]. The case of curvature bounded from above by some constant is treated in [S]. Some examples of the application of the harmonic maps into certain singular spaces are Wolf's [W1],[W2],[W3] investigations of Teichmüller spaces and the actions of fundamental groups of closed surfaces and Hardt and Lin's [HL] study of neumatic liquid crystals. In light of the these successful studies of the harmonic map theory in singular spaces, it is natural to consider an extension of minimal surface theory to these settings.

This paper is organized as follows. In Section 2, we outline Sobolev space theory for metric space targets due to [KS]. In Section 3, we recall the notion of metric spaces with curvature bounds and derive some inequalities for the distance functions that will be important in Section 4. Furthermore, we define area for maps into these spaces which allows us to consider the Plateau Problem. We note that Nikolaev [N2] is the first to consider the Plateau Problem; there he takes a defintion of area that is different from ours. Our definition is a natural extension of the definition of area for maps into smooth Riemannian manifolds and allows us to use a classical approach in the solution of the Plateau Problem. In Section 4, we will prove an inequality satisfied by the energy density function $e(u)$ of an energy minimizing map by a careful consideration of the curvature bound of the target. If $u : M \to N$ is an energy minimizing map between smooth Riemannian manifolds, then the Bochner's formula gives

$$\frac{1}{2}\triangle e(u) = |\nabla du|^2 - \sum_{\alpha,\beta}\langle R^N(u_*e_\alpha, u_*e_\beta)u_*e_\alpha, u_*e_\beta\rangle + \sum_i Ric_M(u^*\theta_i, u^*\theta_i)$$

where $e_1, ..., e_n$ is an orthonormal basis for TM and $\theta_1, ..., \theta_k$ is an orthonormal basis for T^*N. In particular, if M is flat and N has sectional curvature bounded from above by κ, then

$$\triangle e(u) \geq -2\kappa e(u)^2.$$

We will see that the energy density function of energy minimizing maps into metric spaces of curvature bounded from above by κ weakly satisfies the above inequality. This inequality will be the starting point of the proof of the main theorem.

Section 5 is devoted to the proof of the main theorem. In Section 6, we make a geometric interpretation of the analytical result of Section 5; namely, we consider the natural distance function induced by the metric $\lambda(dx^2+dy^2)$ which defines a metric space of curvature bounded from above by κ. In the case when the map minimizes area, Professor Nikolaev has pointed out that this result follows from the works of Reshetnayk [R1],[R2]. We thank him for communicating this observation.

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2. Sobolev Space Theory for Maps to Metric Spaces.

Let Ω be a compact domain in $\mathbf{R}^n$ and (X, d) any complete metric space. In [KS], Korevaar and Schoen develop the space $W^{1,2}(\Omega, X)$. Here we define this space and collect some of their results.

A Borel measurable map $u : \Omega \to X$ is said to be in $L^2(\Omega, X)$ if for $P \in X$,

$$\int_\Omega d^2(u(x), P)dx < \infty.$$

Note that by the triangle inequality, this definition is independent of P chosen. For $u \in L^2(\Omega, X)$, we can construct an ϵ approximate energy function $e_\epsilon : \Omega_\epsilon \to \mathbf{R}$,

$$e_\epsilon(x) = n|\partial B_\epsilon(x)|^{-1} \int_{\partial B_\epsilon(x)} \frac{d^2(u(x), u(y))}{\epsilon^2} d\Sigma.$$

Here Ω_ϵ is the set of points in Ω with distance from the boundary more than ϵ and $B_\epsilon(x)$ is a ball of radius ϵ centered at x. Letting $e_\epsilon(x) = 0$ for $\Omega - \Omega_\epsilon$, we have that $e_\epsilon(x) \in L^1(\Omega)$ and by integrating against continuous functions with compact support, these functions define linear functionals $E_\epsilon : C_c(\Omega) \to \mathbf{R}$. We say $u \in L^2(\Omega, X)$ has finite energy (or that $u \in W^{1,2}(\Omega, X)$) if

$$E^u \equiv \sup_{f \in C_c(\Omega), 0 \leq f \leq 1} \limsup_{\epsilon \to 0} E_\epsilon(f) < \infty.$$

It can be shown that if u has finite energy, the measures $e_\epsilon(x)dx$ converge in the weak* topology to a measure which is absolutely continuous with respect to the Lebesgue measure. Hence, there exists a function $e(x)$, which we call the energy density, so that $e_\epsilon(x)dx \rightharpoonup e(x)dx$. In analogy to the case of real valued functions, we write $|\nabla u|^2(x)$ in place of $e(x)$. In particular,

$$E^u = \int_\Omega |\nabla u|^2 dx.$$

Similarly, the directional energy measures $|u_*(Z)|^2dx$ for $Z \in \Gamma\bar{\Omega}$ can also be defined as the weak* limit of measures ${}^Ze_\epsilon dx$, where

$$^Ze_\epsilon(x) = \frac{d^2(u(x), u(x+\epsilon Z))}{\epsilon^2}.$$

Furthermore, for $Z \in T\bar{\Omega}$,

$$|u_*(Z)|(x) = \lim_{\epsilon \to 0} \frac{d(u(x), u(x+\epsilon\omega)}{\epsilon},$$

a.e. $x \in \Omega$. Finally, we have

$$|\nabla u|^2 = \int_{S^{n-1}} |u_*(Z)|^2 d\sigma(Z).$$

This definition of Sobolev space $W^{1,2}(\Omega, X)$ is consistent with the usual definition when X is a Riemannian manifold. The following theorems allow us to use variational methods in the setting where the target space of maps is a complete metric space. Note that $u_k \to u$ in $L^2(\Omega, X)$ will mean $d(u_k, u)$ converges to 0 in $L^2(\Omega)$, i.e.

$$\lim_{k \to \infty} \int d^2(u_k, u) = 0.$$

Theorem 2.1 ([KS] Theorem 1.6.1). *If $\{u_k\} \subset W^{1,2}(\Omega, X)$ is a sequence with uniformly bounded $W^{1,2}(\Omega, X)$ norms and $u_k \to u$ in $L^2(\Omega, X)$, then $u \in W^{1,2}(\Omega, X)$ and*

$$E(u) = \liminf_{k \to \infty} E(u_k).$$

The following is a generalization of the $W^{1,2}$ trace theory.

Theorem 2.2 ([KS] Theorem 1.12.2). *Any $u \in W^{1,2}(\Omega, X)$ has a well-defined trace map (denoted $tr(u)$), with $tr(u) \in L^2(\partial\Omega, X)$. If $\{u_k\} \subset W^{1,2}(\Omega, X)$ is a sequence with uniformly bounded energies and $u_k \to u$ in $L^2(\Omega, X)$, then $tr(u_k)$ converges to $tr(u)$ in $L^2(\partial\Omega, X)$.*

We also have the following Rellich type precompactness theorem.

Theorem 2.3 ([KS] Theorem 1.13). *Let (X, d) be locally compact. If $\{u_k\} \subset W^{1,2}(\Omega, X)$ satisfy*

$$\int_\Omega d^2(u_k(x), Q)dx + E(u_k) \leq C,$$

where Q is a fixed point in X, then a subsequence of $\{u_k\}$ converges in $L^2(\Omega, X)$ to a finite energy map u.

Using these theorems, one can solve the following.

The Dirichlet Problem. *Let (X, d) be a complete locally compact metric space. Let $\psi \in W^{1,2}(\Omega, X)$. Define*

$$W^{1,2}_{\psi} = \left\{u \in W^{1,2}(\Omega, X) : tr(u) = tr(\psi)\right\}.$$

Let $E_\psi = \inf\{E(v) : v \in W^{1,2}_{\psi}\}$. There exists $u \in W^{1,2}_{\psi}$ such that $E(u) = E_\psi$.

If we assume an upper curvature bound on the target (see Definition 3.1 in the next section), we get nice regularity properties of the solution. In fact, [KS] shows that the solution is Lipschitz when X is non-positively curved and [S1] shows that the same holds in the case when curvature is bounded from above by some constant provided that the boundary data lies in a small geodesic ball. In both cases, the map is Hölder continuous to the boundary.

3. Metric Spaces of Curvature Bounded from Above.

In this section, we will recall the definition of curvature bounds in a metric space, give some technical propositions and define the notion of area for maps into these singular spaces.

3.1. The Definition.

Definition 3.1. A complete metric space (X, d) is said to have curvature bounded from above by κ if the following conditions hold:

(i) (X, d) is a length space; that is, if $P, Q \in X$ there exists a distance realizing curve connecting P and Q. (We call such distance realizing curves geodesics.)

(ii) Let S_κ be a surface of constant curvature κ. For any three points $P, Q, R \in X$ (with $d_{PQ} + d_{QR} + d_{RS} < \frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$) and choices of geodesics γ_{PQ} (of length r), γ_{QR} (of length p) and γ_{PR} (of length q) connecting the respective points, call a triangle $\triangle(\tilde{P}\tilde{Q}\tilde{R})$ in S_κ with vertices $\tilde{P}, \tilde{Q}, \tilde{R}$ and opposite side lengths p, q, r a comparison triangle in S_κ. For any $0 < \lambda < 1$ write Q_λ for the point on γ_{QR} so that $d(Q, Q_\lambda) = \lambda p$ and $d(Q_\lambda, R) = (1 - \lambda)p$ and define $\tilde{Q}_\lambda \in S_\kappa$ analogously to Q_λ, then

$$d(P, Q_\lambda) \leq d_{S_\kappa}(\tilde{P}, \tilde{Q}_\lambda).$$

Remark. These spaces are sometimes defined in terms of an angle excess (see [ABN] for example). The upper angle between geodesics are defined as follows: if γ and σ are geodesics having a common point P with $R \in \gamma, Q \in \sigma$ and $r = d(P,Q)$, $q = d(P,R)$, we let $\alpha^{\kappa}_{\gamma\sigma}(r,q)$ be the angle at $\tilde{P}$ of the comparison triangle $\triangle(\tilde{P}\tilde{Q}\tilde{R})$ of S_κ. The upper angle between γ and σ is

$$\alpha(\gamma,\sigma) = \limsup_{r,q\to 0} \alpha^{\kappa}_{\gamma\sigma}(r,q).$$

This definition is independent of κ. (X,d) is said to be a metric space of curvature bounded from above by κ if for every triangle $\triangle(PQR)$ in X (with $d_{PQ} + d_{QR} + d_{RP} < \frac{\pi}{\sqrt{\kappa}}$),

$$\alpha + \beta + \gamma \leq \alpha_\kappa + \beta_\kappa + \gamma_\kappa$$

where α, β, γ are the upper angles of $\triangle(P,Q,R)$ and $\alpha_\kappa, \beta_\kappa, \gamma_\kappa$ are angles of the comparison triangle in S_κ. This definition is equivalent to the above definition of a curvature bound.

These spaces are referred to as $CAT(\kappa)$ spaces in literature. If $\kappa = 1$, then S_κ is a standard unit sphere S^2. Note that if $\kappa > 0$, we can make X into a $CAT(1)$ space by rescaling the distance function. If $\kappa = -1$, then S_κ is the hyperbolic plane $\mathbf{H}^2$. Again, note that if $\kappa < 0$, then by rescaling the distance function, we can make X into a $CAT(-1)$ space.

3.2. Technical Propositions.

This important result is given in [R1] and will be basis of the propositions that follow.

Theorem 3.2 (Reshetnyak). *Let (X,d) be a metric space of curvature bounded above by κ and Γ be a closed rectifiable curve in X (of length less than or equal to $\frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$). Then there exists a convex domain V in S_κ and a map $\varphi : V \to X$ such that $\varphi(\partial V) = \Gamma$, the lengths of the corresponding arcs coincide, and $d_{S_\kappa}(x,y) \geq d(\varphi(x),\varphi(y))$, for $x, y \in V$.*

Let X be a $CAT(\kappa)$ space and $P,Q,R,S \in X$. If $d_{PS}, d_{QR} < \frac{\pi}{\sqrt{\kappa}}$, then there is a unique geodesic between P and S (Q and R, resp.). We denote by P_t (resp. Q_t) the point on this geodesic such that $d_{PP_t} = t d_{PS}$ (resp. $d_{QQ_t} = t d_{QR}$).

Let (X,d) be a $CAT(1)$ space. Given ordered sequence $\{P,Q,R,S\} \subset X$ with $d_{PQ}+d_{QR}+d_{RS}+d_{SP} < \pi$. Theorem 3.2 asserts that there is an ordered sequence $\{\tilde{P},\tilde{Q},\tilde{R},\tilde{S}\} \subset S^2$ such that the quadrilateral associated with it (i.e. the four ordered points, the geodesics between consecutive points and its interior) is convex and

$$d(P,Q) = d_{S^2}(\tilde{P},\tilde{Q}), \qquad d(Q,R) = d_{S^2}(\tilde{Q},\tilde{R}),$$
$$d(R,S) = d_{S^2}(\tilde{R},\tilde{S}), \qquad d(S,P) = d_{S^2}(\tilde{S},\tilde{P}),$$
$$d(P_t,Q_s) \leq d_{S^2}(\tilde{P}_t,\tilde{Q}_s).$$

We will call $\{\tilde{P},\tilde{Q},\tilde{R},\tilde{S}\}$ a spherical subembedding for $\{P,Q,R,S\}$. Similarly, when (X,d) is a $CAT(-1)$ space, we can define a hyperbolic subembedding

$$\{\hat{P},\hat{Q},\hat{R},\hat{S}\} \subset \mathbf{H}^2.$$

In the propositions below, $0^n(\cdot)$ denotes terms that are nth order in the specified variables.

Proposition 3.3. *Let (X,d) be a metric space of curvature bounded from above by κ with $\kappa = 1$ or -1. Then for $\{P,Q,R,S\} \subset X$ (with $d_{PQ}+d_{QR}+d_{RS}+d_{SP} < \pi$ if $\kappa = 1$), the following inequalities hold:*

For $\kappa = 1$,

$$\begin{aligned} \cos d_{PQ_t} + \cos d_{RQ_{1-t}} \geq{}& \cos d_{PQ}\cos td_{QR} + \sin d_{PQ}\sin td_{QR}\cos\alpha \\ &+ \cos d_{RS}\cos td_{QR} + \sin d_{RS}\sin td_{QR}\cos\beta \end{aligned} \tag{3.1}$$

$$\begin{aligned} \cos d_{P_tQ_t} \geq{}& \frac{\sin(1-t)d_{PS}}{\sin d_{PS}}(\cos d_{PQ}\cos td_{QR} + \sin d_{PQ}\sin td_{QR}\cos\alpha) \\ &+ \frac{\sin td_{PS}}{\sin d_{PS}}(\cos d_{RS}\cos td_{QR} + \sin d_{RS}\sin td_{QR}\cos\beta) \end{aligned} \tag{3.2}$$

where $\alpha = \angle\tilde{P}\tilde{Q}\tilde{R}$ and $\beta = \angle\tilde{S}\tilde{R}\tilde{Q}$ and $\{\tilde{P},\tilde{Q},\tilde{R},\tilde{S}\}$ is a spherical subembedding for $\{P,Q,R,S\}$.

For $\kappa = -1$,

$$\begin{aligned} \cosh d_{PQ_t} + \cosh d_{RQ_{1-t}} \leq{}& \cosh d_{PQ}\cosh td_{QR} - \sinh d_{PQ}\sinh td_{QR}\cos\alpha \\ &+ \cosh d_{RS}\cosh td_{QR} - \sinh d_{RS}\sinh td_{QR}\cos\beta \end{aligned}$$

$$\begin{aligned} \cosh d_{P_tQ_t} \leq{}& \frac{\sinh(1-t)d_{PS}}{\sinh d_{PS}}(\cosh d_{PQ}\cosh td_{QR} - \sinh d_{PQ}\sinh td_{QR}\cos\alpha) \\ &+ \frac{\sinh td_{PS}}{\sinh d_{PS}}(\cosh d_{RS}\cosh td_{QR} - \sinh d_{RS}\sinh td_{QR}\cos\beta) \end{aligned}$$

where $\alpha = \angle \hat{P}\hat{Q}\hat{R}$ *and* $\beta = \angle \hat{S}\hat{R}\hat{Q}$ *and* $\{\hat{P}, \hat{Q}, \hat{R}, \hat{S}\}$ *is a hyperbolic subembedding for* $\{P, Q, R, S\}$.

Proof. We will only prove the case when $\kappa = 1$ by comparing the distance function d to the distance function d_{S^2} of the sphere. The proof of the case when $\kappa = -1$ follows analogously by considering the distance function of the hyperbolic plane instead of the sphere.

Let $\{X, Y, Z\} \in S^2$ such that $d_{XY} + d_{YZ} + d_{ZX} < \pi$. We let Y_t be the point on the geodesic between Y and Z such that $d_{Y_tZ} = td_{YZ}$ and $\theta = \angle_{S^2} YXZ$. We have the following equalities:

$$\cos d_{YZ} = \cos d_{XY} \cos d_{XZ} + \sin d_{XY} \sin d_{XZ} \cos\theta \tag{3.3}$$

$$\cos d_{XY_t} = \frac{\sin(1-t)d_{YZ}}{\sin d_{YZ}} \cos d_{XY} + \frac{\sin td_{YZ}}{\sin d_{YZ}} \cos d_{XZ}. \tag{3.4}$$

Hence we have,

$$\begin{aligned} \cos d_{\tilde{P}\tilde{Q}_t} + \cos d_{\tilde{R}\tilde{Q}_{1-t}} = \cos d_{\tilde{P}\tilde{Q}} \cos td_{\tilde{Q}\tilde{R}} + \sin d_{\tilde{P}\tilde{Q}} \sin td_{\tilde{Q}\tilde{R}} \cos\alpha + \\ \cos d_{\tilde{R}\tilde{S}} \cos td_{\tilde{Q}\tilde{R}} + \sin d_{\tilde{R}\tilde{S}} \sin td_{\tilde{Q}\tilde{R}} \cos\beta, \end{aligned}$$

and

$$\begin{aligned} \cos d_{\tilde{P}_t\tilde{Q}_t} = {} & \frac{\sin(1-t)d_{\tilde{P}\tilde{S}}}{\sin d_{\tilde{P}\tilde{S}}} (\cos d_{\tilde{P}\tilde{Q}} \cos td_{\tilde{Q}\tilde{R}} + \sin d_{\tilde{P}\tilde{Q}} \sin td_{\tilde{Q}\tilde{R}} \cos\alpha) \\ & + \frac{\sin td_{\tilde{P}\tilde{S}}}{\sin d_{\tilde{P}\tilde{S}}} (\cos d_{\tilde{R}\tilde{S}} \cos td_{\tilde{Q}\tilde{R}} + \sin d_{\tilde{R}\tilde{S}} \sin td_{\tilde{Q}\tilde{R}} \cos\beta). \end{aligned} \tag{3.5}$$

The result follows immediately from Theorem 3.2. □

Proposition 3.4. *Let* (X, d) *be a metric space of curvature bounded from above by* κ *with* $\kappa = 1$ *or* -1 *and* $\{P, Q, R, S\} \subset X$ *(with* $d_{PQ} + d_{QR} + d_{RS} + d_{SP} < \pi$ *if* $\kappa = 1$*). Furthermore, let* α, β *as in Proposition* 3.3. *Then*

$$d_{PQ} \cos\alpha + d_{RS} \cos\beta = d_{QR} - d_{PS} + 0^2(d_{PQ}, d_{RS}) \tag{3.6}$$

and for $\kappa = 1$, *we have*

$$\begin{aligned} & d_{PQ} \cos\alpha + d_{RS} \cos\beta \\ & \qquad \geq d_{QR} - d_{PS} - \frac{1}{2} d_{PQ} d_{RS} d_{QR}^2 + 0^2(d_{PQ}, d_{RS}) 0^3(d_{QR}) \end{aligned} \tag{3.7}$$

and for $\kappa = -1$, we have

$$(3.8) \quad d_{PQ} \cos\alpha + d_{RS} \cos\beta \geq d_{QR} - d_{PS} + \frac{1}{2} d_{PQ} d_{RS} d_{QR}^2 + 0^2(d_{PQ}, d_{RS}) 0^3(d_{QR}).$$

Proof. Once again, we will only prove the case when $\kappa = 1$ since the case $\kappa = -1$ follows analogously. Again, let $\{\tilde{P}, \tilde{Q}, \tilde{R}, \tilde{S}\}$ be the spherical subembedding of $\{P, Q, R, S\}$. Assume w.l.o.g., $\tilde{Q} = (1,0,0)$, $\tilde{R} = (\cos\theta, \sin\theta, 0) \in S^2 \subset \mathbf{R}^3$. Note for any $X, Y \in S^2 \subset \mathbf{R}^3$, we have that $\cos d_{S^2}(X,Y) = X \cdot Y$. (Here $\cdot$ denotes the usual dot product in $\mathbf{R}^3$.) In particular, we see that $d_{\tilde{Q}\tilde{R}} = \theta$. Let γ (resp. σ) be a unit speed parameterization of a geodesic on S^2 emanating from $\tilde{Q}$ (resp. $\tilde{R}$) such that, for $t > 0$, $\angle_{S^2}\gamma(t)\tilde{Q}\tilde{R} = \alpha$ (resp. $\angle_{S^2}\sigma(t)\tilde{R}\tilde{Q} = \beta$). If $\varphi(t) = (\cos t, \sin t, 0)$, then γ and σ must satisfy:

$$\begin{aligned} \gamma'(0) \cdot \varphi'(0) &= \cos\alpha, & |\gamma'(0)| &= 1 \\ \sigma'(0) \cdot \varphi'(\theta) &= \cos(\pi - \beta), & |\sigma'(0)| &= 1. \end{aligned}$$

Hence, we have that

$$\begin{aligned} \gamma(0) &= (1,0,0) \\ \gamma'(0) &= (0, \cos\alpha, \sin\alpha) \\ \gamma''(0) &= (-1,0,0) \end{aligned}$$

and that

$$\begin{aligned} \sigma(0) &= (\cos\theta, \cos\theta, 0) \\ \sigma'(0) &= (\cos\beta \sin\theta, -\cos\beta\cos\theta, \sin\beta) \\ \sigma''(0) &= (-\cos\theta, -\sin\theta, 0). \end{aligned}$$

Using the Taylor series expansion,

$$\begin{aligned} \gamma(t) \cdot \sigma(s) = \cos\theta &+ t\cos\alpha\sin\theta + s\cos\beta\sin\theta - \frac{t^2 + s^2}{2}\cos\theta \\ &- ts\cos\alpha\cos\beta\cos\theta + ts\sin\alpha\sin\beta + 0^3(t,s). \end{aligned}$$

Let $f(t,s)=\gamma(t)\cdot\sigma(s)$. Then again using Taylor series expansion,

$$\begin{aligned}
d_{S^2}(\gamma(t),\sigma(s)) &= \arccos f(0,0) + t\frac{\partial}{\partial t}(\arccos f)|_{(0,0)} + s\frac{\partial}{\partial s}(\arccos f)|_{(0,0)} \\
&\quad + \frac{t^2}{2}\frac{\partial^2}{\partial t^2}(\arccos f)|_{(0,0)} + \frac{s^2}{2}\frac{\partial^2}{\partial s^2}(\arccos f)|_{(0,0)} \\
&\quad + ts\frac{\partial^2}{\partial t\partial s}(\arccos f)|_{(0,0)} + 0^3(t,s) \\
&= \theta - t\cos\alpha - s\cos\beta \\
&\quad + \frac{t^2}{2}\frac{\cos\theta - \cos^2\alpha\cos\theta}{\sin\theta} + \frac{s^2}{2}\frac{\cos\theta-\cos^2\beta\cos\theta}{\sin\theta} \\
&\quad - ts\frac{\sin\alpha\sin\beta}{\sin\theta} + 0^3(t,s).
\end{aligned}$$

This shows

$$d_{\tilde P\tilde Q}\cos\alpha + d_{\tilde R\tilde S}\cos\beta = d_{\tilde Q\tilde R} - d_{\tilde P\tilde S} + 0^2(d_{\tilde P\tilde Q}, d_{\tilde R\tilde S}).$$

Hence equation 3.6 follows the above equality. By Cauchy-Schwartz inequality,

$$ts\sin\alpha\sin\beta \le \frac{t^2\sin^2\alpha}{2} + \frac{s^2\sin^2\beta}{2},$$

and thus we obtain

$$\begin{aligned}
&\frac{t^2}{2}(\cos\theta-\cos^2\alpha\cos\theta) + \frac{s^2}{2}(\cos\theta-\cos^2\beta\cos\theta) - ts\sin\alpha\sin\beta \\
&= \frac{t^2}{2}(\cos\theta-\cos^2\alpha\cos\theta) + \frac{s^2}{2}(\cos\theta-\cos^2\beta\cos\theta) - ts\sin\alpha\sin\beta\cos\theta \\
&\quad - ts(1-\cos\theta)\sin\alpha\sin\beta \\
&\ge -ts(1-\cos\theta)\sin\alpha\sin\beta \\
&\ge -ts(1-\cos\theta).
\end{aligned}$$

Since, $1-\cos\theta = \frac{\theta^2}{2} + 0^4(\theta)$ and $\frac{1}{\sin\theta} = \frac{1}{\theta}(1+0^2(\theta))$, we obtain

$$\begin{aligned}
d_{S^2}(\gamma(t),\sigma(s)) &\ge \theta - t\cos\alpha - s\cos\beta - \frac{ts(1-\cos\theta)}{\sin\theta} \\
&\ge \theta - t\cos\alpha - s\cos\beta - \frac{ts\theta}{2} + ts0^3(\theta).
\end{aligned}$$

Hence we obtain,

$$d_{\tilde P\tilde Q}\cos\alpha + d_{\tilde R\tilde S}\cos\beta \ge d_{\tilde Q\tilde R} - d_{\tilde P\tilde S} - \frac{1}{2}d_{\tilde P\tilde Q}d_{\tilde R\tilde S}d_{\tilde Q\tilde R} + 0^2(d_{\tilde P\tilde Q}, d_{\tilde R\tilde S})0^3(d_{\tilde Q\tilde R}).$$

Now inequality 3.7 follows immediately. □

Proposition 3.5. *Let (X,d) be a metric space of curvature bounded from above by κ with $\kappa = 1$ or -1 and $\{P,Q,R,S\} \subset X$ (with $d_{PQ}+d_{QR}+d_{RS}+d_{SP} < \pi$ if $\kappa = 1$). Let $d_0 = d_{QR}$, $d_1 = d_{PS}$, and $l_t = d_{P_tQ_t}$ where, P_t (resp. Q_t) is the point on the unique geodesic between P and S (resp. Q and R) such that $d_{PP_t} = td_{PS}$ (resp. $d_{QQ_t} = td_{QR}$). Then for $\kappa = 1$, we have*

$$\begin{aligned} l_t^2 + l_{1-t}^2 \leq\ & l_0^2 + l_1^2 + 2tl_0l_1d_0^2 + 0^2(t)0^2(l_0,l_1) \\ & + 0^4(l_0,l_1) + t0^2(l_0,l_1)0^3(d_0,d_1) + t0^3(l_0,l_1) \end{aligned} \tag{3.9}$$

and for $\kappa = -1$, we have

$$\begin{aligned} l_t^2 + l_{1-t}^2 \leq\ & l_0^2 + l_1^2 - 2tl_0l_1d_0^2 + 0^2(t)0^2(l_0,l_1) \\ & + 0^4(l_0,l_1) + t0^2(l_0,l_1)0^3(d_0,d_1) + t0^3(l_0,l_1). \end{aligned}$$

Proof. Again, we will only prove the case when $\kappa = 1$. Inequality 3.9 follows immediately from Theorem 3.2 if we can prove the same inequality in the sphere. Hence, let $\{\tilde{P},\tilde{Q},\tilde{R},\tilde{S}\}$ be four points in the sphere with $d_{\tilde{P}\tilde{Q}} + d_{\tilde{Q}\tilde{R}} + d_{\tilde{R}\tilde{S}} + d_{\tilde{S}\tilde{P}} < \pi$ and let $d_0 = d_{\tilde{Q}\tilde{R}}$, $d_1 = d_{\tilde{P}\tilde{S}}$, and $l_t = d_{\tilde{P}_t\tilde{Q}_t}$ where, $\tilde{P}_t$ (resp. $\tilde{Q}_t$) is the point on the unique geodesic between $\tilde{P}$ and $\tilde{S}$ (resp. $\tilde{Q}$ and $\tilde{R}$) such that $d_{\tilde{P}\tilde{P}_t} = td_{\tilde{P}\tilde{S}}$ (resp. $d_{\tilde{Q}\tilde{Q}_t} = td_{\tilde{Q}\tilde{R}}$). From equation 3.5,

$$\begin{aligned} \cos l_t + \cos l_{1-t} =\ & \left(\frac{\sin td_1}{\sin d_1}\cos(1-t)d_0 + \frac{\sin(1-t)d_1}{\sin d_1}\cos td_0\right) \times \\ & \times (\cos l_0 + \cos l_1) \\ & + \left(\frac{\sin td_1}{\sin td_1}\sin(1-t)d_0 + \frac{\sin(1-t)d_1}{\sin d_1}\sin td_0\right) \times \\ & \times (\sin l_0 \cos\alpha + \sin l_1 \cos\beta). \end{aligned}$$

By expanding $\sin d_0$ and $\sin d_1$ using Taylor series and then using inequality 3.7, we have

$$\begin{aligned} \sin l_0\cos\alpha + \sin l_1\cos\beta &= l_0\cos\alpha + l_1\cos\beta + 0^3(l_0,l_1) \\ &\geq d_0 - d_1 - \frac{l_0l_1d_0^2}{2} + 0^2(l_0,l_1)0^3(d_0) + 0^3(l_0,l_1). \end{aligned}$$

Also note that

$$\cos l_t + \cos l_{1-t} = 2 - \frac{l_t^2}{2} - \frac{l_{1-t}^2}{2} + 0^4(l_0,l_1).$$

Hence

$$
\begin{aligned}
2-\frac{l_t^2}{2}-\frac{l_{1-t}^2}{2}+0^4(l_0,l_1)&\\
&=\left(2-\frac{l_0^2}{2}-\frac{l_1^2}{2}+0^4(l_0,l_1)\right)\times\\
&\quad\times\left(\frac{\sin td_1}{\sin d_1}\cos(1-t)d_0+\frac{\sin(1-t)d_1}{\sin d_1}\cos td_0\right)\\
&\quad+\left(d_0-d_1-\frac{l_0l_1d_0^2}{2}+0^2(l_0,l_1)0^3(d_0)+0^3(l_0,l_1)\right)\times\\
&\quad\times\left(\frac{\sin td_1}{\sin d_1}\sin(1-t)d_0+\frac{\sin(1-t)d_1}{\sin d_1}\sin td_0\right).
\end{aligned}
$$

We now use,

$$
\frac{\sin td_1}{\sin d_1}\cos(1-t)d_0+\frac{\sin(1-t)d_1}{\sin d_1}\cos td_0=1+0^2(t)
$$

and

$$
\frac{\sin td_1}{\sin d_1}\sin(1-t)d_0+\frac{\sin(1-t)d_1}{\sin d_1}\sin td_0=2t+0^2(t)
$$

to obtain

$$
\begin{aligned}
-2+\frac{l_t^2}{2}+\frac{l_{1-t}^2}{2}+0^4(l_0,l_1)&\\
&=-2\left(\frac{\sin td_1}{\sin d_1}\cos(1-t)d_0+\frac{\sin(1-t)d_1}{\sin d_1}\cos td_0\right)\\
&\quad+\frac{l_0^2}{2}+\frac{l_1^2}{2}+0^2(t)0^2(l_0,l_1)+0^4(l_0,l_1)\\
&\quad-(d_0-d_1)\left(\frac{\sin td_1}{\sin d_1}\sin(1-t)d_0+\frac{\sin(1-t)d_1}{\sin d_1}\sin td_0\right)\\
&\quad+tl_0l_1d_0^2+0^2(t)0^2(l_0,l_1)+t0^2(l_0,l_1)0^3(d_0,d_1)+t0^3(l_0,l_1).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
-4+l_t^2+l_{1-t}^2&=\frac{-2}{\sin d_1}[2(\sin td_1\cos(1-t)d_0+\sin(1-t)d_1\cos td_0)\\
&\quad+(d_0-d_1)(\sin td_1\sin(1-t)d_0+\sin(1-t)d_1\sin td_0)]\\
&\quad+l_0^2+l_1^2+2tl_0l_1d_0^20^2(t)0^2(l_0,l_1)\\
&\quad+0^4(l_0,l_1)+t0^2(l_0,l_1)0^3(d_0,d_1)+t0^3(l_0,l_1).
\end{aligned}
$$

If we let

$$F(x) = 2[\sin td_1 \cos(1-t)(x+d_1) + \sin(1-t)d_1 \cos t(x+d_1)] \\ + x(\sin td_1 \sin(1-t)(x+d_1) + \sin(1-t)d_1 \sin t(x+d_1)),$$

then the lemma will follow from the following claim with $x = d_0 - d_1$.

Claim. *There exists $\sigma > 0$ such that for $|x| \leq \sigma$, then $F(x) \geq 2\sin d_1$.*

Proof of claim. It is easy to check that $F(0) = 2\sin d_1$, $F'(0) = 0$. Furthermore,

$$\begin{aligned} F''(0) &= -2(1-t)^2 \sin td_1 \cos(1-t)d_1 - 2t^2 \sin(1-t)d_1 \cos td_1 \\ &\quad + 2(1-t)\sin td_1 \cos(1-t)d_1 + 2t\sin(1-t)d_1 \cos td_1 \\ &= 2t(1-t)\sin td_1 \cos(1-t)d_1 + \sin(1-t)d_1 \cos td_1 \\ &= 2t(1-t)\sin d_1 \\ &\geq 0. \end{aligned}$$

Since F is a C^∞ function, the claim follows. □

3.3. The Pull-back Inner Product and the Area.

We make sense of the notion of area for maps $u \in W^{1,2}(D,X)$ when X has an upper curvature bound. We do this by defining an inner product structure on D which generalizes the pull-back metric for a smooth map between smooth Riemannian manifolds. The proof of the existence of such an inner product structure is an easy generalization of the proof in [KS] for maps into NPC space using the following technical lemma.

Lemma 3.6. *Let (X,d) be a $CAT(1)$ space. Let $P,Q,R,S \in X$. Then*

$$d_{PR}^2 + d_{QS}^2 \leq d_{PQ}^2 + d_{QR}^2 + d_{RS}^2 + d_{PS}^2 + o(\sigma),$$

where $\sigma = \max\{d_{PQ}^2, d_{QR}^2, d_{RS}^2, d_{PS}^2\}$.

We use inequality 3.3 to obtain

$$\begin{aligned} \cos d_{PR} + \cos d_{QS} \geq \frac{1}{2}(&\cos d_{PQ} \cos d_{QR} + \cos d_{PS} \cos d_{RS} \\ &+ \cos d_{PQ} \cos d_{PS} + \cos d_{QR} \cos d_{RS}) \\ &+ \sin d_{PQ} \sin d_{QR} \cos \angle Q + \sin d_{PS} \sin d_{RS} \cos \angle S \\ &+ \sin d_{PQ} \sin d_{PS} \cos \angle P + \sin d_{QR} \sin d_{RS} \cos \angle R). \end{aligned}$$

Expanding terms, we obtain,

$$\begin{aligned} d_{PR}^2 + d_{QR}^2 \le d_{PQ}^2 &+ d_{QR}^2 + d_{RS}^2 + d_{PS}^2 \\ &- d_{PQ}d_{QR}\cos\angle Q - d_{PS}d_{RS}\cos\angle S \\ &- d_{PQ}d_{PS}\cos\angle P - d_{QR}d_{RS}\cos\angle R + o(\sigma). \end{aligned}$$

Now consider ordered points $\{\bar{P}, \bar{Q}, \bar{R}, \bar{S}\} \subset \mathbf{R}^2$ such that, $\mathcal{Q}$, the quadrilateral associated with it is convex and $d_{PQ} = |\bar{P} - \bar{Q}|, d_{QR} = |\bar{Q} - \bar{R}|, d_{RS} = |\bar{R} - \bar{S}|, d_{PS} = |\bar{P} - \bar{S}|$. Letting

$$\begin{aligned} \angle\bar{P} &= \angle\bar{S}\bar{P}\bar{Q}, & \angle\bar{Q} &= \angle\bar{P}\bar{Q}\bar{R}, \\ \angle\bar{R} &= \angle\bar{Q}\bar{R}\bar{S}, & \angle\bar{S} &= \angle\bar{R}\bar{S}\bar{P} \end{aligned}$$

in $\mathbf{R}^2$, we have $\angle\bar{P} + \angle\bar{Q} + \angle\bar{R} + \angle\bar{S} = 4\pi$. Furthermore, $\angle\bar{P} \le \angle P, \angle\bar{Q} \le \angle Q, \angle\bar{R} \le \angle R$, and $\angle\bar{S} \le \angle S$. By the Gauss-Bonnet Theorem,

$$\angle P + \angle Q + \angle R + \angle S - 4\pi = \int_{\mathcal{Q}} dA.$$

Hence, $\angle Q - \angle\bar{Q} = 0^2(d_{PQ}, d_{QR}, d_{RS}, d_{PS})$. We can rewrite equation 3.6:

$$\begin{aligned} d_{PR}^2 + d_{QR}^2 - o(\sigma) \le d_{PQ}^2 &+ d_{QR}^2 + d_{RS}^2 + d_{PS}^2 \\ &- d_{PQ}d_{QR}\cos\angle\bar{Q} - d_{PS}d_{RS}\cos\angle\bar{S} \\ &- d_{PQ}d_{PS}\cos\angle\bar{P} - d_{QR}d_{RS}\cos\angle\bar{R}. \end{aligned}$$

Letting A, B, C, D be the oriented vectors pointing to the consecutive vertices of the Euclidean quadrilateral Q, i.e.

$$\begin{aligned} A &= \bar{Q} - \bar{P}, & B &= \bar{R} - \bar{Q} \\ C &= \bar{S} - \bar{R}, & B &= \bar{P} - \bar{S}, \end{aligned}$$

we have that,

$$\begin{aligned} -|\bar{P} - \bar{Q}|\ |\bar{Q} - \bar{R}|\cos\bar{Q} &- |\bar{P} - \bar{S}|\ |\bar{R} - \bar{S}|\cos\bar{S} \\ -|\bar{P} - \bar{Q}|\ |\bar{P} - \bar{S}|\cos\bar{P} &- |\bar{Q} - \bar{R}|\ |\bar{R} - \bar{S}|\cos\bar{R} \\ &= A \cdot B + B \cdot C + C \cdot D + D \cdot A \\ &= (A + C) \cdot (B + C) \\ &= -|A + C|^2. \end{aligned}$$

Here, we have used the fact that $A + B + C + D = 0$. Hence,

$$d_{PR}^2 + d_{QR}^2 \le d_{PQ}^2 + d_{QR}^2 + d_{RS}^2 + d_{PS}^2 + o(\sigma). \qquad \square$$

As a result of the above, the directional energy functions satisfy a parallelogram law:

Lemma 3.7. *Let $\Omega \subset \mathbf{R}^n$ and let X be a $CAT(1)$ space. If $u \in W^{1,2}(\Omega, X)$, then for any $Z, W \in \Gamma(T\bar{\Omega})$, the parallelogram identity*

$$|u_*(Z+W)|^2 + |u_*(Z-W)|^2 = 2|u_*(Z)|^2 + 2|u_*(W)|^2$$

holds for a.e. $x \in \Omega$.

Proof. Use Lemma 3.6 with $P = u(x)$, $Q = u(x+\epsilon Z)$, $R = u(x+\epsilon W)$, $S = u(x+\epsilon(Z+W))$. Divide by ϵ^2 and let $\epsilon \to 0$ to obtain

$$|u_*(Z+W)|^2 + |u_*(Z-W)|^2 \leq 2|u_*(Z)|^2 + 2|u_*(W)|^2.$$

for a.e. $x \in \Omega$. Repeat using $Z+W$ and $Z-W$ in place of Z and W to get the opposite inequality. □

For $Z, W \in \Gamma(T\bar{\Omega})$, we define

$$\pi(Z,W) = \frac{1}{4}|u_*(Z+W)|^2 - \frac{1}{4}|u_*(Z-W)|^2.$$

Proposition 3.8. *The operator π defined above,*

$$\pi : \Gamma(T\bar{\Omega}) \times \Gamma(T\bar{\Omega}) \to L^1(\Omega, \mathbf{R})$$

is continuous, symmetric, bilinear, non-negative and tensorial.

Proof. The proof is the same as the one given in [KS] (Theorem 2.3.2).

Definition 3.9. π as above is the pull back inner product under the map u.

We can now define the area functional $A : W^{1,2}(D, X) \to \mathbf{R}$ by

$$\begin{aligned} A(u) &= \int_D \sqrt{\det \pi_u} dx^1 dx^2 \\ &= \int_D \sqrt{(\pi_u)_{11}(\pi_u)_{22} - (\pi_u)_{12}^2} dx^1 dx^2 \end{aligned}$$

where $(\pi)_{ij} = \pi_u(\partial_i, \partial_j)$.

Thus, we can formulate:

The Plateau Problem. *Let D be a disk and Γ be a closed Jordan curve in X and let*

$$C_\Gamma = \{u \in W^{1,2}(D, X) : u|_{\partial D} \text{ parametrizes } \Gamma \text{ monotonically}\}.$$

There exists $u \in C_\Gamma$ so that $\mathcal{A}(u) = \inf\{\mathcal{A}(v) \mid v \in C_\Gamma\}$. Moreover, u is weakly conformal, i.e. $\pi_{11} = \pi_{22}$ and $\pi_{12} = 0 = \pi_{21}$, and Lipschitz in the interior of D.

We can solve the Plateau Problem for a locally compact $CAT(\kappa)$ space. Since the arguments are essentially the same as the classical approach (see for example [M]), we omit the proof here. Because conformal energy minimizing maps into smooth Riemannian manifolds are minimal, it is natural to define:

Definition 3.10. Let X be a complete metric space of curvature bounded from above by κ. We say $u : D \to X$ is a minimal surface if u is a weakly conformal energy minimizing map. $\lambda = \pi_{11} = \pi_{22}$ is called the conformal factor of the pull back metric under u.

4. The Energy Density Inequality.

Before we can prove our main inequality, we will need to prove another inequality which is of interest in itself. As mentioned in the introduction, this can be seen as a generalization of the Bochner's inequality for harmonic maps between smooth Riemannanian manifolds.

[KS] proves the weak subharmonicity for the energy density of a harmonic map when the target is an NPC space. We generalize their result by proving the following inequality when the target is a space of curvature bounded from above.

Theorem 4.1. *Let $u : D \to X$ be an energy minimizing map into a $CAT(\kappa)$ space (i.e. a metric space of curvature bounded from above by κ). Then for any $\eta \in C^2_c(D)$ with $\eta \geq 0$,*

$$\int_D |\nabla u|^2 \Delta\eta \geq -2\kappa \int_D \eta |\nabla u|^4. \tag{4.1}$$

If u is minimal (i.e. also weakly conformal) with conformal factor λ, then

$$\int_D \lambda \Delta\eta \geq -2\kappa \int_D \eta \lambda^2.$$

Proof. The case $\kappa = 0$ is the result of [KS]. We will first prove the above for the case when $\kappa = 1$. Note that the result for $\kappa > 0$ follows immediately from rescaling the target distance function. The case $\kappa = -1$ is proven analogously.

In the proof below, we prove equation 4.1 for $\eta \in C_c^2(D)$ with $0 \leq \eta \leq \frac{1}{2}$. By rescaling η, we see that equation 4.1 holds for any non-negative C_c^2 function η.

For two given points x, y, define

$$\eta_- = \min\{\eta(x), \eta(y)\}.$$

Let u_0 and u_1 be energy minimizing maps such that

$$\sup_{x \in supp(\eta)} d(u_0, u_1) < \frac{\pi}{2}.$$

We let L_0, L_1 be the Lipschitz constants of u_0 and u_1 in $\text{supp}(\eta)$ and let $L = \max\{L_0, L_1\}$. If $|x-y| \leq \frac{\pi}{2L}$, then $d(u_0(x), u_0(y))$, $d(u_1(x), u_1(y)) < \frac{\pi}{2}$. Let $u_\eta \in W^{1,2}(D, X)$ be defined by taking the geometric interpolation of u_0 and u_1. In other words, let $u_\eta(x)$ be the point on the (unique) geodesic between $u_0(x)$ and $u_1(x)$ such that

$$\frac{d(u_0(x), u_\eta(x))}{d(u_0(x), u_1(x))} = \eta(x) \qquad \text{and} \qquad \frac{d(u_\eta(x), u_1(x))}{d(u_0(x), u_1(x))} = 1 - \eta(x).$$

If $\eta_- = \eta(y)$, we consider the ordered sequence

$$\{u_{\eta_-}(y), u_{\eta_-}(x), u_{1-\eta_-}(x), u_{1-\eta_-}(y)\}$$

and let $t = \frac{\eta(x)-\eta(y)}{1-2\eta(y)}$ and apply Proposition 3.3. If $\eta_- = \eta(x)$, we interchange the roles of x and y and apply Proposition 3.3. Using the shorthand notation,

$$\begin{aligned} l_t &= d(u_t(x), u_t(y)) \\ l_\eta &= d(u_\eta(x), u_\eta(y)) \\ l_{1-\eta} &= d(u_{1-\eta}(x), u_{1-\eta}(y)) \\ d_x &= d(u_0(x), u_1(x)) \\ d_y &= d(u_0(y), u_1(y) \\ \bar{\eta} &= \eta(x) - \eta(y), \end{aligned}$$

we deduce in both cases,

$$\begin{aligned} \cos l_\eta + \cos l_{1-\eta} \geq& [\cos l_{\eta_-} + \cos l_{1-\eta_-}] \cos(\bar{\eta} d_x) \\ &+ [\sin l_{\eta_-} \cos\alpha + \sin l_{1-\eta_-} \cos\beta] \sin(\bar{\eta} d_x) \end{aligned} \tag{4.2}$$

where $\alpha = \angle\tilde{P}\tilde{Q}\tilde{R}$ and $\beta = \angle\tilde{S}\tilde{R}\tilde{Q}$ and $\{\tilde{P},\tilde{Q},\tilde{R},\tilde{S}\}$ is a spherical subembedding for $\{u_{\eta_-}(y), u_{\eta_-}(x), u_{1-\eta_-}(x), u_{1-\eta_-}(y)\}$. By expanding the above, we obtain,

$$\begin{aligned} l_\eta^2 + l_{1-\eta}^2 + 0^4(l_\eta, l_{1-\eta}) \leq\ & l_{\eta_-}^2 + l_{1-\eta_-}^2 - 2(l_{\eta_-}\cos\alpha + l_{1-\eta_-}\cos\beta)\bar{\eta}d_x \\ & + 0^2(\bar{\eta}d_x) + 0^3(l_{\eta_-}, l_{1-\eta_-}) \end{aligned}$$

Applying Proposition 3.5 to $\{u_0(y), u_0(x), u_1(x), u_1(y)\}$ and $t = \eta_-$, we obtain,

$$\begin{aligned} l_{\eta_-}^2 + l_{1-\eta_-}^2 \leq\ & l_0^2 + l_1^2 + 2\eta_- l_0 l_1 d_x^2 + 0^2(\eta_-)0^2(l_0, l_1) \\ & + 0^4(l_0, l_1) + \eta_- 0^2(l_0, l_1)0^3(d_x, d_y) + \eta_- 0^3(l_0, l_1) \end{aligned}$$

Applying Proposition 3.4 to $\{u_{\eta_-}(y), u_{\eta_-}(x), u_{1-\eta_-}(x), u_{1-\eta_-}(y)\}$, we obtain,

$$-(l_{\eta_-}\cos\alpha + l_{1-\eta_-}\cos\beta) \leq (1-2\eta_-)(d_y - d_x) + 0^2(l_{\eta_-}, l_{1-\eta_-})0^2(d_x, d_y)$$

Thus, the Cauchy-Schwartz inequality gives,

$$\begin{aligned} & -(l_{\eta_-}\cos\alpha + l_{1-\eta_-}\cos\beta)\bar{\eta}d_x \\ & \qquad \leq \frac{1}{2}(1-2\eta_-)\bar{\eta}(d_y^2 - d_x^2) + \bar{\eta}0^2(l_{\eta_-}, l_{1-\eta_-})0^3(d_y, d_x) \end{aligned}$$

Hence, inequality 4.2 implies

$$\begin{aligned} l_\eta^2 + l_{1-\eta}^2 + 0^4(l_\eta, l_{1-\eta}) \leq\ & l_0^2 + l_1^2 + 2\eta_- l_0 l_1 d_x^2 + (1-2\eta_-)\bar{\eta}(d_y^2 - d_x^2) \\ & + 0^2(\eta_-)0^2(l_0, l_1) + 0^4(l_0, l_1) \\ & + (\eta_- + \bar{\eta})0^2(l_0, l_1)0^3(d_x, d_y) \\ & + \eta_- 0^3(l_0, l_1) + 0^2(\bar{\eta}d_x) + 0^3(l_0, l_1) \end{aligned}$$

Let $Z \in \Gamma(T\bar{\Omega})$ be a vector field. By taking $y = x + \epsilon Z$, dividing by ϵ^2, and letting $\epsilon \to 0$, we deduce that for a.e. $x \in \Omega$,

$$\begin{aligned} & |(u_\eta)_*(Z)|^2 + |(u_{1-\eta})_*(Z)|^2 \\ & \quad \leq |(u_0)_*(Z)|^2 + |(u_1)_*(Z)|^2 + \eta(|(u_0)_*(Z)|^2 + |(u_1)_*(Z)|^2)d^2(u_0, u_1) \\ & \qquad - (1-2\eta)\eta_*(Z)d^2(u_0, u_1)_*(Z) + 0^2(\eta)0^2(|(u_0)_*(Z)|, |(u_1)_*(Z)|) \\ & \qquad + (\eta + \eta_*(Z))0^2(|(u_0)_*(Z)|, |(u_1)_*(Z)|)0^3(d(u_0, u_1)) + 0^2(\eta_*(Z)d_x) \end{aligned}$$

In the above, substitute η by $t\eta$, divide by t and let $t \to 0$ to obtain,

$$\begin{aligned}(4.3)\quad & |(u_\eta)_*(Z)|^2 + |(u_{1-\eta})_*(Z)|^2 \\ & \le |(u_0)_*(Z)|^2 + |(u_1)_*(Z)|^2 \\ & \quad + \eta(|(u_0)_*(Z)|^2 + |(u_1)_*(Z)|^2)d^2(u_0,u_1) - \eta_*(Z)d^2(u_0,u_1)_*(Z) \\ & \quad + (\eta + \eta_*(Z))0^2(|(u_0)_*(Z)|, |(u_1)_*(Z)|)0^3(d(u_0,u_1))\end{aligned}$$

Adding the above equation with $Z = \partial_x$ to the above with $Z = \partial_y$, we obtain for a.e. $x \in \Omega$,

$$\begin{aligned}|\nabla u_\eta|^2 + |\nabla u_{1-\eta}|^2 \le & |\nabla u_0|^2 + |\nabla u_1|^2 \\ & + \eta(|\nabla u_0|^2 + |\nabla u_1|^2)d^2(u_0,u_1) - \nabla\eta \cdot \nabla d^2(u_0,u_1) \\ & + (\eta + \nabla\eta)0^2(\nabla u_0, \nabla u_1)0^3(d(u_0,u_1))\end{aligned}$$

If u_0 and u_1 are energy minimizers, then integrating over D gives,

$$\begin{aligned}0 \le & \int d^2(u_0,u_1)\Delta\eta + \int \eta(|\nabla u_0|^2 + |\nabla u_1|^2)d^2(u_0,u_1) \\ & + |\eta + \nabla\eta|_{C^\infty} \int_{supp(\eta)} 0^2(|\nabla u_0|, |\nabla u_1|)0^3(d(u_0,u_1))\end{aligned}$$

Let u be an energy minimizing map and $u_\omega(x) = u(x + \delta W)$, with $|W| \le 1$. Then, dividing by δ^2 and letting $\delta \to 0$, we obtain,

$$-2\int \eta(|\nabla u|^2 |u_*(W)|^2 \le \int |u_*(W)|^2 \Delta\eta.$$

Adding the above equation with $W = \partial_x$ to the above with $W = \partial_y$, we obtain

$$-2\int \eta|\nabla u|^4 \le \int |\nabla u|^2 \Delta\eta.$$

Now if u is conformal, then inequality 4.3 implies

$$\begin{aligned}\lambda_\eta + \lambda_{1-\eta} \le & \lambda_0 + \lambda_1 + \eta(\lambda_0 + \lambda_1)d^2(u_0,u_1) \\ & - \eta_*(Z)d^2(u_0,u_1)_*(Z) + (\eta + \eta_*(Z))0^2(\lambda_0,\lambda_1)0^3(d(u_0,u_1)).\end{aligned}$$

Substituting $Z = \frac{\nabla\eta}{|\nabla\eta|}$ if $\nabla\eta \ne 0$ and following the same procedure as above we obtain

$$-2\kappa\int_D \eta\lambda^2 \le \int_D \lambda\Delta\eta. \qquad \Box$$

5. The Curvature Inequality.

In this section, we prove our main result. As mentioned in the introduction, if X is a smooth Riemannian manifold of sectional curvature bounded from above by κ, then the inequality in the following theorem implies that the curvature of the surface is also bounded from above by κ.

Theorem 5.1. *Let $u : D \to X$ be a minimal surface (i.e. a weakly conformal energy minimizing map) with conformal factor λ where (X, d) is a metric space of curvature bounded from above by κ. Then for all non-negative $\varphi \in C_c^\infty(D)$,*

$$\int_D \log \lambda \Delta\varphi \geq -2\kappa \int_D \varphi\lambda. \tag{5.1}$$

Proof. We will prove this for the case of $\kappa = 1$. The result κ arbitrary is obtained in the same manner as below. Before we proceed with the proof of Theorem 5.1, we need the following preliminary lemmas:

Lemma 5.2. *Let λ be a conformal factor of a minimal surface $u : D \to X$ where X is a $CAT(1)$ space. Then $\lambda \in H^1_{loc}(D)$.*

Proof. Let $K \subset\subset D$. Since λ is bounded locally, we let Λ be such that $\lambda \leq \Lambda$ in K. Choose $g \in C^\infty(D)$, non-negative such that $\Delta g \geq \Lambda^2$. Then, by Theorem 4.1 we have

$$\int (\lambda + g)\Delta\phi = \int \phi(-\lambda^2 + \Delta g) \geq 0$$

for any $\phi \in C_c^\infty(K)$. Hence, and $\lambda + g$ is weakly subharmonic in K and is a non-negative function. Thus $\lambda + g \in H^1(K)$ and $\lambda \in H^1_{loc}(D)$. □

Lemma 5.3. *Let λ be a conformal factor of a minimal surface $u : D \to X$ where X is a $CAT(1)$ space. Then for any harmonic function $h : D \to \mathbf{R}$,*

$$\int_D \Delta\varphi(\lambda e^h) \geq -2 \int_D \varphi\lambda^2 e^h.$$

Proof. Let $w(z) : D \to D$ be a conformal change of coordinates. Then $v = u \circ w$ is harmonic. Let $\tilde{\lambda} = |\nabla v|^2$ be the conformal factor for the pull-back metric on D under the map v. By Theorem 4.1,

$$\int_D \tilde{\lambda}\triangle_z\varphi dz = -2\int_D \varphi\tilde{\lambda}^2 dz$$

for all non-negative $\varphi \in C_c^\infty(D)$. Note that

$$\begin{aligned}\triangle_z &= \left|\frac{dw}{dz}\right|\triangle_w,\\ dw &= \left|\frac{dw}{dz}\right| dz,\\ \tilde{\lambda} &= |\nabla(u\circ w)| = |\nabla u|\left|\frac{dw}{dz}\right| = \lambda\left|\frac{dw}{dz}\right|.\end{aligned}$$

Hence,

$$\int \left|\frac{dw}{dz}\right|(\triangle_w\varphi)\lambda\left|\frac{dw}{dz}\right|\left|\frac{dw}{dz}\right|^{-1} dw \geq -2\int \varphi\left|\frac{dw}{dz}\right|^2 \lambda^2\left|\frac{dw}{dz}\right|^{-1} dw,$$

and we get the desired result by choosing w such that $|\frac{dw}{dz}| = e^h$. Hence, let $w = \int e^\psi$ where ψ is an analytic function such that $Re\psi = h$. □

Lemma 5.4. *Let λ be a conformal factor of a minimal surface map $u : D \to X$ where X is a $CAT(1)$ space. Assume $\lambda \geq \lambda_0 > 0$. Then for any harmonic function $h : D \to \mathbf{R}$,*

$$\int_D \triangle\varphi\log\lambda \geq -\int_D \varphi(2\lambda + |\nabla(\log\lambda + h)|^2)$$

Proof. Since λ is bounded away from zero and locally bounded above, we can assume that $\log\lambda \in H^1_{loc}(D)$. Let h be any harmonic function. By Lemma 5.3 and by the fact that C^∞ functions are dense in H^1, for any non-negative $\psi \in H^1_{loc}(D)$,

$$\int_D \triangle\psi(\lambda e^h) \geq -2\int_D \psi\lambda^2 e^h.$$

Let $\varphi \in C_c^\infty(D)$ be a non-negative function, then

$$\begin{aligned}
\int \Delta\varphi \log\lambda &= \int \Delta\varphi(\log\lambda + \log e^h) \\
&= \int \Delta\varphi \log \lambda e^h \\
&= -\int \nabla\varphi \cdot \nabla \log \lambda e^h \\
&= -\int \frac{\nabla\varphi}{\lambda e^h} \cdot \nabla(\lambda e^h) \\
&= -\int \nabla\left(\frac{\varphi}{\lambda e^h}\right) \cdot \nabla(\lambda e^h) - \varphi\frac{|\nabla(\lambda e^h)|^2}{(\lambda e^h)^2} \\
&\geq -2\int \frac{\varphi}{\lambda e^h}\lambda^2 e^h - \int \varphi|\nabla(\log\lambda + h)|^2 \\
&= -2\int \varphi\lambda - \int \varphi|\nabla(\log\lambda + h)|^2.
\end{aligned}$$

□

Now we proceed with the proof of Theorem 5.1 in the special case that $\lambda \geq \lambda_0 > 0$. Let $\delta > 0$ be given. Since $\nabla \log \lambda \in L^2$, by the Lebesque Point Lemma,

$$F = \left\{ x \in D | \lim_{\sigma \to 0} \frac{1}{\pi\sigma^2} \int_{y \in B_\sigma(x)} |\nabla \log \lambda(y) - \nabla \log \lambda(x)|^2 d\mu(y) = 0 \right\}$$

is of full measure in D. For $x \in F$, let σ_x be such that

$$0 < \sigma_x < \frac{1}{5}\text{dist}(x, \partial D)$$

and

$$\int_{y \in B_{5\sigma_x}(x)} |\nabla \log \lambda(y) - \nabla \log \lambda(x)|^2 dy \leq 25\delta\pi\sigma_x^2.$$

Note that $\{B_{\sigma_x}(x)\}_{x \in F}$ is a collection of closed balls such that $\bigcup_{x \in F} B_{\sigma_x}(x)$ is of full measure in D. By the Five Times Covering Lemma, we can choose a disjoint subcollection $\{B_{\sigma_{x_i}}(x_i)\}_{i=1}^\infty$ such that

$$\bigcup_{x \in F} B_{\sigma_x}(x) \subset \bigcup_{i=1}^{\infty} B_{\sigma 5x_i}(x_i).$$

Let $\varphi \in C_c^\infty$ be a non-negative function. Since $|\log\lambda\Delta\varphi|, |\varphi\lambda| \in L^1(D)$, there exists ϵ such that

$$\int_A |\log\lambda\Delta\varphi| < \delta,$$
$$\int_A |\varphi\lambda| < \delta$$

whenever $m(A) < \epsilon$. On the other hand, since $\sum_{i=1}^\infty m(B_{\sigma_{x_i}}(x_i)) \leq \pi$, there exists N such that

$$m(\bigcup_{i=N+1}^\infty B_{5\sigma_{x_i}}(x_i)) < \epsilon.$$

Set $A = \bigcup_{i=N+1}^\infty B_{5\sigma_{x_i}}(x_i)$. Let $\{\chi_i\}_{i=1}^N$ be a partition of unity subordinate to $\{B_{5\sigma_{x_i}}(x_i)\}_{i=1}^N$. Then

$$\begin{aligned}
\int_D \Delta\varphi\log\lambda &= \int_A + \int_{D-A} \log\lambda\Delta\varphi \\
&\geq -\delta + \int_{D-A} \log\lambda\Delta\left(\sum_{i=1}^N \varphi\chi_i\right) \\
&= -\delta + \sum_{i=1}^N \int_{D-A} \log\lambda\Delta(\varphi\chi_i) \\
&\geq -\delta - \sum_{i=1}^N \int_{D-A} (\varphi\chi_i)(2\lambda + |\nabla\log\lambda + \nabla h_i|^2) \\
&\geq -\delta - 2\int_{D-A} \varphi\lambda - |\varphi|_\infty \sum_{i=1}^N \int_{B_{5\sigma_{x_i}}} |\nabla\log\lambda + \nabla h_i|^2 \\
&\geq -2\delta - 2\int_D \varphi\lambda - |\varphi|_\infty \sum_{i=1}^N \int_{B_{5\sigma_{x_i}}} |\nabla\log\lambda + \nabla h_i|^2.
\end{aligned}$$

where $\{h_i\}$ is any collection of harmonic functions in D. For each i, we choose h_i to be a linear function, bounded uniformly away from 0, such that $\nabla h_i = -\nabla\log\lambda(x_i)$. Thus,

$$\begin{aligned}
\sum_{i=1}^N \int_{B_{5\sigma_{x_i}}} |\nabla\log\lambda + \nabla h_i|^2 &= \sum_{i=1}^N \int_{B_{5\sigma_{x_i}}(x_i)} |\nabla\log\lambda - \nabla\log\lambda(x_i)| \\
&\leq 25\delta\pi \sum_{i=1}^N \sigma_{x_i}^2.
\end{aligned}$$

But since $\{B_{\sigma_{x_i}}(x_i)\}$ is a disjoint set, $\sum_{i=1}^{N} \sigma_{x_i}^2 \leq 1$ and thus,

$$\int_D \Delta\varphi \log\lambda \geq -2\delta - 25\delta|\varphi|_\infty \pi - 2\int_D \varphi\lambda$$

Since the choice of δ was arbitrary,

$$\int_D \Delta\varphi \log\lambda \geq -2\int_D \varphi\lambda.$$

Finally, since the choice of $\varphi \in C_c^\infty(D)$ was also arbitrary, we have the desired result. The general case can be handled using the following lemma.

Lemma 5.5. *Let $f_n : D \to \mathbf{R}$ be a decreasing sequence of functions converging to a non-negative function f such that $f_n \leq M$ for all $n = 1, 2, ...$ If*

$$\int_D \log f_n \Delta\varphi \geq -2\int_D \varphi f_n,$$

for all non-negative $\varphi \in C_c^\infty(D)$, we also have that

$$\int_D \log f \Delta\varphi \geq -2\int_D \varphi f,$$

for all non-negative $\varphi \in C_c^\infty(D)$.

Proof. We will show:

$$\lim_{n\to\infty} \int \varphi f_n = \int \varphi f$$
$$\lim_{n\to\infty} \int \log f_n \Delta\varphi = \int \log f \Delta\varphi.$$

The first equality follows immediately from the Lebesgue Convergence Theorem. To prove the second equality, let $g \in C^\infty(D)$ such that $\Delta g \geq 2M$. Then

$$\int (\log f_n + g)\Delta\varphi \geq \int \varphi(-2f_n + \Delta g) \geq 0.$$

Hence $\log f_n + g$ is subharmonic. By the mean value inequality (and assuming w.l.o.g. that $f(0) \neq 0$),

$$\begin{aligned}
-\infty &< \log f(0) + g(0) \\
&\leq \log f_n(0) + g(0) \\
&\leq \frac{1}{\pi}\int_D \log f_n + g \\
&\leq \log M + g.
\end{aligned}$$

In particular, $\int \log f_n$ in uniformly bounded. Let $F_n = \log M - \log f_n$, and $F = \log M - \log f$. Then F_n is an increasing sequence of non-negative functions. Hence by the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \int F_n = \int_D F.$$

In other words, we have that

$$\lim_{n\to\infty} \int_D \log f_n = \int_D \log f.$$

In particular, $\log f \in L^1(D)$. For non-negative $\varphi \in C_c^\infty(D)$, we have that $\log f_n \Delta\varphi \to \log f \Delta\varphi$ a.e. and $|\log f_n \Delta\varphi| \leq |\Delta\varphi|_\infty |\log f_n|$. Hence by the Dominated Convergence Theorem, we have the desired result as we take $\delta \to 0$. □

Now consider the space $X \times D$ endowed with the distance function d_δ defined by

$$d_\delta^2((P,z),(Q,w)) = d^2(P,Q) + \delta|z-w|^2,$$

for $P, Q \in X$ and $z, w \in D$. It can be easily checked that $(X \times D, d_\delta)$ is a $CAT(1)$ space and $u_\delta : D \to (X \times D, d_\delta)$ defined by

$$u_\delta(z) = (u(z), z).$$

is a u_δ is a minimal surface if u is. We let λ_δ be the conformal factor of the pull back metric. Note that λ_δ is a decreasing sequence of functions converging to λ and $\lambda_\delta \geq \delta$. By the special case above, we have that

$$\int_D \Delta\varphi \log \lambda_\delta \geq -2\int_D \varphi\lambda_\delta,$$

for all δ. Hence by Lemma 5.5, we get the desired result when we take $\delta \to 0$. □

6. Surfaces with Conformal Factor λ.

As mentioned in the introduction, when X is a smooth Riemannian manifold of sectional curvature bounded from above by κ, the inequality of Theorem 5.1 implies that the curvature of the minimal surface is also bounded from above by κ. In this section, we will see that this interpretation of Theorem 5.1 also makes sense in the setting where X is a metric space of curvature bounded from above by κ; we show that the conformal factor λ induces a metric space on D which has upper curvature bound of κ.

Theorem 6.1. *Let (X, d) be a complete metric space of curvature bounded from above by κ and let $u : D \to X$ be a minimal surface (i.e. a weakly conformal energy minimizing map) with conformal factor λ. Let $\gamma : [0, 1] \to D$ be a piecewise C^1 curve and let $l(\gamma) = \int_0^1 \sqrt{\lambda(\gamma(t))}|\gamma'(t)|dt$. For $x, y \in D$, we define the distance between x and y as*

$$d_\lambda(x, y) = inf\{l(\gamma) : \gamma \text{ piecewise } C^1 \text{ and } \gamma(0) = x, \gamma(1) = y\}.$$

Then (D, d_λ) is a metric space with curvature bounded from above by κ (locally if $k > 0$). The metric topology is equivalent to the surface topology.

Remark. The fact that $\sqrt{\lambda} \in H^1_{loc}(D)$ follows from the inequality of Theorem 5.1. Hence the definition of $l(\gamma)$ makes sense. The statement that a space has curvature bounded from above by κ locally means that each point is contained in a neighborhood which has an upper curvature bound of κ.

Proof. The fact that d_λ defines a length space and the statement about the equivalence of the topologies follow from the work of Reshetnyak [R3] and the weak inequality of Theorem 5.1. (Reshetnyak considers a metric $\lambda(dx^2 + dy^2)$ where $\log \lambda$ is a difference of two subharmonic functions.) We need to show the curvature bound. It is sufficient to consider the cases $\kappa = -1$, $\kappa = 0$ and $\kappa = 1$. The general case then follows by simply scaling the distance function d of X so that the curvature is either $\kappa = -1, 0$ or 1.

We let λ_σ, $(\log \lambda)_\sigma$ be symmetric mollifications (i.e. mollification by a symmetric mollifier) of λ, $\log \lambda$ and let $\lambda^\sigma = e^{(\log \lambda)_\sigma}$. Also let $D^\sigma = \{z \in D : |z| < 1 - \sigma\}$. By applying Theorem 5.1 with φ the mollifier, we have

$$\begin{aligned} \triangle \log \lambda^\sigma &= \triangle(\log \lambda_\sigma) \\ &\geq -2\kappa\lambda_\sigma \\ &= -2\kappa \left(\frac{\lambda_\sigma}{\lambda^\sigma}\right) \lambda^\sigma \end{aligned} \tag{6.1}$$

for every $z \in D^\sigma$. By Jensen's inequality, $\lambda_\sigma \geq \lambda^\sigma$. Hence, for $\kappa = -1$ or $\kappa = 0$,

$$\frac{-1}{2\lambda^\sigma}\triangle \log \lambda^\sigma \leq \kappa.$$

Thus for $\kappa = -1$ and $\kappa = 0$, $(D^\sigma, \lambda^\sigma(dx^2 + dy^2))$ is a smooth Riemannian surface with curvature bounded from above by 1 and 0, respectively. Furthermore, since λ is subharmonic, $\lambda^\sigma \geq \lambda$. This implies that $d^\sigma \geq d_\lambda$ where d^σ is the distance function induced by $\lambda^\sigma(dx^2 + dy^2)$. Combining this with the fact that $\lambda^\sigma \to \lambda$ in H^1, it is easy to check that $d^\sigma \to d_\lambda$.

From the above discussion, the curvature bound for the case $\kappa = -1$ and $\kappa = 0$ follows easily: Let $x, y, z \in D$ and let y_t (resp. y_t^σ) be the point on the geodesic from y to z with respect to the distance function d_λ (resp. d^σ) so that $d_\lambda(y, y_t) = td_\lambda(y, z)$ (resp. $d^\sigma(y, y_t^\sigma) = td^\sigma(y, z)$).

Claim. *For y_t and y_t^σ defined above, we have $d_\lambda(y_t, y_t^\sigma) \to 0$ as $\sigma \to 0$.*

Proof. Assume $\kappa = -1$. Consider the geodesic triangle $\triangle(y_t, y, z)$ with respect to d^σ. By using the curvature bound of (D^σ, d^σ) we have

$$\begin{aligned}\cosh d^\sigma(y_t, y_t^\sigma) \leq & \frac{\sinh(1-t)d^\sigma(y, z)}{\sinh d^\sigma(y, z)} \cosh d^\sigma(y_t, y) \\ & + \frac{\sinh td^\sigma(y, z)}{\sinh d^\sigma(y, z)} \cosh d^\sigma(y_t, z).\end{aligned}$$

As $\sigma \to 0$, $d^\sigma(y, z) \to d_\lambda(y, z)$, $d^\sigma(y_t, y) \to d_\lambda(y_t, y) = td_\lambda(y, z)$ and $d^\sigma(y_t, z) \to d_\lambda(y_t, z) = (1-t)d_\lambda(y, z)$. Hence the right hand side of the above inequality converges to 1 and since $d_\lambda(y_t, y_t^\sigma) \leq d^\sigma(y_t, y_t^\sigma)$, this proves the claim. The case $\kappa = 0$ is proved analogously. $\square$

For $\kappa = -1$, we want to show that

$$\begin{aligned}\cosh d_\lambda(x, y_t) \leq & \frac{\sinh(1-t)d_\lambda(x, y)}{\sinh d_\lambda(x, y)} \cosh d_\lambda(x, y) \\ & + \frac{\sinh(1-t)d_\lambda(x, z)}{\sinh d_\lambda(x, z)} \cosh d_\lambda(x, z)\end{aligned}$$

which is equivalent to showing the triangle comparison property of Definition 3.1. Since (D^σ, d^σ) has curvature bounded from above by -1, we have

$$\cosh d^\sigma(x, y_t^\sigma) \leq \frac{\sinh(1-t)d^\sigma(x,y)}{\sinh d^\sigma(x,y)} \cosh d^\sigma(x,y) + \frac{\sinh(1-t)d^\sigma(x,z)}{\sinh d^\sigma(x,z)} \cosh d^\sigma(x,z)$$

and the desired inequality follows by taking $\sigma \to 0$ and using the claim. The proof for $\kappa = 0$ follows analogously.

Now we treat the case $\kappa = 1$. First we show (D, d_λ) has curvature bounded from above by 2. Let $D_r(z_0) = \{z : |z - z_0| < r\} \subset D$. Since (D^σ, d^σ) is a smooth Riemannian surface, by the isoperimetric inequality of [Hu],

$$\begin{aligned} \left(\int_{\partial D_r(z_0)} \sqrt{\lambda^\sigma} ds\right)^2 &\geq \left(4\pi - \int_{D_r(z_0)} (\triangle \log \lambda^\sigma)^+ dxdy\right) \int_{D_r(z_0)} \lambda^\sigma dxdy \\ &\geq \left(4\pi - 2\int_{D_r(z_0)} \lambda_\sigma dxdy\right) \int_{D_r(z_0)} \lambda^\sigma dxdy \end{aligned}$$

where $(\triangle \log \lambda^\sigma)^+ = \max\{-\triangle \log \lambda^\sigma, 0\}$. By taking $\sigma \to 0$, we have

$$\left(\int_{\partial D_r(z_0)} \sqrt{\lambda} ds\right)^2 \geq 4\pi \int_{D_r(z_0)} \lambda dxdy - 2\left(\int_{D_r(z_0)} \lambda dxdy\right)^2.$$

[R2] says that if a surface with a metric $\lambda(dx^2 + dy^2)$ has an isoperimetric inequality for disks $D_r(z_0)$ of the form

$$L^2 \geq 4\pi A - \kappa A^2$$

where L is the length of $\partial D_r(z_0)$ and A is area of $D_r(z_0)$, then the surface has an upper curvature bound of κ. This implies (D, d_λ) has an upper curvature bound of 2.

Let us call k the best curvature bound of (D, d_λ) if for every geodesic triangle T with $\mathrm{diam}(T) < \frac{\pi}{\sqrt{2}}$ and angles α, β, γ,

$$\alpha + \beta + \gamma \leq \alpha_k + \beta_k + \gamma_k$$

where α_k, β_k, γ_k are angles of a comparison triangle in S_k. By the above, we know that $k \leq 2$. We wish to show $k \leq 1$. Suppose not, i.e. $1 < k \leq 2$. We need the following claim to obtain a contradiction.

Claim. *Suppose $1 < k' < k$. There exists a constant $C_{k,k'} > 1$ such that for any geodesic triangles $T_k \subset S_k$ and $T_{k'} \subset S_{k'}$ with same side lengths and the sum of side lengths less than $\frac{\pi}{\sqrt{2}}$, we have $area(T_k) \leq C_{k,k'} area(T_{k'})$. Furthermore, $C_{k,k'} \to 1$ as $k' \to k$.*

Proof. Let a, b, c be the side lengths of geodesic triangle $T \subset S_k$. From spherical geometry, as $a \to 0$, $b \to 0$ or $c \to 0$, area(T) approaches the area of a Euclidean triangle with side lengths a, b, c. Thus if $T_{k,i} \subset S_k$ and $T_{k',i} \subset S_{k'}$ are geodesic triangles with side lengths a_i, b_i, c_i and if they form a maximizing sequence of the ratio

$$\frac{\text{area}(T_{k,i})}{\text{area}(T_{k',i})}$$

then we can extract a subsequence so that $a_{i'} \to a > 0$, $b_{i'} \to b > 0$, $c_{i'} \to c > 0$. Thus,

$$C_{k,k'} = \frac{\text{area}(\bar{T}_k)}{\text{area}(\bar{T}_{k'})}$$

where $\bar{T}_k \subset S_k, \bar{T}_{k'} \subset S_{k'}$ are geodesic triangles with side lengths a, b, c. The last assertion is obvious. □

Let T be any geodesic triangle and let α, β, γ be the angles of T. By Gauss-Bonnet (see [R3], Theorem 8.1.7), we have

$$\begin{aligned} \alpha + \beta + \gamma &\leq -\frac{1}{2} \int_T \triangle \log \lambda dx dy + \pi \\ &\leq \int_T \lambda dx dy + \pi \\ &= \text{area}(T) + \pi \end{aligned}$$

Let T_k be a the comparison triangle in S_k. Clearly,

$$\text{area}(T) \leq \text{area}(T_k).$$

By claim, for $1 < k' < k$ and comparison triangle $T_{k'}$ in $S_{k'}$,

$$\text{area}(T_k) \leq C_{k,k'} \text{area}(T_{k'}).$$

We note that $C_{k,k'}$ is independently of T chosen. We choose k' sufficiently close to k so that $C_{k,k'} < k'$. Applying Gauss-Bonnet on $T_{k'} \subset S_{k'}$, we obtain,

$$\text{area}(T_k) \leq k' \text{area}(T_{k'}) \leq \alpha_{k'} + \beta_{k'} + \gamma_{k'} - \pi$$

where $\alpha_{k'}, \beta_{k'}, \gamma_{k'}$ are angles of $T_{k'}$. Thus,

$$\alpha + \beta + \gamma \leq \alpha_{k'} + \beta_{k'} + \gamma_{k'}.$$

Since T can be chosen arbitrarily, this implies that the best curvature bound for (D, d_λ) is k'. This contradiction implies that the best curvature bound is not greater than 1. □

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