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Curvature Estimates for Minimal Annuli and Non-Compact Douglas-Plateau Problem

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In this article we give curvature estimates for minimal annuli with convex boundary $L_1 \cup L_2$ in parallel planes and apply these estimates to solve some kinds of non-compact Douglas-Plateau problem. The estimates for minimal annuli also give various necessary conditions for the existence of minimal surfaces.

1. Introduction.

In this article we give curvature estimates for minimal annuli with convex boundary $L_1 \cup L_2$ in parallel planes and apply these estimates to solve some kinds of non-compact Douglas-Plateau problem. The estimates for minimal annuli also give various necessary conditions for the existence of minimal surfaces.

To state our results, let us fix some notations first.

Let $P_t = \{(x,y,z) \in \mathbb{R}^3;\ z = t\}$, $S(t_1,t_2) = \{(x,y,z) \in \mathbb{R}^3;\ t_1 \le z \le t_2\}$, and $S'(t_1,t_2) = \{(x,y,z) \in \mathbb{R}^3;\ t_1 \le y \le t_2\}$, where $t_1 < t_2$. Let C_R be the solid cylinder $\{(x,y,z) \in \mathbb{R}^3;\ x^2 + y^2 \le R^2\}$.

Let $A \subset S(t_1,t_2)$ be an embedded minimal annulus such that $\partial A = L_1 \cup L_2$, where $L_1 \subset P_{t_1}$ and $L_2 \subset P_{t_2}$ are convex Jordan curves. Denote $\mathcal{P}_\pi$ the perpendicular projection on a plane π perpendicular to the xy-plane. Let $\Omega_\pi := \operatorname{Int}(\mathcal{P}_\pi(A))$, the interior of $\mathcal{P}_\pi(A)$, and $\partial\Omega_\pi = \bigcup_{i=1}^4 \Gamma_i^\pi$, where $\Gamma_i^\pi = \mathcal{P}_\pi(L_i)$, $i = 1$, 2, and Γ_3^π and Γ_4^π are arcs connecting Γ_1^π and Γ_2^π.

In Lemma 2.1 we estimate $|\Gamma_i|$, the length of Γ_i. We have

$$|\Gamma_1^\pi| + |\Gamma_2^\pi| \ge |\Gamma_3^\pi| + |\Gamma_4^\pi|,$$

and

$$|\Gamma_1^\pi||\Gamma_2^\pi| > (t_2 - t_1)^2, \quad |\Gamma_1^\pi| + |\Gamma_2^\pi| > 2(t_2 - t_1).$$

Although the proof of the former is a simple imitation of a technique in [fanghwang-js], and the proof of the latter comes from elementary Euclidean geometry, the more accurate new estimate to Ω leads us to curvature estimates.

In Lemma 2.2 we prove that if $A \subset S(t_1,t_2) \cap S'(-R,R)$ and π is the yz-plane, then for any point $p \in \mathrm{Int}(A)$,

$$\max\{\mathrm{dist}(\mathcal{P}_\pi(p),\Gamma_3^\pi),\ \mathrm{dist}(\mathcal{P}_\pi(p),\Gamma_4^\pi)\} \geq \frac{(\sqrt{2}-1)d^2}{R},$$

where $d = \mathrm{dist}(p, \partial S(t_1,t_2))$. Note that $\partial S(t_1,t_2) = P_{t_1} \cup P_{t_2}$, so if $p = (x,y,z)$, then $d = \min\{z-t_1, t_2-z\} = \min\{\mathrm{dist}(p,P_{t_1}), \mathrm{dist}(p,P_{t_2})\}$.

In Proposition 2.1, using Lemma 2.2 and by projecting on a suitable direction we prove an interior curvature estimate for $A \subset S(t_1,t_2) \cap C_R$, that is, there is a constant $C_0 > 0$, such that for any $p \in \mathrm{Int}(A)$, the Gaussian curvature of p satisfies

$$|K(p)| \leq \frac{C_0 R^2}{d^4},$$

where $d = \mathrm{dist}(p, \partial S(t_1,t_2))$.

This curvature estimate gives immediate generalization of the existence result of Hoffman and Meeks, [fanghwang-hm9], to the continuous convex boundary case, see Theorem 2.2.

Proposition 2.2 gives another curvature estimate. It states that if $A \subset S(t_1,t_2) \cap S'(t_1',t_2')$ is a compact minimal annulus such that $\partial A = L_1 \cup L_2$, where $L_1 \subset P_{t_1}$ and $L_2 \subset P_{t_2}$ are C^2 convex Jordan curves, then there is a constant C_1 such that for any $p \in A$,

$$|K(p)| \leq C_1,$$

where C_1 only depends on the height $t_2 - t_1$, the width $t_2' - t_1'$, and the boundary planar curvature bound of ∂A.

These curvature estimates then lead to compactness theorems, Theorem 2.1 and Theorem 2.3, which state that some sequence of $\{A_n\}$ as in Proposition 2.1 or 2.2 has a convergent subsequence.

As applications of Theorem 2.1 and 2.3, we solve various non-compact Douglas-Plateau problems in Section 3.

Recall that the Douglas-Plateau problem for two contours is as follows: *Let L_1 and L_2 be two disjoint Jordan curves in $\mathbb{R}^3$, find a minimal annulus A such that $\partial A = L_1 \cup L_2$.*

Let S_1 and S_2 be area minimizing disks (when we say disks, we mean that they are homeomorphic to the unit disk in $\mathbb{C}$) such that $\partial S_1 = L_1$,

$\partial S_2 = L_2$. Let $\mathcal{S}$ be the set of rectifiable annulus S such that $\partial S = L_1 \cup L_2$. Douglas [fanghwang-dog1] proved that if

$$\inf_{S \in \mathcal{S}} \{\text{Area}(S)\} < \text{Area}(S_1) + \text{Area}(S_2),$$

then there is an area minimizing (therefore minimal) annulus A such that $\partial A = L_1 \cup L_2$.

If L_1 and L_2 are coaxial unit circles in parallel planes, then it is well-known that there is a constant $h > 0$ such that when the distance between the centres is smaller than h, there are exactly two catenoids bounded by $L_1 \cup L_2$; when the distance between the centres is equal to h, there is only one catenoid bounded by $L_1 \cup L_2$; when the distance between the centres is larger than h, there are no catenoids bounded by $L_1 \cup L_2$. Furthermore, by Shiffman's third theorem [fanghwang-sh1], any minimal annuli bounded by $L_1 \cup L_2$ must be a rotation surface hence is a piece of a catenoid. Thus there are either two, one, or zero minimal annuli bounded by $L_1 \cup L_2$ depending on the distance between their centres.

Meeks and White [fanghwang-mw1] generalized the above observation to minimal annuli bounded by two smooth convex Jordan curves $L_1 \cup L_2$ in different parallel planes, i.e., there are either two, one, or zero minimal annuli bounded by $L_1 \cup L_2$. But unlike the coaxial circles case, there are no simple criteria to tell us when do we have two, one, or zero minimal annuli bounded by $L_1 \cup L_2$.

However, there are some partial conditions, either sufficient or necessary to the existence of a solution to some special Douglas-Plateau problems for two contours. For example, let us consider the Douglas-Plateau problem to the boundary consisting of two Jordan curves $L_1 \cup L_2$ in parallel and different planes, say $L_1 \subset P_0$, $L_2 \subset P_d$, $d > 0$.

If L_1 and L_2 are smooth convex, then besides Douglas's sufficient condition, Hoffman and Meeks in [fanghwang-hm9] gave a sufficient condition to ensure that there are two solutions, i.e., if there is a connected compact non-planar minimal surface (could be branched) whose boundary is contained in open planar disks bounded by L_1 and L_2, then there are two minimal annuli bounded by $L_1 \cup L_2$. For the precise statement, please see Theorem 2.2 below.

The result of Hoffman and Meeks can be also treated as a necessary condition, i.e., let C_1 and C_2 be smooth convex Jordan curves such that L_1 and L_2 are contained in the open planar disks bounded by C_1 and C_2 respectively, then there is a connected compact minimal surface (maybe branched) bounded by $L_1 \cup L_2$ only if there are two minimal annuli bounded by $C_1 \cup C_2$.

There are other necessary conditions. For example, Theorem 2.2 combined with a J. C. C. Nitsche's result [fanghwang-ni2], page 88, implies that the images of perpendicular projection of L_1 and L_2 on the xy-plane must intersect if L_1 and L_2 are convex.

Moreover, Osserman and Schiffer proved in [fanghwang-os1] that if c, d, δ_1, and δ_2 are positive constants and

$$L_1 \subset \left\{(x,y,z); \left(x - \frac{c}{d}z\right)^2 + y^2 \le \delta_1^2,\ z \le 0\right\},$$

$$L_2 \subset \left\{(x,y,z); \left(x - \frac{c}{d}z\right)^2 + y^2 \le \delta_2^2, z \ge d\right\},$$

are closed curves and they bound a minimal annulus, then

$$\delta_1 + \delta_2 > \sqrt{\frac{c^2}{2} + d^2}.$$

Using the basic estimates in Lemma 2.1, in Corollary 2.2 we prove that if Σ is a connected compact non-planar minimal surface (maybe branched) such that $\partial\Sigma = L_1 \cup L_2$ and

$$\begin{aligned} L_1 &\subset \left\{(x,y,z); -\delta_1 \le x - \frac{c}{d}z \le \delta_1,\ z \le 0\right\}, \\ L_2 &\subset \left\{(x,y,z); -\delta_2 \le x - \frac{c}{d}z \le \delta_2,\ z \ge d\right\}, \end{aligned}$$

then

$$2\max\{\delta_1, \delta_2\} > \sqrt{c^2 + d^2}.$$

Furthermore, if

$$\begin{aligned} L_1 &\subset \{(x,y,0); -\delta_1 \le x \le \delta_1\}, \\ L_2 &\subset \{(x,y,d); -\delta_2 \le x - c \le \delta_2\}, \end{aligned}$$

then

$$\delta_1 + \delta_2 > \sqrt{c^2 + d^2}.$$

We define *Non-compact Douglas-Plateau problem of annular type for n boundary curves* as follows:

Let L_i, $i = 1, \cdots, n$, be disjoint, embedded proper complete curves, at least one of them is non-compact, find a minimal annulus A such that $\partial A = \Gamma := \bigcup_{i=1}^n L_i$.

As we have seen, there are many necessary conditions restricting the solvability of even compact Douglas-Plateau problems for two contours, the solvability of the non-compact Douglas-Plateau problem seems should require more hypotheses than the compact case. We will see that in fact in our special cases discussed in Section 3, the same condition that ensures the existence of solutions for compact cases is also enough for non-compact cases.

It is known for more than one hundred years that for some non-compact boundaries we can find minimal annuli solving the corresponding "two contour" Douglas-Plateau problem. A classical example is a minimal annulus bounded by two parallel straight lines, a piece of one of Riemann's examples. Although a straight line is no longer a Jordan curve, it is a proper complete (convex) curve in $\mathbb{R}^3$.

In [fanghwang-f3], it was proved that if L_1 and L_2 are proper non-compact complete smooth planar convex curves in parallel planes with two symmetries, then there are two minimal annuli $\mathcal{A}$ and $\mathcal{B}$ such that $\partial\mathcal{A} = \partial\mathcal{B} = \Gamma$. Furthermore, $\mathcal{A}$ and $\mathcal{B}$ are foliated by strictly convex Jordan curves.

In Section 3, we prove the existence of various types of non-compact Douglas-Plateau problems. We will show that the symmetric conditions in [fanghwang-f3] is redundant, see Theorem 3.1. The proof of Theorem 3.1 is an application of Theorem 2.1, but we must first prove that there are barriers confining the approximate compact minimal annuli such that we can use Theorem 2.1, these barriers are established in Lemma 3.1.

In Theorem 3.2 we prove that there are at least two minimal annuli bounded by four straight lines L_i, $i = 1, 2, 3, 4$, such that $L_1 \subset P_{-1}$ and $L_2 \subset P_{-1}$ are parallel, $L_3 \subset P_1$ and $L_4 \subset P_1$ are parallel, but L_1 and L_3 are not parallel, if the distances between L_1 and L_2, and L_3 and L_4, are sufficiently large.

We will also prove that there are minimal annuli bounded by four parallel straight lines in two different parallel planes, if the boundary satisfies some kind of Hoffman-Meeks condition. See Theorem 3.3.

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2. Curvature Estimates for Minimal Annuli.

Lemma 2.1. *Let $A \subset S(t_1, t_2)$ be a minimal annulus such that $\partial A = L_1 \cup L_2$, where $L_1 \subset P_{t_1}$, $L_2 \subset P_{t_2}$ are convex Jordan curves. Let π be a plane perpendicular to the xy-plane, $\mathcal{P}_\pi$ the perpendicular projection on π. Then $\Omega_\pi := \mathrm{Int}(\mathcal{P}_\pi(A)) = \mathcal{P}_\pi(A) - (\cup_{i=1}^4 \Gamma_i^\pi)$ is a domain in π bounded by $\Gamma_i^\pi = \mathcal{P}_\pi(L_i)$, $i = 1$, 2, and Γ_3^π and Γ_4^π, two curves connecting Γ_1^π and Γ_2^π.*

Let $|\Gamma_i^\pi|$ be the arc length of Γ_i^π, $i = 1, 2, 3, 4$. Then

$$|\Gamma_1^\pi| + |\Gamma_2^\pi| \geq |\Gamma_3^\pi| + |\Gamma_4^\pi|, \tag{2.1}$$

$$|\Gamma_1^\pi||\Gamma_2^\pi| > (t_2 - t_1)^2, \quad |\Gamma_1^\pi| + |\Gamma_2^\pi| > 2(t_2 - t_1). \tag{2.2}$$

In particular, if $A \subset S(t_1, t_2) \cap S'(t_1', t_2')$, take π to be the yz-plane, then from $|\Gamma_i^\pi| \leq t_2' - t_1'$, $i = 1$, 2, we have

$$\min\{|\Gamma_1^\pi|, |\Gamma_2^\pi|\} > \frac{(t_2 - t_1)^2}{(t_2' - t_1')^2} \max\{|\Gamma_1^\pi|, |\Gamma_2^\pi|\}, \tag{2.3}$$

and

$$\min\{|\Gamma_1^\pi|, |\Gamma_2^\pi|\} > \frac{(t_2 - t_1)^2}{t_2' - t_1'}. \tag{2.4}$$

Furthermore, if $A \subset S(t_1, t_2) \cap C_R$, then

$$\min\{|\Gamma_1^\pi|, |\Gamma_2^\pi|\} > \frac{(t_2 - t_1)^2}{4R^2} \max\{|\Gamma_1^\pi|, |\Gamma_2^\pi|\}, \tag{2.5}$$

and

$$\min\{|\Gamma_1^\pi|, |\Gamma_2^\pi|\} > \frac{(t_2 - t_1)^2}{2R}. \tag{2.6}$$

Proof. Select a coordinate system such that π is the yz-plane and

$$\mathcal{P}_\pi(x, y, z) = (y, z).$$

For simplicity, write Ω_π as Ω, $\mathcal{P}_\pi$ as $\mathcal{P}$, etc.

By Shiffman's first theorem in [fanghwang-sh1], every level curve $A \cap P_t$, $t_1 < t < t_2$, is a strictly convex Jordan curve. Thus $\mathcal{P}(A \cap P_t)$ is a line segment and $A \cap P_t$ consists of two graphs on $\mathcal{P}(A \cap P_t)$, $u^+(y,z) \geq u^-(y,z)$ such that $u^+(y,z) = u^-(y,z)$ if and only if (y,z) is one of the two ends of $\mathcal{P}(A \cap P_z)$.

We orient the yz-plane such that $((0,1,0),(0,0,1))$ has positive orientation. Then Ω is bounded by $\Gamma_1 = \mathcal{P}(A \cap P_{t_1})$, $\Gamma_2 = \mathcal{P}(A \cap P_{t_2})$, and Γ_3 and Γ_4 consisting of the set of end points of $\mathcal{P}(A \cap P_t)$ such that for any t, if $(y_1,t) \in \Gamma_3$, $(y_2,t) \in \Gamma_4$, then $y_1 < y_2$, thus we may say that Γ_3 is the left side boundary, Γ_4 is the right side boundary.

Recall that π is the yz-plane. Let S^2 be the unit sphere in $\mathbb{R}^3$ and $S^1_1 := S^2 \cap \pi$. Let $N : \mathrm{Int}(A) \to S^2$ be the Gauss map and p be any interior point of A. Since $\mathcal{P}(p) \in \Gamma_3 \cup \Gamma_4$ if and only if the tangent vector of $A \cap P_t$ at p is in the direction $\pm(1,0,0)$ and since each $A \cap P_t$ is strictly convex, we see that $\mathcal{P}(p) \in \Gamma_3 \cup \Gamma_4$ if and only if $p \in N^{-1}(S^1_1)$. Since it is proved in [fanghwang-mw1] that N is one-to-one and harmonic, we know that $N^{-1}(S^1_1)$ is smooth and its tangent directions are not pointed at $\pm(1,0,0)$, therefore $\Gamma_3 \cup \Gamma_4 = \mathcal{P}(N^{-1}(S^1_1))$ is smooth in its interior.

Note that for an interior point $\mathcal{P}(p)$ of Γ_3, the tangent line of Γ_3 at $\mathcal{P}(p) \in \Gamma_3$ is $\mathcal{P}(T_pA)$, where T_pA is the tangent plane of A at p. Since A is minimal, there are points of A at both sides of T_pA in any neighbourhood of p in $\mathbb{R}^3$. Thus since Γ_3 is the left side boundary of Ω, there are points of $\overline{\Omega}$ on the left side of $\mathcal{P}(T_pA)$ in any neighbourhood of $\mathcal{P}(p)$, hence Γ_3 is locally on the left side of its tangent line at $\mathcal{P}(p)$. If Γ_3 is not convex, then there is another point $\mathcal{P}(p_1) \in \Gamma_3 \cap \mathcal{P}(T_pA)$. Thus there would be another point $\mathcal{P}(q)$ in the interior of Γ_3 and is located between $\mathcal{P}(p)$ and $\mathcal{P}(p_1)$ such that $\mathcal{P}(q)$ is on the left side of $\mathcal{P}(T_pA)$ and the distance from $\mathcal{P}(q)$ to $\mathcal{P}(T_pA)$ is a local maximum, thus T_qA is parallel to T_pA. Therefore in a small neighbourhood of q in $\mathbb{R}^3$ there are no points of A on the left side of $T_q(A)$, a contradiction. Similarly we can prove that Γ_4 is convex.

We now prove that $\mathcal{P}(A) - \bigcup_{i=1}^4 \Gamma_i$ is a domain and $\Omega = \mathcal{P}(A) - \bigcup_{i=1}^4 \Gamma_i$. In fact if $\mathcal{P}(p) \notin \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, then $p \notin N^{-1}(S^1_1)$, there is a neighbourhood U of p in $\mathbb{R}^3$ such that $U \cap N^{-1}(S^1_1) = \emptyset$ and $\mathcal{P}(U \cap A)$ is open in π and $\mathcal{P}(U \cap A) \cap (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4) = \emptyset$, i.e., $\mathcal{P}(p) \in \mathrm{Int}(\mathcal{P}(A))$, hence $\Omega = \mathrm{Int}(\mathcal{P}(A)) = \mathcal{P}(A) - (\bigcup_{i=1}^4 \Gamma_i)$ and $\partial\Omega = \bigcup_{i=1}^4 \Gamma_i$. Finally, Ω is connected since $\mathcal{P}(A)$ is connected and $\bigcup_{i=1}^4 \Gamma_i$ does not separate $\mathcal{P}(A)$. See Figure 1 below.

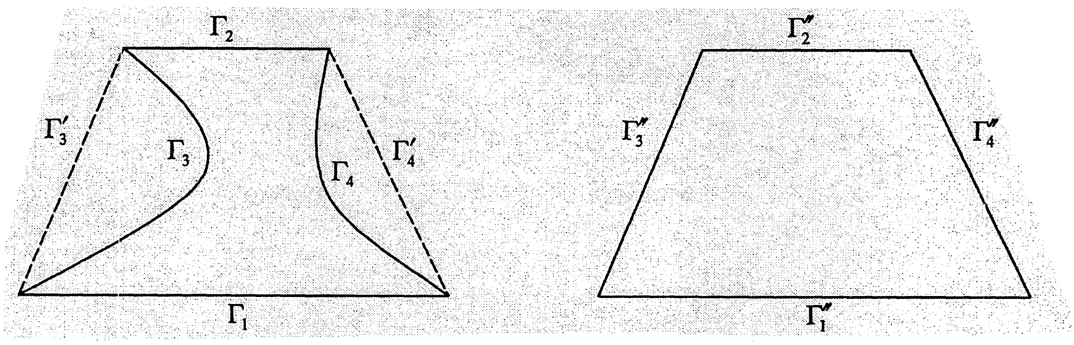

Figure 1

Since Γ_1 and Γ_2 are straight line segments, Ω is a domain with piecewise smooth boundary and $\partial\Omega$ has only four corner points $[\Gamma_1 \cap (\Gamma_3 \cup \Gamma_4)] \cup [\Gamma_2 \cap (\Gamma_3 \cup \Gamma_4)]$. Let ν be the outward unit normal vector of $\partial\Omega$.

Let us consider the graph defined by $u = u^+$. Then u satisfies the minimal surface equation div $Tu = 0$, where $Tu = Du/\sqrt{1+|Du|^2}$. The Gauss map is given by

$$N(x,y,z) = \frac{1}{\sqrt{1+|Du|^2}}\,(1,-Du)\,(y,z), \quad (y,z) \in \Omega, \quad x = u(y,z).$$

If $\mathcal{P}(p)$ is an interior point of $\Gamma_3 \cup \Gamma_4$, then since $N(p) = (0,b,c)$, $\mathcal{P}(N(p)) = N(p)$. Since $N(p)$ is perpendicular to the tangent vector $v(p)$ along $N^{-1}(S_1^1)$ at p, $N(p)$ is also perpendicular to the tangent vector $\tilde{v}((\mathcal{P}(p)) = \mathcal{P}(v(p))$ along Γ_3 or Γ_4 at $\mathcal{P}(p)$, thus $\nu(\mathcal{P}(p)) = \mathcal{P}(N(p))$. We then have that

$$\nu(\mathcal{P}(p)) = \mathcal{P}(N(p)) = N(p) = (0,b,c).$$

Therefore, $\nu \bullet N = 1$ along the interior of Γ_3 and Γ_4. Thus by the expression of N we see that u satisfies the boundary condition

$$\nu \bullet Tu = -\nu \bullet N = -1, \quad \text{on} \quad \Gamma_3 \cup \Gamma_4.$$

First assume that L_1 and L_2 are smooth and strictly convex, then Du exists on $\Gamma_1 \cup \Gamma_2$. We have

$$\int_{\partial\Omega} Tu \bullet \nu = \int_{\Omega} \operatorname{div} Tu = 0.$$

Now since $|Tu \bullet \nu| \leq 1$,

$$|\Gamma_3| + |\Gamma_4| = -\int_{\Gamma_3 \cup \Gamma_4} Tu \bullet \nu = \int_{\Gamma_1 \cup \Gamma_2} Tu \bullet \nu \leq |\Gamma_1| + |\Gamma_2|.$$

If L_1 or L_2 is only continuously convex, then by Shiffman's theorem $A \cap P_t$ is smooth and strictly convex for any $t \in (t_1, t_2)$. Consider $A \cap S(t_1 + \epsilon, t_2 - \epsilon)$, $0 < \epsilon < (t_2 - t_1)/2$, then (2.1) is true to the corresponding Γ_i's of $A \cap S(t_1 - \epsilon, t_2 + \epsilon)$. Since A is continuous up to boundary, letting $\epsilon \to 0$, we have proved (2.1).

To prove (2.2) we replace Γ_3 by the line segment Γ_3' connecting the two end points of Γ_3, replace Γ_4 by the line segment Γ_4' connecting the two end points of Γ_4. Then by comparison principle for minimal surfaces, $|\Gamma_3'| < |\Gamma_3|$, $|\Gamma_4'| < |\Gamma_4|$. Furthermore we replace the four-gon $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3' \cup \Gamma_4'$ by a trapezoid $\Gamma_1'' \cup \Gamma_2'' \cup \Gamma_3'' \cup \Gamma_4''$ such that $|\Gamma_3''| = |\Gamma_4''|$, and $|\Gamma_1''| = |\Gamma_1|$, $|\Gamma_2''| = |\Gamma_2|$ and Γ_1'' and Γ_2'' are parallel.

Note that $|\Gamma_3''| = |\Gamma_4''|$ and $2|\Gamma_3''| \leq |\Gamma_3'| + |\Gamma_4'| < |\Gamma_3| + |\Gamma_4| \leq |\Gamma_1| + |\Gamma_2|$. Now let $h = t_2 - t_1$ and without loss of generality suppose that $|\Gamma_1| \geq |\Gamma_2|$, thus $|\Gamma_2| = c|\Gamma_1|$, $0 < c \leq 1$, and $2|\Gamma_3''| < (1+c)|\Gamma_1|$. Since

$$|\Gamma_3''| = \sqrt{\frac{(|\Gamma_1| - |\Gamma_2|)^2}{4} + h^2} = \sqrt{\frac{(1-c)^2|\Gamma_1|^2}{4} + h^2},$$

we obtain that

$$(1-c)^2|\Gamma_1|^2 + 4h^2 < (1+c)^2|\Gamma_1|^2.$$

Thus

$$h^2 < c|\Gamma_1|^2 = |\Gamma_1||\Gamma_2|. \tag{2.7}$$

By comparison principle for minimal surfaces,

$$2h \leq |\Gamma_3'| + |\Gamma_4'| < |\Gamma_3| + |\Gamma_4| \leq |\Gamma_1| + |\Gamma_2|,$$

(2.2) is proved.

If $A \subset S(t_1, t_2) \cap S'(t_1', t_2')$, then $|\Gamma_1| \leq t_2' - t_1'$, so by (2.7)

$$c > \frac{h^2}{(t_2' - t_1')^2},$$

(2.3) is true. From (2.2),

$$|\Gamma_2| > \frac{h^2}{|\Gamma_1|} \geq \frac{h^2}{t_2' - t_1'},$$

(2.4) is true.

If $A \subset S(t_1,t_2) \cap C_R$, then $t_2' - t_1' \le 2R$. By a rotation, we see that (2.5) and (2.6) are true.

The proof is now complete. □

Similar arguments as in Lemma 2.1 give us further information of the domain $\mathrm{Int}(\mathcal{P}(A))$.

Lemma 2.2. *Suppose $A \subset S(t_1,t_2) \cap S'(-R,R)$ is a minimal annulus such that $\partial A = L_1 \cup L_2$, where $L_1 \subset P_{t_1}$ and $L_2 \subset P_{t_2}$ are convex Jordan curves. Let π be the yz-plane and $\mathcal{P}_\pi$ the perpendicular projection on π. Let $\Omega^\pi = \mathrm{Int}(\mathcal{P}_\pi(A))$ be the domain in π and $\partial\Omega^\pi = \Gamma_1^\pi \cup \Gamma_2^\pi \cup \Gamma_3^\pi \cup \Gamma_4^\pi$ be as in Lemma 2.1. Let p be any interior point of A and $d =$ dist $(p, \partial S(t_1,t_2))$. Then*

$$d' := \max\{\mathrm{dist}(\mathcal{P}_\pi(p), \Gamma_3^\pi),\ \mathrm{dist}(\mathcal{P}_\pi(p), \Gamma_4^\pi)\} > \frac{(\sqrt{2}-1)d^2}{R}.$$

Proof. Let us write $\Omega^\pi = \Omega$ etc.. Since Γ_3 and Γ_4 are compact, there are $q_3 \in \Gamma_3$ and $q_4 \in \Gamma_4$ such that $|\mathcal{P}(p) - q_3| = \mathrm{dist}(\mathcal{P}(p), \Gamma_3)$, $|\mathcal{P}(p) - q_4| = \mathrm{dist}(\mathcal{P}(p), \Gamma_4)$. Connecting $\mathcal{P}(p)$ and q_3, $\mathcal{P}(p)$ and q_4 by line segments l_3 and l_4 to form two subdomains Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2 \cup l_3 \cup l_4$ and $\partial\Omega_1 = \Gamma_1 \cup l_3 \cup l_4 \cup \Gamma_3' \cup \Gamma_4'$, $\partial\Omega_2 = \Gamma_2 \cup l_3 \cup l_4 \cup \Gamma_3'' \cup \Gamma_4''$, where $\Gamma_i' \subset \Gamma_i$, $\Gamma_i'' \subset \Gamma_i$, for $i = 3$, 4. Note that l_3 and l_4 are contained in Ω except at end points on Γ_3 and Γ_4 so that $\Gamma_i' \cup \Gamma_i'' = \Gamma_i$, and $|\Gamma_i'| + |\Gamma_i''| = |\Gamma_i|$, for $i = 3$, 4. See Figure 2 below.

Let $p = (x,y,z)$, $t_1 < z < t_2$. Let T_1 and T_2 be the line segments connecting $\mathcal{P}(p)$ and the two ends of Γ_1, then

$$|T_1| + |T_2| < 2d' + |\Gamma_3'| + |\Gamma_4'|.$$

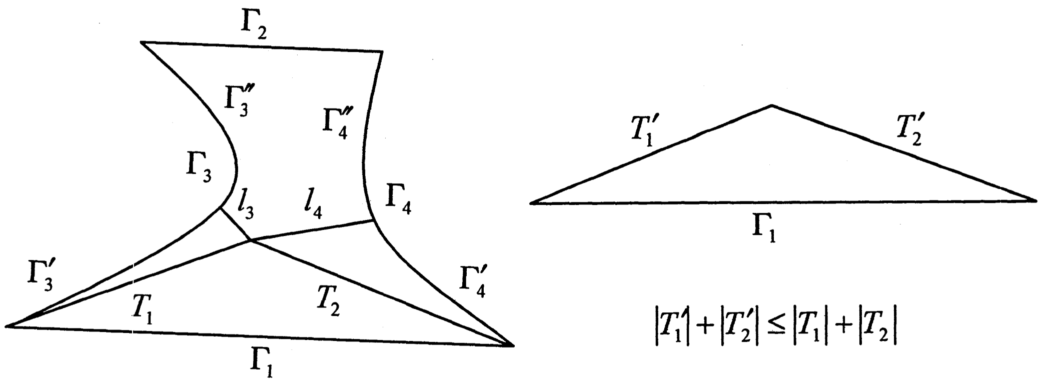

Figure 2

Elementary geometry tells us that

$$|T_1| + |T_2| \geq 2\sqrt{\frac{|\Gamma_1|^2}{4} + (z-t_1)^2},$$

thus

$$2d' + |\Gamma_3'| + |\Gamma_4'| > 2\sqrt{\frac{|\Gamma_1|^2}{4} + (z-t_1)^2} = |\Gamma_1|\sqrt{1 + \frac{4(z-t_1)^2}{|\Gamma_1|^2}}.$$

Similarly, we have

$$2d' + |\Gamma_3''| + |\Gamma_4''| > 2\sqrt{\frac{|\Gamma_2|^2}{4} + (t_2-z)^2} = |\Gamma_2|\sqrt{1 + \frac{4(t_2-z)^2}{|\Gamma_2|^2}}.$$

Now since $d = \min\{(z-t_1), (t_2-z)\} \leq (t_2-t_1)/2$ and $\max\{|\Gamma_1|, |\Gamma_2|\} \leq 2R$, by (2.2) and (2.6),

$$\min\{|\Gamma_1|R, |\Gamma_2|R\} \geq \frac{|\Gamma_1||\Gamma_2|}{2} \geq \frac{(t_2-t_1)^2}{2} \geq 2d^2,$$

thus

$$\max\left\{\frac{2d^2}{|\Gamma_1|R}, \frac{2d^2}{|\Gamma_2|R}\right\} \leq 1.$$

Note that $\sqrt{1+x} \geq 1 + (\sqrt{2}-1)x$ for $0 \leq x \leq 1$, we obtain

$$\begin{aligned}|\Gamma_1|\sqrt{1 + \frac{4(z-t_1)^2}{|\Gamma_1|^2}} &\geq |\Gamma_1|\sqrt{1 + \frac{2d^2}{|\Gamma_1|R}} \geq |\Gamma_1|\left(1 + \frac{2(\sqrt{2}-1)d^2}{|\Gamma_1|R}\right) \\ &= |\Gamma_1| + \frac{2(\sqrt{2}-1)d^2}{R},\end{aligned}$$

$$|\Gamma_2|\sqrt{1 + \frac{4(t_2-z)^2}{|\Gamma_2|^2}} \geq |\Gamma_2| + \frac{2(\sqrt{2}-1)d^2}{R}.$$

Therefore,

$$\begin{aligned}4d' + |\Gamma_3| + |\Gamma_4| &> |\Gamma_1|\sqrt{1 + \frac{4(z-t_1)^2}{|\Gamma_1|^2}} + |\Gamma_2|\sqrt{1 + \frac{4(t_2-z)^2}{|\Gamma_2|^2}} \\ &\geq |\Gamma_1| + |\Gamma_2| + \frac{4(\sqrt{2}-1)d^2}{R}.\end{aligned}$$

But by (2.1)

$$|\Gamma_1| + |\Gamma_2| \geq |\Gamma_3| + |\Gamma_4|,$$

we obtain that

$$d' > \frac{(\sqrt{2}-1)d^2}{R}.$$

□

From Lemma 2.2 we obtain an interior estimate of curvature.

Proposition 2.1 (Interior Curvature Estimate). *If $A \subset S(t_1, t_2) \cap C_R$ is a minimal annulus such that $\partial A = L_1 \cup L_2$, where $L_1 \subset P_{t_1}$ and $L_2 \subset P_{t_2}$ are convex Jordan curves, then there is an absolute constant $C_0 > 0$, such that*

$$|K(p)| \leq \frac{C_0 R^2}{d^4}, \tag{2.8}$$

where $d =$ dist$(p, \partial S(t_1, t_2))$.

Proof. Let $\mathcal{P}_\theta$ be the projection on a plane π_θ with normal

$$v_\theta = (\cos\theta, \sin\theta, 0).$$

Then by Lemma 2.1 Int$(\mathcal{P}_\theta(A))$ is a domain bounded by Γ_i^θ, $i = 1, 2, 3, 4$. We give an orthonormal basis in π_θ, (e_1, e_2), such that (v_θ, e_1, e_2) is an positive basis of $\mathbb{R}^3$. The convention is that under this coordinate system, Γ_3^θ has a smaller first coordinate than that for Γ_4^θ. Define

$$f(\theta) = \text{dist}(\mathcal{P}_\theta(p), \Gamma_3^\theta) - \text{dist}(\mathcal{P}_\theta(p), \Gamma_4^\theta).$$

Then f is continuous and $f(\theta + \pi) = -f(\theta)$. Thus there exists at least one $\theta = \theta_0$, such that $f(\theta_0) = 0$, i.e., $\text{dist}(\mathcal{P}_{\theta_0}(p), \Gamma_3^{\theta_0}) = \text{dist}(\mathcal{P}_{\theta_0}(p), \Gamma_4^{\theta_0}) = d'$.

By Lemma 2.2, $d' > D := (\sqrt{2}-1)d^2/R$, where $d = \text{dist}(p, \partial S(t_1, t_2))$. Thus there is a round disk centred at $\mathcal{P}(p)$ of radius greater than or equal to $r = \min\{D, d\}$ contained in Ω and $p = (u(\mathcal{P}(p)), \mathcal{P}(p))$ is on a minimal graph generated by $u = u^+$ or u^-. A theorem of Osserman [fanghwang-os1], page 107, says that there is an absolute constant $C > 0$ such that

$$|K(p)| \leq \frac{C}{r^2}.$$

Finally by (2.2) and the definition of d,

$$4R \geq |\Gamma_1^{\theta_0}| + |\Gamma_2^{\theta_0}| > 2(t_2 - t_1) \geq 4d > 4(\sqrt{2} - 1)d,$$

we know that $\min\{D, d\} = D$. Taking $C_0 = \frac{C}{(\sqrt{2}-1)^2}$, we complete the proof. □

Recall that the Gaussian curvature bound ensures compactness, as stated in the following summarization appeared in [fanghwang-ands],

Lemma 2.3 (Compactness Lemma). *Let Ω be a bounded domain in a complete Riemannian 3-manifold $\mathbf{N}^3$ and let $\{M_i\}$ be a sequence of minimally immersed surfaces in Ω. Suppose there is a constant C such that the Gauss curvature $K_{M_i}(x)$ satisfies $|K_{M_i}(x)| < C$ for all i. Then a subsequence of $\{M_i\}$ converges smoothly (in the C^k-topology, $k \geq 2$) to an immersed minimal surface M_∞ (with multiplicity) in Ω and $|K_{M_\infty}(x)| \leq C$. If each M_i is embedded, then M_∞ is also embedded.*

Theorem 2.1. *Let $\{A_n\}$ be a sequence of embedded minimal annuli contained in $S(t_1, t_2) \cap C_R$ such that $\partial A_n = L_1^n \cup L_2^n$, where $L_1^n \subset P_{t_1}$, $L_2^n \subset P_{t_2}$ are convex Jordan curves. Then there is a subsequence of $\{A_n\}$ converging to an embedded minimal annulus $A \subset S(t_1, t_2) \cap C_R$ such that $\partial A = L_1 \cup L_2$, where $L_1 \subset P_{t_1}$, $L_2 \subset P_{t_2}$ are convex Jordan curves.*

Proof. First observe that as uniformly bounded convex Jordan curves $\{L_1^n\}$ and $\{L_2^n\}$ are equicontinuous.

In fact, by Lemma 2.1, the arc lengths of $\{L_1^n\}$ and $\{L_2^n\}$ are at least $(t_2 - t_1)^2/R$. Since $\{L_1^n\}$ and $\{L_2^n\}$ are contained in C_R and are convex, their arc lengths have an upper bound too.

Thus a subsequence of $\{L_1^n\}$, still denote by $\{L_1^n\}$, has a convergent arc length, i.e., $l_n := |L_1^n| \to l \geq (t_2 - t_1)^2/R$.

Since L_1^n is convex, it has tangent almost everywhere. Let $\alpha_n : [0, l_n] \to P_{t_1}$ be the embedding of L_1^n such that $|\dot{\alpha}_n| = 1$ almost everywhere. Define $s : [0, l] \to [0, l_n]$ by $s(t) = l_n t/l$ and $\beta_n : [0, l] \to P_{t_1}$ by $\beta_n(t) = \alpha_n(s(t))$, then $|\dot{\beta}_n| = l_n/l$ almost everywhere. Thus $\{\beta_n\}$ is equicontinuous and uniformly bounded.

By Ascoli-Arzela theorem, a subsequence of β_n, still denote by β_n, uniformly converges to a continuous mapping $\beta : [0, l]$. By (2.6) the image of β cannot be a line segment or a point. Since each β_n is one-to-one except at the two ends, β defines a Jordan curve L_1. Let D_1 be the domain enclosed by L_1 and D_1^n the domain enclosed by L_1^n, then $D_1^n \to D_1$. Since each D_1^n is convex, D_1 is convex. Hence L_1 is a convex Jordan curve. Similarly we can treat $\{L_2^n\}$ and we may assume that a subsequence of $\{\partial A_n\}$, still denote by $\{\partial A_n\}$, has a limit $L_1 \cup L_2$, and $L_1 \subset P_{t_1}$ and $L_2 \subset P_{t_2}$ are convex Jordan curves.

Let $\epsilon_m \searrow 0$ as $m \to \infty$ and $A_n^m = A_n \cap S(t_1 + \epsilon_m, t_2 - \epsilon_m) \subset C_R$. Then by Proposition 2.1 for fixed m, $\{A_n^m\}$ has a uniform curvature bound, thus by Lemma 2.3, there is a subsequence $\{A_{m_n}^m\}$ converging in $S(t_1 + \epsilon_m, t_2 - \epsilon_m) \cap C_R$. Thus the subsequence $\{A_{m_m}\}$ converges in the interior of $S(t_1, t_2)$ to an open minimal surface A.

Since each $A_{m_m} \cap P_t$ is a strictly convex Jordan curve in $C_R \cap P_t$ and $\{A_{m_m}\}$ converges in C^k topology, $k \geq 2$, $A \cap P_t$ must be a convex Jordan curve, thus A is a minimal annulus.

Now $\lim_{m\to\infty} \partial A_{m_m} = L_1 \cup L_2$. We only need prove that $\partial A = L_1 \cup L_2$.

In fact, let $\mathcal{P}_\pi$ be the perpendicular projection on a plane π perpendicular to the xy-plane and $\Omega_m^\pi = \mathrm{Int}(\mathcal{P}_\pi(A_{m_m}))$, then by Lemma 2.1, A_{m_m} consists of two simply connected graphs G_m^+ and G_m^- with continuous boundary. Similarly, A consists of two simply connected graphs G^+ and G^- on a domain in the plane π.

Let $X_m^+ : D \to \mathbb{R}^3$ and $X_m^- : D \to \mathbb{R}^3$ be conformal embeddings from the closed unit disk D for G_m^+ and G_m^- respectively, normalized such that $X_m^+(p_i) = q_i^m$, for three fixed points $p_i \in \partial D$ and $\lim_{m\to\infty} q_i^m = q_i \in L_1$. Similarly we require the three points condition for X_m^-.

Since ∂A_{m_m} consists of two convex Jordan curves in $\partial S(t_1, t_2) \cap C_R$, the arc lengths of ∂A_{m_m} is uniformly bounded, hence by isoperimetric inequality, the areas of A_{m_m}, hence of G_m^+, G_m^-, are uniformly bounded. Now since X_m^+ are conformal, $\int_D |DX_m^+|^2$ is uniformly bounded. Therefore, by Courant-Lebesgue Lemma (see Theorem 3 on page 238 of [fanghwang-dhkw]), X_m^+ is uniformly continuous and converges on D.

Similarly we can prove that X_m^- converges on D. Since $G_m^+ \to G^+$ and $G_m^- \to G^-$, and $\partial(G_m^+ \cup G_m^-) \cap \partial S(t_1, t_2) = \partial A_{m_m}$ converges to $L_1 \cup L_2$, and the plane π was arbitrary, we see that $\partial A = L_1 \cup L_2$ and $\overline{A}$ is continuous up to boundary and $\partial A = L_1 \cup L_2$. $\square$

Remark 2.1. The argument used in the proof that $\overline{A}$ is continuous actually gives an alternative proof that a subsequence of $\{A_n\}$ converges to a minimal annulus without curvature estimates and Lemma 2.3, i.e., via the uniformly boundedness of $\int_D |DX_m|^2$ and the Courant-Lebesgue Lemma, with the help of Lemma 2.1 and 2.2.

With Theorem 2.1, we can give a generalization of a theory dealing with smooth convex boundary developed by Hoffman and Meeks in [fanghwang-hm9], and Meeks and White in [fanghwang-mwl], to the continuous convex boundary case.

Theorem 2.2. *Suppose D_1 and D_2 are two open disks lying on parallel planes, and suppose their boundaries L_1 and L_2 are* **continuous** *convex Jordan curves.*

1. *If A' is a connected non-planar compact (maybe branched) minimal surface such that $\partial A' \subset D_1 \cup D_2$, then there exist at least two embedded compact minimal annuli A and B, $\partial A = \partial B = L_1 \cup L_2$.*

2. *A is stable and has the property that for any disks $D' \subset D_1$ and $D'' \subset D_2$ with continuous boundaries, if there is a connected compact (maybe branched) minimal surface N such that $\partial N = \partial D' \cup \partial D''$, then N is contained in the solid V bounded by $A \cup D_1 \cup D_2$. In particular, if $A \neq N$, then $\mathrm{Int}(A) \cap \mathrm{Int}(N) = \emptyset$. On the other hand, B is unstable and $\mathrm{Int}(B) \cap \mathrm{Int}(N) \neq \emptyset$.*

3. *If merely $\partial A' \subset \overline{D_1} \cup \overline{D_2}$, then there exists at least one embedded minimal annulus C such that $\partial C = L_1 \cup L_2$. Such a C is almost stable in the sense that the first eigenvalue of the second variation of C is larger than or equal to zero. Let N be a connected compact (maybe branched) minimal surface such that $\partial N = \partial D' \cup \partial D''$, then N is contained in the solid V bounded by $C \cup D_1 \cup D_2$. In particular, if $C \neq N$, then $\mathrm{Int}(C) \cap \mathrm{Int}(N) = \emptyset$.*

4. *Furthermore, the symmetry groups of A and B, or C, are the same as the symmetry group of $L_1 \cup L_2$.*

Proof. If L_1 and L_2 are smooth, or in the cases of existence of A and C in conclusions 1 or 3 for merely continuous L_1 and L_2, the theorem is a combination of Theorem 1.1, 1.2 of [fanghwang-hm9], and Lemma 2.1 of [fanghwang-mwl], with "exact" replacing "at least" in 1.

In general for the case of L_1 and L_2 are merely continuous, let $\mathrm{Symm}(L_1 \cup L_2)$ be the symmetry group of $L_1 \cup L_2$. We can construct smooth convex Jordan curves L_1^n and L_2^n such that $\lim_{n\to\infty} L_1^n = L_1$, $\lim_{n\to\infty} L_2^n = L_2$, $\mathrm{Symm}(L_1^n \cup L_2^n) = \mathrm{Symm}(L_1 \cup L_2)$, and L_1 and L_2 are enclosed in the disks bounded by L_1^n and L_2^n respectively. Therefore there are corresponding minimal annuli A_n and B_n etc., satisfying all the properties stated in the theorem, and being contained in some C_R. Using Theorem 2.1 we get our limiting minimal annuli A and B bounded by $L_1 \cup L_2$, they satisfy all properties stated in the theorem, especially, $A \neq B$. □

An immediate corollary of Lemma 2.1 and Theorem 2.2 is

Corollary 2.1. *If $\Sigma \subset S(t_1, t_2)$ is a connected compact non-planar minimal surface (maybe branched) such that $\partial\Sigma \subset P_{t_1} \cup P_{t_2}$. Let $L(\theta, t)$ be the length of $\Sigma \cap P_t$ projected on the plane with normal $(\cos\theta, \sin\theta, 0)$, for $(\theta, t) \in [0, \pi] \times [t_1, t_2]$. Then for any $t_1 \leq s < t \leq t_2$ and $\theta \in [0, \pi]$,*

$$L(\theta, s) + L(\theta, t) > 2(t - s), \quad L(\theta, s)L(\theta, t) > (t - s)^2.$$

Thus

$$L(\theta, t) > \max\left\{\frac{(t - t_1)^2}{L(\theta, t_1)}, \frac{(t_2 - t)^2}{L(\theta, t_2)}\right\}, \quad t \in (t_1, t_2).$$

In particular, if $\partial\Sigma \subset S(t_1, t_2) \cap S'(t'_1, t'_2)$, then $t'_2 - t'_1 > t_2 - t_1$. Furthermore, if $\Sigma \subset S(t_1, t_2) \cap C_R$, then

$$L(\theta, t) > \max\left\{\frac{(t - t_1)^2}{2R}, \frac{(t_2 - t)^2}{2R}\right\}, \quad t \in (t_1, t_2).$$

Proof. First observe that by comparison principle for minimal surfaces $\Sigma \cap P_t$ is a variety without isolated points, hence we can apply Theorem 2.2. Thus we can construct a minimal annulus $A(\theta) \subset S(s, t)$ with convex Jordan curves boundary in P_s and P_t respectively, which enclose $\Sigma \cap P_s$ and $\Sigma \cap P_t$. Then by Lemma 2.1

$$|\Gamma_1| + |\Gamma_2| > 2(t - s), \quad |\Gamma_1||\Gamma_2| > (t - s)^2,$$

where Γ_1 and Γ_2 are the projection of $\partial A(\theta)$. Note that we can make $\partial A(\theta)$ such that

$$L(\theta, s) = |\Gamma_1|, \quad L(\theta, t) = |\Gamma_2|.$$

Then all the conclusions are trivial by Lemma 2.1. □

Corollary 2.2. *Let c, d, δ_1, and δ_2 be positive constant numbers. If Σ is a connected compact non-planar minimal surface (maybe branched) such that $\partial\Sigma = L_1 \cup L_2$ and*

$$L_1 \subset D'_1 := \left\{(x, y, z); -\delta_1 \leq x - \frac{c}{d}z \leq \delta_1,\ z \leq 0\right\},$$

$$L_2 \subset D_2' := \left\{(x,y,z); -\delta_2 \le x - \frac{c}{d}z \le \delta_2,\ z \ge d\right\},$$

then

$$2\max\{\delta_1, \delta_2\} > \sqrt{c^2+d^2}. \tag{2.9}$$

Furthermore, if

$$L_1 \subset D_1 := \{(x,y,0); -\delta_1 \le x \le \delta_1\},$$

$$L_2 \subset D_2 := \{(x,y,d); -\delta_2 \le x - c \le \delta_2\}$$

then

$$\delta_1 + \delta_2 > \sqrt{c^2+d^2}. \tag{2.10}$$

Proof. Let $X : M \to \mathbb{R}^3$ be a conformal parametrization of Σ and consider the function

$$\phi = X_1 - \frac{c}{d}X_3,$$

then ϕ is harmonic. Thus let $R := \max\{\delta_1, \delta_2\}$, by maximum principle for harmonic functions,

$$\Sigma \subset \left\{(x,y,z);\ -R \le x - \frac{c}{d}z \le R\right\}.$$

Since $\partial\Sigma$ is contained in $\{z \le 0\} \cup \{z \ge d\}$, $\Sigma \cap S(0,d)$ is a minimal surface whose boundary $L_1' \cup L_2'$ satisfies that

$$L_1' \subset \{(x,y,0); -R \le x \le R\}, \quad L_2' \subset \{(x,y,d); -R \le x - c \le R\}.$$

Let $C_1 \subset P_0$, $C_2 \subset P_d$ be convex Jordan curves such that they enclose L_1' and L_2' respectively and

$$C_1 \subset \{(x,y,0); -R \le x \le R\}, \quad C_2 \subset \{(x,y,d); -R \le x - c \le R\}.$$

By Theorem 2.2, there is a minimal annulus A bounded by C_1 and C_2. Let π be the xz-plane and Γ_i, $i = 1, 2, 3, 4$, be as defined in Lemma 2.1. By comparison principle for minimal surfaces, Γ_3 and Γ_4 are not line segments. Then by Lemma 2.1,

$$4R \ge |\Gamma_1| + |\Gamma_2| \ge |\Gamma_3| + |\Gamma_4| > |l_3| + |l_4|,$$

where l_3 and l_4 are line segments such that $\Gamma_1 \cup \Gamma_2 \cup l_3 \cup l_4$ consists of the boundary of a convex four-gon in the xz-plane. Elementary calculation shows that

$$|l_3| + |l_4| = 2\sqrt{c^2 + d^2}.$$

Similarly we can prove (2.10) by requiring that

$$C_1 \subset \{(x, y, 0); -\delta_1 \leq x \leq \delta_1\}, \quad C_2 \subset \{(x, y, d); -\delta_2 \leq x - c \leq \delta_2\}.$$

Then we have

$$2(\delta_1 + \delta_2) \geq |\Gamma_1| + |\Gamma_2| \geq |\Gamma_3| + |\Gamma_4| > |l_3| + |l_4|,$$

and

$$|l_3| + |l_4| = \sqrt{(\delta_2 - \delta_1 + c)^2 + d^2} + \sqrt{(\delta_1 - \delta_2 + c)^2 + d^2} \geq 2\sqrt{c^2 + d^2}.$$

□

To establish the existence of solutions to non-compact Douglas-Plateau problem with four parallel straight lines as boundary in Section 3, we need another curvature estimate for minimal annuli.

Proposition 2.2. *Let $A \subset S(t_1, t_2) \cap S'(t_1', t_2')$ be a compact embedded minimal annulus such that $\partial A = L_1 \cup L_2$, where $L_1 \subset P_{t_1}$, $L_2 \subset P_{t_2}$ are C^2 convex Jordan curves. Let $E > 0$ such that $|\kappa(p)| \leq E$ for any $p \in \partial A$, where κ is the planar curvature. Then there is a constant $C_1 > 0$ only depending on $t_2 - t_1$, $t_2' - t_1'$ and E, such that*

$$|K(p)| \leq C_1,$$

where $K(p)$ is the Gaussian curvature of A.

Proof. The proof is a generalization of the proof in [fanghwang-mr1] of a special case of this Proposition, the estimates in Lemma 2.1 enables us to make this generalization.

By a homothety, we can assume that $t_1 = -1$, $t_2 = 1$. By a translation we can assume that $t_1' = -R$, $t_2' = R$ for some $R > 0$.

If the Proposition is not true, then there are minimal annuli $B_n \subset S(-1, 1) \cap S'(-R, R)$ such that ∂B_n consists of two convex Jordan curves in P_{-1} and P_1, whose planar curvatures are bounded by E, and $\exists p_n \in B_n$ such that

$$-C_n := K_{B_n}(p_n) \leq K_{B_n}(p), \quad \forall p \in B_n; \quad \lim_{n\to\infty} K_{B_n}(p_n) = -\infty.$$

Let $\hat{B}_n = B_n - p_n := \{p \in \mathbb{R}^3, p + p_n \in B_n\}$. Note that $p_n = (x_n, y_n, z_n)$, $-R \leq y_n \leq R$, $-1 \leq z_n \leq 1$. By a rotation if necessary, we may assume that $-1 \leq z_n \leq 0$. So that $\hat{B}_n = B_n - p_n \subset S(-1,2) \cap S'(-2R, 2R)$ contains the origin, and $\partial \hat{B}_n \subset P_{-1-z_n} \cup P_{1-z_n}$.

Let $\tilde{B}_n = \sqrt{C_n}\hat{B}_n$ be the homothety of $\hat{B}_n$ and $K_{\tilde{B}_n}$ be the Gaussian curvature of $\tilde{B}_n$, then $|K_{\tilde{B}_n}| \leq 1$. Let D_m be the ball centred at origin with radius m. Then by Lemma 2.3 a subsequence of $\{\tilde{B}_n\}$ converges in D_m. By a diagonal argument, a subsequence of $\{\tilde{B}_n\}$, still denote by $\{\tilde{B}_n\}$, converges to an embedded minimal surface $\tilde{M}$ in $\mathbb{R}^3$. $\tilde{M}$ is not a plane, since it has a point (the origin) with Gaussian curvature -1.

Since the Gauss map $N : \tilde{B}_n \to S^2$ is one-to-one and $N \neq \pm(0,0,1)$, [fanghwang-mw1], we have $\int_{\tilde{B}_n} K_{\tilde{B}_n} dA > -4\pi$, see [fanghwang-os1]. It forces that $\tilde{M}$ must have total curvature at least -4π.

Since the boundaries of $\tilde{B}_n$ are on $P_{\sqrt{C_n}(-1-z_n)}$ and $P_{\sqrt{C_n}(1-z_n)}$ and $C_n \to \infty$ as $n \to \infty$, if $\tilde{M}$ has a boundary, it must be that $\lim_{n\to\infty}\sqrt{C_n}(-1-z_n)$ exists in $\mathbb{R}$ or $\lim_{n\to\infty}\sqrt{C_n}(1-z_n)$ exists in $\mathbb{R}$. Since $-1 \leq z_n \leq 0$, it must be $\lim_{n\to\infty}\sqrt{C_n}(-1-z_n) = t_0$ exists and $\partial\tilde{M} \subset P_{t_0}$. Since $B_n \cap P_t$ are convex and have uniform planar curvature bound, $\partial\tilde{M} = \lim_{n\to\infty} \partial\tilde{B}_n \cap P_{\sqrt{C_n}(-1-z_n)}$ exists.

Since ∂B_n has uniform planar curvature bound E, it turns out the planar curvature of $\partial\tilde{M}$ is bounded by $E/\sqrt{C_n} \to 0$, hence $\partial\tilde{M}$ must be a straight line l if $\partial\tilde{M} \neq \emptyset$.

If $\partial\tilde{M} \neq \emptyset$, then rotating $\tilde{M}$ around $l = \partial\tilde{M}$ by π degree, we get a complete minimal surface without boundary, its total curvature is at least -8π, and it contains a straight line. But such a surface does not exist by classification, see for example [fanghwang-lo]. Thus $\partial\tilde{M} = \emptyset$.

It forces that $\tilde{M}$ must be a catenoid since that it is non-flat completely embedded without boundary, and its total curvature is at least -4π.

Thus $\tilde{M} \cap P_0$ is a circle and since $\tilde{B}_n \to \tilde{M}$, the length of $\tilde{B}_n \cap P_0$ should be bounded, i.e., there is an $F > 0$, such that

$$|\tilde{B}_n \cap P_0| \leq F.$$

But since $B_n \cap P_t$ is a convex Jordan curve for each $t \in (-1,1)$, and $B_n \cap S(-1,1) \cap S'(-R,R)$, recalling that $-1 \leq z_n \leq 0$ and applying (2.4) to

$B_n \cap S(z_n, 1)$, we have

$$|\tilde{B}_n \cap P_0| = \sqrt{C_n}|B_n \cap P_{z_n}| > \frac{\sqrt{C_n}(1-z_n)^2}{R} \geq \frac{\sqrt{C_n}}{R} \to \infty.$$

□

Remark 2.2. Since we assume that ∂A is C^2, the estimate is a global one, not just interior. But we can only prove by contradiction that the curvature of $A \subset S(t_1, t_2) \cap S'(t_1', t_2')$ has an a prior bound. The ideal proof is to give an a prior estimate of curvature bound for $A \subset S(t_1, t_2) \cap S'(t_1', t_2')$ explicitly involving the boundary planar curvature of ∂A, the height $t_2 - t_1$, and the width $t_2' - t_1'$. Such a concrete estimate will be useful in many other cases.

Then we have another compactness theorem.

Theorem 2.3. *If $\{A_n\} \subset S(t_1, t_2) \cap S'(t_1', t_2')$ is a sequence of minimal annuli such that $\partial A_n = L_1^n \cup L_2^n$, where $L_1^n \subset P_{t_1}$, $L_2^n \subset P_{t_2}$ are C^2 convex Jordan curves with uniform planar curvature bound E and $\lim_{n\to\infty} L_1^n = L_1$, $\lim_{n\to\infty} L_2^n = L_2$, then there is a subsequence of $\{A_n\}$ which converges to an embedded minimal surface A such that $\partial A = L_1 \cup L_2$.*

Proof. Let D_r be the ball centred at origin with radius r, then $\mathbb{R}^3 = \bigcup_{m=1}^{\infty} D_m$. Since $\lim_{n\to\infty} L_1^n = L_1$, $\lim_{n\to\infty} L_2^n = L_2$, for m and n large enough, $A_n \cap D_m \neq \emptyset$. By Proposition 2.2 $\{A_n\}$ has a uniform curvature bound, so Lemma 2.3 applied in D_m gives a convergent subsequent $\{A_{m_n} \cap D_m\}$, then $\{A_{m_m}\}$ is a subsequence of $\{A_n\}$ which converges to an embedded minimal surface A in any compact set. Since $\lim_{n\to\infty} L_1^n = L_1$, $\lim_{n\to\infty} L_2^n = L_2$, we have $\partial A = L_1 \cup L_2$. □

Remark 2.3. Note that the limit minimal surface A is not necessarily an annulus. In fact, it may be even not connected.

3. Applications to Non-Compact Douglas-Plateau Problem.

First let us define various boundaries for which we want to solve the related Douglas-Plateau problem.

Let $\alpha : \mathbb{R} \to \mathbb{R}^2$ be a properly embedded complete convex curve, and let $L = \alpha(\mathbb{R})$. Suppose that L is not a straight line, then $\mathbb{R}^2 - L$ has two components, only one of them is convex.

Definition 3.1 (Standard Boundary). We call $\Gamma := L_1 \cup L_2$ a **standard boundary** if:

- $L_1 \subset P_{-1}$ and $L_2 \subset P_1$ are two continuously embedded, proper, complete, non-compact, non-flat convex curves.
- Let $Y_1 \subset P_{-1}$ and $Y_2 \subset P_1$ be the two convex domains bounded by L_1 and L_2 respectively. Let $\tilde{Y}_i \subset P_0$ and $\tilde{L}_i \subset P_0$ be the perpendicular projections of Y_i and L_i, for $i = 1, 2$. Then $\tilde{Y}_1 \cap \tilde{Y}_2$ is a bounded convex domain.
- There is a connected compact non-planar (maybe branched) minimal surface Σ such that $\partial\Sigma \subset Y_1 \cup Y_2$.

Remark 3.1. By Corollary 2.2, the last condition of Definition 3.1 implies that $\tilde{Y}_1 \cap \tilde{Y}_2 \neq \emptyset$. We will call this condition **H-M condition to** $Y_1 \cup Y_2$, it first appeared in [fanghwang-hm9] by Hoffman and Meeks.

Let $D_r \subset P_0$ be the disk centred at $(0, 0)$ with radius r. It is well-known that if r is large enough and $D_r \subset \tilde{Y}_1 \cap \tilde{Y}_2$, then there is a piece of catenoid C such that $\partial C \subset Y_1 \cup Y_2$. Hence the H-M condition is satisfied.

Our first existence theorem is:

Theorem 3.1. *Let Γ be a standard boundary. Then there exist two embedded minimal annuli $\mathcal{A}$ and $\mathcal{B}$ such that $\partial\mathcal{A} = \partial\mathcal{B} = \Gamma$. The minimal annuli $\mathcal{A}$ and $\mathcal{B}$ have the following properties:*

1. *For each $t \in (-1, 1)$, $P_t \cap \mathcal{A}$ and $P_t \cap \mathcal{B}$ are strictly convex Jordan curves.*
2. $\mathrm{Int}(\mathcal{A}) \cap \mathrm{Int}(\mathcal{B}) = \emptyset$.
3. *Let N be a connected compact non-planar (maybe branched) minimal surface such that $\partial N \subset \overline{Y}_1 \cup \overline{Y}_2$, then*
$$\mathrm{Int}(\mathcal{A}) \cap \mathrm{Int}(N) = \emptyset, \quad \mathcal{B} \cap N \neq \emptyset.$$
4. *$\mathcal{A}$ and $\mathcal{B}$ have the same symmetry groups as that of Γ.*

Remark 3.2. If we change the last condition in the definition of standard boundary so that $\partial\Sigma \subset \overline{Y}_1 \cup \overline{Y}_2$, then there is at least one embedded minimal annulus $\mathcal{C}$ such that $\partial\mathcal{C} = \Gamma$. Furthermore, $\mathcal{C}$ behaves just like $\mathcal{A}$ in the sense that they satisfy the same properties in 1, 3 and 4 of Theorem 3.1.

A limit case of standard boundary is that of *straight line boundary*,

Definition 3.2 (Straight Line Boundary). A **straight line boundary** is as follows:

- $\Gamma = \bigcup_{i=1}^{4} L_i$, where L_i are straight lines such that L_1 and L_2 are contained in P_{-1} and parallel, while L_3 and L_4 are contained in P_1 and parallel. But L_1 and L_3 are not parallel.
- Let $Y_1 \subset P_{-1}$ be the open strip bounded by L_1 and L_2, $Y_2 \subset P_1$ be the open strip bounded by L_3 and L_4. Then $Y_1 \cup Y_2$ satisfies the H-M condition.

Remark 3.3. Note that since L_1 and L_3 are not parallel, the perpendicular projections $\tilde{Y}_1$ and $\tilde{Y}_2$ of Y_1 and Y_2 have bounded intersection.

Theorem 3.2. *Let Γ be a straight line boundary. Then there exist two embedded minimal annuli $\mathcal{A}$ and $\mathcal{B}$ such that $\partial\mathcal{A} = \partial\mathcal{B} = \Gamma$. The minimal annuli $\mathcal{A}$ and $\mathcal{B}$ have all the properties stated in Theorem* 3.1.

Remark 3.4. The Remark 3.2 also applies to the straight line boundary.

Now let us consider parallel straight lines. It is also an limit case of a standard boundary. Indeed if we consider a standard boundary such that there is a straight line $L \subset P_0$ which intersects $\tilde{L}_1$ and $\tilde{L}_2$ in exactly one point respectively. We may change L_1 and L_2 such that the single intersection points of L with $\tilde{L}_1$ and $\tilde{L}_2$ go to infinity in opposite directions, and $\tilde{L}_1$ and $\tilde{L}_2$ both break into two straight lines parallel to L. Thus we give the following definition of a *parallel line boundary*:

Definition 3.3 (Parallel Boundary). A **parallel boundary** is as follows:

- $\Gamma = \bigcup_{i=1}^{4} L_i$, where L_i are parallel straight lines such that L_1 and L_2 are contained in P_{-1}, while L_3 and L_4 are contained in P_1. The strip bounded by L_1 and L_3 and the strip bounded by L_2 and L_4 are disjoint.

- Let $Y_1 \subset P_{-1}$ be the open strip bounded by L_1 and L_2, $Y_2 \subset P_1$ be the open strip bounded by L_3 and L_4 in P_1. Then $Y_1 \cup Y_2$ satisfies the H-M condition.

Remark 3.5. By Corollary 2.1, a necessary condition for $\Gamma = \bigcup_{i=1}^{4} L_i$ being a parallel boundary is that the product of the widths of the strips Y_1 and Y_2 is larger than 4.

Given a parallel boundary Γ as above, we always assume that the straight lines L_i are parallel to the x-axis.

A special case of parallel boundary is a *lattice boundary* (it defines a lattice in the yz-plane):

Definition 3.4 (Lattice Boundary). Let Γ be a parallel boundary. Let p_i be the intersection points of the L_i with the yz-plane. If the p_i's are the vertices of a parallelogram, then we call the parallel boundary a **lattice boundary**.

Let F be the parallelogram with p_i as vertices. Then we select the bisectrice point (the intersection of the two diagonals) of F as the origin of $\mathbb{R}^3$.

Our third existence theorem is the following:

Theorem 3.3. *Let Γ be a parallel boundary, then there exists an embedded minimal annulus $\mathcal{D}$ such that $\partial\mathcal{D} = \Gamma$. Also $\mathcal{D}$ satisfies:*

1. *For $-1 < t < 1$, $P_t \cap \mathcal{D}$ are strictly convex Jordan curves.*
2. *Let N be a connected compact non-planar (maybe branched) minimal surface such that $\partial N \subset \overline{Y}_1 \cup \overline{Y}_2$, then $\mathcal{D} \cap N \neq \emptyset$.*
3. *$\mathcal{D}$ is invariant under the reflection $(x, y, z) \to (-x, y, z)$.*
4. *If Γ is a lattice boundary, then $\mathcal{D}$ is invariant under the rotation of angle π around the x-axis.*

Remark 3.6. In [fanghwang-mr1], Meeks and Rosenberg gave a proof of the existence of $\mathcal{D}$ with a lattice boundary in order to construct doubly periodic minimal surfaces.

3.1. Proof of Theorem 3.1.

The idea for the proof of Theorem 3.1 is to construct sequences of compact minimal annuli $\{A_n\}$ and $\{B_n\}$ whose convex boundaries approaching the given Γ.

Now we approach L_1 and L_2 by convex Jordan curves $L_n^1 \subset P_{-1}$ and $L_n^2 \subset P_1$ such that for any $R > 0$, there is an $N_R > 0$ such that whenever $n > N_R$,

$$(L_n^1 \cup L_n^2) \cap C_R = (L_1 \cup L_2) \cap C_R. \tag{3.1}$$

Let D_n^1 and D_n^2 be the disks bounded by L_n^1 and L_n^2. We can make L_n^1 and L_n^2 such that

$$\partial\Sigma \subset D_n^1 \cup D_n^2, \quad D_n^1 \subset D_{n+1}^1, \quad D_n^2 \subset D_{n+1}^2,$$

where Σ is the surface in the H-M condition.

By Theorem 2.2 there are two minimal annuli A_n and B_n, such that $\partial A_n = \partial B_n = L_n^1 \cup L_n^2$.

To prove that there are convergent subsequences of $\{A_n\}$ and $\{B_n\}$, we need the following lemma. The proof of this technical lemma is quite involved. In order not to interrupt the main argument, at this moment let us assume the lemma is true.

First let us fix some more notations. Let $Q_a := \{(x, y, z) \in \mathbb{R}^3; x = a\}$, $H_a^+ = \{(x, y, z) \in \mathbb{R}^3; x \geq a\}$, $H_a^- = \{(x, y, z) \in \mathbb{R}^3; x \leq a\}$. Let $W_a = \{(x, y, z); -a \leq x \leq a\}$. Denote the xz-plane by P_0'.

Lemma 3.1. *Let Y_1 and Y_2 be as defined in Definition* 3.1. *Let $A \subset S(-1, 1)$ be a compact minimal surface. Suppose that $\partial A = C_1 \cup C_2$ such that $C_1 \subset Y_1$, $C_2 \subset Y_2$. Then we can choose coordinates (x, y, z) such that for any $a > 0$, there is an $S(a) > 0$ such that $A \cap W_a \subset S'(-S(a), S(a))$. We can choose that if $a > b > 0$, then $S(a) \geq S(b)$.*

And for any $t \in (-1, 1)$, there is an $R(t) > 0$ such that

$$A \cap P_t \subset C_{R(t)}. \tag{3.2}$$

Moreover, we can make that $R(t) = R(-t)$ and $R(s) \leq R(t)$ whenever $|s| < |t|$.

As proved in the proof of Lemma 3.1, we can choose coordinates (x, y, z) of $\mathbb{R}^3$, such that if $a > 0$ is large enough, we have $Q_a \cap L_2 = \emptyset$ and $Q_{-a} \cap L_1 =$

$\emptyset$ or, $Q_a \cap L_1 = \emptyset$ and $Q_{-a} \cap L_2 = \emptyset$, but not both. Without loss of generality, we assume that it is the former and keep this convention in this paper.

By Lemma 3.1, $A_n \cap S(-t,t) \subset C_{R(t)}$ and $B_n \cap S(-t,t) \subset C_{R(t)}$, hence for each $t \in (0,1)$, they are uniformly bounded. By Shiffman's first theorem [fanghwang-sh1], $A_n \cap P_t$ and $B_n \cap P_t$ are strictly convex Jordan curves.

Now we can use Theorem 2.1 to prove subsequences of $\{A_n\}$ and $\{B_n\}$ converge to $\mathcal{A}$ and $\mathcal{B}$ in the interior of $S(-1,1)$. In fact, there are subsequences of $\{A_n\}$ and $\{B_n\}$ which are convergent to embedded compact minimal annuli $A_{t_m} \subset S(-t_m,t_m)$ and $B_{t_m} \subset S(-t_m,t_m)$ in $S(-t_m,t_m)$ for any t_m, where $t_m \nearrow 1$ as $m \to \infty$. By a diagonal argument we see that subsequences of $\{A_n\}$ and $\{B_n\}$ converge to embedded minimal surfaces $\mathcal{A}$ and $\mathcal{B}$. Since for each $s \in (-t_m,t_m)$, $A_n \cap P_s$ and $B_n \cap P_s \subset C_{R(t_m)}$ is uniformly bounded convex Jordan curves and the convergence is smooth, $\mathcal{A} \cap P_s$ and $\mathcal{B} \cap P_s$ are convex Jordan curves. Since for $t \in (-1,1)$, $\mathcal{A} \cap P_t$ and $\mathcal{B} \cap P_t$ are convex, by Shiffman's theorem again, we know that $\mathcal{A} \cap P_t$ and $\mathcal{B} \cap P_t$ are strictly convex, hence $\mathcal{A}$ and $\mathcal{B}$ are minimal annuli.

Still denote these subsequences by $\{A_n\}$ and $\{B_n\}$, we only need prove that $\mathcal{A}$ and $\mathcal{B}$ are continuous up to boundary and $\partial\mathcal{A} = \partial\mathcal{B} = \Gamma$.

Now by Lemma 2.1 A_n consists of two simply connected minimal graphs over a domain $\Omega_n \subset P_0'$, say G_n^+, G_n^-.

Since $A_n = G_n^+ \cup G_n^-$, $A_n \cap Q_a$ and $A_n \cap Q_{-a}$ are the unions of two graphs respectively, hence they are simple curves. Similarly, $\mathcal{A} \cap Q_a$ and $\mathcal{A} \cap Q_{-a}$ are simple curves. Thus $G_n^+ \cap W_a$, $G_n^- \cap W_a$, $G^+ \cap W_a$, and $G^- \cap W_a$ are all simply connected.

Let $\Omega_a = \Omega \cap W_a$, then Ω_a is bounded and has piecewise smooth boundary as proved in Lemma 2.1, hence $\partial\Omega_a$ has finite length. Also by Lemma 3.1, $\mathcal{A} \cap W_a$ is also bounded, thus we know that $G^+ \cap W_a$, $G^- \cap W_a$ have finite area.

Let D be the closed unit disk in $\mathbb{C}$ and $X_n : D \to \mathbb{R}^3$ be a conformal embedding of $G_n^+ \cap W_a$ such that for three fixed points $p_i \in \partial D$, $X_n(p_i) = q_i$, where $q_i \in \partial G_n^+ \cap (L_1 \cup L_2) \cap W_a$, $i = 1, 2, 3$. Since $\partial A_n \to L_1 \cup L_2$, this is alway possible. Since $G_n^+ \to G^+$, the areas of $G_n^+ \cap W_a$ are uniformly bounded and by the conformality of X_n, $\int_D |DX_n|^2$ are uniformly bounded.

By Courant-Lebesgue Lemma, $G_n^+ \cap W_a = X_n(D)$ converges to $(G^+ \cup (L_1 \cup L_2)) \cap W_a$ and is continuous up to boundary. Similar argument for G^- also holds. Thus we see that $\partial(\mathcal{A} \cap W_a) \cap (P_{-1} \cup P_1) = (L_1 \cup L_2) \cap W_a$ for all $a > 0$ large enough. Moreover, it is clear that $\partial\mathcal{A} \subset P_{-1} \cup P_1$. Therefore $\partial\mathcal{A} = L_1 \cup L_2$ and similarly $\partial\mathcal{B} = L_1 \cup L_2$ and they are continuous up to boundary.

Let N be a connected non-planar compact (maybe branched) minimal surface such that $\partial N \subset \overline{Y}_1 \cup \overline{Y}_2$. Let V_n be the solid bounded by $A_n \cup D_n^1 \cup D_n^2$, and V be the solid bounded by $\mathcal{A} \cup \overline{Y}_1 \cup \overline{Y}_2$. We know that $N \subset V_n$ and $\mathrm{Int}(A_n) \cap \mathrm{Int}(N) = \emptyset$. Since $V_n \to V$, $N \subset V$. By the comparison principle for minimal surfaces, either $\mathcal{A} = N$ or $\mathrm{Int}(\mathcal{A}) \cap \mathrm{Int}(N) = \emptyset$. Since N is compact and $\mathcal{A}$ is not compact, $\mathrm{Int}(N) \cap \mathrm{Int}(\mathcal{A}) = \emptyset$. In particular, $\mathrm{Int}(\Sigma) \cap \mathrm{Int}(\mathcal{A}) = \emptyset$.

Since $B_n \cap N \neq \emptyset$, $\lim_{n\to\infty} B_n = \mathcal{B}$, and N is compact, we know that $\mathcal{B} \cap N \neq \emptyset$.

In particular, $\mathrm{Int}(\Sigma) \cap \mathrm{Int}(\mathcal{B}) \neq \emptyset$. Thus $\mathcal{A} \neq \mathcal{B}$.

Let V_n' be the solid bounded by $B_n \cup D_n^1 \cup D_n^2$, and V' be the solid bounded by $\mathcal{B} \cup \overline{Y}_1 \cup \overline{Y}_2$. Then since $V_n' \subset V_n$, $\lim_{n\to\infty} V_n = V$, and $\lim_{n\to\infty} V_n' = V'$, $V' \subset V$. By the comparison principle for minimal surfaces, we have that $\mathrm{Int}(\mathcal{A}) \cap \mathrm{Int}(\mathcal{B}) = \emptyset$.

By Theorem 2.2, we can construct the approaching sequences $\{A_n\}$ and $\{B_n\}$ such that they have the same symmetry groups as that of Γ, thus the limits, $\mathcal{A}$ and $\mathcal{B}$, have the same symmetry groups as that of Γ.

The proof of Theorem 3.1 is complete except that we still need prove Lemma 3.1. □

The idea for the proof of Lemma 3.1 is to construct various barriers and use the comparison principle for minimal surfaces. To establish these barriers, let us quote a Lemma in [fanghwang-chm].

Lemma 3.2 (Lemma 4 in [fanghwang-chm]). *Let L_0 consist of two non-collinear rays emanating from the origin in the plane P_0, and let L_1 be their vertical translation into the plane P_1. Then $L_0 \cup L_1$ is the boundary of a unique properly embedded minimal surface contained in the convex hull of $L_0 \cup L_1$. This minimal surfaces is a graph over an infinite strip and hence is simply connected.*

Remark 3.7. As pointed out in the proof of Lemma 3.2 in [fanghwang-chm], A is asymptotic to a flat strip as it diverges to infinity.

Proof of Lemma 3.1. First we claim that $\tilde{Y}_1 \cup \tilde{Y}_2$ is contained in an unbounded domain Ω with four rays as boundary. And if we adjust the angles between the boundary rays of Ω we can assume that there is a straight line l contained in Ω. Note that this implies that $\mathbb{R}^3 - \overline{\Omega}$ consists of two unbounded convex domains.

In fact, since $\tilde{Y}_1 \cap \tilde{Y}_2$ is convex and bounded, there are exactly two unbounded components on $\tilde{L}_1 - \tilde{L}_2$ and $\tilde{L}_2 - \tilde{L}_1$. Say $\alpha_i \subset \tilde{L}_1 - \tilde{L}_2$, $\beta_i \subset$

$\tilde{L}_1 - \tilde{L}_2$, $i = 1, 2$, are the four unbounded components. Take a point on each of these unbounded components, say $p \in \alpha_1$, $p' \in \alpha_2$, $q \in \beta_1$, $q' \in \beta_2$. Since $\tilde{L}_i$ are convex, there are straight lines passing through these four points such that $\tilde{L}_i$ are on the same side of these lines. Denote these lines by l_p, $l_{p'}$, l_q, and $l_{q'}$.

If l_p and $l_{p'}$ interset, then $\tilde{Y}_1$ is contained in a wedge Ω_1 (a convex domain bounded by two rays issuing from one point) bounded by rays in l_p and $l_{p'}$. If l_p and $l_{p'}$ are parallel, then since $\tilde{L}_1$ is non-compact convex, $\tilde{Y}_1$ is contained in the strip S bounded by l_p and $l_{p'}$. Since $\tilde{L}_1$ is non-flat, we can find a wedge Ω_1 such that $\tilde{Y}_1 \subset \Omega_1$. Similarly, there is a wedge $\Omega_2 \supset \tilde{Y}_2$.

Since $\tilde{Y}_1 \cap \tilde{Y}_2$ is compact, by parallel translations or, if necessary, vary the angles of the wedges, we can assume that $\partial\Omega_1 \cap \partial\Omega_2 = \{P, Q\}$. Take Ω to be the domain bounded by rays in $\partial\Omega_1 \cup \partial\Omega_2$ issuing from P and Q, then clearly $\tilde{Y}_1 \cup \tilde{Y}_2 \subset \Omega$. By enlarging Ω if necessary, we can assume that the straight line l which is the bisector of the line segment $\overline{PQ}$ is contained in Ω. Take l as the x-axis, then the coordinate system of (x, y, z) satisfies that if $a > 0$ large enough, we have $Q_a \cap L_2 = \emptyset$ and $Q_{-a} \cap L_1 = \emptyset$ or, $Q_a \cap L_1 = \emptyset$ and $Q_{-a} \cap L_2 = \emptyset$, but not both.

Denote the two components of $\partial\Omega$ by l^1 and l^2.

Now let $A \subset S(-1, 1)$ be a compact minimal surface such that $\partial A = C_1 \cup C_2$, $C_1 \subset Y_1$, $C_2 \subset Y_2$. Then $C_1 \cap L_2 = \emptyset$ and $C_2 \cap L_1 = \emptyset$. See Figure 3 below.

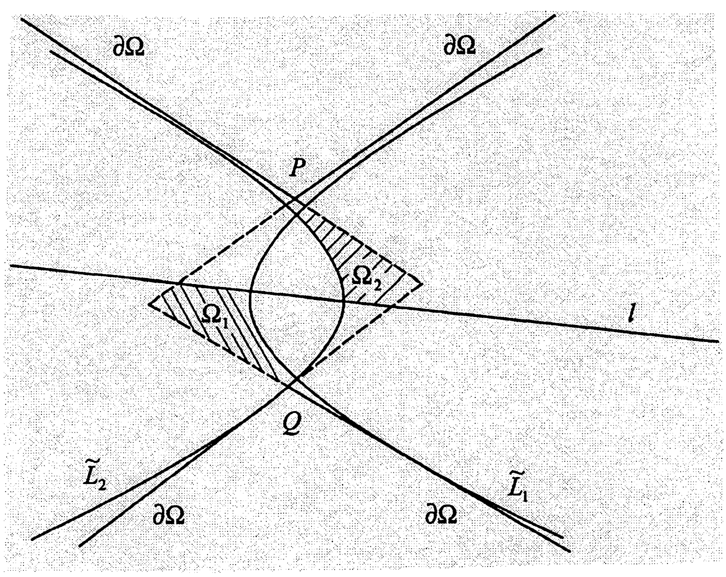

Figure 3

We first prove that for any $a > 0$, there is an $S(a) > 0$ such that $A \cap W_a \subset S'(-S(a), S(a))$.

We use the barrier in Lemma 3.2.

$$l^1 \cap (\tilde{Y}_1 \cup \tilde{Y}_2) = \emptyset. \tag{3.3}$$

Now let us parallel translate l^1 along the z-axis into P_{-1} and P_1 and call them l^1_{-1} and l^1_1 respectively. By Lemma 3.2, there is a minimal graph M bounded by l^1_{-1} and l^1_1. By (3.3), $\partial A \cap (l^1_{-1} \cup l^1_1) = \emptyset$. By the comparison principle for minimal surfaces, $A \cap M = \emptyset$. Thus for any $a \neq 0$, there is an $R^1_a > 0$, such that $A \cap W_a$ is contained in the half-space $\{y \leq R^1_a\}$. Since M is continuous, we can select R^1_a such that it is nondecreasingly continuous respect to a.

Similarly, using l^2 to make minimal graph M', we can find an $R^2_a > 0$ such that $\mathcal{P}(A) \cap W_a$ is contained in the half-space $\{y \geq -R^2_a\}$ and R^2_a is nondecreasingly continuous respect to a. Take $S(a) = \max\{R^1_a, R^2_a\}$, we have proved that $A \cap W_a \subset S'(-S(a), S(a))$ and $S(a)$ is nondecreasingly continuous respect to a.

Next, the minimal graph M bounded by $l^1_{-1} \cup l^1_1$ is contained in the convex hull of $l^1_{-1} \cup l^1_1$. Let $\mathcal{P}$ be the perpendicular projection on the xy-plane. Then $\mathcal{P}(M)$ is contained in the convex domain bounded by l^1. Similarly, $\mathcal{P}(M')$ is contained in the convex domain bounded by l^2. Therefore, $\mathcal{P}(M) \cap (\tilde{Y}_1 \cup \tilde{Y}_2) = \emptyset$ and $\mathcal{P}(M') \cap (\tilde{Y}_1 \cup \tilde{Y}_2) = \emptyset$. Parallel moving l^1 and l^2 to L^1 and L^2 along the $\pm y$-direction and denote the non-convex domain bounded by $L^1 \cup L^2$ by Ω', then by the Remark 3.7 we can make

$$(\mathcal{P}(M) \cup \mathcal{P}(M')) \cap \Omega' = \emptyset, \quad \mathcal{P}(A) \subset \Omega'. \tag{3.4}$$

Still denote L^1 by l^1, L^2 by l^2.

Let $l_u = \{(x, y, z);\ x = u, z = 0\}$. We have proved and made the convention that $\tilde{L}_1 \cap l_{-d} = \emptyset$, $\tilde{L}_2 \cap l_d = \emptyset$ for $d > 0$ large enough.

Let Ω_1 be the unbounded convex domain bounded by part of $l^1 \cup l^2 \cup l_d$. Then $\partial\Omega_1 = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \subset l^1 \cup l^2$ consists of two rays issued from the intersections of l_d with $l^1 \cup l^2$, and Γ_2 is the line segment in l_d between the two intersection points.

Similarly, let Ω_2 be the unbounded convex domain bounded by part of $l^1 \cup l^2 \cup l_{-d}$. Then $\partial\Omega_2 = \Gamma_3 \cup \Gamma_4$, where $\Gamma_3 \subset l^1 \cup l^2$ consists of two rays issued from the intersections of l_{-d} with $l^1 \cup l^2$, and $\Gamma_4 \subset l_{-d}$ is a line segment. See Figure 4 below.

We construct two minimal graphs G_1 and G_2 by solving Dirichlet problems for minimal surface equation on the domains Ω_1 and Ω_2, with the boundary data:

$$\begin{cases} u_1 = -1 & \text{on } \Gamma_1 \\ u_1 = \infty & \text{on } \Gamma_2; \end{cases} \qquad \begin{cases} u_2 = 1 & \text{on } \Gamma_3 \\ u_2 = -\infty & \text{on } \Gamma_4. \end{cases}$$

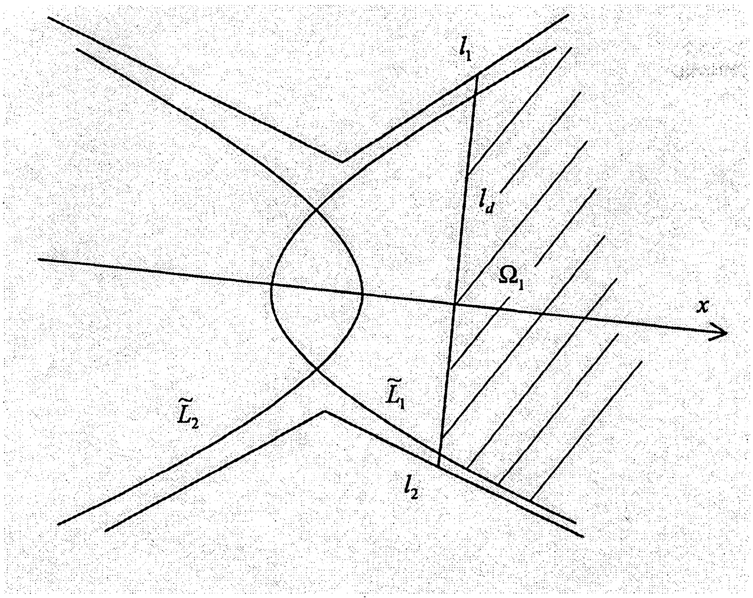

Figure 4

Such minimal graphs exist and are unique. For example, let Ω_a be the domain $\{(x,y) \in \Omega_1;\, x < a\}$, then $\partial\Omega_a = \Gamma_2 \cup \Gamma_a \cup \Gamma_1^a$, where Γ_a is the segment in l_a between the intersection points of l_a with Γ_1; Γ_1^a is $\Gamma_1 \cap \{x < a\}$. Then clearly when a large enough, we have

$$|\Gamma_1^a| > |\Gamma_2| + |\Gamma_a|.$$

By a theorem of Jenkins and Serrin [fanghwang-js], there is a unique solution u_a for the Dirichlet problem of minimal surface equation with the boundary value:

$$u_a = -1 \text{ on } \Gamma_1^a,$$
$$u_a = \infty \text{ on } \Gamma_2 \cup \Gamma_a.$$

Now if $c > a$, then by comparison principle for minimal surfaces, $u_c < u_a$ in Ω_a. For any $p \in \Omega_1$, if a is large enough then $p \in \Omega_a$. Let $c_n \nearrow \infty$ then $\{u_{c_n}\}$ is a decreasing sequence, by the monotone convergence theorem [fanghwang-js], there is a $u = \lim_{n\to\infty} u_{c_n}$ which solves our Dirichlet problem. By the maximum principle in [fanghwang-nil], page 256, $\sup_{x\geq d+1}|u_1(x,y)|$ and $\sup_{x\leq -d-1}|u_2(x,y)|$ are finite. Hence by Theorem 3 of [fanghwang-hwa1], the solution is unique.

An important property for G_i is that $\lim_{|(x,y)|\to\infty} u_i(x,y) = \mp 1$, $i = 1$, 2, see [fanghwang-mr2], Theorem 3.1. Thus by (3.4), for any $t \in (-1,1)$, there is an $R_1(t) > 0$ such that

$$G_1 \cap P_t \subset H^-_{R_1(t)}; \quad G_2 \cap P_t \subset H^+_{-R_1(t)}.$$

By the construction of G_i, $\partial A \cap \partial G_i = \emptyset$ and $\mathcal{P}(A \cap H^+_d) \subset \mathcal{P}(G_1) = \Omega_1$, $\mathcal{P}(A \cap H^-_{-d}) \subset \mathcal{P}(G_2) = \Omega_2$. By the comparison principle for minimal surfaces, we conclude that $A \cap G_i = \emptyset$, for $i = 1, 2$. Since $\mathcal{P}(A) \cap (\tilde{Y}_1 \cap \tilde{Y}_2) \neq \emptyset$ and $\mathcal{P}(G_i) \cap (\tilde{Y}_1 \cap \tilde{Y}_2) = \emptyset$, $i = 1$, 2, by the maximum principle $A \cap P_t \subset H^-_{R_1(t)} \cap H^+_{-R_1(t)}$, $R_1(t) \geq d$, for any $t \in (-1,1)$. Then we have $S(R_1(t)) > 0$, such that $A \cap P_t \subset W_{R_1(t)} \cap S'(-S(R_1(t)), S(R_1(t)))$.

Take $R(t) = \max\{\sqrt{2}R_1(t), \sqrt{2}S(R_1(t))\}$, then $A \cap P_t \subset C_{R(t)}$. Since G_1 and G_2 are continuous, $R_1(t)$ is nondecreasingly continuous respect to t. Thus $R(t)$ is nondecreasingly continuous respect to t.

The proof of Lemma 3.1 is complete. □

3.2. Proof of Theorem 3.2.

The proof of Theorem 3.2 is similar to the proof of Theorem 3.1. The key point is that we can confine approaching minimal annulus sequences $\{A_n\}$ and $\{B_n\}$ by four minimal barriers, i.e., the minimal graphs as in Lemma 3.2, using the eight rays issuing form the four intersection points of $\tilde{L}_i$, $i = 1$, 2, 3, 4. Thus Lemma 3.1 is true for a straight line case. The other arguments are either exactly the same as the arguments in the proof of Theorem 3.1 or are slightly variations of them. □

3.3. Proof of Theorem 3.3.

We start with L^1_n being the convex curve consisting of the two line segments $(L_1 \cup L_2) \cap C_n$, and two round arcs smoothly connecting the two pairs of end points, note that we use the same arcs up to a translation or reflection. Similarly define L^2_n. By this construction, L^1_n and L^2_n are invariant under the reflection about the yz-plane and have uniformly bounded boundary planar curvature.

If Γ is a lattice boundary, then we can make L_n^2 to be the image of L_n^1 under the rotation of angle π around the x-axis.

By Theorem 2.2, there are embedded minimal annuli B_n bounded by $L_n^1 \cup L_n^2$ such that $\mathrm{Int}(B_n) \cap \mathrm{Int}(\Sigma) \neq \emptyset$, where Σ is the minimal surface in the H-M condition of Definition 3.3. Moreover, B_n has the same symmetry group as that of $L_n^1 \cup L_n^2$. Note that there is an $R > 0$ such that $B_n \subset S'(-R, R)$ and by the construction, we see that $L_n^1 \cup L_n^2$ have uniform boundary planar curvature bound and $\lim_{n\to\infty} L_n^1 \cup L_n^2 = \Gamma$. Hence we can apply Theorem 2.3 to conclude that there is a subsequence of $\{B_n\}$ which converges to an embedded minimal surface $\mathcal{D}$ such that $\partial\mathcal{D} = \Gamma$.

Since B_n's satisfy the symmetry conditions in Theorem 3.3, $\mathcal{D}$ also satisfies the symmetry conditions.

It remains to prove that $\mathcal{D}$ is an annulus and satisfies the other properties claimed in Theorem 3.3.

To establish that $\mathcal{D}$ is an annulus, it is sufficient to prove that $\mathcal{D} \cap P_t$ is a strictly convex Jordan curve, for any $-1 < t < 1$.

Since $B_n \cap \Sigma \neq \emptyset$ and Σ is compact, we see that $\mathcal{D} \cap \Sigma \neq \emptyset$.

We observe that $P_t \cap \mathcal{D}$ is the smooth limit of a sequence of strictly convex Jordan curves. If $P_t \cap \mathcal{D}$ is compact, then it must be a convex Jordan curve. Thus the only thing left to be proved is that $\mathcal{D} \cap P_t$ must be compact.

Note that $P_t \cap \mathcal{D}$ is invariant under the reflection about the yz-plane. If $P_t \cap \mathcal{D}$ is not compact, then clearly $P_t \cap \mathcal{D} \cap \{x = \pm s\} \neq \emptyset$, for any $s > 0$, otherwise $P_t \cap \mathcal{D}$ is bounded. This forces that $P_t \cap \mathcal{D}$ consists of two graphs generated by uniformly bounded functions $y_1(x, t)$ and $y_2(x, t)$, $y_1 \geq y_2$, $-\infty < x < \infty$, or in the limit case, $y_1 \equiv y_2$. Let y_1^n and y_2^n be the functions defined by $P_t \cap B_n$, then y_1^n is concave and y_2^n is convex. Since $y_1(x) = \lim_{n\to\infty} y_1^n$, $y_2(x) = \lim_{n\to\infty} y_2^n$, in C^k topology, $\forall k > 0$, $y_i^n(x) = y_i^n(-x)$, $i = 1, 2$, y_1 is concave with maximum $y_1(0, t)$ and y_2 is convex with minimum $y_2(0, t)$. By Lemma 2.1

$$|y_1(0, t) - y_2(0, t)| > \max\left\{\frac{(1-t)^2}{4R^2}, \frac{(t+1)^2}{4R^2}\right\} \geq \frac{1}{4R^2},$$

for $-1 < t < 1$. If $P_t \cap \mathcal{D}$ is not compact, then y_1 and y_2 are both defined on $(-\infty, \infty)$, and y_1 is concave, y_2 is convex. Thus $P_t \cap \mathcal{D}$ is the union of two parallel straight lines which are parallel to the x-axis.

Consider the arc length functions

$$L(t) = \text{the arc length of } P_t \cap \mathcal{D}, \quad \text{and} \quad L_n(t) = \text{the arc length of } P_t \cap B_n.$$

Then $L_n(t) \to L(t)$ when $n \to \infty$. By [fanghwang-osh], L_n is convex with respect to t. We see that if for some $-1 < t_0 < 1$, $L(t_0) = \infty$, then $L_n(t_0) \to \infty$ and there is a closed interval containing t_0 in $(-1,1)$ such that on which $L_n(t) \to \infty$. Thus we can assume that the set

$$\{t \in (-1,1) \,|\, P_t \cap \mathcal{D} \text{ is not compact}\}$$

contains an open interval in $(-1,1)$. Hence there are $-1 < t_1 < t_2 < 1$ such that $P_t \cap \mathcal{D}$ consists of two straight lines parallel to the x-axis for $t_1 < t < t_2$. Thus $\mathcal{D}$ is contained in a ruled minimal surface. Since the only non-planar ruled minimal surface is the Helicoid and its generating straight lines are not parallel, $\mathcal{D}$ is contained in two planes P_1 and P_2 such that $L_1 \cup L_2 \cup L_3 \cup L_4 \subset P_1 \cup P_2$. Now since $\mathcal{D}$ is embedded, we have that P_1 is the plane containing $L_1 \cup L_3$, P_2 is the plane containing $L_2 \cup L_4$. But by the comparison principle for minimal surfaces, $\Sigma \cap (P_1 \cup P_2) = \emptyset$, and $\Sigma \cap \mathcal{D} = \emptyset$, a contradiction. This contradiction proves that $P_t \cap \mathcal{D}$ is compact.

As before, once we know that $P_t \cap \mathcal{D}$ is convex for $-1 < t < 1$, then it is strictly convex by quoting Shiffman's first theorem.

The remaining properties claimed in Theorem 3.3 can be proved in the same way as in the proof of Theorem 3.1.

The proof of Theorem 3.3 is complete now. □

Remark 3.8. There are other cases of boundaries such that similar lemma as Lemma 3.1 is true, thus with H-M condition, there are two solutions. For example, L_1 becomes two parallel straight lines and $\tilde{Y}_1 \cap \tilde{Y}_2$ is bounded, etc.

Is the solution in Theorem 3.3 unique? Similar questions can be asked. For example, are there other solutions besides the two given in Theorem 3.1 and 3.2? is there a theory about non-compact smooth convex boundary as that established by Meeks and White in [fanghwang-mw1]? Furthermore, can the theory of Meeks and White, together with its generalization to non-compact cases (if it is generalizable), be generalized to the continuous case? We would like to know the answers.

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