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Frobenius Manifold Structure on Dolbeault Cohomology and Mirror Symmetry

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We construct a differential Gerstenhaber-Batalin-Vilkovisky algebra from the Dolbeault complex of any closed Kähler manifold, and a formal Frobenius manifold structure on its Dolbeault cohomology.

String theory leads to the mysterious Mirror Conjecture, see Yau [37] for the history. One of the mathematical predictions made by physicists based on this conjecture is the formula due to Candelas-de la Ossa-Green-Parkes [6] on the number of rational curves of any degree on a quintic in $\mathbb{CP}^4$. Recently, it has been proved by Lian-Liu-Yau [20], completing the program of Kontsevich, Manin and Givental.

The theory of quantum cohomology, also suggested by physicists, has lead to a better mathematical formulation of the Mirror Conjecture. As explained in Witten [38], there are two topological conformal field theories on a Calabi-Yau manifold X: the A theory is independent of the complex structure of X, but depends on the Käher form on X, while the B theory is independent of the Kähler form of X, but depends on the complex structure of X. Vafa [33] explained how two quantum rings $\mathcal{R}_x$ and $\mathcal{R}_x{}'$ arise from these theories, and the notion of mirror symmetry could be translated into the equivalence of A theory on a Calabi-Yau manifold X with B theory on another Calabi-Yau manifold $\hat{X}$, called the mirror of X, in the sense that the quantum ring $\mathcal{R}_x$ can be identified with $\mathcal{R}_x{}'$.

Earlier interpretation of mirror symmetry exploits variations of Hodge structures, see e.g. Morrison [25] or Bertin-Peters [4]. There are two natural Frobenius algebras on any Calabi-Yau n-fold,

$$A(X) = \oplus_{k=0}^n H^k(X, \Omega^k), \qquad B(X) = \oplus_{k=0}^n H^k(X, \Omega^{-k}),$$

where Ω^{-k} is the sheaf of holomorphic sections to $\Lambda^k TX$. By Hodge theory, $B(X)$ can be identified with $H^n(X, \mathbb{C})$. By the Bogomolov-Tian-Todorov

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theorem, there is a deformation of the complex structures on X parameterized by an open set in $H^1(X, \Omega^{-1})$. Therefore, one gets a family of Frobenius algebra structures on $H^n(X, \mathbb{C})$. Every Frobenius algebra structure can be characterized by a cubic polynomial Φ (in Physics literature, it is called the Yukawa coupling), so we get a family $\Phi_B(X)$ of cubic polynomials on $H^n(X, \mathbb{C})$ parameterized by an open set in $H^1(X, \Omega^{-1})$. An additional structure provided by algebraic geometry is the Gauss-Manin connection on this family. It is a flat connection with some extra properties. On the other hand, counting of rational curves provides a family $\Psi_A(X)$ of cubic polynomials on $A(X)$ parameterized by an open set in $H^1(X, \Omega^1)$. One version of Mirror Conjecture is the conjectural existence, for a Calabi-Yau 3-fold X, of another Calabi-Yau 3-fold $\hat{X}$, such that one can identify $B(X)$ with $A(\hat{X})$, $\Phi_B(X) = \Psi_A(\hat{X})$, and vice versa.

However, the A theory and the B theory should provide deformations of

$$\widetilde{A}(X) = \oplus_{p,q\geq 0} H^p(X, \Omega^q), \quad \text{and} \quad \widetilde{B}(X) = \oplus_{p,q\geq 0} H^p(X, \Omega^{-q}),$$

which are parameterized by open neighborhood of 0 in $\widetilde{A}(X)$ and $\widetilde{B}(X)$ respectively. According to Dijkgraaf-E. Verlinde-H. Verlinde [10] and Witten [35], the associativity condition can be encoded in a system of nonlinear equations called WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equations. Motivated by this, Dubrovin [11, 12] introduced and extensively studied the notion of Frobenius manifolds. On the A side, under the influence of Witten [35], Ruan-Tian [28] gave a mathematical formulation of quantum cohomology using Gromov-Witten invariants (see also Liu [21] and McDuff-Salamon [22]). They also proved WDVV equations for the quantum cohomology. Witten [38] suggested two kinds of extended moduli spaces, one containing the deformation space of the complex structure, the other containing the complexified Kähler cone. The former provides a natural setting for the B side of the story. Extended moduli space of complex structures on a Calabi-Yau manifold was studied by physicists in the Kodaira-Spencer theory of gravity (Bershadsky-Ceccoti-Ooguri-Vafa [3]). As a generalization of the Bogomolov-Tian-Todorov theorem, they showed that the extended moduli space locally is an open subset of the supermanifold $H^{-*,*}(M)$. For related work, see Ran [27] and Gerstenhaber-Schack [13]. Recently, Barannikov-Kontsevich [2] constructed a structure of formal Frobenius manifold on $H^{-*,*}(M)$ based on the above works. This construction and its application to mirror symmetry was known to the authors of [3]. It was singled out among many impressive results in [3], and given a mathematical treatment. The remark in [2] that this construction can be carried out for any differential Gerstenhaber-Batalin-Vilkovisky (dGBV) algebra with suitable conditions was treated with more details in Manin [24].

The purpose of this paper is to construct a formal Frobenius manifold structure on Dolbeault cohomology $H^{*,*}(M)$ of a closed Kähler manifold by dGBV algebra approach. Comparison with that of Barannikov-Kontsevich [2] suggests that these two kinds of formal Frobenius manifolds might be isomorphic with each other for a pair of mirror manifolds. Now we have two formal Frobenius manifolds on the A-side: one from counting rational curves by Gromov-Witten invariants, the other from our construction. From consideration of spectrum of the Frobenius manifolds, it is known if the formal Frobenius manifolds constructed from a dGBV algebra can be identified with the formal Frobenius manifold structure on the de Rham cohomology of a symplectic manifold M, then M must have torsion c_1. Nevertheless, it is reasonable to conjecture that for a Calabi-Yau manifold M, the Frobenius manifold structure we construct on $H^{*,*}(M)$ can be identified with that from quantum cohomology. If this is true, it should have some applications in enumerative geometry of Calabi-Yau manifolds.

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1. Frobenius algebras and Frobenius manifolds.

In this section, we review the definition of (formal) Frobenius manifold. We will only be concerned with a special case which comes from the consideration of deformations of Frobenius algebras. For the general case, see Dubrovin [12] and Manin [23].

Let H be a finite dimensional commutative associative algebra with 1 over $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$ with multiplication denoted by $\cdot$. Take a basis $\{e_a\}$ of H such that $e_0 = 1$. Then there are constants $\phi_{ab}^c \in \mathbf{k}$ such that

$$e_a \cdot e_b = \phi_{ab}^c e_c.$$

An inner product on H is a nondegenerate quadratic form

$$(\cdot, \cdot) : H \times H \to \mathbf{k}.$$

It is called invariant if

$$(a \cdot b, c) = (a, b \cdot c), \tag{1}$$

for any $a, b, c \in H$. A commutative associative algebra with 1 together with an invariant inner product is called a *Frobenius algebra.* A simple observation is that the structure of a Frobenius algebra H can be encoded in a cubic polynomial $\phi : H \to \mathbf{k}$ as follows. Let $\eta_{ab} = (e_a, e_b)$ and (η^{ab}) be the inverse matrix of (η_{ab}). Set $\phi_{abc} = \phi^p_{ab}\eta_{pc}$. Then

$$\phi_{abc} = (e_a \cdot e_b, e_c).$$

Hence (1) implies that ϕ is symmetric in the three indices. One can recover the inner product and multiplication by

$$\eta_{ab} = \phi_{0ab}, \qquad \phi^c_{ab} = \phi_{abp}\eta^{pc}.$$

The associativity of the multiplication is equivalent to the following system of equations

$$\phi_{abp}\eta^{pq}\phi_{qcd} = \phi_{bcp}\eta^{pq}\phi_{aqd}. \tag{2}$$

Consider a smooth (analytic) family of associativity multiplications $\{\cdot_\alpha : \alpha \in H\}$ such that $\cdot_0 = \cdot$. Then we get a family of structure constants $\phi_{abc}(\alpha)$ such that $\phi_{abc}(0) = \phi_{abc}$. Denote by $\{x^a\}$ the linear coordinates in the basis $\{e_a\}$. If for any $\alpha \in H$,

$$\phi_{abc,d}(\alpha) = \frac{\partial}{\partial x^d}\phi_{abc}(\alpha)$$

is symmetric in all four indices, then there is a function $\Phi : H \to \mathbf{k}$, such that

$$\phi_{abc}(\alpha) = \frac{\partial^3 \Phi}{\partial x^a \partial x^b \partial x^c}(\alpha).$$

The function Φ is called the *potential function* of the family $(H, \{\cdot_\alpha\})$ which is then called a *potential family.* It is clear that

$$\left(\frac{\partial^3 \Phi}{\partial x^0 \partial x^a \partial x^b}(\alpha)\right) = (\eta_{ab}) \tag{3}$$

is a constant symmetric nondegenerate matrix if 1 is an identity for all α. Furthermore, from (2), we know that the associativity of the multiplications $\cdot_\alpha$ implies that Φ satisfies the WDVV equations:

$$\frac{\partial^3 \Phi}{\partial x^a \partial x^b \partial x^p}\eta^{pq}\frac{\partial^3 \Phi}{\partial x^q \partial x^c \partial x^d} = \frac{\partial^3 \Phi}{\partial x^b \partial x^c \partial x^p}\eta^{pq}\frac{\partial^3 \Phi}{\partial x^a \partial x^q \partial x^d}. \tag{4}$$

Conversely, given a Frobenius algebra $(H, \cdot, (\cdot,\cdot))$, a function Φ satisfying (3), (4) and such that

$$\phi_{abc} = \frac{\partial^3 \Phi}{\partial x^a \partial x^b \partial x^c}(0),$$

we have a potential family of Frobenius algebra structures on H with the fixed inner product. This gives an example of a Frobenius manifold structure on H. If Φ is just a formal power series with the above properties, then it yield a formal Frobenius manifold structure.

Remark 1.1. One can also consider $\mathbb{Z}_2$-graded version of the above discussion. For details, cf. Manin [23, 24].

2. A construction of formal Frobenius (super)manifolds.

In this section, we review a construction of formal Frobenius supermanifolds. For details, the reader should consult the papers by Tian [31], Todorov [32], Bershadsky-Ceccoti-Ooguri-Vafa [3], Barannikov-Kontsevich [2] and Manin [24]. Here, we follow the formulation by Manin [24].

Let $(\mathcal{A}, \wedge)$ be a supercommutative associative algebra with identity over a field $\mathbf{k}$, i.e., $(\mathcal{A}, \wedge)$ is an algebra with identity over $\mathbf{k}$, furthermore, $\mathcal{A}$ has $\mathbb{Z}_2$-grading, such that for any homogeneous elements $a, b \in \mathcal{A}$ with degrees $|a|$, $|b|$ respectively, we have

$$a \wedge b = (-1)^{|a|\cdot|b|} b \wedge a.$$

Assume that there is a $\mathbf{k}$-linear map δ of odd degree on $\mathcal{A}$, such that $\delta^2 = 0$, and δ is a derivation: $\delta(a \wedge b) = (\delta a) \wedge b + (-1)^{|a|} a \wedge (\delta b)$. Assume that the cohomology $H = H(\mathcal{A}, \delta)$ is finite dimensional. Then there is an induced multiplication $\wedge$ on H. We will be interested in the deformation of this multiplication obtained by deforming the operator δ.

For this idea to work, one needs the notion of a *differential Gerstenhaber-Batalin-Vilkovisky (dGBV) algebra* (see e.g. [24], §5). More precisely, one needs another $\mathbf{k}$-linear map Δ of odd degree on $\mathcal{A}$, such that

- $\delta\Delta + \Delta\delta = 0$, $\Delta^2 = 0$.
- If $[a \bullet b] = (-1)^{|a|}(\Delta(a \wedge b) - \Delta a \wedge b - (-1)^{|a|} a \wedge \Delta b)$, then $[a \bullet \cdot] : \mathcal{A} \to \mathcal{A}$ is a derivation of degree $|a| + 1$, i.e.,

$$[a \bullet (b \wedge c)] = [a \bullet b] \wedge c + (-1)^{(|a|+1)|b|} b \wedge [a \bullet c].$$

The quintuple $(\mathcal{A}, \wedge, \delta, \Delta, [\cdot \bullet \cdot])$ with the above properties is called a dGBV algebra. As in Manin [24], §5, we have

$$\begin{aligned} [a \bullet b] &= -(-1)^{(|a|+1)(|b|+1)}[b \bullet a], \\ [a \bullet [b \bullet c]] &= [[a \bullet b] \bullet c] + (-1)^{(|a|+1)(|b|+1)}[b \bullet [a \bullet c]], \\ \Delta[a \bullet b] &= [\Delta a \bullet b] + (-1)^{|a|+1}[a \bullet \Delta b], \\ \delta[a \bullet b] &= [\delta a \bullet b] + (-1)^{|a|+1}[a \bullet \delta b]. \end{aligned}$$

See also Koszul [18].

An *integral* on a dGBV algebra $\mathcal{A}$ is an even linear functional $\int : \mathcal{A} \to \mathbf{k}$, such that

$$\int (\delta a) \wedge b = (-1)^{|a|+1} \int a \wedge (\delta b), \tag{5}$$

$$\int (\Delta a) \wedge b = (-1)^{|a|} \int a \wedge (\Delta b), \tag{6}$$

for any homogeneous $a, b \in \mathcal{A}$. It follows from (5) that $\int$ induces a well-defined $\mathbf{k}$-bilinear functional

$$\begin{aligned} &(\cdot, \cdot) : H(\mathcal{A}, \delta) \otimes H(\mathcal{A}, \delta) \to \mathbf{k}, \\ &([\alpha], [\beta]) = \int \alpha \wedge \beta, \end{aligned}$$

where $[\alpha]$ and $[\beta]$ are cohomology classes represented by δ-closed α and β respectively. If $(\cdot, \cdot)$ is nondegenerate, we say that the integral is *nice*. It is obvious that

$$([\alpha] \wedge [\beta], [\gamma]) = ([\alpha], [\beta] \wedge [\gamma]).$$

By definition, $(H(\mathcal{A}, d), \wedge, (\cdot, \cdot))$ is then a *Frobenius algebra*, when $\mathcal{A}$ has a nice integral.

To find a deformation of the ring structure on H, consider $\delta_a = \delta + [a \bullet \cdot] : \mathcal{A} \to \mathcal{A}$ for even $a \in \mathcal{A}$. When a satisfies

$$\begin{aligned} &\delta a + \frac{1}{2}[a \bullet a] = 0, \\ &\Delta a = 0, \end{aligned} \tag{7}$$

then $(\mathcal{A}, \wedge, \delta_a, \Delta, [\cdot \bullet \cdot])$ is also a dGBV algebra. Such deformations appeared also in e.g. Akman [1], §5.3. If $\int$ is an integral for $(\mathcal{A}, \wedge, \delta, \Delta, [\cdot \bullet \cdot])$, so is it for $(\mathcal{A}, \wedge, \delta_a, \Delta, [\cdot \bullet \cdot])$. If there is a natural way to identify $H(\mathcal{A}, \delta_a)$ with $H = H(\mathcal{A}, \delta)$ which preserves $(\cdot, \cdot)$, then we get another Frobenius algebra structure on H. The construction of Frobenius manifold structure is based on the existence of a solution $\Gamma = \sum \Gamma_n$ to (7), such that $\Gamma_0 = 0$, $\Gamma_1 = \sum x^j e_j$, $\{e_j \in \operatorname{Ker} \delta \cap \operatorname{Ker} \Delta\}$ induce a basis of H. For $n > 1$, $\Gamma_n \in \operatorname{Im} \Delta$ is a homogeneous super polynomial of degree n in x^j's, such that the total degree of Γ_n is even. Furthermore, x^0 only appears in Γ_1. Such a solution is called a *normalized universal solution.* Its existence can be established inductively. This is how Tian [30] and Todorov [32] proved that the deformation of complex structures on a Calabi-Yau manifold is unobstructed. This was generalized by Bershadsky *et al* [3] to the case of extended moduli space of complex structures of a Calabi-Yau manifold. An interesting observation in [3] is that the action functional of the Kodaira-Spencer theory of gravity provides the potential function for a formal structure on $H^{-*,*}(x)$. As mentioned earlier, Barannikiv and Kontesevich [2] gave a mathematical treatment of this observation. They also remarked that this construction can be carried out for dGBV algebra with suitable condition. See Manin [24] for a detailed account. The result can be stated as follows:

Theorem 2.1. *Let* $(\mathcal{A}, \wedge, \delta, \Delta, [\cdot \bullet \cdot])$ *be a dGBV algebra which satisfies the following conditions:*

1. $H = H(\mathcal{A}, \delta)$ *is finite dimensional.*

2. *There is a nice integral on* $\mathcal{A}$.

3. *The inclusions* $i : (\operatorname{Ker} \Delta, \delta) \hookrightarrow (\mathcal{A}, \delta)$ *and* $j : (\operatorname{Ker} \delta, \Delta) \hookrightarrow (\mathcal{A}, \Delta)$ *induce isomorphisms on cohomology.*

Then there is a structure of formal Frobenius manifold on the formal spectrum of $\mathbf{k}[[H']]$*, the algebra of formal power series generated by* H'*, where* H' *is the dual* $\mathbf{k}$*-vector space of* H.

3. Frobenius manifold structures on Dolbeault cohomology.

In this section, we show that there is a natural dGBV algebra structure on the Dolbeault complex of a Kähler manifold. Furthermore, it is shown that all the conditions in Theorem 2.1 are satisfied.

Let (X, g, J) be a closed Kähler manifold with Kähler form ω. Consider the quadruple $(\Omega^{*,*}(X), \wedge, \delta = \bar\partial, \Delta = \partial^*)$. It is well-known that $\bar\partial^2 = 0$, $(\partial^*)^2 = 0$, and $\bar\partial\partial^* + \partial^*\bar\partial = 0$. Furthermore, $\bar\partial$ is a derivation. Set

$$[a \bullet b]_{\partial^*} = (-1)^{|a|}(\partial^*(a \wedge b) - \partial^* a \wedge b - (-1)^{|a|} a \wedge \partial^* b).$$

A formula of this type in complex geometry was discovered by Tian [30] to prove the important result that deformations of Calabi-Yau manifolds are unobstructed (cf. Todorov [32] and Bershadsky *et al* [3]). In Tian's situation, he used the holomorphic volume form on a Calabi-Yau manifold to get an operator on $\Omega^{-*,*}(M)$ from ∂ on $\Omega^{*,*}(M)$ and showed that the corresponding bracket can be identified with the Schouten-Nijenhuis bracket. Such a method of defining bracket also appeared in Koszul's work on Poisson geometry [18]. Conjugation of the exterior differential d by the isomorphism induced by a volume form was used in Witten [38] and Schwarz [29] to construct BV algebras.

Since both ω and J are parallel with respect to the Levi-Civita connection, it follows that near each $x \in X$, one can find a local frame $\{e_1, \cdots, e_n\}$ of $T^{1,0}X$ with dual frame $\{e^1, \cdots, e^n\}$, such that $\omega = e^1 \wedge \bar{e}^1 + \cdots + e^n \wedge \bar{e}^n$. Furthermore, $\nabla_{e_j} e_k = \nabla_{\bar{e}_j} e_k = \nabla_{e_j} \bar{e}_k = \nabla_{\bar{e}_j} \bar{e}_k = 0$ at x. Then at x, for any $\alpha \in \Omega^{*,*}(X)$, we have (Griffiths-Harris [15], p. 113):

$$\begin{aligned} \partial\alpha &= e^j \wedge \nabla_{e_j}\alpha, & \bar{\partial}\alpha &= \bar{e}^j \wedge \nabla_{\bar{e}_j}\alpha, \\ \partial^*\alpha &= -e_j \vdash \nabla_{\bar{e}_j}\alpha, & \bar{\partial}^*\alpha &= -\bar{e}_j \vdash \nabla_{e_j}\alpha, \end{aligned}$$

Lemma 3.1. $[a \bullet (b \wedge c)]_{\partial^*} = [a \bullet b]_{\partial^*} \wedge c + (-1)^{(|a|+1)|b|} b \wedge [a \bullet c]_{\partial^*}$.

Proof. We first express $[a \bullet b]_{\partial^*}$ in terms of the Levi-Civita connection:

$$\begin{aligned} [a \bullet b]_{\partial^*} &= (-1)^{|a|}(\partial^*(a \wedge b) - (\partial^* a) \wedge b - (-1)^{|a|} a \wedge \partial^* b) \\ &= (-1)^{|a|}(\partial^*(a \wedge b) - (\partial^* a) \wedge b - (-1)^{|a|} a \wedge \partial^* b) \\ &= (-1)^{|a|}(-(e_j \vdash \nabla_{\bar{e}_j}(a \wedge b)) + (e_j \vdash \nabla_{\bar{e}_j} a) \wedge b \\ &\quad + (-1)^{|a|} a \wedge (e_j \vdash \nabla_{\bar{e}_j} b)) \\ &= -(-1)^{|a|} e_j \vdash (\nabla_{\bar{e}_j} a \wedge b + a \wedge \nabla_{\bar{e}_j} b) + (-1)^{|a|}(e_j \vdash \nabla_{\bar{e}_j} a) \wedge b \\ &\quad + a \wedge (e_j \vdash \nabla_{\bar{e}_j} b) \\ &= -\nabla_{\bar{e}_j} a \wedge (e_j \vdash b) - (-1)^{|a|}(e_j \vdash a) \wedge \nabla_{\bar{e}_j} b. \end{aligned}$$

The left hand side is then given by

$$[a \bullet (b \wedge c)]_{\partial^*} = -\nabla_{\bar{e}_j} a \wedge e_j \vdash (b \wedge c) - (-1)^{|a|}(e_j \vdash a) \wedge \nabla_{\bar{e}_j}(b \wedge c).$$

The right hand side is computed in a similar fashion:

$$\begin{aligned}
&[a \bullet b]_{\partial^*} \wedge c + (-1)^{(|a|+1)|b|} b \wedge [a \bullet c]_{\partial^*} \\
&\quad = (-\nabla_{\bar{e}_j} a \wedge (e_j \vdash b) - (-1)^{|a|} (e_j \vdash a) \wedge \nabla_{\bar{e}_j} b) \wedge c \\
&\qquad + (-1)^{(|a|+1)|b|} b \wedge (-\nabla_{\bar{e}_j} a \wedge (e_j \vdash c) - (-1)^{|a|} (e_j \vdash a) \wedge \nabla_{\bar{e}_j} c) \\
&\quad = -\nabla_{\bar{e}_j} a \wedge (e_j \vdash b) \wedge c - (-1)^{|a|} (e_j \vdash a) \wedge \nabla_{\bar{e}_j} b \wedge c \\
&\qquad - (-1)^{|b|} \nabla_{\bar{e}_j} a \wedge b \wedge (e_j \vdash c) - (-1)^{|a|} (e_j \vdash a) \wedge b \wedge \nabla_{\bar{e}_j} c \\
&\quad = -\nabla_{\bar{e}_j} a \wedge e_j \vdash (b \wedge c) - (-1)^{|a|} (e_j \vdash a) \wedge \nabla_{\bar{e}_j} (b \wedge c).
\end{aligned}$$

This completes the proof. □

Corollary 3.1. *For any Kähler manifold (X, g), $(\Omega^{*,*}(X), \wedge, \delta = \bar{\partial}, \Delta = \partial^*)$ is a dGBV algebra.*

Remark 3.1. It is actually more natural to take

$$\Delta = -\sqrt{-1}\,\partial^*.$$

Let $\int_X : \Omega^{*,*}(X) \to \mathbb{C}$ be the ordinary integration of differential forms. It is easy to show the following

Lemma 3.2. *The following identity holds on closed Kähler manifolds:*

$$\int_X \bar{\partial} a \wedge b = (-1)^{|a|+1} \int_X a \wedge \bar{\partial} b,$$
$$\int_X \partial^* a \wedge b = (-1)^{|a|} \int_X a \wedge \partial^* b.$$

Theorem 3.1. *For any closed Kähler manifold X, if K is the algebra of formal power series generated by the dual space of the Dolbeault cohomology $H^*_{\bar{\partial}}(X)$, then there is a structure of formal Frobenius manifold on the formal spectrum of K obtained from the dGBV algebra $(\Omega^{*,*}(X), \wedge, \delta = \bar{\partial}, \Delta = \partial^*)$.*

Proof. By the method of Deligne-Griffiths-Morgan-Sullivan [9], the two natural inclusions $i : (\mathrm{Ker}\, \partial^*, \bar{\partial}) \to (\Omega^{*,*}(X), \bar{\partial})$ and $j : (\mathrm{Ker}\, \bar{\partial}, \partial^*) \to (\Omega^{*,*}(X), \partial^*)$ induce isomorphisms on cohomology. Then the theorem follows from Lemma 3.2 and Theorem 2.1. □

Remark 3.2. The same construction carries through if we take $\delta = \partial$, $\Delta = \bar{\partial}^*$.

4. Comparison with Barannikov-Kontsevich's Frobenius manifold.

In this section, we compare our construction with that of Barannikov-Kontsevich [2], and make some speculations based on their similarity.

Assume that X is a closed Calabi-Yau n-manifold; fix a nowhere vanishing holomorphic n-form $\Omega \in \Gamma(X, \Lambda^n T^* X)$. Barannikov-Kontsevich [2] considered the dGBV algebra with

$$\mathcal{B} = \oplus \mathcal{B}_k, \qquad \mathcal{B}_k = \oplus_{q+p=k} \Gamma(X, \Lambda^p TX \otimes \Lambda^q \overline{T}^* X).$$

Here $\mathcal{B}$ is $\mathbb{Z}$-graded. It also has an induced $\mathbb{Z}_2$-grading. This grading is different from that in [2] (shifted by 1); we adopt this grading to be compatible with the notations of Manin [24]. The multiplication $\wedge$ on $\mathcal{B}$ is given by the ordinary wedge products on $\Lambda^* \overline{T}^* X$ and $\Lambda^* TX$. The derivation is $\delta = \bar{\partial}$. Notice that for any two integers $0 \leq p, q \leq n$, Ω defines an isomorphism

$$\Lambda^p TX \otimes \Lambda^q \overline{T}^* X \to \Lambda^{n-p} T^* X \otimes \Lambda^q \overline{T}^* X, \qquad \gamma \mapsto \gamma \vdash \Omega,$$

defined by the contraction of the p-vector with Ω to get a form of type $(n-p, 0)$. Then Δ is defined by

$$(\Delta \gamma) \vdash \Omega = \partial(\gamma \vdash \Omega).$$

Tian's formula shows that the bracket, defined by

$$[\alpha \bullet \beta] = (-1)^{|\alpha|}(\Delta(\alpha \wedge \beta) - \Delta\alpha \wedge \beta - (-1)^{|\alpha|} \alpha \wedge \Delta\beta),$$

is given by the wedge product on type $(0, *)$-forms and the Schouten-Nijenhuis bracket on type $(*, 0)$ polyvector fields. Hence $(\mathcal{B}, \wedge, \delta, \Delta)$ as above is a dGBV algebra. Furthermore, the linear functional

$$\int : \mathcal{B} \to \mathbb{C}, \qquad \int \gamma = \int_X (\gamma \vdash \Omega) \wedge \Omega$$

is a nice integral on X.

Since the complex $(\mathcal{B}, \delta)$ contains the deformation complex

$$\Omega^{0,0}(TX) \xrightarrow{\bar{\partial}} \Omega^{0,1}(TX) \xrightarrow{\bar{\partial}} \Omega^{0,2}(TX),$$

it is called the extended deformation complex. Barannikov-Kontsevich [2] (Lemma 2.1) showed that the deformation functor associated with the graded differential Lie algebra $(\mathcal{B}[-1], \delta, [\cdot \bullet \cdot])$ is represented by the formal spectrum of formal power series generated by H', the dual of $H = H(\mathcal{B}, \delta)$. (This is the extended moduli space of complex structures in Witten [38].) Their method is that used in [31], [32] and [3].

There are remarkable similarities between the dGBV algebras used by us and in [2]. One can also express Δ on $\mathcal{B}$ in terms of Levi-Civita connection:

$$\Delta = e^j \vdash \nabla_j.$$

We have the following interesting comparison of the two dGBV algebras:

$$\text{(BK)}\quad \mathcal{B} = \oplus_{p,q}\Gamma(X, \Lambda^p TX \otimes \Lambda^q \overline{T}^* X), \qquad \delta = \bar{\partial}, \qquad \Delta = e^j \vdash \nabla_j,$$

$$\text{(CZ)}\quad \mathcal{A} = \oplus_{p,q}\Gamma(X, \Lambda^p T^* X \otimes \Lambda^q \overline{T}^* X), \qquad \delta = \bar{\partial}, \quad \Delta = \partial^* = -e_j \vdash \nabla_j,$$

(The multiplications on both algebras are given by the exterior products!) It is reasonable to conjecture that the two corresponding Frobenius manifolds are isomorphic to each other for a pair of mirror manifolds.

References.

[1] F. Akman, *On some generalizations of Batalin-Vilkovisky algebras*, J. Pure Appl. Algebra, **120** (1997), no. 2, 105–141.

[2] S. Barannikov, M. Kontsevich, *Frobenius Manifolds and Formality of Lie Algebras of Polyvector Fields*, Internat. Math. Res. Notices 1998, no. 4, 201-215, alg-geom/9710032.

[3] M. Bershadsky, S. Ceccoti, H. Ooguri, C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes* Comm. Math. Phys. **165** (1994), no. 2, 311–427.

[4] J. Bertin, C. Peters, *Variations de structures de Hodge, variétés de Calabi-Yau et symétrie miroir*, in *Introduction à la théorie de Hodge*, 169–256, Panor. Synthèses, 3, Soc. Math. France, Paris, 1996.

[5] J.-L. Brylinski, *A differential complex for Poisson manifolds*, J. Differential Geom. **28** (1988), no. 1, 93–114.

[6] P. Candelas, X. C. de la Ossa, p. S. Green, L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nuclear Phys. B, **359** (1991), no. 1, 21–74.

[7] H.-D. Cao, J. Zhou, *On Quantum de Rham Cohomology Theory*, preprint, April, 1998.

[8] H.-D. Cao, J. Zhou, *Identification of Two Frobenius Manifolds*, Math. Res. Lett. 6 (1999), no. 1, 17-29.

[9] P. Deligne, P. Griffiths, J. Morgan, D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), no. 3, 245–274.

[10] R. Dijkgraaf, H. Verlinde, E. Verlinde, *Notes on topological string theory and 2D quantum gravity*, in *String theory and quantum gravity* (Trieste, 1990), 91–156, World Sci. Publishing, River Edge, NJ, 1991.

[11] B. Dubrovin, *Integrable systems in topological field theory*, Nuclear Phys. B, **379** (1992), no. 3, 627–689.

[12] B. Dubrovin, *Geometry of 2D topological field theories*, in *Integrable systems and quantum groups* (Montecatini Terme, 1993), 120–348, Lecture Notes in Math., 1620, Springer, Berlin, 1996.

[13] M. Gerstenhaber, S.D. Schack, *Algebraic cohomology and deformation theory*, in *Deformation theory of algebras and structures and applications (Il Ciocco, 1986)*, 11–264, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., **247**, Kluwer Acad. Publ., Dordrecht, 1988.

[14] W.M. Goldman, J.J. Millson, *The deformation theory of representations of fundamental groups of compact Khler manifolds*, Inst. Hautes tudes Sci. Publ. Math. **67** (1988), 43–96.

[15] P. Griffiths, J. Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York, 1978.

[16] M. Kontsevich, *Homological algebra of mirror symmetry*, in *Proceedings of the International Congress of Mathematicians*, Vol. 1, 2 (Zrich, 1994), 120–139, Birkhuser, Basel, 1995.

[17] M. Kontsevich, *Deformation quantization of Poisson manifolds, I*, preprint, q-alg/9709040.

[18] J.-L. Koszul, *Crochet de Schouten-Nijenhuis et cohomologie*, *The mathematical heritage of Élie Cartan (Lyon, 1984)*, Astrisque 1985, Numero Hors Serie, 257–271.

[19] J. Li, G. Tian, *Comparison of the algebraic and the symplectic Gromov-Witten invariants*, preprint, alg-geom/9712035.

[20] B. Lian, K. Liu, S.T. Yau, *Mirror principle I*, Asian J. Math 1 (1997), no. 4, 729-763, alg-geom/9712011.

[21] G. Liu, *Associativity of quantum multiplication*, Comm. Math. Phys. 191 (1998), no. 2, 265-282.

[22] D. McDuff, D. Salamon, *J-holomorphic curves and quantum cohomology*, University Lecture Series, **6**, American Mathematical Society, Providence, RI, 1994.

[23] Yu. Manin, *Frobenius manifolds, quantum cohomology, and moduli spaces*, AMS Colloquium Publications, 47, AMS, Providence, RI, 1999.

[24] Y. Manin, *Three constructions of Frobenius manifolds: a comparative study*, preprint, math.QA/9801006.

[25] D.R. Morrison, *Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians*, J. Amer. Math. Soc. **6** (1993), no. 1, 223–247.

[26] J. Morrow, K. Kodaira **Complex manifolds**, Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London, 1971.

[27] Z. Ran, *Thickening Calabi-Yau moduli spaces*, in **Mirror symmetry, II**, 393–400, AMS/IP Stud. Adv. Math., 1, Amer. Math. Soc., Providence, RI, 1997.

[28] Y. Ruan, G. Tian, *A mathematical theory of quantum cohomology*, J. Differential Geom. **42** (1995), no. 2, 259–367.

[29] A. Schwarz, *Geometry of Batalin-Vilkovisky quantization*, Comm. Math. Phys. **155** (1993), no. 2, 249–260.

[30] G. Tian, *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric*, in *Mathematical aspects of string theory* (San Diego, Calif., 1986), 629–646, Adv. Ser. Math. Phys., 1, World Sci. Publishing, Singapore, 1987.

[31] G. Tian, *Quantum cohomology and its associativity*, in *Current developments in mathematics*, 1995 (Cambridge, MA), 361–401, Internat. Press, Cambridge, MA, 1994.

[32] A.N. Todorov, *The Weil-Petersson geometry of the moduli space of* $\mathrm{SU}(n \geq 3)$ *(Calabi-Yau) manifolds. I.*, Comm. Math. Phys. **126** (1989), no. 2, 325–346.

[33] C. Vafa, *Topological mirrors and quantum rings*, in *Essays on mirror manifolds*, S.T. Yau ed., 96–119, Internat. Press, Hong Kong, 1992.

[34] I. Vaisman, *Lectures on the geometry of Poisson manifolds*. Progress in Mathematics, **118**, Birkhuser Verlag, Basel, 1994.

[35] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, in *Surveys in differential geometry* (Cambridge, MA, 1990), 243–310, Lehigh Univ., Bethlehem, PA, 1991.

[36] E. Witten, *Mirror manifolds and topological field theory*, in *Essays on mirror manifolds*, S. T. Yau ed., International Press Co., Hong Kong, 1992.

[37] S.T. Yau ed., *Essays on mirror manifolds*, International Press, Hong Kong, 1992.

[38] E. Witten, *A note on antibracket formalism*, Modern Phys. Lett. A 5 (1990), no. 7, 487-494.

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