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The geometry and topology of toric hyperkähler manifolds

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We study hyperkähler manifolds that can be obtained as hyperkähler quotients of flat quaternionic space by tori, and in particular, their relation to toric varieties and Delzant polytopes. When smooth, these hyperkähler quotients are complete. We also show that for smooth projective toric varieties X the cotangent bundle of X carries a hyperkähler metric, which is complete only if X is a product of projective spaces. Our hyperkähler manifolds have the homotopy type of a union of compact toric varieties intersecting along toric subvarieties. We give explicit formulas for the hyperkähler metric and its Kähler potential.

1. Introduction.

A $4n$-dimensional manifold is hyperkähler if it possesses a Riemannian metric g which is Kähler with respect to three complex structures J_1, J_2, J_3 satisfying the quaternionic relations $J_1J_2 = -J_2J_1 = J_3$ etc. To date the most powerful technique for constructing such manifolds is the hyperkähler quotient method of Hitchin, Karlhede, Lindström and Roček [HKLR]. The power of this method lies in the fact that a flat hyperkähler space may have highly nontrivial quotients.

In this paper we shall make a detailed study of a class of hyperkähler quotients of flat quaternionic space $\mathbb{H}^d$ by subtori of T^d. The geometry of these spaces turns out to be closely connected with the theory of *toric varieties*, that is, varieties of complex dimension n admitting an action of $(\mathbb{C}^*)^n$ with an open dense orbit. The toric varieties we shall be concerned with have a Kähler metric preserved by the action of $T^n \leq (\mathbb{C}^*)^n$.

If $4n$ is the dimension of our hyperkähler quotient there is an isometric action of T^n which is holomorphic with respect to all the complex structures. We shall refer to our manifolds as *toric hyperkähler manifolds* (cf. [Go]).

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We shall study various topological and metric properties of toric hyperkähler manifolds. First we give necessary and sufficient conditions for a hyperkähler quotient M of quaternionic space by our torus actions to be smooth (Theorem 3.2) or an orbifold (Theorem 3.3). When smooth, M is complete as a Riemannian manifold. We show that the hyperkähler moment map ϕ for the induced torus action on M is a surjection onto $\mathbb{R}^{3n}$ with connected fibers. This can be viewed as an analogue of the convexity theorem for compact toric varieties. We also explain how to read off the singular orbits and fixed points of the T^n action (Theorem 3.1).

Our discussion is influenced by the work of Delzant [De] and Guillemin [Gu1],[Gu2], who have shown that a large class of toric varieties can be obtained as Kähler quotients of $\mathbb{C}^d$ by subtori of T^d. A guiding principle of our work is that, while a compact Kähler toric variety is determined by a convex polytope, a complete toric hyperkähler orbifold is determined by an *arrangement of affine subspaces.*

In section 4 we discuss how the existence of a large family of compact 3-Sasakian manifolds found by Boyer, Galicki and Mann [BGM 1],[BGMR] can be read off from our results.

In section 5 we show that the generic complex structure of a toric hyperkähler orbifold is that of an affine variety (Theorem 5.1). In section 6 we discuss the topology of toric hyperkähler orbifolds M, and show that it depends only the torus used to obtain M and not on the moment map (Theorem 6.1). We identify the homotopy type of our orbifolds as that of a union of finitely many toric varieties intersecting along toric subvarieties (Theorem 6.5). We also give a combinatorial formula for the Betti numbers of toric hyperkähler orbifolds (Theorem 6.7).

If X is a toric variety arising from Delzant's construction, we show in section 7 that the cotangent bundle T^*X carries a natural hyperkähler metric whose restriction to the zero section is the Kähler metric on X. This hyperkähler metric is complete only when X is a product of projective spaces. We also discuss when the metric on T^*X can be smoothly completed.

The last two sections deal with the Kähler geometry of our manifolds. We give an explicit formula for the Kähler form (Theorem 8.3), generalizing the formula of Guillemin [Gu1] for compact toric varieties. We also give an explicit description of the Riemannian metric (Theorem 9.1), which corresponds to finding a solution of generalized Bogomolny equations of Pedersen and Poon [PP].

We refer the reader to [HKLR] for a thorough discussion of Kähler and hyperkähler quotients. Let us remark here that a particular class of our manifolds was studied by Goto [Go] (see Remark 3.6). Even for this class our point of view is different from Goto's as we particularly stress the relation with algebraic toric varieties. There is also some relation with the work of Nakajima [Na].

2. Toric varieties.

In this section we shall give a quick overview of Kähler quotients of $\mathbb{C}^d$ by tori and in particular of Delzant's construction of certain toric varieties from polytopes [De]. We follow the exposition of Guillemin [Gu1],[Gu2].

The real torus $T^d = \{(t_1, \dots, t_d) \in \mathbb{C}^d : |t_i| = 1\}$ acts diagonally on $\mathbb{C}^d$ preserving the flat Kähler metric whose Kähler form is

$$\frac{\sqrt{-1}}{2} \sum_{k=1}^{d} dz_k \wedge d\bar{z}_k. \tag{2.1}$$

The moment map for this action is

$$\mu(z) = \frac{1}{2} \sum_{k=1}^{d} |z_k|^2 e_k + c, \tag{2.2}$$

where the e_i are the standard basis vectors of $\mathbb{R}^d$ and c is an arbitrary constant in $\mathbb{R}^d$. If N is a subtorus of T^d whose Lie algebra $\mathfrak{n} \subset \mathbb{R}^d$ is generated by rational vectors, then we can perform the Kähler quotient construction with respect to N. Such a subtorus is determined by a collection of nonzero integer vectors $\{u_1, \dots, u_d\}$ (which we shall always take to be primitive) generating $\mathbb{R}^n$. For then we obtain exact sequences of vector spaces

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\imath} \mathbb{R}^d \xrightarrow{\beta} \mathbb{R}^n \longrightarrow 0, \tag{2.3}$$

$$0 \longrightarrow \mathbb{R}^n \xrightarrow{\beta^*} \mathbb{R}^d \xrightarrow{\imath^*} \mathfrak{n}^* \longrightarrow 0, \tag{2.4}$$

where the map β sends e_i to u_i. There is a corresponding exact sequence of groups

$$1 \to N \to T^d \to T^n \to 1. \tag{2.5}$$

In order to obtain a smooth Kähler quotient one has to make certain assumptions on N. We will not discuss these in full generality (but see below for the case when the u_i come from a polytope). In the next section we shall give necessary and sufficient conditions for the corresponding hyperkähler quotient to be smooth.

The torus N acts on $\mathbb{C}^d$ preserving the Kähler form (2.1), and the moment map for N is, from (2.2),

$$\mu(z) = \frac{1}{2}\sum_{k=1}^{d} |z_k|^2 \alpha_k + c, \tag{2.6}$$

where $\alpha_k = \imath^*(e_k)$. The constant c is of the form

$$c = \sum_{k=1}^{d} \lambda_k \alpha_k, \tag{2.7}$$

for some scalars $\lambda_1, \dots, \lambda_k \in \mathbb{R}$. If 0 is a regular value of the moment map (2.6), we obtain a smooth Kähler quotient $X = \mu^{-1}(0)/N$ which is a toric variety. The torus $T^n = T^d/N$ of (2.5) acts on X and gives rise to a moment map $\phi : X \to \mathbb{R}^n$. If X is compact, the image of this map is a convex polytope Δ called the *Delzant polytope* of X. (Note that its vertices are not required to lie on an integer lattice and in this respect the Delzant polytope differs from the Newton polytope of algebraic toric varieties).

Conversely, any smooth compact toric variety X of complex dimension n, with a Kähler metric invariant under $T^n \leq (\mathbb{C}^*)^n$, comes from Delzant's construction. For the T^n action induces a moment map as above, whose image is a convex polytope Δ in $\mathbb{R}^n$. The smoothness of X corresponds to the properties that precisely n edges meet at each vertex of Δ (that is, Δ is *simple*), and that the directions of these n edges are given by a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. The polytope Δ is defined by a system of inequalities of the form

$$\langle x, u_i \rangle \geq \lambda_i, \qquad (i = 1, \dots, d), \tag{2.8}$$

where u_i is the inward-pointing normal vector to the i-th $(n-1)$-dimensional face of Δ. Now X is produced by the Kähler quotient construction described above, where the vectors u_i and the scalars λ_i are those in (2.8).

The Kähler quotient $X = \mu^{-1}(0)/N$ can be identified as follows with the quotient of an open subset of $\mathbb{C}^d$ by the complexified torus $N^{\mathbb{C}}$. Every orbit in $\mathbb{C}^d$ of $(T^d)^{\mathbb{C}}$ is of the form

$$\mathbb{C}^d_I = \{(z_1, \dots, z_d) \, : \, z_i = 0 \ \text{ iff } \ i \notin I\} \tag{2.9}$$

for some multi-index $I = (i_1, \dots, i_r)$, $1 \leq i_1 < \dots < i_r \leq d$ (we allow $r = 0$). If F is a face of Δ of codimension r, then F is defined by replacing the inequalities of (2.8) by equalities for i belonging to the complement of some multi-index I of length $d-r$. If we let $\mathbb{C}^d_F = \mathbb{C}^d_I$ then the set

$$\mathbb{C}^d_\Delta = \bigcup_F \mathbb{C}^d_F \tag{2.10}$$

is open, and X is biholomorphic to $\mathbb{C}^d_\Delta / N^{\mathbb{C}}$. (Note that to be consistent with the notation later in this paper our definition of $\mathbb{C}^d_I$ is dual to that of Guillemin).

Example 2.1. Consider the following $n+1$ vectors in $\mathbb{R}^n$: $u_i = e_i$, $1 \leq i \leq n$, and $u_{n+1} = -(e_1 + \ldots + e_n)$. For any negative scalars $\lambda_1, \ldots, \lambda_{n+1}$, the polytope Δ defined by (2.8) is similar to the standard simplex in $\mathbb{R}^n$ (see Fig.1 for $n=2$). Here $\mathbb{C}^d_\Delta = \mathbb{C}^{n+1} - \{0\}$ and $N^{\mathbb{C}}$ is the diagonal $\mathbb{C}^*$, so X is $\mathbb{C}P^n$.

Example 2.2. In this example the vectors u_i are not determined by a polytope. We take $u_1 = -e_1$, $u_2 = u_3 = e_1$ in $\mathbb{R}$ and $\lambda_1 = -1$, $\lambda_2 = \frac{1}{2}$ and $\lambda_3 = 0$. This time $\mathfrak{n}$ is spanned by $(1,1,0)$ and $(1,0,1)$ and the zero set of the moment map (2.6) is described by the equations: $|z_1|^2 + |z_2|^2 = 1$, $-|z_2|^2 + |z_3|^2 = 1$. In this case $\left(\mathbb{C}^d\right)^{\min} = \left(\mathbb{C}^2 - \{0\}\right) \times \mathbb{C}^*$ and X is $\mathbb{C}P^1$.

If we fix $u_1, \ldots, u_d$, the resulting variety still depends on the choice of the moment map, that is, on the scalars λ_i. In particular the topology may change when we pass through a critical value of $c = \sum \lambda_k \alpha_k$. This change corresponds to a proper birational morphism of the toric varieties ([Od,Gu2]).

Example 2.3. Consider the vectors $u_1 = e_1$, $u_2 = e_2$, $u_3 = -e_1$, $u_4 = -ae_1 - e_2$ in $\mathbb{R}^2$, where a is a positive integer. Figure 2 shows the case $a=1$. For large λ_3 the polytope Δ is a trapezoid and the corresponding surface X is the Hirzebruch surface $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a))$. Moving the line orthogonal to u_3 beyond the intersection point of lines orthogonal to u_2 and u_4 corresponds to blowing down the divisor D with $D \cdot D = -a$. The blown-down surface is the weighted projective space $\mathbb{C}P^2(1,1,a)$, which is nonsingular only for $a = 1$.

A toric variety is also determined by a *fan* $\mathcal{F}$, that is, a collection of rational strongly convex polyhedral cones in $\mathbb{R}^n$ such that each face of a cone in $\mathcal{F}$ is also a cone in $\mathcal{F}$ and the intersection of two cones in $\mathcal{F}$ is a face of each [Fu]. A convex polytope Δ described by (2.8) determines a fan

$\mathcal{F}$ as follows: the cone $\{\sum_{i\in I} t_i u_i : t_i \geq 0\}$ belongs to $\mathcal{F}$ if and only if the $(n-1)$-dimensional faces of Δ corresponding to u_i, $i \in I$, meet in Δ. The passage from a polytope to the fan is equivalent to forgetting the Kähler metric of X.

3. Toric hyperkähler manifolds.

We shall now discuss hyperkähler quotients of $\mathbb{H}^d$ by subtori of T^d. The quaternionic vector space $\mathbb{H}^d$ is a flat hyperkähler manifold with complex structures J_1, J_2, J_3 given by right multiplication by i, j, k. The real torus T^d acts on $\mathbb{H}^d$ by left diagonal multiplication, preserving the hyperkähler structure. If we choose one complex structure, say J_2, and identify $\mathbb{H}^d$ with $\mathbb{C}^d \times \mathbb{C}^d$, then the action can be written as

$$t \cdot (z, w) = (t \cdot z, t^{-1} \cdot w). \tag{3.1}$$

On the other hand, taking the complex structure J_1 identifies $\mathbb{H}^d$ with $T^*\mathbb{C}^d$, with the natural torus action induced from that on $\mathbb{C}^d$.

The three moment maps μ_1, μ_2, μ_3 corresponding to the complex structures can be written as

$$\mu_1(z, w) = \frac{1}{2} \sum_{k=1}^{d} \left(|z_k|^2 - |w_k|^2\right) e_k + c_1, \tag{3.2a}$$

$$(\mu_2 + \sqrt{-1}\mu_3)(z, w) = \sum_{k=1}^{d} (z_k w_k) e_k + c_2 + \sqrt{-1} c_3, \tag{3.2b}$$

where c_1, c_2, c_3 are arbitrary constant vectors in $\mathbb{R}^d$. Notice, that unlike in the Kähler case, the hyperkähler moment map (μ_1, μ_2, μ_3) is surjective for any choice of c_1, c_2, c_3, and in fact gives a homeomorphism $\mathbb{H}^d/T^d \to \mathbb{R}^{3d}$.

Now, let u_i $(i = 1, \ldots, d)$, define a subtorus N of T^d by (2.3) and (2.5). As before we assume that the vectors u_i are integer, primitive and generate $\mathbb{R}^n$. The moment maps for the action of N are (cf. (2.6))

$$\mu_1(z, w) = \frac{1}{2} \sum_{k=1}^{d} \left(|z_k|^2 - |w_k|^2\right) \alpha_k + c_1, \tag{3.3a}$$

$$(\mu_2 + \sqrt{-1}\mu_3)(z, w) = \sum_{k=1}^{d} (z_k w_k) \alpha_k + c_2 + \sqrt{-1} c_3\,. \tag{3.3b}$$

The constants c_j are of the form

$$c_j = \sum_{k=1}^{d} \lambda_k^j \alpha_k, \qquad (j = 1, 2, 3). \tag{3.3c}$$

where $\lambda_k^j \in \mathbb{R}$. We shall adopt the notation

$$\lambda_k = (\lambda_k^1, \lambda_k^2, \lambda_k^3), \quad (k = 1, \ldots, d).$$

We shall denote the hyperkähler quotient $\mu^{-1}(0)/N$ corresponding to $\underline{u} = (u_1, \ldots, u_d)$ and $\underline{\lambda} = (\lambda_1, \ldots, \lambda_d)$ by $M(\underline{u}, \underline{\lambda})$, or sometimes just M.

It will be important to consider the hyperplanes in $\mathbb{R}^n$

$$H_k^j = \{y \in \mathbb{R}^n : \langle y, u_k \rangle = \lambda_k^j\}, \quad (j = 1, 2, 3, \ \ k = 1, \ldots, d) \tag{3.4}$$

and the codimension 3 flats (affine subspaces) in $\mathbb{R}^{3n}$

$$H_k = H_k^1 \times H_k^2 \times H_k^3. \tag{3.5}$$

It is these flats, rather than the intersection of half-spaces as for toric varieties, that determine the structure of toric hyperkähler manifolds.

The action of $T^n = T^d/N$ on $M(\underline{u}, \underline{\lambda})$ preserves the hyperkähler structure and gives rise to a hyperkähler moment map $\phi = (\phi_1, \phi_2, \phi_3)$. The following result describes its essential properties.

Theorem 3.1. *Let $u_1, \ldots, u_d \in \mathbb{Z}^n$ be primitive and span $\mathbb{R}^n$ and let $\lambda_1, \ldots, \lambda_d \in \mathbb{R}^3$. Then:*

(i) *The hyperkähler moment map $\phi : M \to \mathbb{R}^{3n}$ for the action of T^n defines a homeomorphism $M/T^n \to \mathbb{R}^{3n}$. Therefore M is connected.*

(ii) *If $x \in \mathbb{R}^{3n}$, then the T^n-stabiliser of a point in $\phi^{-1}(x)$ is the torus whose Lie algebra is spanned by the vectors u_k for which $x \in H_k$.*

Proof. We claim that (z, w) is in the zero set of (3.3) if and only if there exist $a \in \mathbb{R}^n, b \in \mathbb{C}^n$ such that

$$z_k w_k + \lambda_k^2 + \sqrt{-1}\lambda_k^3 = \langle b, u_k \rangle, \quad \frac{1}{2}(|z_k|^2 - |w_k|^2) + \lambda_k^1 = \langle a, u_k \rangle \tag{3.6}$$

for $k = 1, \ldots, d$. (The first inner product is complex). Indeed the complex equation (3.3b) means that the real and imaginary parts of $\sum_{k=1}^{d}(z_k w_k + \lambda_k^2 + \sqrt{-1}\lambda_k^3)e_k$ are in $\operatorname{Ker} \imath^*$, which from (2.4) equals $\operatorname{Im} \beta^*$. Now

$$\beta^*(s) = \sum_{k=1}^{d} \langle s, u_k \rangle e_k, \tag{3.7}$$

yielding the first equation of (3.6). The same argument works for μ_1.

As remarked after equation (3.2), the moment map for the action of T^d on $\mathbb{H}^d$ defines a homeomorphism from $\mathbb{H}^d/T^d$ onto $\mathbb{R}^{3d}$. Since the vectors u_k generate $\mathbb{R}^n$, (3.6) shows that the map $(z, w) \mapsto (a, b)$ gives a homeomorphism of the quotient by T^d of the zero-set of (3.3) onto $\mathbb{R}^{3n}$. We therefore obtain a homeomorphism of M/T^n onto $\mathbb{R}^{3n}$. We see from (2.4) that (a, b) is the value of ϕ at the point in M with representative (z, w), so we have proved (i).

The T^n-stabiliser of the point in M represented by (z, w) is just the quotient of the T^d-stabiliser of (z, w) by the N-stabiliser of (z, w). Now $z_k = w_k = 0$ if and only if both $\langle a, u_k \rangle = \lambda_k^1$ and $\langle b, u_k \rangle = \lambda_k^2 + \sqrt{-1}\lambda_k^3$, that is, if and only if $(a, b) \in H_k$. Therefore the T^d-stabilizer of (z, w) is the subtorus of T^d whose Lie algebra is generated by the vectors e_k for which $(a, b) \in H_k$. Part (ii) of the theorem now follows from (2.3). □

This result shows at once that, even if u_k, λ_k^1 define a polytope Δ by (2.8) corresponding to a toric variety X, our manifold $M(\underline{u}, \underline{\lambda})$ need not be T^n-equivariantly diffeomorphic to T^*X. We can see this by considering the fixed points of T^n on T^*X. For the fixed points of T^n on X correspond to the vertices of Δ and are therefore isolated. It follows that these are the only fixed points of T^n on T^*X. If, however, some n faces of Δ corresponding to linearly independent u_i meet outside Δ, then we get additional fixed points of T^n on $M(\underline{u}, \underline{\lambda})$.

We shall see in section 6 that $M(\underline{u}, \underline{\lambda})$ is typically not homeomorphic to T^*X, even non-equivariantly. This is essentially due to the fact that the hyperkähler moment map $\phi : M \to \mathbb{R}^{3n}$ is surjective, unlike in the Kähler case treated in §2, where the image of the Kähler moment map ϕ is a polytope in $\mathbb{R}^n$. This difference between the Kähler and hyperkähler picture will be important at several points in the paper.

We shall now give necessary and sufficient conditions for $\mu^{-1}(0)/N$ to be smooth or an orbifold. We shall assume that the flats are distinct.

Theorem 3.2. *Suppose we are given primitive vectors $u_1, \dots, u_d \in \mathbb{Z}^n$ generating $\mathbb{R}^n$ and elements $\lambda_1, \dots, \lambda_d$ of $\mathbb{R}^3$ such that the flats H_k are distinct. Then the hyperkähler quotient $M(\underline{u}, \underline{\lambda})$ is smooth if and only if every $n+1$ flats among the H_k have empty intersection and whenever some n flats $H_{k_1}, \dots, H_{k_n}$ have nonempty intersection, then the set $\{u_{k_1}, \dots, u_{k_n}\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.*

Theorem 3.3. *With the assumptions of Theorem 3.2 $M(\underline{u}, \underline{\lambda})$ is an orb-*

ifold, with at worst abelian quotient singularities, if and only if every $n+1$ *flats among the* H_k *have empty intersection.*

Proof. **(a).** We begin by noting that if J is a maximal set of indices satisfying $\bigcap_{k\in J} H_k \neq \emptyset$, then the set $\{u_k : k \in J\}$ spans $\mathbb{R}^n$. For if $t \notin J$, then by maximality $\bigcap_{k\in J\cup\{t\}} H_k$ is empty, so u_t is in the span of $\{u_k : k \in J\}$. As we always suppose that the set of all u_k spans $\mathbb{R}^n$, the claim follows.

Now we consider the following statements:

1) for all $x \in \mathbb{R}^{3n}$, the set $\{u_k : x \in H_k\}$ is contained in a $\mathbb{Z}$-basis for $\mathbb{Z}^n$,

2) for all $x \in \mathbb{R}^{3n}$, the set $\{u_k : x \in H_k\}$ is linearly independent.

We claim that 1) is equivalent to the condition of Theorem 3.2 and 2) to that of Theorem 3.3. It is obvious that 1) and 2) imply the respective conditions. Conversely, let $x \in \mathbb{R}^{3n}$ and let I be the set of indices k such that $x \in H_k$. Let J be a maximal element of the set of indices containing I and satisfying $\bigcap_{k\in J} H_k \neq \emptyset$. By the observation made at the beginning of the proof, the set $\{u_k : k \in J\}$ spans $\mathbb{R}^n$ and in particular $\#J \geq n$. The claim now easily follows.

(b). Next, we shall show that 1), 2) are equivalent to the action of N on the zero level set of μ being free or locally free respectively.

Let $(z,w) \in \mu^{-1}(0)$ and let $(a,b) \in \mathbb{R}^n \times \mathbb{C}^n$ be $\phi(z,w)$, as in (3.6). We also regard (a,b) as a point $x \in \mathbb{R}^{3n}$ in the obvious way. We observe from 3.1(i) that any $(a,b) \in \mathbb{R}^n \times \mathbb{C}^n$, and hence any $x \in \mathbb{R}^{3n}$, can occur in (3.6).

If $I = \{k : x \in H_k\}$, we let $\mathbb{R}^d_I$ denote the span of $\{e_k : k \in I\}$, and T_I be the associated subtorus of T^d. The proof of Theorem 3.1(ii) shows that T_I is the stabilizer of (z,w) for the T^d action.

The work of Delzant and Guillemin now shows that 1), 2) are equivalent to $N \cap T_I$ always being trivial or finite respectively. For example, notice that $\mathfrak{n} \cap \mathbb{R}^d_I$ is zero if and only if the kernel of β on $\mathbb{R}^d_I$ is zero, that is, if and only if the set $\{u_k : k \in I\} = \{u_k : x \in H_k\}$ is linearly independent.

(c). Standard results of symplectic geometry show that freeness or local freeness of the action of N on $\mu^{-1}(0)$ imply that the quotient is smooth or an orbifold respectively. In both cases, $\mu^{-1}(0)$ is smooth.

We shall now show the necessity of the condition of Theorem 3.3. Suppose that $M(\underline{u},\underline{\lambda})$ is an orbifold and let J be a maximal set of indices satisfying $\bigcap_{k\in J} H_k \neq \emptyset$. Therefore $\{u_k : k \in J\}$ spans $\mathbb{R}^n$ and $\bigcap_{k\in J} H_k$ is a point, say x.

It follows from Theorem 3.1 that $m = \phi^{-1}(x)$ is fixed by T^n. Since M is an orbifold, it has a well defined tangent space at m of the form $\mathbb{R}^{4n}/\Gamma$

for some finite linear group Γ, and pulling back to $\mathbb{R}^{4n}$ we obtain a linear representation of T^n with a finite kernel. (The dimension of M must be $4n$ because of 3.1(i).) As the T^n action preserves the hyperkähler structure, we see that we have the standard representation of T^n as the maximal torus in $Sp(n)$.

Moreover some T^n-invariant neighbourhood of m is T^n-equivariantly homeomorphic to a neighbourhood of zero in $\mathbb{R}^{4n}/\Gamma$. Theorem 3.1(ii) now shows that $\#J \leq n$, establishing the necessity of the condition of Theorem 3.3.

In particular, if M is a manifold then the condition of 3.3 holds and hence the action of N on $\mu^{-1}(0)$ is locally free, so, as mentioned above, the zero set of μ is smooth. As the (quaternionic) action of N is generically free, smoothness of M now implies that the action of N on $\mu^{-1}(0)$ is free. From above, we have now shown the necessity of the condition of Theorem 3.2. □

Remark 3.4. It follows that for any fixed set of vectors u_k, the hyperkähler quotient $M(\underline{u}, \underline{\lambda})$ is an orbifold for a generic choice of vectors λ_k. On the other hand, this quotient is a *manifold* for a generic choice of vectors λ_k if and only if any set of n independent vectors among the u_i is a $\mathbb{Z}$-basis for $\mathbb{Z}^n$. Furthermore, if the latter condition is satisfied, then the set of λ_k for which $M(\underline{u}, \underline{\lambda})$ is singular has codimension 3 in $\mathbb{R}^{3d}$ and hence the set of λ_k for which $M(\underline{u}, \underline{\lambda})$ is smooth is path-connected. Therefore we expect the topology of smooth $M(\underline{u}, \underline{\lambda})$ to be independent of the vectors λ_k. We shall show in section 6 that this is indeed the case.

Theorems 3.1 and 3.3 imply

Corollary 3.5. *Suppose that $M(\underline{u}, \underline{\lambda})$ is an orbifold (with all H_k distinct). Then:*

(i) *the set of fixed points for the action of T^n is finite and in one-to-one correspondence with the set of intersection points of n among the flats.*

(ii) *if $x \in \mathbb{R}^{3n}$ lies in exactly r flats, then the T^n-stabiliser of a point in $\phi^{-1}(x)$ is an r-dimensional torus.* □

If the condition of Theorem 3.2 or Theorem 3.3 is satisfied, we shall refer to $M(\underline{u}, \underline{\lambda})$ as a *toric hyperkähler manifold* or *toric hyperkähler orbifold* respectively. In the former case it is a complete $4n$-dimensional Riemannian manifold with a hyperkähler action of T^n. Not all hyperkähler manifolds with such an action can be obtained as a hyperkähler quotient of $\mathbb{H}^d$ by a torus. Examples are provided by the Taub-NUT metric and various higher-dimensional analogues (see [HKLR] for the Roček metrics, and [GR] for some more recent constructions). This is a consequence of the fact that T^n is not the only maximal abelian group preserving the hyperkähler structure of $\mathbb{H}^n$.

Remark 3.6. Goto [Go] considers a special class of hyperkähler quotients of $\mathbb{H}^d$ by tori. In his case $n = m_1+m_2+\ldots+m_k$, $d = n+k$ and the u_i are the vectors e_i of the standard basis of $\mathbb{R}^n$ together with the k vectors $-\sum_{i=1}^{s_j} e_i$, $s_j = m_1 + m_2 + \ldots + m_j$, $j = 1, \ldots, k$. For this class of toric hyperkähler manifolds Goto obtains statements essentially equivalent to Corollary 3.5 and Theorems 3.2 and 6.5. On the other hand, Nakajima [Na] studies very general properties of a class of quotients of flat quaternionic spaces by unitary groups. In the abelian case, his class of subtori of T^d, while larger than that of Goto, is still quite special - when $n = 2$, for instance, it does not include tori from Example 2.3 for $a \neq 0, 1$.

As examples of toric hyperkähler manifolds, consider the hyperkähler quotients corresponding to examples 2.1 and 2.2. In the first case we obtain the Calabi metric [Ca] on $T^*\mathbb{C}P^n$, while the second case yields the Gibbons-Hawking metric on the resolution of the Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_3$ [GH,Hi,Kr].

The following example illustrates the dependence of $M(\underline{u}, \underline{\lambda})$ on the arrangement of flats (3.5) and not on the intersection of half-spaces (2.8)

Example 3.7. Let $n = 2$ and $u_1 = e_1$, $u_2 = e_2$, $u_3 = -e_1 + e_2$. For negative scalars $\lambda_1, \lambda_2, \lambda_3$ with $\lambda_2 > \lambda_1 + \lambda_3$ the intersection of half-spaces (2.8) is illustrated in Figure 3. The corresponding toric variety is the line bundle $O(1)$ over $\mathbb{C}P^1$. Now consider the hyperkähler orbifold $M(\underline{u}, \underline{\lambda})$ with the same u_k and $\lambda_k^1 = \lambda_k$, $\lambda_k^2 = \lambda_k^3 = 0$. Figure 4 shows the hyperplanes H_k^1 $(k = 1, 2, 3)$. This is the same hyperplane arrangement as for the projective space $\mathbb{C}P^2$ (see Fig. 1). In fact $M(\underline{u}, \underline{\lambda})$ is $T^*\mathbb{C}P^2$ because the hyperkähler quotient of $\mathbb{H}^3$ by $N = \{(t, -t, t) \, : \, t \in S^1\}$ is the same as that by $\{(t, t, t) \, : \, t \in S^1\}$.

4. Compact 3-Sasakian manifolds.

We shall briefly discuss how the ideas of the previous sections can be used to produce a large family of compact 3-Sasakian manifolds considered in [BGM1,2] and [BGMR]. We recall here that 3-Sasakian manifolds are a special class of Einstein manifolds with positive scalar curvature. Also, a 3-Sasakian manifold admits a locally free action of $Sp(1)$, and the quotient is a quaternionic Kähler orbifold. A Riemannian manifold $(\mathcal{S}, g)$ is 3-Sasakian if and only if the Riemannian cone $C(\mathcal{S}) = (\mathbb{R}^+ \times \mathcal{S}, dr^2 + r^2 g)$ is hyperkähler.

Theorem 4.1. *Let $\underline{u} = (u_1, \dots, u_d)$ be a collection of mutually nonparallel primitive vectors in $\mathbb{Z}^n$ generating $\mathbb{R}^n$. Then the hyperkähler quotient $M(\underline{u}, \underline{0})$ is the Riemannian cone over a compact 3-Sasakian manifold $\mathcal{S} = \mathcal{S}(\underline{u})$ if and only if the following two conditions hold:*

(i) *every subset of $\underline{u}$ with n elements is linearly independent,*

(ii) *every subset of $\underline{u}$ with less than n elements is a part of a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.*

Proof. Let us first show that these conditions are necessary and sufficient for $M(\underline{u}, \underline{0})$ to have only one singularity, the point corresponding to $z = w = 0$. From the proof of Theorems 3.1 and 3.2 it follows that $(z, w) \in \mathbb{H}^d$ will yield a singular point of $M(\underline{u}, \underline{0})$ precisely when there exists $(a, b) \in \mathbb{R}^n \times \mathbb{C}^n$, such that $z_k w_k = \langle b, u_k \rangle$, $|z_k|^2 - |w_k|^2 = 2\langle a, u_k \rangle$ for $k = 1, \dots, d$, and the set $\{u_k : \langle b, u_k \rangle = \langle a, u_k \rangle = 0\}$ is not a part of a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Assumption (ii) means that this can only happen if this set has at least n elements, but assumption (i) implies that in this case $a = b = 0$ and so $z = w = 0$. Necessity of (i),(ii) follows easily from Theorem 3.2.

Now we recall that $\mathbb{H}^d$ is the Riemannian cone over the standard sphere S^{4d-1} and S^{4d-1} is a 3-Sasakian manifold. The 3-Sasakian structure of S^{4d-1} is given by the right diagonal action of $Sp(1)$ on $\mathbb{H}^d$. Since we have chosen all λ_i to be zero, the zero-set of the moment map (3.3) is invariant under the action of both $\mathbb{R}^+$ and $Sp(1)$. As the action of N commutes with that of $\mathbb{R}^+$, and as the only singularity is at the origin, $M(\underline{u}, \underline{0})$ is a Riemannian cone over a *manifold* $\mathcal{S}$. The action of $Sp(1)$ also commutes with N, and induces an action on $\mathcal{S}$ defining a 3-Sasakian structure. Finally $\mathcal{S}$ is compact since $M(\underline{u}, \underline{0})$ is complete (as a stratified manifold) and the cone is complete only if its base is. This implies that $\mathcal{S}$ is complete and so compact by Myers's theorem. Alternatively we could realize $\mathcal{S}$ as the 3-Sasakian quotient [BGM2] of S^{4d-1}. □

Remark 4.2. Usually the $Sp(1)$ action on $\mathcal{S} = \mathcal{S}(\underline{u})$ has many different orbit types and so the quotient of $\mathcal{S}$ by $Sp(1)$ is only a quaternionic-Kähler orbifold. It is a manifold only when N is the circle acting diagonally on $\mathbb{H}^{n+1}$, which gives the homogenous quaternionic-Kähler manifold $\mathrm{Gr}_2(\mathbb{C}^{n+1})$.

Remark 4.3. For $n = 1$ the conditions of Theorem 4.1 are void. When $n = 2$ the conditions are satisfied if each pair of the vectors u_k is linearly independent and each u_k has relatively prime coordinates. The resulting quaternionic-Kähler orbifolds are 4-dimensional and admit an action of T^2.

5. Complex structures.

We shall now describe the generic complex structure of our toric hyperkähler orbifolds.

Theorem 5.1. *Let $M = M(\underline{u}, \underline{\lambda})$ be a toric hyperkähler orbifold and suppose that every $n+1$ flats $H_k^2 \times H_k^3$ have empty intersection in $\mathbb{R}^{2n}$. Then M, equipped with the complex structure J_1, is biholomorphic to the affine variety $\operatorname{Spec} A[W]^{N^{\mathbb{C}}}$ where $W \subset \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}^n$ is defined by the equations*

$$z_k w_k = \langle b, u_k \rangle - (\lambda_k^2 + \sqrt{-1}\lambda_k^3), \qquad (k = 1, \dots, d), \tag{5.1}$$

and $N^{\mathbb{C}}$ acts on $\mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}^n$ by $t \cdot (z, w, b) = (t \cdot z, t^{-1} \cdot w, b)$.

Proof. By (3.6), the variety W is precisely the zero-set of the complex moment map (3.3b). We have to show that the action of $N^{\mathbb{C}}$ on W has at most discrete stabilizers and that each $N^{\mathbb{C}}$-orbit meets the zero-set of the moment map μ_1. This will prove that the variety W is smooth (since W is the zero-set of the moment map for the complex-symplectic $N^{\mathbb{C}}$ action) and that M, the Kähler quotient of W by N, can be identified with the complex quotient $W/N^{\mathbb{C}}$. The argument we use is a slight modification of the one used for the construction of toric varieties as Kähler quotients (see [Gu2]).

Let $(z, w) \in \mathbb{C}^d \times \mathbb{C}^d$. Then the image of the $N^{\mathbb{C}}$-orbit of (z, w) under the moment map μ_1 is the set

$$\left\{ \sum_{\{i; z_i \neq 0\}} t_i \alpha_i - \sum_{\{i; w_i \neq 0\}} s_i \alpha_i + c_1 : t_i, s_i > 0 \right\} \subset \mathfrak{n}^*. \tag{5.2}$$

The proof of this is essentially the same as in [Gu2;Appendix 1]. The moment map restricted to the orbit is given in our case by the Legendre transform of the function $F : \mathfrak{n}^{\mathbb{C}} \cap \mathbb{R}^d \to \mathbb{R}$ defined by

$$F(y) = \frac{1}{4} \sum_{\{i; z_i \neq 0\}} a_i e^{2\alpha_i \cdot y} + \frac{1}{4} \sum_{\{i; w_i \neq 0\}} b_i e^{-2\alpha_i \cdot y} + c_1 \cdot y, \tag{5.3}$$

where a_i, b_i are positive constants. This is a strictly convex function and all the arguments of Guillemin go through.

If (z, w) is a point of W then, from the proof of 3.1, we know that $\sum_{k=1}^{d}(z_k w_k + \lambda_k^2 + \sqrt{-1}\lambda_k^3)e_k = \beta^*(b)$ for some $b \in \mathbb{C}^n$. If for all k we have $\langle b, u_k\rangle \neq \lambda_k^2 + \sqrt{-1}\lambda_k^3$, then $z_k w_k$ is nonzero for all k and so the full group $(T^d)^{\mathbb{C}}$ acts freely at (z, w). From (5.2), as the vectors α_i span $\mathfrak{n}^*$, the restriction of μ_1 to $N^{\mathbb{C}}(z, w)$ is surjective.

On the other hand, if $\langle b, u_k\rangle = \lambda_k^2 + \sqrt{-1}\lambda_k^3$ precisely when $k \in I$, where I is some multi-index, then $z_k w_k = 0$ if and only if $k \in I$. In particular the stabiliser group of (z, w) for the action of $N^{\mathbb{C}}$ is a subgroup of $T_I^{\mathbb{C}}$. Since the flats $H_k^2 \times H_k^3$, $k \in I$, now have nonempty intersection, the assumption of the theorem and the argument at the beginning of the proof of Theorem 3.2 imply that the vectors u_k, $k \in I$, are independent. Therefore the map β sending e_i to u_i must be injective on $\mathbb{R}_I^d$ =Lie(T_I). and, from (2.3),(2.4), we see that $\mathfrak{n} \cap \mathbb{R}_I^d = 0$. The analogous statement for complex vector spaces is proved similarly, so the stabiliser for the $N^{\mathbb{C}}$ action is discrete. We also see that $\mathbb{R}^d = (\mathbb{R}_I^d)^{\perp} + \mathfrak{n}^{\perp} = (\mathbb{R}_{I^c}^d) + \mathfrak{n}^{\perp}$, and, since i^* is just the orthogonal projection onto $\mathfrak{n} \equiv \mathfrak{n}^*$, it follows that $\mathfrak{n}^*$ is spanned by the set $\{\alpha_i; i \notin I\}$. Therefore, from (5.2), μ_1 is still surjective on $N^{\mathbb{C}}(z, w)$. This proves Theorem 5.1. □

Example 5.2. Consider the hyperkähler quotient corresponding to Example 2.1 with $n = 1$ (the Eguchi-Hanson space). The variety W is described by the two equations $z_1 w_1 = b - \nu_1$ and $z_2 w_2 = -b - \nu_2$, where $\nu_k = \lambda_k^2 + \sqrt{-1}\lambda_k^3$. The assumption of Theorem 5.1 is satisfied if $\nu_1 \neq -\nu_2$. Eliminating b, we can view W as the hypersurface in $\mathbb{C}^4$ with equation $z_1 w_1 + z_2 w_2 = \tau$, where $\tau \neq 0$. The ring of invariant polynomials for the action of $N^{\mathbb{C}} \cong \mathbb{C}^*$ is generated by $z_1 w_2, z_2 w_1, z_1 w_1$. We find that (M, J_1) is biholomorphic to the variety $xy = z(\tau - z)$ which can be viewed as a deformation of the Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_2$ [Hi].

6. Topology of toric hyperkähler orbifolds.

Our next task is to show how the vectors $u_1, \ldots, u_d \in \mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_d \in \mathbb{R}^3$ determine the topology of $M(\underline{u}, \underline{\lambda})$. First of all we have

Theorem 6.1. *If $M(\underline{u}, \underline{\lambda})$ and $M(\underline{u}', \underline{\lambda}')$ are toric hyperkähler orbifolds, and $\underline{u} = \underline{u}'$, then $M(\underline{u}, \underline{\lambda})$ is homeomorphic to $M(\underline{u}', \underline{\lambda}')$.*

In other words the homeomorphism type of M depends only on the torus N and not on the moment map (3.3). Before proving this let us establish a few related facts.

Proposition 6.2. *Let $M(\underline{u}, \underline{\lambda})$ be a toric hyperkähler orbifold such that every $n+1$ hyperplanes H^1_k have empty intersection. Then $M(\underline{u}, \underline{\lambda})$ is diffeomorphic to $M(\underline{u}, \underline{\lambda}')$, where $\lambda'_k = (\lambda^1_k, 0, 0)$ for each k.*

Proof. Let us write $M(\underline{\lambda}^1, \underline{\lambda}^2, \underline{\lambda}^3)$ for $M(\underline{u}, \underline{\lambda})$. Applying Theorem 5.1 with respect to the complex structure J_3 shows that $M(\underline{\lambda}^1, \underline{\lambda}^2, \underline{\lambda}^3)$ is diffeomorphic to $M(\underline{\lambda}^1, \underline{\lambda}^2, 0)$. Applying it again, with respect to J_2, shows that $M(\underline{\lambda}^1, \underline{\lambda}^2, 0)$ is diffeomorphic to $M(\underline{\lambda}^1, 0, 0)$. □

If the condition of Theorem 3.2 or Theorem 3.3 is satisfied, then the hypothesis of 6.2 holds for a generic direction in $\mathbb{R}^3$. More precisely:

Lemma 6.3. *Suppose that we are given vectors $u_1, \dots, u_d$ generating $\mathbb{R}^n$ and elements $\lambda_1, \dots, \lambda_d$ of $\mathbb{R}^3$ such that every $n+1$ flats H_k defined by (3.4)-(3.5) have empty intersection. Then for a generic element (a, b, c) of the 2-sphere in $\mathbb{R}^3$, every $n+1$ of the hyperplanes $\{y \in \mathbb{R}^n : \langle y, u_k \rangle = a\lambda^1_k + b\lambda^2_k + c\lambda^3_k\}$ have empty intersection.*

Proof. If not, then there is a particular set of $n+1$ indices, say $1, \dots, n+1$, such that the set S of (a, b, c) for which the corresponding $n+1$ hyperplanes intersect spans $\mathbb{R}^3$. Now for each $(a, b, c) \in S$ there exists x_{abc} such that $\langle x_{abc}, u_k \rangle = a\lambda^1_k + b\lambda^2_k + c\lambda^3_k$ for $k = 1, \dots, n+1$. As S spans $\mathbb{R}^3$, by taking linear combinations of various x_{abc} we can easily find a common point of the flats $H_1, \dots, H_{n+1}$, contradicting the assumption of the lemma. □

Lemma 6.4. *Let $M(\underline{u}, \underline{\lambda})$ and $M(\underline{u}, \underline{\lambda}')$ be two toric hyperkähler orbifolds such that there is an element A of $SO(3)$ with $A\lambda_k = \lambda'_k$ for $k = 1, \dots, d$. Then $M(\underline{u}, \underline{\lambda})$ and $M(\underline{u}, \underline{\lambda}')$ are T^n-equivariantly diffeomorphic.*

Proof. The right diagonal action on $\mathbb{H}^d$ of an element of $Sp(1)$ covering A induces a T^d-equivariant diffeomorphism of the two level sets. □

Proof of Theorem 6.1. Because of Proposition 6.2 and Lemmas 6.3, 6.4 we can assume that all λ_k and λ'_k lie on the x_1-axis. Let U be the set of vectors

$\lambda^1 = (\lambda_1^1, \dots, \lambda_d^1) \in \mathbb{R}^d$ such that $M(\underline{u}, \underline{\lambda})$ is an orbifold. The complement of U is the set of λ^1 for which $n+1$ of the hyperplanes

$$H_k^1 = \{x \in \mathbb{R}^n : \langle x, u_k \rangle = \lambda_k^1\} \tag{6.1}$$

have nonempty intersection. We shall first show that the topology of M does not change as long as we stay within a connected component of U. If λ^1 and $\lambda^{1\prime}$ lie in the same component of U, then there is a homeomorphism h of $\mathbb{R}^n$ onto itself mapping each half-space $\{x \in \mathbb{R}^n : \langle x, u_k \rangle \le \lambda_k^1\}$ onto the corresponding half-space $\{x \in \mathbb{R}^n : \langle x, u_k \rangle \le \lambda_k^{1\prime}\}$ and similarly for the opposite half-spaces. We consider now, as in [Go], the homeomorphism τ between $\mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0}$ and $\mathbb{R} \times \mathbb{R}_{\ge 0}$ given by

$$\tau(x, y) = \left(\frac{1}{2}(x^2 - y^2), xy\right), \tag{6.2}$$

which we extend diagonally to a homeomorphism, also denoted by τ, between $(\mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0})^d$ and $(\mathbb{R} \times \mathbb{R}_{\ge 0})^d$. Let $V(\underline{\lambda})$ be the subset of $\mathbb{R}^d$ consisting of vectors $p = (p_1, \dots, p_d)$ such that there is an $a \in \mathbb{R}^n$ with $p_k = \langle a, u_k \rangle - \lambda_k^1$, $k = 1, \dots, d$. Since the vectors u_k generate $\mathbb{R}^n$, the map $v : V(\underline{\lambda}) \to \mathbb{R}^n$ sending p to a is a homeomorphism. We define $V(\underline{\lambda}')$ and v' similarly . Let us extend h, v, v' to homeomorphisms $h : \mathbb{R}^n \times (\mathbb{R}_{\ge 0})^d \to \mathbb{R}^n \times (\mathbb{R}_{\ge 0})^d$, $v : V(\underline{\lambda}) \times (\mathbb{R}_{\ge 0})^d \to \mathbb{R}^n \times (\mathbb{R}_{\ge 0})^d$ and $v' : V(\underline{\lambda}') \times (\mathbb{R}_{\ge 0})^d \to \mathbb{R}^n \times (\mathbb{R}_{\ge 0})^d$ by putting the identity map on the second factor. The composition $\Phi = \tau^{-1} \circ (v')^{-1} \circ h \circ v \circ \tau$ gives a homeomorphism between $\tau^{-1}\left(V(\underline{\lambda}) \times (\mathbb{R}_{\ge 0})^d\right)$ and $\tau^{-1}\left(V(\underline{\lambda}') \times (\mathbb{R}_{\ge 0})^d\right)$. Finally we define $\pi : \mathbb{C}^d \times \mathbb{C}^d \to (\mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0})^d$ by $\pi(z, w) = (|z_1|, |w_1|, \dots, |z_d|, |w_d|)$. Let us write $\Phi \circ \pi = (\Phi_1^1, \Phi_1^2, \dots, \Phi_d^1, \Phi_d^2)$. We can now define a T^d-equivariant homeomorphism Ψ between the zero level sets for $\underline{\lambda}$ and $\underline{\lambda}'$ by putting

$$\begin{aligned}&\Psi(z, w)\\ &= \left(\Phi_1^1(z, w)\frac{z_1}{|z_1|}, \dots, \Phi_d^1(z, w)\frac{z_d}{|z_d|}, \Phi_1^2(z, w)\frac{w_1}{|w_1|}, \dots, \Phi_d^2(z, w)\frac{w_d}{|w_d|}\right).\end{aligned}$$

This map induces a T^n-equivariant homeomorphism between $M(\underline{u}, \underline{\lambda})$ and $M(\underline{u}, \underline{\lambda}')$.

We have shown that as long as λ^1 does not pass through a critical value, i.e. a value for which $n+1$ hyperplanes (6.1) have nonempty intersection, then the topology of $M(\underline{u}, \underline{\lambda})$ does not change. We shall now show that it does not change even when we pass through a critical value. Let $\lambda^1 = (\lambda_1^1, \dots, \lambda_d^1)$ be a critical value. We can assume that it is the hyperplanes

$H_1^1, \ldots, H_{n+1}^1$ that have a nonempty intersection. We can also assume that $\{H_1^1, \ldots, H_{n+1}^1\}$ is a *maximal* set of hyperplanes with nonempty intersection, because the configurations with more than $n+1$ hyperplanes intersecting form a codimension 2 set in $\mathbb{R}^d$. Moreover, we can take $u_2, \ldots, u_{n+1}$ to be linearly independent, so that small perturbations of λ_1 will make $M(\underline{u}, \underline{\lambda})$ an orbifold.

Let U_- (resp. U_+) denote the component of U to which $(\lambda_1^1 - \delta, \lambda_2^1, \ldots, \lambda_d^1)$ belongs for a small positive (resp. negative) δ. It will be enough to show that the topology of M does not change as we pass from U_- to U_+. Let us consider an orbifold $M(\underline{u}, \underline{\lambda})$ where $\underline{\lambda}$ is obtained from $(\lambda_1, \ldots, \lambda_d)$ by replacing $\lambda_1 = (\lambda_1^1, 0, 0)$ with $(\lambda_1^1, \delta_2, \delta_3)$ for small δ_2, δ_3. Using Lemma 6.3 we can obtain a toric hyperkähler orbifold by projecting $\underline{\lambda}$ onto the subspace $\mathbb{R}(a, b, c) \otimes \mathbb{R}^d$ for generic small b, c and a close to 1. Now we can use Lemma 6.4 to obtain an element $\lambda(b, c)$ of $\mathbb{R}^d = \mathbb{R}(1, 0, 0) \otimes \mathbb{R}^d$ such that the corresponding $M(\underline{u}, \underline{\lambda})$ is an orbifold. Moreover Proposition 6.2 and Lemma 6.4 show that the topology of this orbifold does not depend on (b, c). However, by changing the signs of b and c we can guarantee that $\lambda(b, c)$ belongs to U_- for some choices of (b, c), and belongs to U_+ for other choices. This proves Theorem 6.1. □

We shall now discuss the homotopy type of $M(\underline{u}, \underline{\lambda})$. Because of Theorem 6.1 we can assume that the vectors λ_k are of the form $(\lambda_k^1, 0, 0)$.

In what follows we shall use a similar argument to that of Goto [Go]. We shall consider the hyperplanes H_k^1 defined by (6.1). These hyperplanes divide $\mathbb{R}^n$ into a finite family of closed convex polyhedra, some unbounded. Let $\mathcal{A}$ be the *polyhedral complex* consisting of all faces of all dimensions of these polyhedra. We recall that a polyhedral complex is a family of polyhedra such that every face of a member of $\mathcal{A}$ is itself a member of $\mathcal{A}$ and the intersection of any two members of $\mathcal{A}$ is a face of each of them. We define the polyhedral (in fact polytopal) complex $\mathcal{C}$ to consist of all *bounded* polyhedra in $\mathcal{A}$. This complex is nonempty since, as the vectors u_k generate $\mathbb{R}^n$, $\mathcal{C}$ must contain a vertex corresponding to the intersection of n hyperplanes H_k^1. We index the elements of $\mathcal{C}$ by some set I and denote the polyhedra in $\mathcal{C}$ by Δ_s, $s \in I$. Finally, we denote by $|\mathcal{C}|$ the support $\bigcup_{s \in I} \Delta_s$ of the complex.

Recall that $\phi = (\phi_1, \phi_2, \phi_3) : M \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ is the moment map for the action of T^n on M. We define subsets X_s of M by

$$X_s = \phi^{-1}(\Delta_s, 0, 0), \quad s \in I. \tag{6.3}$$

The following result describes the topology of $M(\underline{u}, \underline{\lambda})$.

Theorem 6.5. *Let $M = M(\underline{u}, \underline{\lambda})$ be a toric hyperkähler orbifold, where $\lambda_k = (\lambda_k^1, 0, 0)$ for each k. Then:*

(i) $\bigcup_{s\in I} X_s = \phi^{-1}(|\mathcal{C}|, 0, 0)$ *is a T^n-equivariant deformation retract of M.*

(ii) *Each X_s is a Kähler subvariety of (M, J_1, ω_1), isotropic with respect to the form $\omega_2 + i\omega_3$ and invariant under the T^n-action.*

(iii) *Each X_s is T^m-equivariantly isometric and biholomorphic to the toric variety determined by the polytope Δ_s, where T^m is the subtorus of T^n acting effectively on X_s.*

Proof. Once more we consider the homeomorphism τ between $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ and $\mathbb{R} \times \mathbb{R}_{\geq 0}$ defined by (6.2). Let j_t be the deformation map of $\mathbb{R} \times \mathbb{R}_{\geq 0}$ defined by $j_t(u, v) = (u, tv)$. Then the composite map $\jmath_t = \tau^{-1} \circ j_t \circ \tau$ is a deformation of $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. Let us write $\jmath_t(x, y) = (\jmath_t^1(x, y), \jmath_t^2(x, y))$. Now we define a deformation of $\mathbb{C}^2$ by the map $h : [0, 1] \times \mathbb{C}^2 \to \mathbb{C}^2$ where

$$h(t, z, w) = \left(\jmath_t^1(|z|, |w|)\frac{z}{|z|}, \jmath_t^2(|z|, |w|)\frac{w}{|w|} \right),$$

and extend this diagonally to a deformation of $\mathbb{C}^d \times \mathbb{C}^d$. We observe that h is T^d-equivariant and the moment map (3.2) satisfies

(6.4)
$$\mu_1 \circ h_t(z, w) = \mu_1(z, w), \quad ((\mu_2 + \sqrt{-1}\mu_3) \circ h_t)(z, w) = t(\mu_2 + \sqrt{-1}\mu_3)(z, w)$$

for any $t \in [0, 1]$ (recall that we are setting $c_2 = c_3 = 0$).

Therefore h preserves the zero-set of (3.3). Since h is T^d-equivariant, we obtain a T^n-equivariant deformation of M. Moreover $h_0(M) = (\phi_2 + \sqrt{-1}\phi_3)^{-1}(0)$, because of (3.6) and the fact that $b = (\phi_2 + \sqrt{-1}\phi_3)(z, w)$.

We have now deformed M to $(\phi_2 + \sqrt{-1}\phi_3)^{-1}(0)$, which, by (3.6), corresponds to the quotient by N of the set of $(z, w, a) \in \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{R}^n$ such that

$$(6.5) \qquad z_k w_k = 0, \quad \frac{1}{2}(|z_k|^2 - |w_k|^2) + \lambda_k^1 = \langle a, u_k \rangle, \quad (k = 1, \dots, d).$$

Let us recall once more that $a = \phi_1(z, w)$. We claim that there is a deformation map $p : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$, such that $p(1, a) = a$, the map $a \mapsto p(0, a)$ is a retraction onto $|\mathcal{C}| = \bigcup_{s\in I} \Delta_s$, and, if a lies on a hyperplane H_k^1, then $p(t, a)$ lies on this hyperplane for all $t \in [0, 1]$. To see this we observe that the complement of $|\mathcal{C}| = \bigcup_{s\in I} \Delta_s$ in $\mathbb{R}^n$ is a union of convex unbounded polyhedra K_i with non-empty interior such that the intersection of any two of them will be a common face (of positive codimension) of each. Moreover each K_i is line-free (if K_i contains a line, spanned by a vector v, then v is parallel to all hyperplanes H^1_k hence orthogonal to all u_k, contradicting the assumption that the vectors u_k span $\mathbb{R}^n$). Therefore we can think of each K_i as a convex polytope P whose unique face at infinity F_0 has been removed. We can find a deformation retraction of $K_i = P - F_0$ onto the part of the boundary consisting of bounded faces. Moreover we can assume that this deformation of $P - F_0$ is an extension of any given deformation of $\partial P - F_0$. Therefore, by doing it first on the intersections of K_i's and then extending to their interiors, we can define the desired deformation map p.

For $k = 1, \dots, d$, put $p_t^k(a) = 2\langle p(t,a), u_k\rangle - 2\lambda_k^1$. We now define a T^d-equivariant deformation f_t of the set given by (6.5):

$$f_t(z_k, w_k) = \begin{cases} (0,0) & \text{if } \ z_k = w_k = 0, \\ \left(\frac{\sqrt{p_t^k(a)}}{|z_k|} z_k, 0\right) & \text{if } \ z_k \neq 0, \\ \left(0, \frac{\sqrt{-p_t^k(a)}}{|w_k|} w_k\right) & \text{if } \ w_k \neq 0, \end{cases}$$

and

$$f_t(a) = p(t,a).$$

Observe that $\phi_1(f_t(z,w)) = p(t,a)$. The deformation f_t induces a T^n-equivariant deformation of $(\phi_2 + \sqrt{-1}\phi_3)^{-1}(0)$ onto $\bigcup_{s\in I} X_s$, proving part (i).

For (ii)-(iii) we observe, as in [Go], that each $\mathcal{D}_s$ can be obtained as a Kähler quotient of a submanifold of $\mathbb{C}^d \times \mathbb{C}^d$ by the construction of section 2 for the polytope Δ_s. These submanifolds are Kähler with respect to ω_1 and isotropic with respect to $\omega_2 + \sqrt{-1}\omega_3$, so all statements of (ii)-(iii) follow. □

Recall that an arrangement of hyperplanes in $\mathbb{R}^n$ is *simple* if no $n+1$ of them intersect.

Corollary 6.6. *If the arrangment of hyperplanes (6.1) is simple, then the homotopy type of the compact variety of Theorem 6.5(i) depends only on the vectors* u_k.

There is a very simple formula relating the Betti numbers of a compact toric orbifold to the combinatorics of the corresponding convex polytope [Fu]. It turns out that a similar formula holds for toric hyperkähler orbifolds.

Theorem 6.7. *Let $M = M(\underline{u}, \underline{\lambda})$ be a toric hyperkähler orbifold of dimension $4n$. Then M is simply connected, $H^j(M, \mathbb{Q}) = 0$ if j is odd or $j > 2n$, and the Betti numbers in the remaining even dimensions are given by*

$$b_{2k} = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} d_i, \qquad (k = 1, \dots, n) \tag{6.6}$$

where d_i denotes the number of i-dimensional elements of the complex $\mathcal{C}$. The Poincaré polynomial of M is therefore

$$\sum_{k=0}^{n} d_k (t^2 - 1)^k. \tag{6.7}$$

In particular the Euler characteristic of M is d_0, the number of vertices of $\mathcal{C}$.

Remark. For toric hyperkähler *manifolds*, $H^*(M, \mathbb{Z})$ has no torsion, so Theorem 6.7 tells us the cohomology over the integers.

We need first the following result.

Proposition 6.8. *Let $\mathcal{C}$ be a polytopal complex determined by a simple arrangement of hyperplanes* (6.1) *and suppose that $|\mathcal{C}|$ is not contained in any hyperplane. Then every element of $\mathcal{C}$ which is maximal with respect to inclusion has dimension n.*

Proof. Suppose, for a contradiction, that there is a maximal element F of $\mathcal{C}$ with $\dim F = m \leq n - 1$. We can suppose that $m \geq 1$. Let us choose one of the vertices of F to be the origin, and the edges meeting at this vertex to be generated by an orthonormal basis $e_1, \dots, e_n$ of $\mathbb{R}^n$. In other words we put $u_i = e_i$ and $\lambda_i^1 = 0$ for $i = 1, \dots, n$. We can further suppose that $F \subset \bigcap_{i=1}^m \{x \in \mathbb{R}^n : \langle x, e_i \rangle \geq 0\} \cap \bigcap_{i=m+1}^n \{x \in \mathbb{R}^n : \langle x, e_i \rangle = 0\}$. The hyperplanes (6.1) divide $\mathbb{R}^n$ into finitely many closed convex n-dimensional polyhedra and, by assumption, F is a face only of unbounded ones. Each of the polyhedra having F for a face is determined by a function $\epsilon : \{m+1, \dots, n\} \to \{-1, +1\}$. The polyhedron P^ϵ is then defined by having nonempty intersection with $\bigcap_{i=1}^m \{x : \langle x, e_i \rangle = 0\} \cap \bigcap_{i=m+1}^n \{x : \langle x, e_i \rangle \cdot \epsilon(i) > 0\}$. Since these are unbounded polyhedra, each of them contains a ray. In fact, for each function ϵ there is a y^ϵ with $\langle y^\epsilon, e_i \rangle \cdot \epsilon(i) > 0$, $(i > m)$, and such that the ray $R_x^\epsilon = \{x + ty^\epsilon : t \geq 0\}$ is contained in P^ϵ for all $x \in F$. Note that $\langle y^\epsilon, e_i \rangle \geq 0$ for $i \leq m$.

Suppose first that for all $i \leq m$, and for all ϵ, we have $\langle y^\epsilon, e_i\rangle = 0$. It is straightforward to show that if $k > n$ then $u_k \in \mathrm{Span}\{e_1, \ldots, e_m\}$ (otherwise one of the rays meets H_k^1, giving a contradiction). So $|\mathcal{C}| \subset \mathrm{Span}\{e_1, \ldots, e_m\}$, which contradicts the assumption of the proposition.

Let us assume, then, that there exist $i_0 \leq m$, and ϵ, such that $\langle y^\epsilon, e_{i_0}\rangle > 0$. If we remove the hyperplanes $H_{m+1}^1, \ldots, H_n^1$, the remaining hyperplanes still divide $\mathbb{R}^n$ into convex n-dimensional polyhedra P_j. It follows that a neighborhood of any interior point of F belongs now to a single P_{j_0} and, consequently, the rays $R_x^\epsilon, (x \in F)$, are all contained in P_{j_0}. Since ϵ ranges over all sign combinations of the last $n-m$ coordinates, we can find z in the interior of the convex hull of the y^ϵ such that $\langle z, e_i\rangle = 0$ for $i > m$. Observe that $\langle z, e_i\rangle$ is nonnegative for $i \leq m$, and is positive for $i = i_0$.

Since P_{j_0} is convex, we see that for any x in the interior of F, the ray $R = \{x + tz : t > 0\}$ is contained in the interior of P_{j_0}. As $\langle z, e_i\rangle = 0$ for $i > m$, this ray must meet an $(m-1)$-dimensional face of F, and so meets a hyperplane H_k^1 with $k > n$, contradicting the fact that R lies in the interior of P_{j_0}. □

Proof of Theorem 6.7. We observe first that both sides of (6.6) depend only on the vectors u_k. Indeed, Theorem 6.1 shows that it is so for the Betti numbers. On the other hand, moving one of the hyperplanes (6.1) in the direction of its orthogonal does not change the number of k-dimensional faces of $|\mathcal{C}|$, as long as the initial and final hyperplane arrangments are simple. Also, Theorem 6.1 shows that the other statements of Theorem 6.7 depend only on the u_k.

We proceed now by induction on n. The result is easily verified if $n = 1$. Suppose that $n > 1$, and that the theorem holds for $k < n$. In dimension n we proceed by induction on the number d of hyperplanes. Our statements hold for n hyperplanes with a nonempty intersection. Suppose that they hold for q hyperplanes in $\mathbb{R}^n$ whenever $n \leq q \leq d-1$. Now consider a toric hyperkähler orbifold $M(\underline{u}, \underline{\lambda})$ corresponding to hyperplanes $H_1^1, \ldots, H_d^1$. Because of Proposition 6.8 we can suppose that $\dim |\mathcal{C}| = n$. By the remark above we can move the hyperplane H_d^1 until all of $|\mathcal{C}|$ lies to one side of H_d^1, say $|\mathcal{C}| \subset \{x : \langle x, u_d\rangle \geq \lambda_d^1\}$. The intersections of H_d^1 with the H_k^1, $k < d$, determine a simple arrangement of hyperplanes in $H_d^1 \cong \mathbb{R}^{n-1}$ which gives a toric hyperkähler orbifold Y of real dimension $4n-4$. Let us denote its polytopal complex by $\mathcal{E}$. On the other hand the hyperplanes $H_1^1, \ldots, H_{d-1}^1$ also determine a toric hyperkähler orbifold W with a polytopal complex $\mathcal{G}$. By the inductive hypothesis, the theorem holds for Y and W. We observe that, as every maximal element of $\mathcal{C}$ has dimension n, every i-dimensional element of $\mathcal{E}$ is a face of an $(i+1)$-dimensional element of $\mathcal{C}$. This implies that, if e_k, g_k denote the number of k-dimensional faces of $\mathcal{E}, \mathcal{G}$ respectively, then $d_0 = g_0 + e_0$ and $d_k = g_k + e_k + e_{k-1}$ for k positive.

Let us now pick a suitably small, positive, δ, and consider the neigbourhoods of $|\mathcal{E}|$ and $|\mathcal{G}|$ in $|\mathcal{C}|$ defined by $U_1 = |\mathcal{C}| \cap \{x \in \mathbb{R}^n : \langle x, u_d \rangle < \lambda_d^1 + 2\delta\}$ and $U_2 = |\mathcal{C}| \cap \{x \in \mathbb{R}^n : \langle x, u_d \rangle > \lambda_d^1 + \delta\}$. We also consider the deformation retract X of M given by Theorem 6.5(i). We have $X = V_1 \cup V_2$ where $V_1 = \phi^{-1}(U_1, 0, 0)$ and $V_2 = \phi^{-1}(U_2, 0, 0)$. Now, by the argument used in the proof of 6.5, V_1 can be deformed onto the deformation retract of Y defined by Theorem 6.5(i) and so V_1 is homotopy equivalent to Y. Similarly V_2 is homotopy equivalent to W. Moreover, we see using 6.8 that $V_1 \cap V_2$ is homotopy equivalent to an S^1-bundle E over Y. We deduce that M is simply-connected. Note that $E \to Y$ is an orientable bundle, as Y is simply-connected.

We now consider the Mayer-Vietoris sequence for V_1, V_2 and the Gysin sequence for $E \to Y$. The cohomology here is rational, but it can be taken integer if M is a manifold. By the inductive hypothesis, the odd Betti numbers of Y and W vanish, so the Mayer-Vietoris and the Gysin sequences split off at each even level as

$$0 \to H^{2k-1}(E) \to H^{2k}(M) \to H^{2k}(Y) \oplus H^{2k}(W) \to \\ \to H^{2k}(E) \to H^{2k+1}(M) \to 0,$$

$$0 \to H^{2k-1}(E) \to H^{2k-2}(Y) \to H^{2k}(Y) \to H^{2k}(E) \to 0.$$

The Gysin sequence implies that the map $H^{2k}(Y) \to H^{2k}(E)$ is onto, so the odd cohomology of M vanishes. Comparing the two short sequences, we find that the even Betti numbers satisfy the relation $b_{2k}(M) = b_{2k}(W) + b_{2k-2}(Y)$ for $k > 0$ and $b_0(M) = b_0(W)$.

The result now easily follows from these relations, together with the inductive hypothesis, the above formulae relating d_k, e_k, g_k, and standard identities for binomial coefficients. □

Example 6.9. Consider $M(\underline{u}, \underline{\lambda})$ where u_k, λ_k are as in Example 2.2. The polytopes Δ_s of Theorem 6.5 are just two intervals with a common point, so the deformation retract of $M(\underline{u}, \underline{\lambda})$ given by this theorem is the union of two copies of $\mathbb{C}P^1$ intersecting at a point. This retract is the exceptional divisor of the resolution of $\mathbb{C}^2/\mathbb{Z}_3$.

Example 6.10. Suppose u_k, λ_k are as in Example 2.3, with $a = 1$. For a suitable choice of λ_3, we see that $|\mathcal{C}|$ is the union of a trapezoid and an isosceles right triangle intersecting along a line segment (see Fig. 5). The deformation retract X of M, given by Theorem 6.5, is the union of $\mathbb{CP}^2$ and the blowup of $\mathbb{CP}^2$, intersecting along the exceptional divisor. We calculate, using (6.6), that $b_2 = b_4 = 2$. If we decrease λ_3, then $|\mathcal{C}|$ becomes the union of two isosceles right triangles meeting in a point (Fig. 6). The deformation retract X' of M is the union of two copies of $\mathbb{C}P^2$ intersecting at a point. Corollary 6.6 implies that X and X' are homotopy equivalent.

Proposition 6.8 implies that the deformation retract of M is always a pseudomanifold.

7. Toric hyperkähler manifolds from polytopes.

In this section we shall discuss the toric hyperkähler manifolds corresponding to a convex polytope Δ in $\mathbb{R}^n$. That is, we shall consider $M(\underline{u}, \underline{\lambda})$ where $\underline{u} = (u_1, \dots, u_d)$, $\underline{\lambda} = (\lambda_1, \dots, \lambda_d)$, $\lambda_k = (\lambda_k^1, 0, 0)$ and Δ is the intersection of half-spaces

$$\langle x, u_k \rangle \geq \lambda_k^1, \qquad (k = 1, \dots, d), \tag{7.1}$$

as in §2. We shall always assume that Δ is simple, that is, there are precisely n edges meeting at each vertex of Δ. In this situation we shall write M_Δ for $M(\underline{u}, \underline{\lambda})$. It is useful to observe that with this choice of λ_k, a collection of flats H_k intersect if and only if the corresponding collection of hyperplanes H_k^1 intersect.

We shall be particularly interested in the relation between M_Δ and the Kähler toric variety X_Δ obtained by the construction of section 2. First of all we shall show that the cotangent bundle of a toric manifold always carries a hyperkähler metric (usually incomplete).

Theorem 7.1. *Let X_Δ be a smooth compact toric variety corresponding to a Delzant polytope Δ. Then T^*X_Δ with its natural complex-symplectic structure is T^n-equivariantly isomorphic to an open subset U_Δ of the (usually singular) space $(M_\Delta, J_1, \omega_2 + \sqrt{-1}\omega_3)$. If we identify U_Δ with T^*X_Δ, the hyperkähler metric of M_Δ restricted to the zero section of T^*X_Δ is the Kähler metric on X_Δ determined by Δ.*

Proof. Consider the open subset $Y = \mathbb{C}^d_\Delta \times \mathbb{C}^d$ of $\mathbb{H}^d \simeq \mathbb{C}^d \times \mathbb{C}^d$, where $\mathbb{C}^d_\Delta$ is given by (2.10). Now Y is a hyperkähler T^d-invariant submanifold of $\mathbb{H}^d$ so in particular is N-invariant, where N denotes the torus of (2.5). Moreover the action of N on Y is free, because it is free on $\mathbb{C}^d_\Delta$. Therefore we can perform the hyperkähler quotient construction on Y and obtain a smooth manifold U_Δ which is an open subset of M_Δ. Note that U_Δ is preserved by the T^n action on M_Δ.

We want to identify U_Δ, the hyperkähler quotient of Y by N, with the complex-symplectic quotient of Y by $N^{\mathbb{C}}$ (with respect to the complex structure J_1). For this we have to show that every $N^{\mathbb{C}}$ orbit in the intersection of Y with the zero-set of (3.3b) (where $c_2 = c_3 = 0$) meets the zero-set of (3.3a). Let (z, w) be in the zero-set of (3.3b), where $z \in \mathbb{C}^d_\Delta$. From the proof of Theorem 5.1 we know that the image of the $N^{\mathbb{C}}$-orbit of (z, w) under (3.3a) is $S = \left\{\sum_{\{i;z_i\neq 0\}} t_i\alpha_i - \sum_{\{i;w_i\neq 0\}} s_i\alpha_i + c_1 : t_i, s_i > 0\right\}$. We also know [Gu2] that the image under (2.6) is the set $\left\{\sum_{\{i;z_i\neq 0\}} t_i\alpha_i + c_1 : t_i > 0\right\}$ and that for $z \in \mathbb{C}^d_\Delta$ this set is open. However, since $z \in \mathbb{C}^d_\Delta$, this last set contains 0, so S contains 0.

We have shown that (U_Δ, J_1) is the complex-symplectic quotient of Y by $N^{\mathbb{C}}$, and so is $\{(z, w) \in \mathbb{C}^{2d} : \sum_{k=1}^d (z_k w_k)\alpha_k = 0,\ z \in \mathbb{C}^d_\Delta\}/N^{\mathbb{C}}$. The equation in z, w simply says that the vector $w \in T^*_z\mathbb{C}^d_\Delta$ annihilates the vertical tangent vectors of the projection $\mathbb{C}^d_\Delta \to \mathbb{C}^d_\Delta/N^{\mathbb{C}} = X_\Delta$. This shows that (U_Δ, J_1) is biholomorphic to T^*X_Δ. It is also clear that the symplectic forms are the same, since the form $\omega_2 + \sqrt{-1}\omega_3$ on $T^*\mathbb{C}^d_\Delta$ is just $\sum dz_k \wedge dw_k$. The statement about the metrics follows as in (iii) of Theorem 6.5. □

The metric on T^*X_Δ is complete precisely when $U_\Delta = M_\Delta$. However our next result shows that this occurs only when X_Δ is the product of projective spaces.

Theorem 7.2. *Let X_Δ be a smooth compact toric variety as in Theorem 7.1. Then the following conditions are equivalent:*

(i) $U_\Delta = M_\Delta = T^*X_\Delta$,

(ii) *if some collection of hyperplanes containing $(n-1)$-dimensional faces of Δ do not meet in Δ, then they do not meet outside Δ,*

(iii) *X_Δ is the product of projective spaces.*

Proof. Observe that (ii) is equivalent to requiring that all vertices (intersections of n hyperplanes) lie in Δ. It is also clear from Theorem 6.5 or the remark after Theorem 3.1 that (ii) is necessary for (i). Let us show that it is sufficient.

Without loss of generality we can assume that Δ contains 0 in its interior, so all the λ_k^1 must be negative. Now, $\mu_1^{-1}(0)$ can be written as the set of (z, w) satisfying

$$\frac{1}{2}\sum_{k=1}^{d}|z_k|^2\alpha_k = \sum_{k=1}^{d}\left(\frac{1}{2}|w_k|^2 - \lambda_k^1\right)\alpha_k.$$

It follows that if $(z,w) \in \mu_1^{-1}(0)$, then z lies in $(\mu_1')^{-1}(0)$ where μ_1' is the moment map (2.6) with a different choice of c. Hence z belongs to $\mathbb{C}^d_{\Delta'}$ where Δ' is the intersection of half-spaces (7.1) with λ_k^1 of possibly *larger* absolute value (note that (ii) implies that Δ' is Delzant). Condition (ii) also shows that in fact $z \in \mathbb{C}^d_\Delta$. Hence the hyperkähler quotient of $\mathbb{H}^d$ by N is the same as the hyperkähler quotient of $Y = \mathbb{C}^d_\Delta \times \mathbb{C}^d$ by N. The proof of Theorem 7.1 now shows that $U_\Delta = M_\Delta$.

The implication (iii) $\Rightarrow$ (ii) is obvious. Let us now show the converse. As usual, we denote by u_{k_i} the vectors defining Δ. We consider the fan $\mathcal{F}$ corresponding to the polytope Δ and defined at the end of section 2. Condition (ii) implies that for any independent set of vectors $\{u_{k_1}, \dots, u_{k_s}\}$ the cone $\{\sum t_i u_{k_i} \ : \ t_i \geq 0\}$ belongs to $\mathcal{F}$. Indeed, since the vectors are independent, the hyperplanes orthogonal to them must intersect, so by (ii) they intersect in Δ.

From this two facts follow: 1) any vector in $\mathbb{R}^n$ can be written uniquely as $\sum t_i u_{k_i}$ with $t_i > 0$ and $u_{k_1}, \dots, u_{k_s}$ linearly independent; 2) if Δ' is another Delzant polytope, then there are no nontrivial equivariant birational morphisms $X_\Delta \to X_{\Delta'}$. For 1) notice that if a vector could be written thus in two ways, then then the cones spanned by the two sets of u_{k_i} would intersect in their interior, contradicting the definition of the fan. For 2) we first recall [Od] that such a morphism corresponds to removing a number of $(n-1)$-dimensional walls in cones of the fan $\mathcal{F}$ of X_Δ to obtain the fan $\mathcal{F}'$ of $X_{\Delta'}$. Consider an n-dimensional cone σ in $\mathcal{F}'$ that is not in $\mathcal{F}$. If σ is a cone over a simplex, then the vectors generating σ are linearly independent and we get a contradiction as $\sigma \notin \mathcal{F}$. If σ has more than n generating vectors, then taking two independent n-element sets such that the cones spanned by them have n-dimensional intersection we obtain a contradiction with the fact that the intersection of two cones in $\mathcal{F}$ is a face of each of them.

We appeal now to Reid's version [Re] of Mori's theory for projective toric varieties (see also the exposition in [Od]). We can conclude from fact 2) above, and Corollary 2.28(1) and Theorem 2.27(2) in [Od], that $\mathbb{R}^n = \sum V_i$

where each V_i is a vector space of positive dimension and each 1-dimensional cone of $\mathcal{F}$ lies in some V_i. Moreover, each V_i is spanned by the cones it contains. (In Oda's terminology, the V_i are the spaces $\pi_+(R)$ where R ranges over the extremal rays of $NE(X_\Delta)$). We denote by $\mathcal{F}_i$ the restriction of $\mathcal{F}$ to V_i, that is, the cones of $\mathcal{F}_i$ are precisely the cones of $\mathcal{F}$ contained in V_i. Now Corollary 2.6 of [Re] shows that each $\mathcal{F}_i$ is a fan of a projective space of an appropriate dimension. It remains to show that the sum $\sum V_i$ is direct. Suppose that the sum $V_1 + \ldots + V_s$ is direct and that V_{s+1} intersects $\bigoplus_1^s V_i$ nontrivially. If v lies in the intersection, then, because of the definition of the spaces V_i, it can be written as $\sum t_i u_{k_i}$ with $t_i > 0$, $u_{k_i} \in \bigoplus_1^s V_i$ and also as $\sum s_j u_{l_j}$ with $s_j > 0$, $u_{l_j} \in V_{s+1}$, where the u_{k_i} and the u_{l_j} are linearly independent. By fact 1) the two sets $\{u_{k_i}\}$ and $\{u_{l_j}\}$ are equal and the vectors u_{k_i} must belong to both $\bigoplus_1^s V_i$ and to V_{s+1}. The vector $-u_{k_1}$ also belongs to both $\bigoplus_1^s V_i$ and to V_{s+1}. Moreover, since the fan $\mathcal{F}_{s+1}$ is the fan of a projective space, $-u_{k_1}$ belongs to the open cone in $\mathcal{F}_{s+1}$ generated by all 1-dimensional cones of $\mathcal{F}_{s+1}$ except u_{k_1} and so it can be written as their combination with all coefficients positive. Repeating the previous argument with $v = -u_{k_1}$ shows that all 1-dimensional cones of $\mathcal{F}_{s+1}$ belong to $\bigoplus_1^s V_i$ and so $V_{s+1} \subset \bigoplus_1^s V_i$. In fact we have shown that any 1-dimensional cone of $\mathcal{F}_{s+1}$ is a 1-dimensional cone of some $\mathcal{F}_i$, $i \leq s$. However, each of these fans is the fan of a projective space, and the only way that all generators of a fan of a projective space can lie among generators of fans of other projective spaces lying in a direct sum of the relevant vector spaces is when $\mathcal{F}_{s+1}$ is equal to $\mathcal{F}_i$, for some $i \leq s$. Such a repetition does not alter the conclusion that $\mathcal{F}$ is the fan of a product of projective spaces. □

We can also ask when M_Δ is smooth. This is equivalent to asking whether the hyperkähler metric on T^*X_Δ can be smoothly completed. Delzant's work shows that the toric variety X_Δ obtained by the construction of section 2 is smooth if and only if whenever n of the defining hyperplanes meet at a vertex of the simple polytope Δ, the corresponding vectors u_i form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. This condition is not, however, sufficient for M_Δ to be smooth. Indeed, Theorem 3.2 requires that the Delzant condition holds at *any* intersection of n hyperplanes even if the intersection is outside Δ. In particular each of the varieties X_s of Theorem 6.5 must be smooth.

Proposition 7.3. *Let X be a smooth projective toric variety of complex dimension n. Then the following statements are equivalent:*

(i) *X carries a T^n-invariant Kähler metric such that, if Δ denotes the corresponding Delzant polytope, then M_Δ is smooth,*

(ii) *every set of n independent generators u_i of the fan of X is a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.*

Proof. The above discussion shows that (i) implies (ii). As X is projective and toric it can be embedded equivariantly in projective space so admits a T^n-invariant Kähler metric, so can be obtained from the Delzant construction. As in Remark 3.4, by adjusting λ_i we can choose an invariant Kähler metric on X so that no $n+1$ flats intersect. Condition (ii), together with the argument at the beginning of 3.2, now shows that the condition of 3.2 holds, so M_Δ is smooth. □

Remark 7.4. The argument of 3.4 shows that every smooth projective toric variety carries a T^n-invariant Kähler metric such that M_Δ is a hyperkähler completion of T^*X_Δ with at worst abelian quotient singularities.

Condition (ii) of Proposition 7.3 is rather restrictive. Let us choose an n-dimensional cone of $\mathcal{F}$, which we can take to be generated by vectors e_i, $(i = 1, \dots, n)$. Then any other generator u_i of $\mathcal{F}$ must have coordinates in $\{-1, 0, 1\}$. In particular the number of 1-dimensional cones of $\mathcal{F}$ is bounded by $3^n - 2$ (we exclude the zero vector and $e_1 + \ldots + e_n$) and so there only finitely many such varieties in each dimension. Proposition 7.3 can be used to show the following results.

Proposition 7.5. *Let X be a smooth compact toric variety of complex dimension 2 satisfying assumption (ii) of Proposition 7.3. Then X is either $\mathbb{C}P^1 \times \mathbb{C}P^1$ or the equivariant blow-up of $\mathbb{C}P^2$ at k points, where $0 \leq k \leq 3$.*
□

Proposition 7.6. *Let $X_\Delta = \prod X_i$, where each X_i is the equivariant blow-up of $\mathbb{C}P^{n_i}$ at k_i points, and $0 \leq k_i \leq n_i + 1$. Then M_Δ is smooth.*

8. Kähler potentials.

Guillemin has derived a formula for the Kähler form of a toric variety in terms of the associated polytope [Gu1]. We shall now find an expression in terms of $\underline{u}, \underline{\lambda}$ for the Kähler form, say ω_1, on the hyperkähler manifold $M(\underline{u}, \underline{\lambda})$.

The Kähler form ω (and so the metric) of a Kähler manifold X is locally determined by a Kähler potential, a real-valued function K locally defined on X such that $2\sqrt{-1}\,\partial\bar{\partial}K = \omega$. In general, finding the Kähler potential of a hyperkähler manifold is a complicated problem. It is much simpler, however, when the $4n$-dimensional hyperkähler manifold M admits a free Hamiltonian action of an n-dimensional abelian group G preserving the hyperkähler structure. Then M described as a principal G-bundle over an open subset of $\mathbb{R}^n \otimes \mathbb{R}^3$ where is the projection is just the moment map $\phi = (\phi_1, \phi_2, \phi_3) : M \to \mathbb{R}^n \otimes \mathbb{R}^3$. Since the group action preserves the hyperkähler structure, the Kähler potential with respect to any complex structure does not depend on the fiber coordinate. It is convenient to introduce the map $\pi = (2\phi_1, \phi_2 + \sqrt{-1}\phi_3) : M \to \mathbb{R}^n \times \mathbb{C}^n$.

Theorem 8.1 [HKLR]. *In the above situation the Kähler potential for the form ω_1 on M is*

$$K = \pi^* \left(F - \sum_{i=1}^{n} s_i \frac{\partial F}{\partial s_i} \right), \tag{8.1}$$

where $F = F(s_i, v_i, \bar{v}_i)$ is a real-valued function on $\mathbb{R}^n \times \mathbb{C}^n$ satisfying the linear equations $F_{s_i s_j} + F_{v_i \bar{v}_j} = 0$, $(1 \le i, j \le n)$. □

Our manifolds fall into this class of examples with $G = T^n$, provided we restrict to an open dense subset.

Example 8.2. Consider the open subset of $\mathbb{H}^d$ on which the diagonal action of the torus T^d is free. We have [HKLR]

$$F(s, v, \bar{v}) = \frac{1}{4} \sum_{i=1}^{d} (r_i - s_i \ln(s_i + r_i)), \tag{8.2}$$

where $r_i^2 = s_i^2 + 4v_i\bar{v}_i$. Here v_i, s_i are related to our standard coordinates z_i, w_i by $v_i = z_i w_i$ and $s_i = |z_i|^2 - |w_i|^2$. The Kähler potential is given by $K = \frac{1}{4} \sum r_i$.

We now want to calculate the Kähler potential for the form ω_1 on our hyperkähler quotient $M = M(\underline{u}, \lambda)$ (or more precisely on the open dense subset where T^n acts freely). Our metric was given as the hyperkähler quotient of the metric of Example 8.2 by some subtorus of T^d. In the coordinates s_i, v_i, the equations definining the zero-set of the moment map (3.3) become linear:

$$\sum_{k=1}^{d} (s_k + 2\lambda_k^1) \alpha_k = 0, \tag{8.3a}$$

(8.3b)
$$\sum_{k=1}^{d}(v_k + \lambda_k^2 + \sqrt{-1}\lambda_k^3)\alpha_k = 0.$$

The function F given by (8.2) restricts to the flats defined by(8.3), and gives the Kähler potential on M via formula (8.1) [HKLR, section 2(C)]. Therefore all that remains to be done is to express this restricted function in terms of the coordinates $(a,b) = (\phi_1(m), (\phi_2 + \sqrt{-1}\phi_3)(m))$, where (ϕ_1, ϕ_2, ϕ_3) is the hyperkähler moment map for the action of T^n on M.

Let $m \in M$, and suppose that the image of m in $\mathbb{R}^d \times \mathbb{C}^d$ is a point (s, v) satisfying (8.3). Using (3.6) we obtain:

(8.4)
$$s_k = 2\langle a, u_k\rangle - 2\lambda_k^1, \quad v_k = \langle b, u_k\rangle - \lambda_k^2 - \sqrt{-1}\lambda_k^3$$

and so we have the function F for (M, ω_1). We calculate the Kähler potential according to (8.1) and obtain

$$F - \sum_{i=1}^{n} a_i \frac{\partial F}{\partial a_i} = F - \sum_{i=1}^{n}\sum_{k=1}^{d} a_i \frac{\partial F}{\partial s_k}\frac{\partial s_k}{\partial a_i} = F - 2\sum_{i=1}^{n}\sum_{k=1}^{d} a_i \frac{\partial F}{\partial s_k}(u_k)_i$$
$$= F - \sum_{k=1}^{d}(s_k + 2\lambda_k^1)\frac{\partial F}{\partial s_k} = \frac{1}{4}\sum_{k=1}^{d}\left(r_k + 2\lambda_k^1 \ln(s_k + r_k)\right),$$

where at the last step we use the equation

$$\frac{\partial F}{\partial s_k} = \frac{1}{4}\sum_{k=1}^{d}\left(\frac{s_k}{r_k} - \ln(s_k + r_k) - \frac{s_k}{s_k + r_k}\left(1 + \frac{s_k}{r_k}\right)\right) = -\frac{1}{4}\sum_{k=1}^{d}\ln(s_k + r_k).$$

This gives the next theorem, in which $\pi : M \to \mathbb{R}^n \times \mathbb{C}^n$ is the projection defined above, and ∂_1 is the Dolbeault operator corresponding to the complex structure J_1.

Theorem 8.3. *On the open dense subset where the action of T^n is free, the Kähler form ω_1 on the toric hyperkähler manifold $M = M(\underline{u}, \underline{\lambda})$ is given by:*

(8.5)
$$\omega_1 = \frac{\sqrt{-1}}{2}\partial_1\bar{\partial}_1\pi^*\left(\sum_{k=1}^{d}\left(r_k + 2\lambda_k^1 \ln(s_k + r_k)\right)\right),$$

where s_k and v_k are given by (8.4) *and* $r_k^2 = s_k^2 + 4v_k\bar{v}_k$. □

In the situation of Theorem 7.1, restricting (8.5) to U_Δ and then to X_Δ, that is, the subset of U_Δ where $v_1 = \ldots = v_d = 0$, gives the formula of Guillemin [Gu1] for the Kähler form of the toric variety X_Δ.

9. The metric and generalized monopoles.

Pedersen and Poon [PP] have given an explicit formula for the metric of a hyperkähler $4n$-manifold M with a free action of T^n preserving the hyperkähler structure. Using the coordinate system a_i, b_i, they find that if F is the function of Theorem 8.1 for M, then by putting

$$(\Phi_{ij}, A_j) = \left(2F_{a_i a_j}, \sum_l \sqrt{-1}(F_{a_j b_l} db_l - F_{a_j \bar{b}_l} d\bar{b}_l)\right) \tag{9.1}$$

we obtain a solution to the generalized Bogomolny equations with gauge group T^n. We call such a solution a monopole. More precisely, we can define a pair (A, Φ) by putting $A = (A_1, \dots, A_n)$ and $\Phi = (\Phi_1, \dots, \Phi_n)$ where $\Phi_i = (\Phi_{i1}, \dots, \Phi_{in})$. Then A is a 1-form on $\mathbb{R}^3 \otimes \mathbb{R}^n$ with values in $\mathbb{R}^n$ and Φ_i are Higgs fields $\mathbb{R}^3 \otimes \mathbb{R}^n \to \mathbb{R}^n$. If we put $w_j^1 = a_j, w_j^2 = \operatorname{Re} b_j, w_j^3 = \operatorname{Im} b_j$, then (A, Φ) satisfy the linear system of PDEs

$$\begin{aligned} R_{w_i^\alpha w_j^\beta} &= \sum_\gamma \epsilon_{\alpha\beta\gamma} \nabla_{w_i^\gamma} \Phi_j \,, \\ \nabla_{w_i^\alpha} \Phi_j &= \nabla_{w_j^\beta} \Phi_i \,, \end{aligned} \tag{9.2}$$

where ϵ is the alternating symbol, $\nabla = d + A$ is a connection on the trivial $\mathbb{R}^n$ bundle over $\mathbb{R}^3 \otimes \mathbb{R}^n$ and R is its curvature.

The hyperkähler metric g on M is given by

$$g = -\sum_{i,j} \left[\Phi_{ij}(da_i da_j + db_i d\bar{b}_j) + \Phi_{ij}^{-1}(dy_i + A_i)(dy_j + A_j)\right], \tag{9.3}$$

where $dy_i = \sqrt{-1}(\bar{\partial}_1 F_{a_i} - \partial_1 F_{a_i})$ are the fiber coordinates given by Killing vector fields corresponding to the action of T^n.

We shall now find the monopole corresponding to the metric on the toric hyperkähler manifold $M = M(\underline{u}, \underline{\lambda})$. We have to calculate partial derivatives $F_{a_i a_j}, F_{a_i b_j}, F_{a_i \bar{b}_j}$ where F is given by (8.2) and (8.4). We have

$$F_{a_i} = \sum_{k=1}^d \frac{\partial F}{\partial s_k} \frac{\partial s_k}{\partial a_i} = -\frac{1}{2} \sum_{k=1}^d \ln(s_k + r_k)(u_k)_i, \tag{9.4}$$

and then

$$F_{a_i a_j} = -\sum_{k=1}^d \frac{\frac{\partial s_k}{\partial a_j} + \frac{\partial r_k}{\partial a_j}}{s_k + r_k} (u_k)_i = -\sum_{k=1}^d \frac{(u_k)_j (u_k)_i}{r_k}, \tag{9.5}$$

$$F_{a_i b_j} = -\frac{1}{2}\sum_{k=1}^{d} \frac{\frac{\partial r_k}{\partial b_j}}{s_k + r_k}(u_k)_i = -\sum_{k=1}^{d} \frac{\bar{v}_k (u_k)_j (u_k)_i}{(s_k + r_k) r_k}, \tag{9.6}$$

and

$$F_{a_i \bar{b}_j} = -\sum_{k=1}^{d} \frac{v_k (u_k)_j (u_k)_i}{(s_k + r_k) r_k}. \tag{9.7}$$

This gives us the monopole and therefore the following explicit formula for the metric on M in terms of the moment map.

Theorem 9.1. *On the open dense subset where the action of T^n is free, the hyperkähler metric g on M is given by* (9.3), *where*

$$(\Phi_{ij}, A_j) = \left(-2\sum_{k=1}^{d} \frac{(u_k)_j (u_k)_i}{r_k},\ \sqrt{-1}\sum_{l=1}^{n}\sum_{k=1}^{d} \frac{(u_k)_j (u_k)_l}{(s_k + r_k) r_k} (v_k d\bar{b}_l - \bar{v}_k d b_l) \right),$$

and $\frac{\partial}{\partial y_j}$ *are infinitesimal isometries given by*

$$dy_i = \frac{\sqrt{-1}}{2}(\partial_1 - \bar{\partial}_1) \sum_{k=1}^{d} \ln(s_k + r_k)(u_k)_i. \qquad \square$$

Once more, in the situation of Theorem 7.1 restricting to $v_k = 0$ gives a formula for the Kähler metric on the toric variety X_Δ.

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