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Positive Mass Theorem for Hypersurface in 5-Dimensional Lorentzian Manifolds

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1. Introduction.

It is well-known that Positive Mass Theorem has a fundamental importance in Einstein's general relativity. The positive mass theorem for 5-dimensional Lorentzian manifolds is therefore interesting in the context of Kaluza-Klein theory which provides a 5-dimensional general relativity containing both Einstein's 4-dimensional theorey of gravity and Maxwell's theory of electromagnetism. This idea of Kaluza-Klein was enthusiastically received by unified-field theorists and was extended to higher dimensions to include the strong and weak forces (i.e., 11-dimensional supergravity theories and 10-dimensional superstrings). We refer to review article [OW] for higher-dimensional unified theories from the general relativity side. Mathematically, the existence of $Spin^c$ structures on orientable 4-manifolds provides a unified treatment on gravity and electromagnetism. In this paper we adapt Witten's method and the analytic arguments of Parker and Taubes to such a $Spin^c$ structure. This yields a Positive Mass Theorem (Theorem 1.6 below) for hypersurfaces in 5-dimensional Lorentzian manifolds.

Let N be a 5-dimensional Lorentzian manifold with Lorentzian metric $\widetilde{g}$ of signature $(-1, 1, 1, 1, 1)$, which satisfies the Einstein equations

$$\widetilde{R}_{\alpha\beta} - \frac{\widetilde{R}}{2}\,\widetilde{g}_{\alpha\beta} = T_{\alpha\beta}, \tag{1.1}$$

where $\widetilde{R}_{\alpha\beta}$, $\widetilde{R}$ are the Ricci and scalar curvatures of $\widetilde{g}$ respectively, $T_{\alpha\beta}$ is a symmetric tensor field which is interpreted physically as the energy-momentum tensor of matter.

Definition 1.1. A spacelike hypersurface M of N is called asymptotically flat of order τ if there is a compact set $K \subset M$ such that $M - K$ is the disjoint union of a finite number of subsets $M_1, \cdots, M_k$ – called the "ends" of M – each diffeomorphic to the complement of a contractible compact set in R^4. Under the diffeomorphism the metric of $M_l \subset M$ is of the form

$$g_{ij} = \delta_{ij} + a_{ij} \tag{1.2}$$

in the standard coordinates $\{x^i\}$ on R^4, where a_{ij} satisfies

$$a_{ij} = O(r^{-\tau}), \qquad \partial_k a_{ij} = O(r^{-\tau-1}), \qquad \partial_l \partial_k a_{ij} = O(r^{-\tau-2}). \tag{1.3}$$

Furthermore, the second fundamental form of M satisfies

$$h_{ij} = O(r^{-\tau-1}), \qquad \partial_k h_{ij} = O(r^{-\tau-2}). \tag{1.4}$$

A $U(1)$ line bundle L over M is called asymptotically flat of order τ if there is a trivialization of L over the end and a $u(1)$-value 1-form A such that on end M_l, the connection on L can be written as

$$d_{Aj} = \partial_j + A_j \mathbf{i}, \tag{1.5}$$

where A_j is real, and satisfies

$$A_j = O(r^{-\tau-1}), \qquad \partial_k A_j = O(r^{-\tau-2}). \tag{1.6}$$

We will often identify the end $M_l \subset M$ with the corresponding set $M_l \subset R^4$.

The curvature $F_A = dA$ of such a connection on L may be interpreted physically as the electromagnetic field. For spacelike asymptotically flat hypersurface M and asymptotically flat line bundle L, we can define the total energy, the total linear momentum and the total electromagnetic momentum. They are defined in each asymtotic end M_l as limits over the sphere $S_{R,l}$ of radius R in $M_l \subset R^4$.

Definition 1.2. Total energy of end M_l is defined as

$$E_l = \lim_{R\to\infty} C_4^{-1} \int_{S_{R,l}} (\partial_j g_{ij} - \partial_i g_{jj}) d\Omega^i, \tag{1.7}$$

total linear momentum of end M_l is defined as

$$p_{lk} = \lim_{R\to\infty} C_4^{-1} \int_{S_{R,l}} 2(h_{ik} - \delta_{ik} h_{jj}) d\Omega^i, \tag{1.8}$$

total electromagnetic momentum of end M_l is defined as

$$q_{lij} = \lim_{R\to\infty} C_4^{-1} \left(\int_{S_{R,l}} 2A_j d\Omega^i - \int_{S_{R,l}} 2A_i d\Omega^j \right), \tag{1.9}$$

where $C_4 = 12\omega_3$ and ω_3 is the volume of unit sphere S^3 with standard metric.

Definition 1.3. The current matrix of electromagnetic field on end M_l is defined by

$$\Omega_l = (\omega_{lij}),$$

where

$$\begin{aligned}
\omega_{l11} &= 2^{-1}(-q_{l12}^2 - q_{l13}^2 - q_{l14}^2 + q_{l34}^2 + q_{l42}^2 + q_{l23}^2),\\
\omega_{l22} &= 2^{-1}(-q_{l12}^2 + q_{l13}^2 + q_{l14}^2 + q_{l34}^2 - q_{l42}^2 - q_{l23}^2),\\
\omega_{l33} &= 2^{-1}(q_{l12}^2 - q_{l13}^2 + q_{l14}^2 - q_{l34}^2 + q_{l42}^2 - q_{l23}^2),\\
\omega_{l44} &= 2^{-1}(q_{l12}^2 + q_{l13}^2 - q_{l14}^2 - q_{l34}^2 - q_{l42}^2 + q_{l23}^2),\\
\omega_{lij} &= \sum_{k\neq\{i,j\}} q_{lik}q_{lkj}, \qquad 1 \leq i,j \leq 4,\ i \neq j.
\end{aligned}$$

When the asymptotic order $\tau > 1$, these quantities are finite, independent on the choice of asymptotic coordinates. Since $q_{lij} = -q_{lji}$, Ω_l is real symmetric. Moreover, Ω_l is traceless.

The following Positive Mass Conjecture was proved first by R. Schoen and S.T. Yau [SY1, SY2, SY3], then by E. Witten [W, PT].

Theorem 1.4 (Schoen-Yau, Witten). *Let N be a 4-dimensional Lorentzian manifold with Lorentzian metric $\tilde{g}$ of signature $(-1,1,1,1)$, which satisfies the Einstein equations (1.1), $M \subset N$ be a spacelike asymptotically flat hypersurface of order $\tau > \frac{1}{2}$. If M satisfies the dominant energy condition*

$$T_{00} \geq \sqrt{\sum_i T_{0i}^2}, \quad \text{and} \quad T_{00} \geq |T_{\alpha\beta}|,$$

then, for each end M_l, we have

$$E_l \geq \sqrt{\sum_i p_{li}^2}.$$

If $E_{l_0} = 0$ for some l_0, then M has only one end and N is flat over M.

One key point in Witten's argument is to prove that there is a positive definite Hermitian metric on $Spin(3,1)$ spinors. This fact was verified by T. Parker and C. Taubes [PT] in terms of representation theory of spin group $SL(2,C)$, and was extended to $Spin(4,1)$ spinors by the author in terms of representation theory of spin group $HU(1,1)$. Consequently, Positive Mass

Conjecture can be proved for spin spacelike hypersurface in 5-dimensional Lorentzian manifolds [Z1]. It should be true for all spin group $Spin(n,1)$, an issue we will address elsewhere.

Now since N is 5-dimensional and M is an orientable hypersurface in N, M has a $Spin^c$ structure. It means that there is a $U(1)$ line bundle L on N such that $S \otimes L^{\frac{1}{2}}$ is globally-defined over M, where S is (locally) spinor bundle of N, which is not globally-defined on N except that N is spin. Denote $W = S \otimes L^{\frac{1}{2}}$. W is called the complex Witten-Dirac spinor bundle, and L is called $Spin^c$ structure. Let A be a $U(1)$ connection 1-form on L, and denote F_A^M as the curvature of L restricted on M. The corresponding connection on $L^{\frac{1}{2}}$ is $\tilde{d}_A = d + \frac{1}{2}A$. Let ∇ be the metric connection on S. Then the globally-defined connection ∇_A and the metric on W are defined as follows: write $\phi = s_1 \otimes \sigma_1$, $\psi = s_2 \otimes \sigma_2$ locally, where $s_1, s_2 \in S$, $\sigma_1^2, \sigma_2^2 \in L$, then

$$\begin{aligned} \nabla_A \phi &= \nabla s_1 \otimes \sigma_1 + s_1 \otimes \tilde{d}_A \sigma_1, \\ \langle \phi, \psi \rangle_W &= \langle s_1, s_2 \rangle_S \cdot \langle \sigma_1, \sigma_2 \rangle_L \, . \end{aligned}$$

Obviously, ∇_A is compatible with the metric $\langle \, , \, \rangle_W$. At each $p \in M$, we fix an orthonormal frame $\{e_\alpha | \alpha = 0,1,2,3,4\}$ with e_0 normal to M and e_1, e_2, e_3, e_4 tangent to M. (Here, and henceforth, repeated indices are summed with Latin indices running from 1 to 4 and Greek indices running from 0 to 4.) Denote $\{e^\alpha | \alpha = 0,1,2,3,4\}$ as its dual frame.

Definition 1.5. The above M satisfies the charged dominant energy condition if

$$T_{00} \geq \sqrt{\sum_i T_{0i}^2} + \sqrt{\sum_{i,j} F_{Aij}^2}, \quad \textit{and} \quad T_{00} \geq |T_{\alpha\beta}| + |F_{A\alpha\beta}| \, . \tag{1.10}$$

Theorem 1.6. *Let N be a 5-dimensional Lorentzian manifold with Lorentzian metric $\tilde{g}$ of signature $(-1,1,1,1,1)$, which satisfies the Einstein equations (1.1), $M \subset N$ be a spacelike asymptotically flat hypersurface of order $\tau > 1$. Let L be the $Spin^c$ structure of complex Witten-Dirac spinor bundle of M with $U(1)$ connection A, which is also asymptotically flat of order $\tau > 1$. If M satisfies the charged dominant energy condition (1.10), then, for each end M_l, we have*

$$E_l \geq \begin{cases} \sqrt{|Q_l|^2 + 2\,|q_{l12}q_{l34} + q_{l13}q_{l42} + q_{l14}q_{l23}|} & if \;\; |P_l| = 0, \\ |P_l| + \sqrt{2^{-1}\,|Q_l|^2 + \vec{P}_l^t \Omega_l \vec{P}_l} & if \;\; |P_l| \neq 0, \end{cases}$$

where $|P_l| = \sqrt{\sum_i p_{li}^2}$, $|Q_l| = \sqrt{\sum_{i<j} q_{lij}^2}$ *and* $\vec{P}_l = |P_l|^{-1}(p_{l1}, p_{l2}, p_{l3}, p_{l4})^t$ *if* $|P_l| \neq 0$. *If* $E_{l_0} = 0$ *for some* l_0, *then* M *has only one end and* N, L *are flat over* M. *Moreover,* $p_{l_0k} = 0$, $q_{l_0ij} = 0$.

We also prove an analogous theorem for 4-dimensional Lorentzian manifolds in the appendix. Namely,

Theorem 1.7. *Let* N *be a 4-dimensional Lorentzian manifold with Lorentzian metric* $\tilde{g}$ *of signature* $(-1, 1, 1, 1)$, *which satisfies the Einstein equations (1.1),* $M \subset N$ *be a spacelike asymptotically flat hypersurface of order* $\tau > \frac{1}{2}$. *Let* L *be the* $Spin^c(3,1)$ *structure of* N *with* $U(1)$ *connection* A, *which is also asymptotically flat of order* $\tau > \frac{1}{2}$ *over* M. *If* M *satisfies the charged dominant energy condition (1.10), then, for each end* M_l, *we have*

$$E_l \geq \sqrt{|P_l|^2 + |Q_l|^2 + 2\,|p_{l1}q_{l23} + p_{l2}q_{l31} + p_{l3}q_{l12}|},$$

where $|P_l| = \sqrt{\sum_i p_{li}^2}$, $|Q_l| = \sqrt{\sum_{i<j} q_{lij}^2}$. *If* $E_{l_0} = 0$ *for some* l_0, *then* M *has only one end and* N, L *are flat over* M. *Moreover,* $p_{l_0k} = 0$, $q_{l_0ij} = 0$.

2. Spinors.

Let N be a 5-dimensional Lorentzian manifold, and M be a spacelike hypersurface in N. Denote H as the field of quaternions. The hyper-unitary group $HU(1,1) = Spin^0(4,1)$ is the double covering group of connected Lorentz group $SO(4,1)$ (see [Ha], p272). A $Spin^c$ structure on N is a globally defined $HU(1,1) \times_{Z_2} U(1)$ bundle W over M locally of the form $W = S \otimes L^{\frac{1}{2}}$. For any $X \in End(W)$, denote X^* the adjoint of X under $HU(1,1) \times_{Z_2} U(1)$ Hermitian structure. Denote

$$\aleph = \{X \in End(W), X = X^*, Trace(X) = 0\}.$$

There is an invariant metric on $\aleph$ defined for $X, Y \in \aleph$ by,

$$\langle X,\ Y\rangle = -\frac{1}{2}\Re e(Trace(XY)).$$

Moreover, for any $X \in T^*N$ with coordinate $(x_0, x_1, x_2, x_3, x_4)$, we have a canonical identification of X to an element in $\aleph$, i.e.,

$$X \mapsto \begin{pmatrix} x_0 & x \\ -\bar{x} & -x_0 \end{pmatrix}, \tag{2.1}$$

where $x = x_1 + x_2 I + x_3 J + x_4 K$. As in [Z1] one can prove that this defines an isometry $T^*N \equiv \aleph$.

The spinor bundle W has a $HU(1,1) \times_{Z_2} U(1)$ invariant Hermitian metric defined by

$$(\phi, \psi) = \bar{\xi}_1 \cdot \eta_1 - \bar{\xi}_2 \cdot \eta_2$$

for $\phi = (\xi_1, \xi_2)^t \in W, \psi = (\eta_1, \eta_2)^t \in W$. This metric is not positive definite.

The Clifford multiplication is the map $T^*N \otimes W \longrightarrow W$ that sends $X \otimes \phi$ to $X\phi$, where $X\phi$ means that spinor ϕ is mutiplied by the corresponding matrix (2.1) of covector X. Obviously, $XY + YX = -2\tilde{g}(X,Y) \cdot Id$. The choice of a timelike covector e^0 gives another Hermitian metric on W by

$$\langle \phi, \psi \rangle = (e^0 \phi, \psi) = \bar{\xi}_1 \cdot \eta_1 + \bar{\xi}_2 \cdot \eta_2$$

for $\phi = (\xi_1, \xi_2)^t \in W$, $\psi = (\eta_1, \eta_2)^t \in W$. This new metric is positive definite and $Sp(1) \times Sp(1) \times_{Z_2} U(1)$ invariant. Furthermore, for any $X \in T_p^* N$, $x \in T_p^* M$, spinors $\phi, \psi \in W$, we have

(2.2)
$$(X\phi,\ \psi) = (\phi,\ X\psi), \quad \langle x\phi,\ \psi \rangle = -\langle \phi,\ x\psi \rangle, \quad \langle e^0\phi,\ \psi \rangle = \langle \phi,\ e^0\psi \rangle.$$

The proofs of above facts are similar to those in [Z1]. By (2.1), we get a canonical representation of the coframe

$$\begin{aligned}
&e^0 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e^1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
&e^2 \mapsto \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad e^3 \mapsto \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}, \quad e^4 \mapsto \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}.
\end{aligned} \tag{2.3}$$

Now we derive the Pauli representation. We identify $H \cong C^2$ as follows: For any $x_H = x_1 + x_2 I + x_3 J + x_4 K = (x_1 + x_2 I) + J(x_3 - x_4 I) \in H$, we identify it to $x_C = (x_1 + x_2 \mathbf{i}, x_3 - x_4 \mathbf{i})^t \in C^2$. Since $I \cdot x_H = I(x_1 + x_2 I) + J(-I)(x_3 - x_4 I)$, $J \cdot x_H = J(x_1 + x_2 I) - (x_3 - x_4 I)$, and $K \cdot x_H = J(-I)(x_1 + x_2 I) - I(x_3 - x_4 I)$. We can obtain the following canonical Pauli representation

$$I \mapsto \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}, \quad J \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad K \mapsto \begin{pmatrix} 0 & -\mathbf{i} \\ -\mathbf{i} & 0 \end{pmatrix}. \tag{2.4}$$

For any $x_H, y_H \in H$, we have $\Re e(\bar{x}_H y_H) = \Re e(\bar{x}_C^t y_C)$. This fact implies that, for any $\phi, \psi \in W$, $\Re e \langle \phi, \psi \rangle_H = \Re e \langle \phi, \psi \rangle_C$, where $\langle\ ,\ \rangle_H$ is quaternions

Hermitian metric on W and $\langle\ ,\ \rangle_C$ is the corresponding complex Hermitian metric on W while W is viewed as a complex rank-4 bundle.

Obviously, $W = W^+ \oplus W^-$ over M, where $W^\pm = \{\phi \in W : *\phi = \pm\phi\}$ $(* = -e^1e^2e^3e^4)$. The 'half spinor bundles' $W^\pm$ are orthogonal w.r.t. metrics $(\ ,\ )$ and $\langle\ ,\ \rangle$. Moreover, since $e^0* = *e^0$, e^0 preserves $W^\pm$. Now the space of 2-forms of M splits as the self-dual part $\bigwedge^+$ and the anti-self-dual part $\bigwedge^-$, $\bigwedge^\pm = span\{e^I_\pm, e^J_\pm, e^K_\pm\}$, where

$$
\begin{aligned}
e^I_\pm &= e^1 \wedge e^2 \pm e^3 \wedge e^4,\\
e^J_\pm &= e^1 \wedge e^3 \pm e^4 \wedge e^2,\\
e^K_\pm &= e^1 \wedge e^4 \pm e^2 \wedge e^3.
\end{aligned}
\tag{2.5}
$$

Define the Clifford multiplication of 2-form on W by: $(e^i \wedge e^j) = e^ie^j$ $(i \neq j)$. A straightforward computation shows $\bigwedge^\pm W^\mp = 0$. Furthurmore,

$$e^I_-e^1 = -e^1e^I_+,\ e^I_-e^2 = -e^2e^I_+,\ e^I_-e^3 = e^3e^I_+,\ e^I_-e^4 = e^4e^I_+, \tag{2.6}$$

$$e^J_-e^1 = -e^1e^J_+,\ e^J_-e^2 = e^2e^J_+,\ e^J_-e^3 = -e^3e^J_+,\ e^J_-e^4 = e^4e^J_+, \tag{2.7}$$

$$e^K_-e^1 = -e^1e^K_+,\ e^K_-e^2 = e^2e^K_+,\ e^K_-e^3 = e^3e^K_+,\ e^K_-e^4 = -e^4e^K_+, \tag{2.8}$$

and

$$e^I_+e^J_+ = 2e^K_+,\ e^J_+e^K_+ = 2e^I_+,\ e^K_+e^I_+ = 2e^J_+, \tag{2.9}$$

$$e^I_-e^J_- = 2e^K_-,\ e^J_-e^K_- = 2e^I_-,\ e^K_-e^I_- = 2e^J_-. \tag{2.10}$$

3. Hypersurface *Spin^c* Dirac operator.

Let N be a 5-dimensional Lorentzian manifold, and M be a spacelike hypersurface in N. Fix a point $p \in M$ and an orthonormal basis $\{e_\alpha\}$ of T_pN with e_0 normal and e_1, e_2, e_3, e_4 tangent to M. Extend e_1, e_2, e_3, e_4 to an orthonormal frame in a neighbourhood of p in M such that $(\nabla_ie_j)_p = 0$. Extend this to a local orthonormal frame $\{e_\alpha\}$ for N with $(\widetilde{\nabla}_0e_j)_p = 0$. Let $\{e^\alpha\}$ be the dual frame. Then $(\widetilde{\nabla}_ie^j)_p = -h_{ij}e^0$, $(\widetilde{\nabla}_ie^0)_p = -h_{ij}e^j$, where $h_{ij} = \left\langle \widetilde{\nabla}_ie_0, e_j\right\rangle$, $1 \leq i,j \leq 4$, are the components of the second fundamental form at p. The metric connection $\widetilde{\nabla}$ and ∇, together with a $U(1)$ connection A on L, induce two connections on W. These induced connections on W, which we denote by $\widetilde{\nabla}_A$, ∇_A respectively, are related by

$$\widetilde{\nabla}_{Ai} = \nabla_{Ai} + \frac{1}{2}h_{ij}e^0e^j. \tag{3.1}$$

By definition, $\widetilde{\nabla}_A$ is compatible with the metric $(\ ,\ )$, i.e.,

$$d\Big((\phi,\psi)*e_i\Big)=\Big(\big(\widetilde{\nabla}_{Ai}\phi,\psi\big)+\big(\phi,\widetilde{\nabla}_{Ai}\psi\big)\Big)*1.$$

Using (2.2) and (3.1), we can prove that ∇_A is also compatible with the metrics $(\ ,\ )$ and $\langle\ ,\ \rangle$, i.e.,

$$d\Big((\phi,\psi)*e_i\Big)=\Big(\big(\nabla_{Ai}\phi,\psi\big)+\big(\phi,\ \nabla_{Ai}\psi\big)\Big)*1.$$
$$d\Big(\langle\phi,\psi\rangle*e_i\Big)=\Big(\big\langle\nabla_{Ai}\phi,\psi\big\rangle+\big\langle\phi,\nabla_{Ai}\psi\big\rangle\Big)*1.$$

In a local orthonormal coframe $\{e^i\}$ of M, *Spin*^{c} Dirac operator D_A and the hypersurface *Spin*^{c} Dirac operator $\widetilde{D}_A$ are defined by

$$D_A=e^i\nabla_{Ai},\qquad \widetilde{D}_A=e^i\widetilde{\nabla}_{Ai},$$

respectively. Obviously, D_A is self-adjoint with respect to the metric $\langle\ ,\ \rangle$. We also have the following standard Weitzenböck formula:

$$D_A^2=\nabla_A^*\nabla_A+\frac{R}{4}+\frac{1}{2}F_A^M,$$

where R is the scalar curvature of M, and F_A^M is the restriction on M of the curvature of L. From (3.1), we have

$$\widetilde{D}_A=D_A+\frac{H}{2}e^0,$$

where $H=\sum h_{ii}$ is the mean curvature of M. Moreover,

$$\begin{aligned}d\Big(\langle e^i\phi,\psi\rangle*e^i\Big)&=\Big(\big\langle D_A\phi,\psi\big\rangle-\big\langle\phi,D_A\psi\big\rangle\Big)*1\\&=\Big(\big\langle\widetilde{D}_A\phi,\psi\big\rangle-\big\langle\phi,\widetilde{D}_A\psi\big\rangle\Big)*1.\end{aligned}$$

and

$$d\Big(\big\langle\phi,\widetilde{\nabla}_{Ai}\psi\big\rangle*e^i\Big)=\Big(\big\langle\widetilde{\nabla}_{Ai}\phi,\widetilde{\nabla}_{Ai}\psi\big\rangle-\big\langle\phi,(-\widetilde{\nabla}_{Ai}+h_{ij}e^0e^j)\widetilde{\nabla}_{Ai}\psi\big\rangle\Big)*1.$$

It follows that the adjoints under the metric $\langle\ ,\ \rangle$ are $D_A^*=D_A$, $\widetilde{D}_A^*=\widetilde{D}_A$, $\widetilde{\nabla}_{Ai}^*=-\widetilde{\nabla}_{Ai}+h_{ij}e^0e^j$. With the information, we can easily derive (as in [Z1]) the following two Weitzenböck formulas,

$$\widetilde{D}_A^2=\nabla_A^*\nabla_A+\frac{1}{4}(R+H^2)-\frac{1}{2}\nabla_iHe^0e^i+\frac{1}{2}F_A^M \tag{3.2}$$

$$=\widetilde{\nabla}_A^*\widetilde{\nabla}_A+\frac{1}{2}(T_{00}+T_{0i}e^0e^i+F_A^M). \tag{3.3}$$

The integral form of Weitzenböck formula (3.3) is

$$(3.4)\qquad \int_M \left|\widetilde{\nabla}_A\phi\right|^2 + \left\langle \phi, \widetilde{R}\phi \right\rangle - \left|\widetilde{D}_A\phi\right|^2 = \frac{1}{2}\int_{\partial M} \left\langle \phi, [e^i, e^j]\widetilde{\nabla}_{Aj}\phi \right\rangle * e^i,$$

where $\widetilde{R} = \frac{1}{2}(T_{00} + T_{0i}e^0e^i + F_A^M)$, and $[e^i, e^j] = e^ie^j - e^je^i$.

Now recall that M and L are asymptotically flat of order $\tau > 1$ with asymptotic coordinates $\{dx^i\}$ on the end. Orthonormalizing $\{dx^i\}$ yields an orthonormal coframe

$$e^i = dx^i + \frac{1}{2}a_{ik}dx^k + O(r^{-\tau-1}).$$

Denote e^0 as dx^0. Then, on each end,

$$\begin{aligned}
\nabla_{Aj} &= \partial_j - \frac{1}{4}\Gamma_{kjl}dx^kdx^l + \frac{1}{2}A_j\mathbf{i} + O(r^{-2\tau-1}),\\
\widetilde{D}_A &= dx^j\partial_j - \frac{1}{4}\Gamma_{kjl}dx^jdx^kdx^l + \frac{H}{2}dx^0 + \frac{1}{2}dx^jA_j\mathbf{i} + O(r^{-2\tau-1}),
\end{aligned}$$

where $\Gamma_{kjl} = \frac{1}{2}(\partial_jg_{kl} + \partial_lg_{kj} - \partial_kg_{jl}) = O(r^{-\tau-1})$. Therefore $\widetilde{D}_A$ gives the maps for the weighted Hölder spaces

$$C^{2,\alpha}_{-\tau}(W) \xrightarrow{\widetilde{D}_A} C^{1,\alpha}_{-\tau-1}(W) \xrightarrow{\widetilde{D}_A} C^{0,\alpha}_{-\tau-2}(W)$$

defined by connection ∇_A on W. Here we are using the weighted spaces defined in the papers of Bartnik [B] and Lee-Parker [LP]. For constant spinor ϕ_0, $\partial_j\phi_0 = 0$, we have $\widetilde{D}_A\phi_0 \in C^{1,\alpha}_{-\tau-1}(W)$, and $\widetilde{D}^2_A\phi_0 \in C^{0,\alpha}_{-\tau-2}(W)$.

The following lemma can be easily proved in the spirit of [PT].

Lemma 3.1. *Suppose M, L are asymptotically flat of order $\tau > 1$ and ϕ, $\{\phi_i\} \in W$ are C^1 spinors which satisfy $\widetilde{\nabla}_A\phi = 0$, $\widetilde{\nabla}_A\phi_i = 0$ for each i,*

(i) *If $\lim_{x\to\infty}\phi(x) = 0$, where the limit is taken along M in one asymptotic end, then $\phi = 0$.*

(ii) *If $\{\phi_i\}$ are linearly independent in some end, then they are linearly independent everywhere on M.*

Proof. By the assumption, we have $\nabla_{Ai}\phi = -\frac{1}{2}h_{ij}e^0e^j\phi$. Then

$$\left|d|\phi|^2\right| = 2\left|\Re e\langle\nabla_A\phi, \phi\rangle\right| \leq C|h||\phi|^2.$$

Therefore the lemma can be proved in the same way as Lemma 4.1, [Z1].□

Lemma 3.2. *If M, L are asymptotically flat of order $\tau > 1$ and the charged dominant energy condition (1.10) holds on M, then the map*

$$\widetilde{D}_A^2 : C_{-\tau}^{2,\alpha}(W) \longrightarrow C_{-\tau-2}^{0,\alpha}(W)$$

is an isomorphism.

Proof. First note that the lower order term in (3.2)

$$\left(\frac{1}{4}(R + H^2) - \frac{1}{2}\nabla_i H e^0 e^i + \frac{1}{2}F_A^M\right)$$

lies in $C_{-\tau-2}^{0,\alpha}(W)$. Consequently, Theorem 9.2(d) of [LP] shows that $\widetilde{D}_A^2$ is an isomorphism provided it is injective. To show injectivity, suppose that $\phi \in C_{-\tau}^{2,\alpha}(W)$ satisfies $\widetilde{D}_A^2\phi = \widetilde{\nabla}_A^*\widetilde{\nabla}_A\phi + \widetilde{R}\phi = 0$. Integrating over the region $M_r \subset M$ inside radius r in asymptotic coordinates, we have

$$\int_{M_r} \left|\widetilde{\nabla}_A\phi\right|^2 + \left\langle \widetilde{R}\phi, \phi\right\rangle = \int_{\partial M_r} \left\langle \phi, \widetilde{\nabla}_{Ai}\phi\right\rangle * e^i.$$

But

$$\left\langle \phi,\, \widetilde{\nabla}_{Ai}\phi\right\rangle = \left\langle \phi,\, \left(\nabla_{Ai}\phi + \frac{1}{2}h_{ij}e^0.e^j.\phi\right)\right\rangle = O(r^{-2\tau-1}),$$

and $Vol(\partial M_r) = O(r^3)$ by (1.2), (1.3). Hence the right hand side of the above integral vanishes in the limit as $r \to \infty$. Therefore $\widetilde{\nabla}_A\phi = 0$ on M. Hence $\phi = 0$ by Lemma 3.1 (i), and the proof of the lemma is complete. □

Theorem 3.3. *If M, L are asymptotically flat of order $\tau > 1$ and the charged dominant energy condition (1.10) holds on M, then for any constant spinor ϕ_0 on ends, the following boundary value problem has a unique solution $\phi \in C^{2,\alpha}(W)$,*

$$\left\{ \begin{array}{rcl} \widetilde{D}_A\phi & = & 0, \\ \lim_{r\to\infty}\phi & = & \phi_0. \end{array} \right. \tag{3.5}$$

Proof. Since $\widetilde{D}_A^2\phi_0 \in C_{-\tau-2}^{0,\alpha}(W)$, Lemma 3.2 show that there is unique $\phi_1 \in C_{-\tau}^{2,\alpha}(W)$ such that $\widetilde{D}_A^2\phi_1 = -\widetilde{D}_A^2\phi_0$. Then $\phi = \phi_1 + \phi_0$ satisfies $\widetilde{D}_A^2\phi = 0$. Let $\psi = \widetilde{D}_A\phi \in C_{-\tau-1}^{1,\alpha}(W)$, then

$$\int_{M_r} \left|\widetilde{\nabla}_A\psi\right|^2 + \left\langle \widetilde{R}\psi, \psi\right\rangle = \int_{\partial M_r} \left\langle \psi, \widetilde{\nabla}_{Ai}\psi\right\rangle * e^i = \int_{\partial M_r} O(r^{-2\tau-3}) \to 0$$

as $r \to \infty$. Therefore $\widetilde{\nabla}_A\psi = 0$ on M. Hence $\psi = 0$ by Lemma 3.1 (i) and ϕ is the unique solution of (3.5). □

4. Positive Mass Theorem.

In this section, we will prove Positive Mass Theorem.

Proof of Theorem 1.6. Fix a constant spinor $\phi_0 \neq 0$ on M_l and $\phi_0 = 0$ on the other ends. Let $\phi = \phi_0 + \phi_1$ be the solution of (3.5) with $\phi_1 \in C^{2,\alpha}_{-\tau}(W)$. As in [Z1] we have

$$
\begin{aligned}
(4.1)\quad & \int_M \left|\widetilde{\nabla}_A \phi\right|^2 + \left\langle \phi, \widetilde{R}\phi \right\rangle \\
&= \frac{1}{2}\int_{\partial M_\infty} \left\langle \phi_0, [dx^i, dx^j]\widetilde{\nabla}_{Aj}\phi_0 \right\rangle \\
&= \frac{1}{2}\int_{\partial M_\infty} \left\langle \phi_0, [dx^i, dx^j]\widetilde{\nabla}_j\phi_0 \right\rangle * dx^i + \frac{1}{4}\int_{\partial M_\infty} \left\langle \phi_0, [dx^i, dx^j]A_j \mathbf{i}\phi_0 \right\rangle * dx^i \\
&= \frac{C_4}{4}\sum_l \left(\left\langle \phi_0,\ E_l\phi_0 \right\rangle + \left\langle \phi_0,\ p_{lk} dx^0 dx^k \phi_0 \right\rangle + \sum_{i<j} \left\langle \phi_0,\ dx^i dx^j q_{lij}\mathbf{i}\phi_0 \right\rangle \right).
\end{aligned}
$$

We next simplify these terms algebraically. For this we temporarily drop the subscript on ϕ_0, writing $\phi_0 = (\phi^+, \phi^-) \in W^+ \oplus W^-$. Similarly, we drop the subscript l from E_l, P_l, Q_l, Ω_l, p_{li} and q_{lij}. When $|P| \neq 0$, we choose ϕ^- so that $p_k dx^0 dx^k \phi^+ = -|P|\phi^-$. Then

$$
\begin{aligned}
\left\langle \phi_0,\ p_k dx^0 dx^k \phi_0 \right\rangle &= \left\langle \phi^+,\ p_k dx^0 dx^k \phi^- \right\rangle + \left\langle \phi^-,\ p_k dx^0 dx^k \phi^+ \right\rangle \\
&= -|P|\,|\phi_0|^2 .
\end{aligned}
$$

Denote the self-dual part of total electromagnetic momentum of end M_l by

$$
q_1^+ = 2^{-1}(q_{12} + q_{34}),\quad q_2^+ = 2^{-1}(q_{13} + q_{42}),\quad q_3^+ = 2^{-1}(q_{14} + q_{23}),
$$

and anti-self-dual part of total electromagnetic momentum of end M_l by

$$
(4.2)\qquad q_1^- = 2^{-1}(q_{12} - q_{34}),\quad q_2^- = 2^{-1}(q_{13} - q_{42}),\quad q_3^- = 2^{-1}(q_{14} - q_{23}).
$$

Let $q^+ = e_+^I q_1^+ + e_+^J q_2^+ + e_+^K q_3^+$, $q^- = e_-^I q_1^- + e_-^J q_2^- + e_-^K q_3^-$, then

$$
\begin{aligned}
(4.3)\qquad \sum_{i<j} \left\langle \phi_0,\ dx^i dx^j q_{ij}\mathbf{i}\phi_0 \right\rangle &= \left\langle \phi^+,\ q^+\mathbf{i}\phi^+ \right\rangle + \left\langle \phi^-,\ q^-\mathbf{i}\phi^- \right\rangle \\
&= \left\langle \phi^+,\ (q^+ - |P|^{-2} p_k p_j dx^k q^- dx^j)\mathbf{i}\phi^+ \right\rangle .
\end{aligned}
$$

Using (2.6), (2.7), (2.8), (2.9) and (2.10), we obtain

$$\begin{aligned}
p_k p_j dx^k e_-^I dx^j &= \Big(- p_k p_1 dx^k dx^1 - p_k p_2 dx^k dx^2 + p_k p_3 dx^k dx^3 \\
&\quad + p_k p_4 dx^k dx^4 \Big) e_+^I \\
&= \Big(p_1^2 + p_2^2 - p_3^2 - p_4^2 + 2p_1p_3 dx^1 dx^3 - 2p_4p_2 dx^4 dx^2 \\
&\quad + 2p_1p_4 dx^1 dx^4 + 2p_2p_3 dx^2 dx^3 \Big) e_+^I \\
&= \Big[p_1^2 + p_2^2 - p_3^2 - p_4^2 + (p_1p_3 - p_4p_2) e_+^J \\
&\quad + (p_1p_4 + p_2p_3) e_+^K \Big] e_+^I \\
&= \Big(p_1^2 + p_2^2 - p_3^2 - p_4^2 \Big) e_+^I + 2\Big(p_1p_4 + p_2p_3 \Big) e_+^J \\
&\quad + 2\Big(p_2p_4 - p_1p_3 \Big) e_+^K .
\end{aligned}$$

Similarly, one finds that

$$\begin{aligned}
p_k p_j dx^k e_-^J dx^j &= 2\Big(p_2p_3 - p_1p_4 \Big) e_+^I + \Big(p_1^2 - p_2^2 + p_3^2 - p_4^2 \Big) e_+^J \\
&\quad + 2\Big(p_1p_2 + p_3p_4 \Big) e_+^K , \\
p_k p_j dx^k e_-^K dx^j &= 2\Big(p_1p_3 + p_2p_4 \Big) e_+^I + 2\Big(p_3p_4 - p_1p_2 \Big) e_+^J \\
&\quad + \Big(p_1^2 - p_2^2 - p_3^2 + p_4^2 \Big) e_+^K .
\end{aligned}$$

Denote

$$\begin{aligned}
c_1 &= q_1^+ - |P|^{-2} \Big((p_1^2 + p_2^2 - p_3^2 - p_4^2) q_1^- + 2(p_2p_3 - p_1p_4) q_2^- \\
&\quad + 2(p_1p_3 + p_2p_4) q_3^- \Big), \\
c_2 &= q_2^+ - |P|^{-2} \Big(2(p_1p_4 + p_2p_3) q_1^- + (p_1^2 - p_2^2 + p_3^2 - p_4^2) q_2^- \\
&\quad + 2(p_3p_4 - p_1p_2) q_3^- \Big), \\
c_3 &= q_3^+ - |P|^{-2} \Big(2(p_2p_4 - p_1p_3) q_1^- + 2(p_1p_2 + p_3p_4) q_2^- \\
&\quad + (p_1^2 - p_2^2 - p_3^2 + p_4^2) q_3^- \Big).
\end{aligned}$$

When $|P| = 0$, we set

$$\begin{aligned}
c_1 &= q_1^+ - |S|^{-2}\Big((s_1^2 + s_2^2 - s_3^2 - s_4^2)q_1^- + 2(s_2 s_3 - s_1 s_4)q_2^- \\
&\quad + 2(s_1 s_3 + s_2 s_4)q_3^-\Big), \\
c_2 &= q_2^+ - |S|^{-2}\Big(2(s_1 s_4 + s_2 s_3)q_1^- + (s_1^2 - s_2^2 + s_3^2 - s_4^2)q_2^- \\
&\quad + 2(s_3 s_4 - s_1 s_2)q_3^-\Big), \\
c_3 &= q_3^+ - |S|^{-2}\Big(2(s_2 s_4 - s_1 s_3)q_1^- + 2(s_1 s_2 + s_3 s_4)q_2^- \\
&\quad + (s_1^2 - s_2^2 - s_3^2 + s_4^2)q_3^-\Big),
\end{aligned}$$

where s_1, s_2, s_3, s_4 are arbitrary real numbers such that $|S| = \sqrt{\sum_i s_i^2} \neq 0$. We choose ϕ^- by $s_k dx^0 dx^k \phi^+ = -|S|\,\phi^-$. Then we can repeat the above calculation, replacing p_k by s_k. Therefore,

$$\begin{aligned}
\sum_{i<j} \Big\langle \phi_0,\ dx^i dx^j q_{ij} \mathbf{i}\phi_0 \Big\rangle &= \Big\langle \phi^+,\ (e_+^I c_1 + e_+^J c_2 + e_+^K c_3)\mathbf{i}\phi^+ \Big\rangle \\
&= 2\Big\langle \phi^+,\ (Ic_1 + Jc_2 + Kc_3)\mathbf{i}\phi^+ \Big\rangle.
\end{aligned}$$

By the Pauli representation (2.4), we have

$$\Re e \sum_{i<j} \Big\langle \phi_0,\ dx^i dx^j q_{ij} \mathbf{i}\phi_0 \Big\rangle = 2\Re e\Big\langle \phi^+,\ C\phi^+ \Big\rangle,$$

where

$$C = \begin{pmatrix} -c_1 & -\mathbf{i}c_2 + c_3 \\ \mathbf{i}c_2 + c_3 & c_1 \end{pmatrix},$$

which has real eigenvalues $\lambda = \pm|C|$, $|C| = \sqrt{\sum_i c_i^2}$. Now we take ϕ^+ to be the eigenspinor of eigenvalue $-|C|$ with $|\phi^+|^2 = \frac{1}{2}$. We obtain

$$E - |P| - |C| = 4C_4^{-1} \int_M \left|\widetilde{\nabla}_A \phi\right|^2 + \Big\langle \phi, \widetilde{R}\phi \Big\rangle \geq 0.$$

Next we compute $|C|$. Denote

$$t_k = \begin{cases} |S|^{-1} s_k & if\ \ |P| = 0, \\ |P|^{-1} p_k & if\ \ |P| \neq 0. \end{cases}$$

Obviously, $\sum_k t_k^2 = 1$. A straightforward computation gives

$$\begin{aligned}
&\Big((t_1^2+t_2^2-t_3^2-t_4^2)q_1^- + 2(t_2t_3-t_1t_4)q_2^- + 2(t_1t_3+t_2t_4)q_3^-\Big)^2 \\
&+\Big(2(t_1t_4+t_2t_3)q_1^- + (t_1^2-t_2^2+t_3^2-t_4^2)q_2^- + 2(t_3t_4-t_1t_2)q_3^-\Big)^2 \\
&+\Big(2(t_2t_4-t_1t_3)q_1^- + 2(t_1t_2+t_3t_4)q_2^- + (t_1^2-t_2^2-t_3^2+t_4^2)q_3^-\Big)^2 \\
&=(q_1^-)^2+(q_2^-)^2+(q_3^-)^2.
\end{aligned} \tag{4.4}$$

Therefore

$$\begin{aligned}
|C|^2 &= (q_1^+)^2+(q_2^+)^2+(q_3^+)^2+(q_1^-)^2+(q_2^-)^2+(q_3^-)^2 \\
&\quad -2(t_1^2+t_2^2-t_3^2-t_4^2)q_1^+q_1^- - 4(t_2t_3-t_1t_4)q_1^+q_2^- - 4(t_1t_3+t_2t_4)q_1^+q_3^- \\
&\quad -4(t_1t_4+t_2t_3)q_2^+q_1^- - 2(t_1^2-t_2^2+t_3^2-t_4^2)q_2^+q_2^- - 4(t_3t_4-t_1t_2)q_2^+q_3^- \\
&\quad -4(t_2t_4-t_1t_3)q_3^+q_1^- - 4(t_1t_2+t_3t_4)q_3^+q_2^- - 2(t_1^2-t_2^2-t_2^3+t_4^2)q_3^+q_3^- \\
&= \frac{1}{2}|Q|^2 + \vec{T}^t\Omega\vec{T},
\end{aligned}$$

where $\vec{T} = (t_1,t_2,t_3,t_4)^t$. Now we show when $|P| = 0$, there is an another choice of constant spinor ϕ_0 such that the third term in (4.1) has sharper value. First, by mean value inequality and (4.4),

$$|C|^2 \le 2((q_1^+)^2+(q_2^+)^2+(q_3^+)^2+(q_1^-)^2+(q_2^-)^2+(q_3^-)^2) = |Q|^2. \tag{4.5}$$

On the other hand,

$$\begin{aligned}
\Re e\sum_{i<j}\Big\langle \phi_0,\ dx^i dx^j q_{ij}\mathbf{i}\phi_0\Big\rangle &= \Re e\Big\langle \phi^+,\ q^+\mathbf{i}\phi^+\Big\rangle + \Re e\Big\langle \phi^-,\ q^-\mathbf{i}\phi^-\Big\rangle \\
&= 2\Re e\Big\langle \phi^+,\ Q^+\phi^+\Big\rangle - 2\Re e\Big\langle \phi^-,\ Q^-\phi^-\Big\rangle,
\end{aligned}$$

where

$$Q^+ = \begin{pmatrix} -q_1^+ & -\mathbf{i}q_2^+ + q_3^+ \\ \mathbf{i}q_2^+ + q_3^+ & q_1^+ \end{pmatrix}, \quad Q^- = \begin{pmatrix} -q_1^- & -\mathbf{i}q_2^- + q_3^- \\ \mathbf{i}q_2^- + q_3^- & q_1^- \end{pmatrix}.$$

When $|P| = 0$, we can choose ϕ^+, ϕ^- freely. So we choose ϕ^+ to be the eigenspinor of eigenvalue $-|Q^+|$ of Q^+, and choose ϕ^- to be the eigenspinor of eigenvalue $|Q^-|$ of Q^- such that $|\phi^+|^2 + |\phi^-|^2 = 1$, $|Q^+| = \sqrt{\sum_i (q_i^+)^2}$, $|Q^-| = \sqrt{\sum_i (q_i^-)^2}$. Then,

$$-\Re e\sum_{i<j}\Big\langle \phi_0,\ dx^i dx^j q_{ij}\mathbf{i}\phi_0\Big\rangle = 2|Q^+||\phi^+|^2 + 2|Q^-||\phi^-|^2.$$

We choose $\phi^- = 0$ if $|Q^+| \geq |Q^-|$, and $\phi^+ = 0$ if $|Q^+| \leq |Q^-|$. Thus

$$-\Re e \sum_{i<j} \left\langle \phi_0,\ dx^i dx^j q_{ij} \mathbf{i} \phi_0 \right\rangle = 2 \max \left\{ |Q^+|, |Q^-| \right\}$$
$$= \sqrt{|Q|^2 + 2 \left| q_{12}q_{34} + q_{13}q_{42} + q_{14}q_{23} \right|}.$$

By (4.5), we know to get a sharper result by choose constant spinor in this way when $|P| = 0$. The proof of the first part of Theorem 1.6 is complete.

Now suppose $E_1 = 0$. Then $p_{1k} = 0$, $1 \leq k \leq 4$, $c_{1j} = 0$, $1 \leq j \leq 3$ and $q_{1ij} = 0$, $1 \leq i, j \leq 4$. Take constant spinor $\{\psi_{1\mu} : \mu = 1, 2, 3, 4\}$ which form a basis of W on M_1 and $\psi_{1\mu} = 0$ on all other ends M_l, where we take W as complex bundle. Let ψ_μ be the solutions of $\widetilde{D}_A \psi_\mu = 0$ constructed from this data by Theorem 3.3. The vanishing of E_1 then implies $\widetilde{\nabla}_A \psi_\mu = 0$ and $\psi_\mu \to 0$ uniformly on each end except M_1. But this contradicts Lemma 3.1 (i) unless M_1 is the only end of M. By Lemma 3.1 (ii), $\{\psi_\mu : \mu = 1, 2, 3, 4\}$ are linearly independent everywhere on M, so in a local frame $\{e_i\}$ of M,

$$-\frac{1}{4}\widetilde{R}_{\alpha\beta ij} e^\alpha e^\beta \psi_\mu + \frac{1}{2} F_{Aij} \psi_\mu = (\widetilde{\nabla}_{Ai}\widetilde{\nabla}_{Aj} - \widetilde{\nabla}_{Aj}\widetilde{\nabla}_{Ai} - \widetilde{\nabla}_{A[e_i, e_j]})\psi_\mu = 0.$$

In terms of (2.3), (2.4), we obtain

$$\begin{pmatrix} \widetilde{R}^+_{ij1}\mathbf{i} - F_{Aij} & -\widetilde{R}^+_{ij2} - \widetilde{R}^+_{ij3}\mathbf{i} & \widetilde{R}_{ij01} + \widetilde{R}_{ij02}\mathbf{i} & -\widetilde{R}_{ij03} - \widetilde{R}_{ij04}\mathbf{i} \\ \widetilde{R}^+_{ij2} - \widetilde{R}^+_{ij3}\mathbf{i} & -\widetilde{R}^+_{ij1}\mathbf{i} - F_{Aij} & \widetilde{R}_{ij03} - \widetilde{R}_{ij04}\mathbf{i} & \widetilde{R}_{ij01} - \widetilde{R}_{ij02}\mathbf{i} \\ \widetilde{R}_{ij01} - \widetilde{R}_{ij02}\mathbf{i} & -\widetilde{R}_{ij03} + \widetilde{R}_{ij04}\mathbf{i} & -\widetilde{R}^-_{ij1}\mathbf{i} - F_{Aij} & \widetilde{R}^-_{ij2} + \widetilde{R}^-_{ij3}\mathbf{i} \\ \widetilde{R}_{ij03} + \widetilde{R}_{ij04}\mathbf{i} & \widetilde{R}_{ij01} + \widetilde{R}_{ij02}\mathbf{i} & -\widetilde{R}^-_{ij2} + \widetilde{R}^-_{ij3}\mathbf{i} & \widetilde{R}^-_{ij1}\mathbf{i} - F_{Aij} \end{pmatrix} \psi_\mu = 0,$$

where

$$\widetilde{R}^\pm_{ij1} = \widetilde{R}_{ij12} \pm \widetilde{R}_{ij34}, \qquad \widetilde{R}^\pm_{ij2} = \widetilde{R}_{ij13} \pm \widetilde{R}_{ij42}, \qquad \widetilde{R}^\pm_{ij3} = \widetilde{R}_{ij14} \pm \widetilde{R}_{ij23}.$$

This immediately implies that, over M, $\widetilde{R}_{ij\alpha\beta} = 0$, $F_{Aij} = 0$. Therefore $T_{00} = 0$ by the Einstein equations, and $\widetilde{R}_{0i0j} = 0$, $F_{A0i} = 0$ by the charged dominant energy condition. Thus the proof of Theorem 1.6 is complete. □

5. Appendix: Analogue on 4-Lorentzian Manifolds.

In this appendix, we assume N is a 4-dimensional Lorentzian manifold with Lorentzian metric $\widetilde{g}$ of signature $(-1, 1, 1, 1)$, which satisfies the Einstein equations (1.1), and M is a spacelike hypersurface in N which is asymptotically flat of order $\tau > \frac{1}{2}$. Let L be a $U(1)$ line bundle which is a $Spin^c(3, 1)$

structure of N. We assume L is also asymptotically flat of order $\tau > \frac{1}{2}$ over M. The total energy, the total linear momentum and the total electromagnetic momentum of each end of M can be defined same as (1.7), (1.8), (1.9) except that we integrate over the sphere in 3-dimensional asymptotically flat ends.

Proof of Theorem 1.7. Let V be (locally) $SL(2,C)$ bundle. The complex spinor bundle W of N is equal to $(\bar{V} \otimes L^{\frac{1}{2}}) \oplus (V^* \otimes L^{\frac{1}{2}})$. Note that $L^{\frac{1}{2}}$ is globally-defined over M since every orientable 3-manifold is spin. Now the Clifford multiplication can be defined as follows: For any $X \in T^*N$ with coordinate (x_0, x_1, x_2, x_3), we identify it to the corresponding elements $X \in Hom(\bar{V} \otimes L^{\frac{1}{2}} \to V^* \otimes L^{\frac{1}{2}})$ and $X^\sigma \in Hom(V^* \otimes L^{\frac{1}{2}} \to \bar{V} \otimes L^{\frac{1}{2}})$,

$$X \mapsto \begin{pmatrix} x_0 - x_1 & x_2 + \mathbf{i}x_3 \\ x_2 - \mathbf{i}x_3 & x_0 + x_1 \end{pmatrix}, \qquad X^\sigma \mapsto \begin{pmatrix} x_0 + x_1 & -x_2 - \mathbf{i}x_3 \\ -x_2 + \mathbf{i}x_3 & x_0 - x_1 \end{pmatrix}.$$

Then the Clifford multiplication $T^*N \otimes W \longrightarrow W$ is defined by $X \otimes (\xi, \eta)^t = (X\eta, X^\sigma \xi)^t$. We refer to [PT, Z2] for details. Now let $\phi_0 = (\xi_0, \eta_0)^t$,

$$p_{lk} dx^0 dx^k \phi_0 + \sum_{i<j} dx^i dx^j q_{lij} \mathbf{i} \phi_0 = \Big(C_{l1}\xi_0, C_{l2}\eta_0\Big)^t,$$

where

$$C_{l\xi} = \begin{pmatrix} p_{l1} - q_{l23} & -p_{l2} + q_{l31} - \mathbf{i}(p_{l3} - q_{l12}) \\ -p_{l2} + q_{l31} + \mathbf{i}(p_{l3} - q_{l12}) & -p_{l1} + q_{l23} \end{pmatrix},$$

$$C_{l\eta} = \begin{pmatrix} -(p_{l1} + q_{l23}) & p_{l2} + q_{l31} + \mathbf{i}(p_{l3} + q_{l12}) \\ p_{l2} + q_{l31} - \mathbf{i}(p_{l3} + q_{l12}) & p_{l1} + q_{l23} \end{pmatrix}.$$

Note $C_{l\xi}$ has eigenvalues $\pm\lambda_{l\xi}$,

$$\lambda_{l\xi} = \sqrt{(p_{l1} - q_{l23})^2 + (p_{l2} - q_{l31})^2 + (p_{l3} - q_{l12})^2},$$

and $C_{l\eta}$ has eigenvalues $\pm\lambda_{l\eta}$,

$$\lambda_{l\eta} = \sqrt{(p_{l1} + q_{l23})^2 + (p_{l2} + q_{l31})^2 + (p_{l3} + q_{l12})^2}.$$

We choose spinor $\phi_0 = (\xi_0, \eta_0)$ such that ξ_0 is the eigenspinor of eigenvalue $-\lambda_{l\xi}$ and η_0 is the eigenspinor of eigenvalue $-\lambda_{l\eta}$. Moreover, $|\xi_0|^2 + |\eta_0|^2 = 1$. Then

$$\Big\langle \phi_0,\ p_{lk} dx^0 dx^k \phi_0 \Big\rangle + \sum_{i<j} \Big\langle \phi_0,\ dx^i dx^j q_{lij} \mathbf{i} \phi_0 \Big\rangle = -\lambda_{l\xi} \Big|\xi_0\Big|^2 - \lambda_{l\eta} \Big|\eta_0\Big|^2.$$

We choose $\eta_0 = 0$ if $\lambda_{l\xi} \geq \lambda_{l\eta}$, and $\xi_0 = 0$ if $\lambda_{l\xi} \leq \lambda_{l\eta}$. Thus, if M satisfies the charged dominant energy condition, then

$$E_l \geq \sqrt{|P_l|^2 + |Q_l|^2 + 2\,|p_{l1}q_{l23} + p_{l2}q_{l31} + p_{l3}q_{l12}|}.$$

If $E_{l_0} = 0$ for some l_0, then M has only one end, $p_{l_0 k} = 0$, $q_{l_0 ij} = 0$, and $\widetilde{R}_{\alpha\beta\gamma\delta} = 0$, $F_{A\alpha\beta} = 0$ over M. Thus the proof of Theorem 1.7 is complete.□

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