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Rigidity of area minimizing tori in 3-manifolds of nonnegative scalar curvature

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The following conjecture arises from remarks in Fischer-Colbrie-Schoen ([FCS], Remark 4, p. 207): If (M, g) is a complete Riemannian 3-manifold with nonnegative scalar curvature and if Σ is a two-sided torus in M which is suitably of least area then M is flat. Such a result, as Fischer-Colbrie and Schoen commented, would be an interesting analogue of the Cheeger-Gromoll splitting theorem. Here we present a proof of this conjecture assuming Σ is of least area in its isotopy class. The proof is a consequence of the following local result, which is the main result of the paper.

Theorem 1. *Let (M, g) be a C^∞ 3-manifold with nonnegative scalar curvature, $S \geq 0$. If Σ is a two-sided torus in M which is locally of least area (see Section 2), then M is flat in a neighborhood of Σ.*

It follows that Σ is flat and totally geodesic and that locally M splits along Σ. A partial infinitesimal version of Theorem 1 was observed in [FCS], namely, if Σ is a stable minimal two-sided torus in M with nonnegative scalar curvature then Σ must be flat and totally geodesic, and the scalar curvature and normal Ricci curvature of M vanishes along Σ. In [CG] the authors proved Theorem 1 under the assumption that M is analytic. The result in the analytic case follows as an immediate consequence of a more general result which holds for C^∞ manifolds, see Theorem B in [CG]. Here we will make use of Theorem B to present a proof of Theorem 1.

We note that, under the assumptions of Theorem 1, M need not be globally flat. Consider, for example, $S^1 \times S^2$, where S^2 is a sphere which is flattened near the equator E. Then $S^1 \times E$ is a torus which is locally of least area in $S^1 \times S^2$.

The idea of the proof of Theorem 1 is as follows. It is first shown that Σ cannot be locally *strictly* of least area. If it were, then under a sufficiently small perturbation of the metric to a metric of (strictly) positive scalar

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curvature, Σ would be perturbed to a torus which is still locally of least area. But this would contradict the fact that a compact two-sided stable minimal surface in a 3-manifold with strictly positive scalar curvature must be a sphere, cf. Theorem 5.1 in [SY1]. It is then shown that on each side of Σ there is a torus which is locally of least area. By cutting out the region bounded by these two tori and pasting it appropriately to a second copy, one obtains, using Theorem B, a smooth 3-torus with nonnegative scalar curvature. By Schoen and Yau [SY], this 3-torus must be flat, and Theorem 1 follows. We now proceed to a detailed proof of Theorem 1.

In all that follows we work in the C^∞ category. For simplicity, all surfaces are assumed to be embedded. However, by pulling back to the normal bundle of Σ, it is clear that a version of Theorem 1 holds for immersed surfaces, as well. By definition, a compact two-sided surface Σ in a 3-manifold M is locally of least area provided in some normal neighborhood V of Σ, $A(\Sigma) \leq A(\Sigma')$ for all Σ' isotopic to Σ in V, where A is the area functional. If the inequality is strict for $\Sigma' \neq \Sigma$, we say that Σ is locally strictly of least area. Note that "locally of least area" implies "stable minimal".

Let V be a normal neighborhood of a compact two-sided surface Σ in a 3-manifold M. Then, via the normal exponential map, $V = (-\ell, \ell) \times \Sigma$, and the metric $g = ds^2$ takes the form,

$$ds^2 = dt^2 + \sum_{i,j=1}^{2} g_{ij}(t,x)\, dx^i dx^j \,. \tag{1}$$

The following is a restatement of part of Theorem B in [CG].

Lemma 1. *Let (M,g) be a 3-manifold with nonnegative scalar curvature, $S \geq 0$. Suppose Σ is a two-sided torus in M which is locally of least area. Then with respect to geodesic normal coordinates along Σ (see equation 1),*

$$\frac{\partial^n g_{ij}}{\partial t^n}(0,x) = 0\,,$$

for all n and all $x \in \Sigma$.

Lemma 1 is used below to ensure that after certain cut and paste operations the resulting metric is smooth.

Lemma 2. *Suppose Σ is a compact two-sided surface in a 3-manifold (M,g) with nonnegative scalar curvature, $S(g) \geq 0$. Then there exists a neighborhood U of Σ and a sequence of metrics $\{g_n\}$ on U such that $g_n \to g$ in C^∞ topology on U, and each g_n has strictly positive scalar curvature, $S(g_n) > 0$.*

Proof. Let $V = (-\ell, \ell) \times \Sigma$ be a normal neighborhood of Σ, so that the metric g takes the form (1). Consider the sequence of metrics $g_n = e^{-2n^{-1}t^2} g$. A straight forward computation gives,

$$S(g_n) = e^{2n^{-1}t^2}(S(g) + 8n^{-1}(1 + tH_t - n^{-1}t^2)),$$

where H_t is the mean curvature (in the metric g) of $\Sigma_t = \{t\} \times \Sigma$. It is clear that by taking ℓ sufficiently small and n sufficiently large we have $S(g_n) > 0$. □

In the next lemma we show that if $\Sigma \subset M$ is locally *strictly* of least area then by perturbing the metric of M slightly, Σ gets perturbed to a surface which is still locally of least area.

Lemma 3. *Suppose Σ is a compact two-sided surface in (M^3, g) which is locally strictly of least area. Let $\{g_n\}$ be any sequence of metrics such that $g_n \to g$ in C^∞ topology. Then for any neighborhood U of Σ and any positive integer N there exists, for some $n \geq N$, a surface $\Sigma_n \subset U$ isotopic to Σ in U which is locally of least area in (M, g_n).*

Proof. The proof makes use of basic existence and convergence results for least area surfaces. Let $V = [-\ell, \ell] \times \Sigma$ be a compact normal neighborhood of Σ contained in U, and restrict attention to the compact Riemannian manifold-with-boundary (V, g). Since Σ is locally strictly of least area, we can choose ℓ sufficiently small so that,

$$A_g(\Sigma) < A_g(\Sigma') \quad \text{for all } \Sigma' \in \mathcal{I}(\Sigma),\ \Sigma' \neq \Sigma,$$

where $\mathcal{I}(\Sigma)$ is the isotopy class of Σ in V, and A_g is the area functional in the g metric.

Set $V_0 = [-\frac{\ell}{2}, \frac{\ell}{2}] \times \Sigma$. Let $f = f(t)$ be a smooth nonnegative function on $[-\ell, \ell]$ such that $f = 0$ on $[-\frac{\ell}{2}, \frac{\ell}{2}]$. By making the derivatives $f'(\pm\ell)$ sufficiently large in absolute value, with $f'(\ell) > 0$ and $f'(-\ell) < 0$, we obtain a conformally related metric $\bar{g} = e^f g$ with the following properties.

(1) $\bar{g}|_{V_0} = g|_{V_0}$.

(2) $(V, \bar{g})$ has strictly mean convex boundary, i.e., ∂V has positive mean curvature.

(3) For all $\Sigma' \in \mathcal{I}(\Sigma)$ such that $\Sigma' \neq \Sigma$, $A_{\bar{g}}(\Sigma) < A_{\bar{g}}(\Sigma')$.

For each n, set $\bar{g}_n = e^f g_n$. Then the metrics $\bar{g}_n$ satisfy: (1) $\bar{g}_n|_{V_0} = g_n|_{V_0}$, (2) $\bar{g}_n \to \bar{g}$ in C^∞ topology and (3) for n sufficiently large, $(V, \bar{g}_n)$ has mean convex boundary. For each such n let,

$$\alpha_n = \inf_{\Sigma' \in \mathcal{I}(\Sigma)} A_{\bar{g}_n}(\Sigma') .$$

Then by Theorem 5.1 and Section 6 in [HS] (see also [MSY]) there exists for each n a surface $\Sigma_n \in \mathcal{I}(\Sigma)$ such that $A_{\bar{g}_n}(\Sigma_n) = \alpha_n$. In applying the results from [HS] we have used the fact that V is P^2-irreducible (provided $\Sigma \neq S^2, P^2$) and that V does not contain any compact one-sided surfaces. (If $\Sigma = S^2$ or P^2, one may appeal to Theorem 5.2 in [HS] and use specific features of the topology of $[-\ell, \ell] \times \Sigma$).

For each n, Σ_n is a compact stable minimal surface in $(V, \bar{g}_n)$, and the sequence $\{\alpha_n\}$ is bounded. It then follows by well-known convergence arguments that there is a subsequence of surfaces, call it again $\{\Sigma_n\}$, which converges locally in C^∞ topology to a compact (embedded) minimal surface $\bar{\Sigma}$ in $(V, \bar{g})$; see especially Section 2.2 in [M], which applies fairly directly to the situation considered here. By the nature of the convergence, $\{\Sigma_n\}$ is eventually contained in any tubular neighborhood of $\bar{\Sigma}$, and for n sufficiently large, Σ_n will be transverse to the normal geodesics of $\bar{\Sigma}$. It follows that Σ_n covers $\bar{\Sigma}$ via projection along the normal geodesics. Since $\bar{\Sigma}$ is necessarily two-sided (again, because V does not contain any compact one-sided surfaces), it follows that the covering of $\bar{\Sigma}$ by Σ_n must be one-sheeted, i.e., projection along the normal geodesics of $\bar{\Sigma}$ provides a diffeomorphism of Σ_n onto $\bar{\Sigma}$; see e.g., [S].

Thus, $\bar{\Sigma}$ is isotopic to Σ since each Σ_n is. Furthermore, we have,

$$A_{\bar{g}}(\bar{\Sigma}) = \lim_{n\to\infty} \alpha_n \leq \lim_{n\to\infty} A_{\bar{g}_n}(\Sigma) = A_{\bar{g}}(\Sigma) .$$

Since Σ is strictly of least area in its isotopy class in $(V, \bar{g})$, we conclude that $\bar{\Sigma} = \Sigma$. But by the above convergence, this means that for n large enough, Σ_n is contained in $\operatorname{int} V_0$, in which $\bar{g}_n = g_n$. It follows that, for such n, Σ_n is locally of least area in (V, g_n). This concludes the proof of Lemma 3. □

Proof of Theorem 1. Let $V = (-\ell, \ell) \times \Sigma$ be a normal neighborhood of Σ with metric g as in equation (1). Choose ℓ sufficiently small so that Σ is of least area in its isotopy class in V. For technical reasons we modify the metric g as follows. Let $\hat{g}$ be the metric on V of the form (1) but with component functions $\hat{g}_{ij}$, $1 \leq i, j \leq 2$, defined by,

$$\hat{g}_{ij}(t,x) = \begin{cases} g_{ij}(t,x), & \text{for } t \in [0,\ell) \\ g_{ij}(-t,x), & \text{for } t \in (-\ell,0]. \end{cases}$$

$(V, \hat{g})$ is a smooth Riemannian manifold (by Lemma 1) such that $S(\hat{g}) \geq 0$ and reflection across Σ, $(t,x) \mapsto (-t,x)$, is an isometry. Further, Σ is of least area in its isotopy class in $(V, \hat{g})$. By choosing ℓ even smaller if necessary, we guarantee that Lemma 2 holds for the neighborhood $U = V$.

If Σ were *strictly* of least area in its isotopy class in $(V, \hat{g})$ then Lemmas 2 and 3 would imply that there is a two-sided stable minimal torus Σ' near Σ with respect to some metric of strictly positive scalar curvature on V. This would contradict Theorem 5.1 in [SY]. Thus, there exists a surface $\bar{\Sigma} \in \mathcal{I}(\Sigma)$, $\bar{\Sigma} \neq \Sigma$ such that $A_{\hat{g}}(\bar{\Sigma}) = A_{\hat{g}}(\Sigma)$. Hence, $\bar{\Sigma}$ is also of least area in its isotopy class.

We claim that $\bar{\Sigma}$ is contained in one of the components of $V \setminus \Sigma$. If not, then $\bar{\Sigma}$ and Σ must meet. Since $\bar{\Sigma}$ and Σ are stable minimal tori in $(V, \hat{g})$ they must be totally geodesic (cf. [FCS]). Since they are totally geodesic and distinct, they must meet transversally. Thus, the intersection of $\bar{\Sigma}$ and Σ will consist of a finite number of circles. By reflecting the portion of $\bar{\Sigma}$ in $(-\ell, 0] \times \Sigma$ across Σ to $[0,\ell) \times \Sigma$ and smoothing out the resulting ridge along the circles of intersection, we obtain a surface isotopic to Σ with less area than Σ, which is a contradiction. Thus, $\bar{\Sigma}$ lies to one side of Σ and does not meet Σ.

These arguments imply that in the original Riemannian manifold (V, g) there exist two tori Σ^+ and Σ^-, one on each side of Σ, each isotopic to Σ and each locally of least area. Let W be the region in V bounded by Σ^+ and Σ^-. Standard properties of isotopies [H] imply that W has topology $[-1,1] \times T^2$. By taking two copies of W and gluing them appropriately along their boundaries, we obtain, by Lemma 1, a smooth Riemannian manifold with nonnegative scalar curvature which is diffeomorphic to a 3-torus. By Schoen and Yau [SY], this 3-torus, and hence W must be flat. □

By fairly standard continuation arguments, Theorem 1 can be globalized as follows.

Theorem 2. *Let M be a complete connected 3-manifold of nonnegative scalar curvature whose boundary (possibly empty) is mean convex. If M contains a two-sided torus Σ which is of least area in its isotopy class then M is flat.*

Proof. By the maximum principle, either Σ is a boundary component of M or Σ is in the interior of M. If Σ is a boundary component, let $M_0 = M$. If Σ is in the interior and disconnects M, let $M_0 = \overline{U}_0$, where U_0 is one of the components of $M \setminus \Sigma$. If Σ is in the interior and does not disconnect M, let M_0 be the manifold with boundary obtained by "separating" M along Σ. In all cases, Σ is a boundary component of M_0. To prove Theorem 2 it suffices to show that M_0 is flat. Consider the normal exponential map $\Phi : [0, \infty) \times \Sigma \to M_0$ along Σ defined by $\Phi(t, x) = \exp_x tN$, where N is the inward pointing unit normal along Σ. (Note Φ need not be defined on all of $[0, \infty) \times \Sigma$.)

By Theorem 1, M_0 is flat in a neighborhood of Σ. (It is easily seen that Theorem 1 is still valid if Σ is a boundary component.) Then, by standard arguments (which require only nonnegative Ricci curvature), since Σ is locally of least area there exists $a > 0$ such that $\Phi : [0, a) \times \Sigma \to \Phi([0, a) \times \Sigma)$ is an isometry. (Here $[0, a) \times \Sigma$ carries the standard product metric and hence is flat since Σ is). Let ℓ be the largest number (possibly ∞) such that $\Phi : [0, \ell) \times \Sigma \to \Phi([0, \ell) \times \Sigma)$ is an isometry. Consider first the case $\ell = \infty$. Using that the limit of a sequence of normal geodesics to Σ is a normal geodesic to Σ, one easily verifies that $\Phi([0, \infty) \times \Sigma)$ is both open and closed in M_0. Hence, $\Phi([0, \infty) \times \Sigma) = M_0$ and M_0 is flat.

Now consider the case $\ell < \infty$. Since M_0 is complete, each normal geodesic to Σ, $\gamma_x : t \mapsto \Phi(t, x)$, $0 \le t < \ell$, extends to $t = \ell$. Suppose that $\Phi : [0, \ell] \times \Sigma \to \Phi([0, \ell] \times \Sigma)$ is an isometry. Then $\Sigma_\ell = \Phi(\{\ell\} \times \Sigma)$ is an embedded totally geodesic torus in M_0 which is locally of least area. By the maximality of ℓ, Σ_ℓ must meet some component Σ' of ∂M_0. By the maximum principle for hypersurfaces, Σ_ℓ and Σ' must agree, $\Sigma_\ell = \Sigma'$. One can now argue that $\Phi([0, \ell] \times \Sigma)$ is both open and closed in M_0. Hence, $M_0 = \Phi([0, \ell] \times \Sigma)$ is flat.

Now suppose $\Phi : [0, \ell] \times \Sigma \to \Phi([0, \ell] \times \Sigma)$ is not an isometry. The only way this can happen is if two normal geodesics to Σ, γ_{x_i}, $i = 1, 2$, meet at $t = \ell$, $\gamma_{x_1}(\ell) = \gamma_{x_2}(\ell)$. Since there can be no focal points to Σ along $\gamma_{x_i}|_{[0,\ell]}$, there exists a neighborhood U_i of x_i in Σ such that $\Phi : [0, \ell] \times U_i \to \Phi([0, \ell] \times U_i)$ is an isometry. Hence, $\Phi(\{\ell\} \times U_i)$ is an embedded totally geodesic hypersurface in M_0 which (by the choice of ℓ) is a constant distance ℓ from Σ. It follows that $\Phi(\{\ell\} \times U_1)$ and $\Phi(\{\ell\} \times U_2)$ must agree near the common end point $\gamma_{x_1}(\ell) = \gamma_{x_2}(\ell)$. By a straight forward continuation argument we conclude that each geodesic segment γ_x, $x \in \Sigma$, of length 2ℓ meets Σ orthogonally at both end points. It is now easily argued that $\Phi([0, \ell] \times \Sigma)$ is both open and closed in M_0. Hence, $M_0 = \Phi([0, \ell] \times \Sigma)$ is flat, and the proof of Theorem 2 is complete. □

We make some concluding remarks.

1. The results presented here were motivated in part by certain problems concerning the topology of black holes in general relativity, cf., [CG] [G1], [G2].

2. The example mentioned after Theorem 1, $M = S^1 \times S^2$, with S^2 flattened near the equator E, and $\Sigma = S^1 \times E$, shows that stability is not sufficient to imply flatness. Assume the S^1 factor and E have the same radius. Cutting M along Σ we obtain two solid tori, the boundary of each of which is a copy of Σ. Gluing the two solid tori back together along their toroidal boundaries after a suitable twist we obtain a manifold M' diffeomorphic to S^3 with nonnegative scalar curvature which contains a stable minimal torus. Applying Theorem A in [GL], which is proved by a local construction, we can add an asymptotically flat end to M' to obtain an asymptotically flat manifold diffeomorphic to $\mathbb{R}^3$ with nonegative scalar curvature which contains a stable minimal torus. In the language of general relativity, we have obtained an asymptotically flat time symmetric initial data set on $\mathbb{R}^3$ satisfying the constraint equations which contains a stable toroidal apparent horizon. However, we know of no such vacuum (scalar flat) examples, and conjecture that there are none.

3. Using the higher dimensional work of Schoen and Yau [SY2] it appears that Theorem 1 can be extended to higher dimensions as follows: Let M^n have nonnegative scalar curvature. If Σ is a compact two-sided hypersurface in M^n which does not admit a metric of positive scalar curvature and which is locally of least area then a neighborhood of Σ splits. We are grateful to a referee for suggesting an alterative proof of Lemma 3, valid in higher dimensions, which makes this generalization possible. Further aspects of this will be discussed elsewhere.

4. In [FCS], Fischer-Colbrie and Schoen proved that a complete stable minimal surface in an orientable 3-manifold with nonnegative scalar curvature must be conformal to the complex plane or the cylinder A. In the latter case it has been shown that A is flat and totally geodesic, cf., [FCS] and [CM]. The example $M = \mathbb{R} \times S^2$, where S^2 is flattened near the equator, shows that M need not be flat. However, in view of the results cited and the results presented here, it seems reasonable to conjecture that if the cylinder A is actually area minimizing (in a suitable sense) then M is flat (cf., Remark 4 in [FCS]).

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