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Integral curvature bounds and bounded diameter

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We prove an analogue of Myers' diameter bound for Riemannian manifolds in the case where the Ricci curvature below a positive constant is small in an averaged sense. This improves several previous results for manifolds with small amounts of nonpositive curvature.

1. Introduction.

One of the earliest and most fundamental theorems relating local geometry to global geometry/topology is that of Myers [10], which states that a complete Riemannian manifold with Ricci curvature bounded from below by $(n-1)k > 0$ is compact, with diameter $\leq \pi/\sqrt{k}$, and finite fundamental group. There have been several attempts to generalize this in various directions, the closest in spirit to the following being those of Elworthy–Rosenberg [5],[6], Rosenberg–Yang [14], and Wu [15] which extend the theorem to manifolds with small "wells of negative curvature". In these cases, the wells are assumed to be of either small diameter ([5], [15]), or small volume ([6], [14]). (Note that with this type of theorem it is also necessary to impose a restriction on the "depth" of the wells, such as a fixed but arbitrary lower Ricci curvature bound, as can be seen by attaching a very small handle to a sphere, or by attaching two spheres by a very small neck.) In addition, in [14] there are related generalizations of Bochner's vanishing theorem and Myers' π_1-finiteness result for wells bounded in L^1-norm.

Here, we show using an inequality of Cheeger–Colding that in fact for a complete Riemannian manifold with (nonpositive) lower Ricci curvature bounds, one has bounded diameter and finite fundamental group provided that the Ricci curvature below some positive constant is small in a suitable integral sense. This generalizes several of the results in the above papers. In particular, all theorems on the finiteness of $\pi_1(M)$ in the above are consequences of Theorem 1.3 below. Furthermore, [14, Theorem 11.1] and [15, Theorem 1] should be compared with Corollary 3.2 and Theorem 1.2, where we are able to bound the diameter of M by a value arbitrarily close to π without restricting the set of points on which the Ricci curvature can be nonpositive.

For many recent results on manifolds with integral curvature bounds, see [11], [12], [13] and the references therein.

Notation: Let $\mathrm{Ric}_-(x)$ denote the lowest eigenvalue of the Ricci tensor, Ric_x. (M^n, g) will always be a complete n-dimensional Riemannian manifold. For an arbitrary function f on M, $f_+(x) = \max\{f(x), 0\}$.

Theorem 1.1. *Let (M,g) be a compact Riemannian manifold with $\mathrm{Ric} \geq 0$. Then for any $\delta > 0$ there exists $\varepsilon = \varepsilon(n,\delta)$ such that if*

$$\frac{1}{\mathrm{vol}(M)} \int_M ((n-1) - \mathrm{Ric}_-)_+ dV < \varepsilon(n,\delta), \tag{1.1}$$

then $\mathrm{diam}(M) < \pi + \delta$.

In the case that (M,g) is noncompact or does not possess nonnegative Ricci curvature, one can achieve a similar result by averaging the 'bad' part of Ric over metric balls.

Theorem 1.2. *Let (M,g) be a complete Riemannian manifold with $\mathrm{Ric} \geq (n-1)k$ $(k \leq 0)$. Then for any $R, \delta > 0$, there exists $\varepsilon = \varepsilon(n,k,R,\delta)$ such that if*

$$\sup_x \frac{1}{\mathrm{vol}(B(x,R))} \int_{B(x,R)} ((n-1) - \mathrm{Ric}_-)_+ dV < \varepsilon(n,k,R,\delta), \tag{1.2}$$

then (M,g) is compact, with $\mathrm{diam}(M) < \pi + \delta$.

Finally, we show that under similar conditions, one can also restrict the topology of M as in Myers' Theorem.

Theorem 1.3. *Let (M,g) be a complete Riemannian manifold with $\mathrm{Ric} \geq (n-1)k$ $(k \leq 0)$. Then for any $R > 0$, there exists $\tilde{\varepsilon} = \tilde{\varepsilon}(n,k,R)$ such that if*

$$\sup_x \frac{1}{\mathrm{vol}(B(x,R))} \int_{B(x,R)} ((n-1) - \mathrm{Ric}_-)_+ dV < \tilde{\varepsilon}(n,k,R), \tag{1.3}$$

then the universal cover of M is compact, and hence $\pi_1(M)$ is finite.

We note that in the aforementioned papers, all theorems on the finiteness of $\pi_1(M)$ are proved in the class of Riemannian manifolds with $\mathrm{Ric} \geq (n-1)k$, $\mathrm{diam} < D$, and $\mathrm{vol} > v$. By a theorem of Anderson [1], this implies that the number of isomorphism classes possible for $\pi_1(M)$ is finite. However, in the above we merely require that our curvature quantity is small in an averaged sense, rather than assuming a strict lower volume bound on (M, g). Hence there are an infinite number of possible isomorphism classes for $\pi_1(M)$. For instance, $S^3/\mathbb{Z}_q$ with constant sectional curvature 1 metric will always satisfy the hypotheses of Theorem 1.3. On the other hand, in the noncollapsing case one can also extend a result of [14] to achieve finiteness of $\pi_1(M)$ when no pointwise curvature bounds are assumed at all (Theorem 4.2 below).

As with [15], [14, Theorem 11.1], the proofs of Theorems 1.1-1.3 are geometric, being for the most part faithful to Myers' method in [10]. This is in contrast with [6], which depends on more analytic techniques. For recent probabilistic arguments, see [2] for an entirely different proof of the original theorem of Myers, and also [9] for results related to [14].

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2. Compact manifolds of nonnegative Ricci curvature.

Let A_1, A_2, W be open subsets of M such that $A_1, A_2 \subset W$, and all minimal geodesics $\gamma_{x,y}$ from $x \in A_1$ to $y \in A_2$ lie in W. f will be any nonnegative integrable function on M.

In order to convert integral curvature bounds on M into integral bounds along geodesics we will use the following estimate of Cheeger and Colding ([4, Theorem 2.11]):

$$\int_{A_1 \times A_2} \int_{\gamma_{x,y}} f(\gamma(s)) ds dV_{A_1 \times A_2} \leq C(n, k, R) \left(\operatorname{diam}(A_2) \operatorname{vol}(A_1) + \operatorname{diam}(A_1) \operatorname{vol}(A_2)\right) \int_W f dV. \tag{2.1}$$

Where for $k \leq 0$,

$$C(n, k, R) = \frac{\operatorname{area}(\partial B_k(x, R))}{\operatorname{area}(\partial B_k(x, \frac{R}{2}))}, \tag{2.2}$$

$$R \geq \sup\{d(x, y) | (x, y) \in (A_1 \times A_2)\}, \tag{2.3}$$

and $B_k(x, r)$ denotes the ball of radius r in the simply-connected space of constant sectional curvature k. We will assume henceforth in this section that $\operatorname{Ric} \geq 0$ on M, and thus $C(n, k, R) = C(n)$.

A_1 and A_2 will be metric balls of small radius, $W = M$, and $f = ((n-1) - \mathrm{Ric}_-)_+$. We assume that all geodesics are parameterized by arclength and, by possibly removing a set of measure 0 in $A_1 \times A_2$, that there is a unique minimal geodesic from x to y for all $(x, y) \in (A_1, A_2)$.

Proof of Theorem 1.1. Let $p, q \in M$ be such that $d(p, q) = \mathrm{diam}(M) = D$, $r > 0$, $A_1 = B(p, r)$, $A_2 = B(q, r)$.

Then (2.1) gives

$$\text{(2.4)} \quad \int_{A_1 \times A_2} \int_{\gamma_{x,y}} ((n-1) - \mathrm{Ric}_-)_+ ds dV_{A_1 \times A_2} \\ \leq 2rC(n) \left(\mathrm{vol}(A_1) + \mathrm{vol}(A_2)\right) \int_M ((n-1) - \mathrm{Ric}_-)_+ dV,$$

which implies

$$\text{(2.5)} \quad \inf_{(x,y) \in A_1 \times A_2} \int_{\gamma_{x,y}} ((n-1) - \mathrm{Ric}_-)_+ ds \\ \leq 2rC(n) \left(\frac{1}{\mathrm{vol}(A_1)} + \frac{1}{\mathrm{vol}(A_2)} \right) \int_M ((n-1) - \mathrm{Ric}_-)_+ dV \\ \leq 4rC(n) \frac{D^n}{r^n} \frac{1}{\mathrm{vol}(M)} \int_M ((n-1) - \mathrm{Ric}_-)_+ dV,$$

where the final inequality follows from relative volume comparison. We can then find a minimizing unit-speed geodesic γ from $x \in \overline{A}_1$ to $y \in \overline{A}_2$ which realizes this infimum, and will show that for $L = d(x, y)$ much larger than π, γ cannot be minimizing if the right hand side of (2.5) is small enough.

Let $E_1(t), ..., E_n(t) = \gamma'(t)$ be parallel, pointwise orthonormal vector fields along γ, $Y_i = \sin(\frac{\pi t}{L}) E_i(t)$, $i = 1, ..., n-1$. Then denoting by $L_i(s)$ the length functional of a fixed-endpoint variation of curves through γ with variational vector field Y_i, we have by the second variation formula for arclength

$$\text{(2.6)} \quad \sum_{i=1}^{n-1} \frac{d^2 L_i}{ds^2}\bigg|_{s=0} = \sum_{i=1}^{n-1} \int_0^L g(\nabla_{\gamma'} Y_i, \nabla_{\gamma'} Y_i) - R(\gamma', Y_i, \gamma', Y_i) ds \\ = \int_0^L (n-1) \frac{\pi^2}{L^2} \cos^2(\frac{\pi t}{L}) - \sin^2(\frac{\pi t}{L}) \mathrm{Ric}(\gamma', \gamma') ds \\ = (n-1) \int_0^L \frac{\pi^2}{L^2} \cos^2(\frac{\pi t}{L}) - \sin^2(\frac{\pi t}{L}) ds$$

$$+\int_0^L \sin^2(\frac{\pi t}{L})((n-1)-\mathrm{Ric}(\gamma',\gamma'))ds$$
$$=-\frac{(n-1)L}{2}\left(1-\frac{\pi^2}{L^2}\right)$$
$$+\int_0^L \sin^2(\frac{\pi t}{L})((n-1)-\mathrm{Ric}(\gamma',\gamma'))ds.$$

And if the above quantity is negative, then there is a fixed endpoint variation $\gamma_s(t)$, satisfying

$$\frac{d^2}{ds^2}\mathrm{length}(\gamma_s)\Big|_{s=0}<0, \tag{2.7}$$

and thus $\gamma=\gamma_0$ cannot minimize arclength.

But then,

$$\begin{aligned}\int_0^L \sin^2(\frac{\pi t}{L})&((n-1)-\mathrm{Ric}(\gamma',\gamma'))ds \\ &\le \int_0^L \sin^2(\frac{\pi t}{L})((n-1)-\mathrm{Ric}_-)_+ds \\ &\le \int_0^L ((n-1)-\mathrm{Ric}_-)_+ds \\ &\le 4rC(n)\frac{D^n}{r^n}\frac{1}{\mathrm{vol}(M)}\int_M((n-1)-\mathrm{Ric}_-)_+dV.\end{aligned} \tag{2.8}$$

So suppose that we want to assure $D=\mathrm{diam}(M)<\pi+\delta$. Then letting $r=\frac{D}{N}$, choose $N=N(\delta)$ such that

$$\frac{1}{1-\frac{2}{N}}<\frac{\pi+\delta}{\pi+\frac{\delta}{2}}. \tag{2.9}$$

By the triangle inequality,

$$L=d(x,y)\ge d(p,q)-2r=D\left(1-\frac{2}{N}\right). \tag{2.10}$$

So then showing that L must be less than $\pi+\frac{\delta}{2}$ will finish the proof. By (2.6), γ cannot be minimal if

$$4rC(n)\frac{D^n}{r^n}\frac{1}{\mathrm{vol}(M)}\int_M((n-1)-\mathrm{Ric}_-)_+dV<\frac{(n-1)L}{2}\left(1-\frac{\pi^2}{L^2}\right). \tag{2.11}$$

This, together with (2.10) implies that γ cannot be minimizing if

(2.12)
$$4C(n)\frac{1}{1-\frac{2}{N}}\frac{N^{n-1}}{\mathrm{vol}(M)}\int_M((n-1)-\mathrm{Ric}_-)_+dV < \frac{(n-1)}{2}\left(1-\frac{\pi^2}{L^2}\right).$$

So then choosing

$$\varepsilon = \frac{(n-1)(1-\frac{2}{N})}{8C(n)N^{n-1}}\left(1-\frac{\pi^2}{(\pi+\frac{\delta}{2})^2}\right), \tag{2.13}$$

we have that if

$$\frac{1}{\mathrm{vol}(M)}\int_M((n-1)-\mathrm{Ric}_-)_+dV < \varepsilon, \tag{2.14}$$

then

$$D = \mathrm{diam}(M) \leq \frac{L}{1-\frac{2}{N}} < \pi+\delta. \tag{2.15}$$

□

3. Complete Manifolds of Ricci Curvature Bounded Below.

We first assume that $R > \pi$. Then we will in fact show the following:

Theorem 3.1. *Let (M,g) be a complete Riemannian manifold with* Ric $\geq (n-1)k$ $(k \leq 0)$. *Then for any fixed $R > \pi$, there exists $\varepsilon = \varepsilon(n,k,R,\delta)$ such that if*

$$\frac{1}{\mathrm{vol}(B(p,R))}\int_{B(p,R)}((n-1)-\mathrm{Ric}_-)_+dV < \varepsilon(n,k,R,\delta), \tag{3.1}$$

for some $B(p,R) \subset M$ then $M = B(p,R) \subset B(p,\pi+\delta)$.

Proof. Again we will use estimate (2.1). Fix $p \in M$, $W = B(p,R)$. Then q will be any point in W such that $\pi + 4r < d(p,q) < R - 3r$, where $0 < r < \frac{1}{8}(R-\pi)$ is to be determined, and $A_1 = B(p,r)$, $A_2 = B(q,r)$. From the triangle inequality, all minimal geodesics from $x \in \overline{A}_1$ to $y \in \overline{A}_2$ lie in W.

As in the proof of Theorem 1.1, there is a geodesic γ of length L from $x \in \overline{A}_1$ to $y \in \overline{A}_2$ with

$$
\begin{aligned}
(3.2)\quad & \int_\gamma ((n-1) - \mathrm{Ric}_-)_+ ds \\
& \le 2rC(n,k,R)\left(\frac{1}{\mathrm{vol}(B(p,r))} + \frac{1}{\mathrm{vol}(B(q,r))}\right)\int_{B(p,R)} ((n-1)-\mathrm{Ric}_-)_+ dV \\
& \le 2rC(n,k,R)\left(\frac{v_k(R)}{v_k(r)}\frac{1}{\mathrm{vol}(B(p,R))} + \frac{v_k(2R)}{v_k(r)}\frac{1}{\mathrm{vol}(B(q,2R))}\right)\int_{B(p,R)} ((n-1)-\mathrm{Ric}_-)_+ dV \\
& \le 2rC(n,k,R)\left(\frac{v_k(R)}{v_k(r)} + \frac{v_k(2R)}{v_k(r)}\right)\frac{1}{\mathrm{vol}(B(p,R))}\int_{B(p,R)} ((n-1)-\mathrm{Ric}_-)_+ dV.
\end{aligned}
$$

Now suppose we want to show that $B(p,R) \subset B(p,\pi+\delta)$, where $\delta < \frac{1}{2}(R-\pi)$. Then fixing $r = \frac{1}{4}\delta$, and proceeding as in Theorem 1.1, we have that (3.2) implies that γ cannot be minimizing if

$$
(3.3)\quad \frac{1}{2}\delta C(n,k,R)\frac{v_k(R)+v_k(2R)}{v_k(\frac{1}{4}\delta)}\frac{1}{\mathrm{vol}(B(p,R))}\int_{B(p,R)} ((n-1)-\mathrm{Ric}_-)_+ dV \le \frac{(n-1)L}{2}\left(1-\frac{\pi^2}{L^2}\right).
$$

So setting

$$
(3.4)\quad \varepsilon = \frac{2}{\delta C(n,k,R)}\frac{v_k(\frac{1}{4}\delta)}{v_k(R)+v_k(2R)}\frac{(n-1)(\pi+\frac{1}{2}\delta)}{2}\left(1-\frac{\pi^2}{(\pi+\frac{1}{2}\delta)^2}\right),
$$

we have that if

$$
(3.5)\quad \frac{1}{\mathrm{vol}(B(p,R))}\int_{B(p,R)} ((n-1)-\mathrm{Ric}_-)_+ dV < \varepsilon,
$$

then the minimizing geodesic γ from $B(p,r)$ to $B(q,r)$ must have length $L \le \pi+\frac{1}{2}\delta$. So then $D = d(p,q) \le \pi+\delta$. Hence for all points $q \in B(p,R-3r)$

we have $d(p,q) \leq \pi + \delta$. And since $\pi + \delta < R - 3r$, this implies that no geodesic emanating from p of length greater than $\pi + \delta$ can be length minimizing. Therefore $M = B(p,R) = B(p, R-3r) \subset B(p, \pi + \delta)$. □

By the triangle inequality, one also has:

Corollary 3.2. *Let (M,g) be a complete Riemannian manifold with* $\mathrm{Ric} \geq (n-1)k$ $(k \leq 0)$. *Then for any fixed $R > \pi$, there exists $\varepsilon = \varepsilon(n,k,R,\delta)$ such that if*

$$\frac{1}{\mathrm{vol}(B(p,R))} \int_{B(p,R)} ((n-1) - \mathrm{Ric}_-)_+ dV < \varepsilon(n,k,R,\delta) \tag{3.6}$$

for some $B(p,R) \subset M$, then $\mathrm{diam}(M) < 2(\pi + \delta)$.

Theorem 3.1 shows that Theorem 1.2 holds for $R > \pi$. We are then left with showing the result for $R \leq \pi$.

Proof of Theorem 1.2.

Let $R' > \pi$ be fixed. Then for any $R \leq \pi$, we have as is standard ([8]) that there is $N = N(k,R,R')$ such that any R'-ball in M can be covered by N or fewer R-balls, $B(x_i, R)$. So then,

$$\begin{aligned}
&\frac{1}{\mathrm{vol}(B(z,R'))} \int_{B(z,R')} ((n-1) - \mathrm{Ric}_-)_+ dV \\
&\leq N(k,R,R') \frac{1}{\mathrm{vol}(B(z,R'))} \sup_{x_i} \int_{B(x_i,R)} ((n-1) - \mathrm{Ric}_-)_+ dV \\
&\leq N(k,R,R') \frac{v_k(R+R')}{v_k(R')} \frac{1}{\mathrm{vol}(B(z,R+R'))} \\
&\qquad \cdot \sup_{x_i} \int_{B(x_i,R)} ((n-1) - \mathrm{Ric}_-)_+ dV \\
&\leq N(k,R,R') \frac{v_k(R+R')}{v_k(R')} \\
&\qquad \cdot \sup_{x_i} \frac{1}{\mathrm{vol}(B(x_i,R))} \int_{B(x_i,R)} ((n-1) - \mathrm{Ric}_-)_+ dV.
\end{aligned} \tag{3.7}$$

And thus we have

$$\begin{aligned}
&\sup_x \frac{1}{\mathrm{vol}(B(x,R'))} \int_{B(x,R')} ((n-1) - \mathrm{Ric}_-)_+ dV \\
&\leq N(k,R,R') \frac{v_k(R+R')}{v_k(R')} \sup_x \frac{1}{\mathrm{vol}(B(x,R))} \int_{B(x,R)} ((n-1) - \mathrm{Ric}_-)_+ dV. \ \square
\end{aligned} \tag{3.8}$$

4. Integral curvature bounds and the fundamental group.

We now show that choosing

$$\sup_x \frac{1}{\operatorname{vol}(B(x,R))}\int_{B(x,R)} ((n-1)-\operatorname{Ric}_-)_+ dV \tag{4.1}$$

small enough will imply compactness of the universal cover of M. As in the proof of Theorem 1.2, we will actually show something slightly more general than this. Theorem 1.3 will then follow from (3.8).

Theorem 4.1. *Let (M,g) be a complete Riemannian manifold with $\operatorname{Ric} \geq (n-1)k$ $(k \leq 0)$. Then for any $R > 2\pi$, there exists $\tilde{\varepsilon} = \tilde{\varepsilon}(n,k,R)$ such that if*

$$\frac{1}{\operatorname{vol}(B(x,R))}\int_{B(x,R)} ((n-1)-\operatorname{Ric}_-)_+ dV < \tilde{\varepsilon}(n,k,R), \tag{4.2}$$

for some $B(x,R) \subset M$ then the universal cover of M is compact, and hence $\pi_1(M)$ is finite.

Proof. Let $R > 2\pi$. By Corollary 3.2 we can assume that $\operatorname{diam}(M) < R$. So it is then sufficient to show that

$$\frac{1}{\operatorname{vol}(B(\tilde{x},R))}\int_{B(\tilde{x},R)} ((n-1)-\operatorname{Ric}_-)_+ dV \to 0 \tag{4.3}$$

for $\tilde{x}$ in $\tilde{M}$ as

$$\frac{1}{\operatorname{vol}(M)}\int_{M} ((n-1)-\operatorname{Ric}_-)_+ dV \to 0. \tag{4.4}$$

Let N denote the minimal number of fundamental domains in $\tilde{M}$ necessary to cover $B(\tilde{x},R)$. Then we have that

$$\begin{aligned}
&\frac{1}{\operatorname{vol}(B(\tilde{x},R))}\int_{B(\tilde{x},R)} ((n-1)-\operatorname{Ric}_-)_+ dV \\
&\qquad\leq \frac{N}{\operatorname{vol}(B(\tilde{x},R))}\int_{M} ((n-1)-\operatorname{Ric}_-)_+ dV \\
&\qquad\leq \frac{v_k(3R)}{v_k(R)}\frac{N}{\operatorname{vol}(B(\tilde{x},3R))}\int_{M} ((n-1)-\operatorname{Ric}_-)_+ dV \\
&\qquad\leq \frac{v_k(3R)}{v_k(R)}\frac{N}{N\operatorname{vol}(M)}\int_{M} ((n-1)-\operatorname{Ric}_-)_+ dV \\
&\qquad= \frac{v_k(3R)}{v_k(R)}\frac{1}{\operatorname{vol}(M)}\int_{M} ((n-1)-\operatorname{Ric}_-)_+ dV,
\end{aligned} \tag{4.5}$$

and the result follows. □

It is worth pointing out that given the results in [7], [12], and [14] it is very possible that Theorems 1.2-1.3 are true with pointwise Ricci-curvature bound replaced by smallness in the L^p norm of $(-\operatorname{Ric}_-)_+$, $p > \frac{n}{2}$. However (2.1) does not readily generalize to this case. We do have at least one obvious extension of a result in [14] to the case of L^p curvature bounds, which states that given (M, g) with upper diameter bound, lower volume bound, and lower sectional curvature bound, smallness of the L^1 norm of the Ricci curvature below a positive constant implies finite fundamental group. This is a consequence of Theorem 1.3 above, which does not require a sectional curvature bound, but it can also be slightly generalized in a different direction. Namely, in [14] the lower sectional curvature bound is used to control the 1-systole by Cheeger's method ([3]), which has been extended to L^p curvature bounds in [11]. Therefore combining the proof of [14, Theorem 5] with [11] one has:

Theorem 4.2. *Suppose* (M, g) *satisfies* $\operatorname{diam}(M) < D$, $\operatorname{vol}(M) > v$. *Let* $\Lambda \in \mathbb{R}$, $\lambda > 0$, $p > n-1$. *Then there is* $\varepsilon_1 = \varepsilon_1(n, p, v, D, \Lambda)$, $\varepsilon_2 = \varepsilon_2(n, p, v, D, \lambda, \Lambda)$ *such that if*

$$\int_M (\Lambda - \sec_-)_+^p dV < \varepsilon_1 \tag{4.6}$$

and

$$\int_M (\lambda - \operatorname{Ric}_-)_+ dV < \varepsilon_2, \tag{4.7}$$

then M *has finite fundamental group. Here,* $\sec_-(x)$ *denotes the infimum of sectional curvatures of 2-planes at* x.

Proof. We follow the method of [14].

Let A and B be constants such that the Sobolev inequality

$$\|f\|^2_{2q/(q-1)} \leq A\|\nabla f\|_2^2 + B\|f\|_2^2 \tag{4.8}$$

holds, where $\frac{n}{2} < q < p$. By the proof of [14, Theorem 10.1], if

$$\int_M (\lambda - \operatorname{Ric}_-)_+^q dV < \min(A^{-1}, \lambda B^{-1}), \tag{4.9}$$

then the first homology group $H^1(\overline{M}, \mathbb{R})$ vanishes for every finite cover $\overline{M}$ of M. If we take $B = v^{-\frac{1}{q}}$, then by [7] A can be taken to be $A(q, v, D)$, under the assumption that

$$\int_M (\lambda - \mathrm{Ric}_-)_+^q dV < \varepsilon_3(n, q, v, D). \tag{4.10}$$

Given that $H^1(\overline{M}, \mathbb{R}) = 0$ for all $\overline{M}$, it is shown in [6] that if $\pi_1(M)$ is of polynomial growth, then $\pi_1(M)$ is in fact finite. By [14, Theorem 6], we have that this is true if

$$\int_M (\lambda - \mathrm{Ric}_-)_+^q dV < \varepsilon_4(n, q, v, D, s), \tag{4.11}$$

where s is a lower bound for $\mathrm{sys}_1(M)$. To bound $\mathrm{sys}_1(M)$ from below, we use [11, Theorem 1.2], which shows that there is $\varepsilon_1(n, p, v, D, \Lambda)$ such that if

$$\int_M (\Lambda - \sec_-)_+^p dV < \varepsilon_1, \tag{4.12}$$

then $\mathrm{sys}_1(M) \geq s(n, p, v, D, \Lambda)$. So then assuming (4.12) holds, we can choose

$$\varepsilon_5 = \min(A^{-1}, \lambda B^{-1}, \varepsilon_3, \varepsilon_4) = \varepsilon_5(n, p, q, v, D, \lambda, \Lambda). \tag{4.13}$$

Such that if

$$\int_M (\Lambda - \sec_-)_+^p dV < \varepsilon_1 \tag{4.14}$$

and

$$\int_M (\lambda - \mathrm{Ric}_-)_+^q dV < \varepsilon_5, \tag{4.15}$$

then M has finite fundamental group. But then we also have

$$
\begin{aligned}
(4.16) \quad & \int_M (\lambda - \mathrm{Ric}_-)_+^q dV \\
&= \int_{\{\mathrm{Ric}_- > \Lambda\}} (\lambda - \mathrm{Ric}_-)_+^q dV + \int_{\{\mathrm{Ric}_- \leq \Lambda\}} (\lambda - \mathrm{Ric}_-)_+^q dV \\
&\leq (\lambda - \Lambda)^{q-1} \int_{\{\mathrm{Ric}_- > \Lambda\}} (\lambda - \mathrm{Ric}_-)_+ dV + 2^q (\lambda - \Lambda)^q \operatorname{vol}(\{\mathrm{Ric}_- \leq \Lambda\}) \\
&\quad + 2^q \int_{\{\mathrm{Ric}_- \leq \Lambda\}} (\Lambda - \mathrm{Ric}_-)_+^q dV \\
&\leq (\lambda - \Lambda)^{q-1} \int_{\{\mathrm{Ric}_- > \Lambda\}} (\lambda - \mathrm{Ric}_-)_+ dV + 2^q (\lambda - \Lambda)^q \operatorname{vol}(\{\mathrm{Ric}_- \leq \Lambda\}) \\
&\quad + 2^q \operatorname{vol}(\{\mathrm{Ric}_- \leq \Lambda\})^{\frac{p-q}{p}} \left(\int_M (\Lambda - \mathrm{Ric}_-)_+^p dV\right)^{\frac{q}{p}} \\
&\leq (\lambda - \Lambda)^{q-1}(1 + 2^q) \int_M (\lambda - \mathrm{Ric}_-)_+ dV \\
&\quad + 2^q (\lambda - \Lambda)^{\frac{q-p}{p}} \left(\int_M (\lambda - \mathrm{Ric}_-)_+ dV\right)^{\frac{p-q}{p}},
\end{aligned}
$$

where in the final inequality we have assumed that ε_1 was chosen to be less than 1. So then we fix a $q > \frac{n}{2}$, and choose $\varepsilon_2 = \varepsilon_2(n, p, v, D, \lambda, \Lambda)$ such that for

$$
(4.17) \qquad \int_M (\lambda - \mathrm{Ric}_-)_+ dV < \varepsilon_2,
$$

the right hand side of the above equation is less than ε_5. □

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