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Hermitian-Einstein metrics and Chern number inequalities on parabolic stable bundles over Kähler manifolds

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Let $\overline{X}$ be a compact complex manifold with a smooth Kähler metric and $D = \sum_{i=1}^{m} D_i$ a divisor in $\overline{X}$ with normal crossings. Let E be a holomorphic vector bundle over $\overline{X}$ with a stable parabolic structure along D. We prove that there exists a Hermitian-Einstein metric on $E' = E|_{\overline{X}\backslash D}$ and obtain a Chern number inequality for a stable parabolic bundle.
Without the assumption that the irreducible components D_i of D meet transversely, using Hironaka's theorem on the resolution of singularities, we also get a Chern number inequality for a more general stable parabolic bundle.

1. Introduction.

Let $\overline{X}$ be a compact Kähler manifold of complex dimension n with a Kähler form ω. Let E be a rank r holomorphic vector bundle over $\overline{X}$. It is proved that E is stable if and only if E admits a Hermitian-Einstein metric, under the assumption that E is indecomposable ([N-S], [D1], [D2], [D3], [U-Y]). The theorem yields Bogomolov-Gieseker inequality easily, which says that, if E is stable,

$$(2C_2(E) - \frac{r-1}{r}C_1(E)^2) \cdot [\omega]^{n-2} \geq 0.$$

Let $D = \sum_{i=1}^{m} D_i$ be a divisor in $\overline{X}$ with normal crossings, we introduce a parabolic structure of E with respect to D which consists of flags of $E|_{D_i}$ and weights attached to the flags, we define the notion of parabolic stability of a parabolic structure (see Section 2). Set $X = \overline{X}\backslash D, E' = E|_X$. In section 3, we construct a metric K_0 on E' with the property that $|F_{K_0}|_{K_0} \in L^p(X)$ $(p > 1)$. In Section 4, we construct Kähler metrics ω_α $(0 < \alpha < 2)$ on X and show that (X, ω_α) satisfies the three assumptions in Section 2 of [S1]. It is proved in Section 5 (Proposition 5.10) that the parabolic structure is parabolic stable if and only if (E, K_0) is analytic stable (Definition 5.2, also see [S1],p.877), which yields one of our main results (Theorem 6.3) that the parabolic stability of a parabolic structure is essentially equivalent to the existence of a Hermitian-Einstein metric on E' with respect to ω_α for some $0 < \alpha < 2$. Furthermore, we prove a Chern number inequality in section 7 (Theorem 7.5) for a stable parabolic structure.

In section 8, we prove a Chern number inequality (Theorem 8.3) for a more general stable parabolic structure. In this case, we do not assume that the irreducible components D_i of D meet transversely, and we need not suppose that the flags of $E|_{D_i}$ satisfy the compatibility condition (Definition 2.1 and Definition 8.1). We use a theorem of Hironaka [H] on the resolution of singularities to get complex manifold $\widetilde{X}$ by successively blowing up submanifolds such that the proper transforms D_i^* of $D_i, (i = 1, \cdots, m)$ do not meet each other. Let $q : \widetilde{X} \to \overline{X}$ be the canonical map, and let $\widetilde{\omega}$ be a Kähler form on $\widetilde{X}$. We show (Lemma 8.5) that the stable parabolic structure of E along $D = \sum_{i=1}^m D_i$ induces a stable parabolic structure of q^*E along $D^* = \sum_{i=1}^m D_i^*$ on $\widetilde{X}$ using the Kähler form $q^*\omega + \varepsilon\widetilde{\omega}$ for sufficiently small $\varepsilon > 0$. Then Theorem 7.5 yields another main result of this paper, the Chern number inequality Theorem 8.3.

If D is a smooth divisor, Li-Narasimhan [L-N] construct a metric K_0 on E' with $|F_{K_0}|_{K_0} \in L^p(X)$ $(p > 2)$. If X is of complex dimension 2, they show the equivalence between the stability of a parabolic structure and the existence of a Hermitian-Einstein metric on the bundle, using the restriction of the Kähler metric ω to X.

Parabolic bundles over Riemann surfaces is treated in [MS, B, K, Na-St, P, S1, S2].

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2. Parabolic stability over Kähler manifolds.

Let $\overline{X}$ be a compact Kähler manifold of complex dimension n with a Kähler metric ω, D a divisor in $\overline{X}$ with normal crossings. Let $X = \overline{X} \setminus D$, the restriction of ω to X gives a Kähler metric on X, we fix it once for all. Set $D = \sum_{i=1}^m D_i$ where the irreducible components D_i of D are smooth and meet transversely.

Let E be a holomorphic vector bundle over $\overline{X}, E' = E|_X$. Define I to be the set of all tuples of integers $(k_1, \cdots, k_j)$ with $1 \le j \le n$ and $1 \le k_i \le m$. For each $J = (k_1, \cdots, k_j) \in I$, denote by X_J the smooth variety defined as the intersection of $D_{k_1} \cdots D_{k_j}$.

Definition 2.1. A parabolic structure on E with respect to D consists of

a) flags of $E|_{D_i} (1 \le i \le m)$:

$$E|_{D_i} = F_1^i \supset F_2^i \supset \cdots \supset F_{m_i}^i \supset \{0\} = F_{m_i+1}^i$$

where F_{l+1}^i is a proper subbundle of F_l^i $(1 \le l \le m_i - 1)$, and the flags satisfy the following compatibility condition: For every $J = (k_1, \cdots, k_j) \in I, \{F_l^{k_i}|_{D_{k_1} \cdots D_{k_j}}, 1 \le i \le j, 1 \le l \le m_{k_i}\}$ yields a flag of $E|_{D_{k_1} \cdots D_{k_j}}$ which is a refined flag of $\{F_l^{k_i}|_{D_{k_1} \cdots D_{k_j}}, 1 \le l \le m_{k_i}\}$ for each $i \in \{1, \cdots, j\}$.

b) weights $\alpha_1^i, \alpha_2^i, \cdots, \alpha_{m_i}^i$ attached to $F_1^i, F_2^i, \cdots, F_{m_i}^i$, satisfying $0 \le \alpha_1^i < \alpha_2^i < \cdots < \alpha_{m_i}^i < 1$.

A holomorphic vector bundle E with a parabolic structure is called parabolic bundle.

Definition 2.2. We define the parabolic degree of a parabolic bundle E by

$$par \deg E = \deg E + \sum_{i=1}^{m} \sum_{l=1}^{m_i} rank(F_l^i / F_{l+1}^i) \alpha_l^i \deg[D_i]$$

where $[D_i]$ is the line bundle defined by the divisor D_i, $\deg E$ (resp. $\deg[D_i]$) is the degree of E (resp. the degree of $[D_i]$) in the usual sense using the Kähler form ω.

Suppose that V is a proper coherent subsheaf of E with quotient torsion free. Then there is a natural flag of $V|_{D_i}$ by coherent subsheaves

$$V|_{D_i} = F_1^i V \supset \cdots \supset F_{n_i}^i V \supset \{0\} = F_{n_i+1}^i V$$

induced by $F_1^i \cap V \supseteq \cdots \supseteq F_{m_i}^i \cap V \supseteq \{0\}$, clearly, $n_i \le m_i$. Let us define the weights attached to the flag by $\alpha_l^i(V) =$ the largest α_k^i such that $F_l^i V \subseteq F_k^i \cap V$, i.e., the subscript k is the largest integer with the property that $F_l^i V \subseteq F_k^i$, $1 \le l \le n_i$, $1 \le i \le m$.

Definition 2.3. We define the parabolic degree of V by

$$par\deg V = \deg V + \sum_{i=1}^{m}\sum_{l=1}^{n_i} rank(F_l^iV/F_{l+1}^iV)\alpha_l^i(V)\deg[D_i]$$

Definition 2.4. We say that a parabolic bundle E is parabolic stable if for every proper coherent subsheaf V of E with quotient torsion free we have

$$\frac{par\deg V}{rankV} < \frac{par\deg E}{rankE}$$

3. Construction of metrics on vector bundles.

Let E be a parabolic bundle over $\overline{X}$ as given in Section 2. At a point $p \in D$ through which $j(1 \le j \le n)$ of the D_i pass, we may choose local holomorphic coordinates in a neighborhood $U = \triangle^n = \{|z_i| < 1, i = 1, \cdots, n\}$ of $p = (0, \cdots, 0)$ such that $D \cap U = \{z_1 \cdots z_j = 0\}$ is the union of coordinate hyperplanes. The complement $U^* = U \backslash U \cap D = (\triangle^*)^j \times \triangle^{n-j}$ is a punctured polycylinder $P^*(j, n)$ given by $\{(z_1, \cdots, z_n) | |z_i| < 1, z_1 \cdots z_j \ne 0\}$.

Definition 3.1. If $\{e_1, \cdots, e_r\}$ $(r = rankE)$ is a holomorphic basis of E in $\triangle^n$ satisfying the property that, for any $i \in \{1, \cdots, j\}$:

$$\left\{e_{r-r^i_{m_i}+1}, \cdots, e_r\right\} \text{ is a basis of } F^i_{m_i} \text{ over } U \cap D_i,$$
$$\left\{e_{r-r^i_{m_i-1}+1}, \cdots, e_r\right\} \text{ is a basis of } F^i_{m_i-1} \text{ over } U \cap D_i,$$
$$\vdots$$
$$\left\{e_{r-r^i_2+1}, \cdots, e_r\right\} \text{ is a basis of } F^i_2 \text{ over } U \cap D_i$$

where $r^i_l = rankF^i_l, l = 1, \cdots, m_i$, we say that it preserves the flags on $U \cap D$.

The following lemma is clear.

Lemma 3.2. *There is a holomorphic basis $\{e_1, \cdots, e_r\}$ of E in $\triangle^n$ such that it preserves the flags on $U \cap D$.*

Proof. We choose a holomorphic basis $\{e_1, \cdots, e_r\}$ of $E|_{D_1 \cdots D_j}$, such that it preserves the flag F yielded by $\{F^i_l|_{D_1 \cdots D_j}, 1 \le l \le m_i, 1 \le i \le j\}$. Since

F is a refined flag of $\{F_l^i|_{D_1\cdots D_j}, 1 \le l \le m_i\}$ for each $i \in \{1, \cdots, j\}$, the basis $\{e_1, \cdots, e_r\}$ can be extended naturally so that it preserves the flags on $U \cap D$. This proves the lemma.

Let $\{e_1, \cdots, e_r\}$ be a holomorphic basis of E in a neighborhood U_{p_1} of p_1, $\{f_1, \cdots, f_r\}$ a holomorphic basis of E in a neighborhood U_{p_2} of p_2. Suppose that $(f_1, \cdots, f_r) = (e_1, \cdots, e_r)g$, i.e. $f_\beta = e_\alpha g_{\alpha\beta}$ in $U_{p_1} \cap U_{p_2} \neq \emptyset$, $U_{p_1} \cap U_{p_2} \cap D = \cup_{i=1}^j D_{k_i}$ $(1 \le j \le n)$. Assume that $\{e_1, \cdots, e_r\}$ (resp. $\{f_1, \cdots, f_r\}$) preserves the flags in U_{p_1} (resp. in U_{p_2}). Then on each $D_{k_i} (i = 1, 2, \cdots, j)$,

$$(1) \qquad g_{\alpha\beta} = 0 \text{ if } r - r_{l-1}^{k_i} + 1 \le \beta \le r - r_l^{k_i}, \quad \alpha \le r - r_{l-1}^{k_i}$$

here $r_l^{k_i}$ =rank $F_l^{k_i}, 2 \le l \le m_{k_i} + 1$.

For each $i = 1, \cdots, m$, we choose a metric on the line bundle $[D_i]$ defined by the divisor D_i. Let σ_i be the canonical section of D_i which vanishes on D_i. We may assume that its length $|\sigma_i| < 1$. We put $\sigma = \sigma_i \otimes \cdots \otimes \sigma_m$, which is a section of $[D]$, then $|\sigma| = \Pi_i |\sigma_i| < 1$.

Put $\beta_l^i = \alpha_j^i$ if $r - r_j^i < l \le r - r_{j+1}^i$, where $r_j^i = rank F_j^i, j = 1, \cdots, m_i$. Set

$$\beta^i = \begin{pmatrix} \sigma_i^{-\beta_1^i} & & \\ & \ddots & \\ & & \sigma_i^{-\beta_r^i} \end{pmatrix}$$

$$S^i = \begin{pmatrix} |\sigma_i|^{\beta_1^i} & & \\ & \ddots & \\ & & |\sigma_i|^{\beta_r^i} \end{pmatrix}$$

Now we construct a metric H on $E|_{P^*(j,n)}$. Let $\{e_1, \cdots, e_r\}$ be a holomorphic basis of E in U preserving the flags on $U \cap D$. Assume that $U \cap D = \cup_{i=1}^j D_{k_i}$, we define the metric H so that its matrix with respect to $\{e_1, \cdots, e_r\}$ is $(S^{k_1})^2 \cdots (S^{k_j})^2$. Set $(e_1^{k_1\cdots k_j}, \cdots, e_r^{k_1\cdots k_j}) = (e_1, \cdots, e_r)\beta^{k_1} \cdots \beta^{k_j}$, it is well defined in a small neighborhood of any point $x \in P^*(j,n)$, and it is a holomorphic basis of E' there. It is clear that with respect to the basis $(e_1^{k_1\cdots k_j}, \cdots, e_r^{k_1\cdots k_j})$ the matrix of H is identity.

There are finite neighborhoods U_i and $V_i (i = 1, \cdots, N)$ such that, 1) $U_i \supset\supset V_i$ and $\cup_i V_i \supset D$; 2) associated to each U_i there is a unique j-tuple $(k_1, \cdots, k_j)$ with $1 \le j \le n$ such that $U_i \cap D_{k_1} \cap \cdots \cap D_{k_j}$ is an open coordinate chart of $D_{k_1} \cap \cdots \cap D_{k_j}$ disjoint from any other D_k $(k \neq k_l$, $l = 1, \cdots, j)$. Let $U_0 = \overline{X} \setminus \cup_i V_i$. Then $\{U_i \mid i = 0, \cdots, N\}$ is a finite covering of $\overline{X}$. Suppose that $\{\psi_i \in C_0^\infty(U_i) \mid i = 0, \cdots, N\}$ is a partition of unity corresponding to the covering. In each U_i $(i = 1, \cdots, N)$ we choose a metric H_i on E' as above and in U_0 we choose a smooth metric on E. We define a metric K_0 on E' by

$$K_0 = \sum_{i=0}^{N} \psi_i H_i \tag{2}$$

Lemma 3.3. *Let K_0 be the metric on E' defined above, then its curvature form satisfied that $|F_{K_0}|_{K_0} \in L^p(X)(p > 1)$.*

Proof. It is clear that, in $U_k \setminus (\cup_{i\neq k} U_i)(k = 1, \cdots, N), |F_{K_0}|_{K_0} \in L^\infty$. Suppose that $U_{i_1} \cap U_{i_2} \supset D_{k_1}(1 \leq i_1 \leq N, 1 \leq i_2 \leq N), D_{k_1} \cdots D_{k_{j_1}}$ is assoiated to $U_{i_1}, D_{k_1} D_{l_2} \cdots D_{l_{j_2}}$ is associated to U_{i_2}. Let $\{e_1, \cdots, e_r\}$ be a holomorphic basis of E preserving the flags on $U_{i_1} \cap D$, and let $\{f_1, \cdots, f_r\}$ be a holomorphic basis of E preserving the flags on $U_{i_2} \cap D$. We write $f_\beta = e_\alpha g_{\alpha\beta}$, then

$$\begin{aligned}
&(f_1^{k_1 l_2 \cdots l_{j_2}}, \cdots, f_r^{k_1 l_2 \cdots l_{j_2}}) \\
&\quad = (f_1 \cdots f_r) \beta^{k_1} \beta^{l_2} \cdots \beta^{l_{j_2}} = (e_1 \cdots e_r) g \beta^{k_1} \beta^{l_2} \cdots \beta^{l_{j_2}} \\
&\quad = (e_1^{k_1 \cdots k_{j_1}}, \cdots, e_r^{k_1 \cdots k_{j_1}}) (\beta^{k_{j_1}})^{-1} \cdots (\beta^{k_2})^{-1} (\beta^{k_1})^{-1} g \beta^{k_1} \beta^{l_2} \cdots \beta^{l_{j_2}}
\end{aligned}$$

where g is the matrix with elements $g_{\alpha\beta}$.

Set

$$G = (\beta^{k_1})^{-1} g \beta^{k_1} = (G_{\alpha\beta})$$

By (1) we can see that $G_{\alpha\alpha}$ are able to be extended holomorphically to D_{k_1} and that $G_{\alpha\beta} = H_{\alpha\beta} \sigma_{k_1}^{\gamma_{\alpha\beta}}$ $(\alpha \neq \beta)$ where

$$\gamma_{\alpha\beta} \geq \min\{ \alpha_i^{k_1} - \alpha_{i-1}^{k_1},\ 1 - (\alpha_i^{k_1} - \alpha_{i-1}^{k_1}) \mid i = 2, \cdots, m_{k_1} + 1 \} > 0$$

and $H_{\alpha\beta}$ are able to be extended holomorphically to D_{K_1}. So, using the holomorphic basis $(e_1^{k_1 \cdots k_{j_1}}, \cdots, e_r^{k_1 \cdots k_{j_1}})$, applying the fact that $G_{\alpha\beta}$ are holomorphic away from the divisor, we can calculate the curvature form and obtain

$$F_{K_0} = a_{k_1 \overline{k}_1} |\sigma_{k_1}|^{2(\gamma - 1)} dz^{k_1} d\overline{z}^{k_1} + A$$

where

$$\gamma \geq \min\{ \alpha_i^{k_1} - \alpha_{i-1}^{k_1},\ 1 - (\alpha_i^{k_1} - \alpha_{i-1}^{k_1}) \mid i = 2, \cdots, m_{k_1} + 1 \} > 0,$$

A is smooth, $a_{k_1\overline{k_1}}$ is smooth.

Similarly, if $\cap_{k=1}^{k_0} U_{i_k} \supset \cup_{j=1}^{l} D_{k_j}$, we can show that, in $\cap_{k=1}^{k_0} U_{i_k}$,

$$F_{K_0} = \sum_{j=1}^{l} a_{k_j\overline{k}_j} |\sigma_{k_j}|^{2(\gamma_j - 1)} dz^{k_j} d\overline{z}^{k_j} + A \tag{3}$$

where

$$\gamma_j \geq \min\{ \alpha_i^{k_j} - \alpha_{i-1}^{k_j},\ 1 - (\alpha_i^{k_j} - \alpha_{i-1}^{k_j}) \mid i = 2, \cdots, m_{k_j} + 1 \} > 0,$$

$a_{k_j\overline{k}_j}$ is smooth ($j = 1, \cdots, l$), A is smooth. Therefore the lemma follows.

4. Singular metrics on manifolds.

Recall that ω is a Kähler metric of $\overline{X}$. For $0 < \alpha < 2$ we define

$$\omega_\alpha = \sqrt{-1}\left(\frac{2}{2-\alpha}\right) \sum_{i=1}^{m} \partial\overline{\partial}|\sigma_i|^{2-\alpha} + C_\alpha \omega$$

where C_α is a constant large enough so that ω_α is a Kähler metric on X. We set $\omega_0 = \omega$.

For any point $p \in D$, we choose a neighborhood U_p of p. Assume that $(z_1, \cdots, z_n)$ is a coordinate system in U_p such that $U_p \cap D = \{z_1 \cdots z_j = 0\}$ Then we can see that, in $U_p \setminus D$, ω_α is quasi isomeric to

$$\sum_{i=1}^{j} |\sigma_i|^{-\alpha} dz_i \wedge d\overline{z}_i + \sum_{i=j+1}^{n} dz_i \wedge d\overline{z}_i$$

Applying the weighted Sobolev inequality proved in [St] (Theorem 2.2.56), we obtain, for any $f \in C^\infty(\overline{X})$

$$\left(\int_M |f|^r |\sigma|^{-\alpha} dV\right)^{\frac{1}{r}} \leq C_\alpha \left(\left(\int_M |\bigtriangledown f|^2 dV\right)^{\frac{1}{2}} + \left(\int_M |f|^2 |\sigma|^{-\alpha} dV\right)^{\frac{1}{2}}\right)$$

if $2 \leq r \leq \frac{2n-\alpha}{n-1}$, where dV is the volume element of $\omega, \bigtriangledown$ is the gradient with respect to ω.

Since in $U_p \setminus D, |\bigtriangledown f|^2 = \sum_{i=1}^{n} |\frac{\partial f}{\partial z_i}|^2$,

$$\begin{aligned} |\bigtriangledown_\alpha f|^2 &\geq C_\alpha \left(\sum_{i=1}^{j} |\sigma_i|^\alpha \left|\frac{\partial f}{\partial z_i}\right|^2 + \sum_{i=j+1}^{n} \left|\frac{\partial f}{\partial z_i}\right|^2\right) \\ &\geq C_\alpha |\sigma|^\alpha |\bigtriangledown f|^2 \end{aligned} \tag{4}$$

we have the Sobolev inequality:

$$\left(\int_M |f|^r dV_\alpha\right)^{\frac{1}{r}} \leq C_\alpha\left(\left(\int_M |\nabla_\alpha f|^2 dV_\alpha\right)^{\frac{1}{2}} + \left(\int_M |f|^2 dV_\alpha\right)^{\frac{1}{2}}\right)$$

if $2 \leq r \leq \frac{2n-\alpha}{n-1}$, where dV_α is the volume element of ω_α, and ∇_α is the gradient with respect to ω_α.

Therefore, we have the following proposition.

Proposition 4.1. *(X, ω_α) satisfies the three assumptions in [S1] (Section 2), that is, (1) (X, ω_α) has finite volume; (2) there exists an exhaustion function ϕ with $\Delta_\alpha \phi$ bounded; (3) if f is a nonnegative bounded function on X with $\Delta_\alpha f \geq -B$ $(B \in L^p(X), p > n)$, then $||f||_{L^\infty(X)} \leq C(||B||_{L^p(X)} + ||f||_{L^1(X)})$ where Δ_α is the Laplace operator with respect to ω_α.*

Proof. Assumption (1) is clearly satisfied. We set $\phi = \log|\sigma|^2$. Then $\Delta_\alpha \phi = -\sqrt{-1}\Lambda_\alpha \bar{\partial}\partial\phi$. The Poincaré-Lelong formula ([SABK], Ch. II, Section 1, Theorem 2) yields that it satisfies Assumption (2). To prove Assumption (3) is satisfied, it suffices to show that the Moser's iterative argument [Mo] (also see [G-T], Ch. 8) works on the manifold (X, ω_α), which is guaranteed by the Sobolev inequality that we proved above.

5. Analytic stability and parabolic stability.

Let $E' = E|_X$, K a Hermitian metric on E', let $d_K = \partial_K + \overline{\partial}$ be the Hermitian connection of K, F_K the curvature of d_K.

If $|\Lambda_\alpha F_K|_K \in L^1(X, \omega_\alpha), (0 \leq \alpha < 2)$, where Λ_α is the contraction with respect to the Kähler form ω_α, we can define (see [L-N], [S1]) the analytic degree of (E, K) by

$$\begin{aligned} d_\alpha(E, K) &= \frac{\sqrt{-1}}{2\pi}\int_X tr(\Lambda_\alpha F_K) dV_\alpha \\ &= \frac{\sqrt{-1}}{2\pi}\int_X trF_K \wedge *\omega_\alpha \\ &= \frac{\sqrt{-1}}{2\pi}\int_X trF_K \wedge \frac{\omega_\alpha^{n-1}}{(n-1)!}, \end{aligned}$$

If V is a proper coherent subsheaf of E' with quotient torsion free, we can define the analytic degree of V (see [L-N], [S1]) by

$$\begin{aligned} d_\alpha(V,K) &= \frac{\sqrt{-1}}{2\pi}\int_X tr(\Lambda_\alpha F_{K|_V})dV_\alpha \\ &= \frac{\sqrt{-1}}{2\pi}\int_X tr(\pi\Lambda_\alpha F_K)dV_\alpha - \frac{1}{2\pi}\int_X |\bar\partial\pi|^2_{K,\omega_\alpha}dV_\alpha \end{aligned}$$

where π is the orthogonal projection with respect to the metric K onto V in the complement of an analytic set. The analytic set is of codimension ≥ 2, outside which V is a proper subbundle of E'.

Let $S_K = S_K(E')$ denote the vector bundle of self-adjoint endmorphisms of E' with respect to K.

Definition 5.1 ([L-N]). Suppose that K, H are Hermitian metrics on E' with the property that $|\Lambda_\alpha F_K|_K \in L^1(X,\omega_\alpha)$ and $|\Lambda_\alpha F_H|_H \in L^1(X,\omega_\alpha)$. Let $H = Kh$. If

a) H, K are mutually bounded;

b) $|\bar\partial h|_{K,\omega_\alpha} \in L^2(X,\omega_\alpha)$
then we say that H and K are compatible with respect to ω_α.

It is proved in [L-N] (Lemma 3.4) that, if K and H are compatible with respect to ω_α, then $d_\alpha(E,K) = d_\alpha(E,H)$.

Definition 5.2. Suppose that K is a Hermitian metric on E' with the property that $|\Lambda_\alpha F_K|_K \in L^1(X,\omega_\alpha)$. We say that (E,K) is analytic stable with respect to ω_α, if for every proper coherent subsheaf V of E' with quotient torsion free,

$$\frac{d_\alpha(V,K)}{rankV} < \frac{d_\alpha(E,K)}{rankE}$$

We compute the analytic degree of (E,K_0) with respect to ω_α where K_0 is defined in Section 3. For this purpose, we introduce a metric K_1 of E over $\overline{X}$. We adopt the same notations as that in Section 3. Let H' be a metric on $E|_{\Delta^n}$ whose matrix with respect to the basis $\{e_1,\cdots,e_r\}$ is identity. Then we choose, in each U_i $(i=1,\cdots,N)$, a metric H_i' on E as above, and define the metric K_1 by

$$K_1 = \psi_0 h_0 + \sum_{i=1}^N \psi_i h_i'$$

Proposition 5.3. $d_0(E,K_0) = par\deg E$

Proof. Set $K_1^{-1}K_0 = h$, then tr F_{K_0} =tr$F_{K_1} + \bar{\partial}\partial$log det h.

$$d_0(E, K_0) = \deg E + \frac{\sqrt{-1}}{2\pi}\int_X \bar{\partial}\partial \log \det h \wedge *\omega$$

We have

$$d_0(E, K_0) = \deg E + \frac{\sqrt{-1}}{2\pi}\int_X \bar{\partial}\partial \log \det \left(\prod_i (S^i)^2\right) \wedge *\omega$$
$$+ \frac{\sqrt{-1}}{2\pi}\int_X \bar{\partial}\partial \log \det \left(\prod_i (S^i)^{-2} h\right) \wedge *\omega$$

By the Poincaré-Lelong formula ([SABK], Ch.II, Section 1, Theorem 2), we have

$$d_0(E, K_0) = par \deg E + \frac{\sqrt{-1}}{2\pi}\int_X \bar{\partial}\partial \log \det \left(\prod_i (S^i)^{-2} h\right) \wedge *\omega$$

By the construction of the metrics K_0 and K_1, it is not difficult to see that $\log\det(\prod_i (S^i)^{-2}h)$ can be extended smoothly to $\overline{X}$, so the last term on the right hand side of the identity vanishes, this proves the proposition.

It is clear that

$$\int_X trF_{K_1} \wedge \frac{\left(\sqrt{-1}\left(\frac{2}{2-\alpha}\right)\sum_{i=1}^m \partial\bar{\partial}|\sigma_i|^{2-\alpha} + C_\alpha\omega\right)^{n-1}}{(n-1)!}$$
$$= C_\alpha^{n-1}\int_X trF_{K_1} \wedge \frac{\omega^{n-1}}{(n-1)!}$$

and

$$\int_X \bar{\partial}\partial \log \det h \wedge \frac{\left(\sqrt{-1}\left(\frac{2}{2-\alpha}\right)\sum_{i=1}^m \partial\bar{\partial}|\sigma_i|^{2-\alpha} + C_\alpha\omega\right)^{n-1}}{(n-1)!}$$
$$= C_\alpha^{n-1}\int_X \bar{\partial}\partial \log \det h \wedge \frac{\omega^{n-1}}{(n-1)!}$$

So, we have the following proposition

Proposition 5.4.

$$d_\alpha(E, K_0) = C_\alpha^{n-1} par \deg E$$

Then we consider the parabolic degree and the analytic degree of a coherent subsheave of E.

Proposition 5.5. *Let V be a proper coherent subsheaf of E with quotient torsion free.*

1) *If K_1 is a metric on E*

$$\deg V = \frac{\sqrt{-1}}{2\pi} \int_X tr(\Lambda F_{K_1|_V}) dV$$

2) *If K_0 is the metric defined in Section 3, we have*

$$par \deg V = d_0(V, K_0).$$

Consequently, $|\overline{\partial}\pi|_{K_0} \in L^2(X)$.

Proof. By a theorem of Hironaka ([H],p.145, Corollary 2), we can find a complex manifold $\widetilde{X}$, which is obtained by successively blowing up complex submanifolds, such that the following holds. Let $q : \widetilde{X} \to \overline{X}$ be the canonical map and let $\{P_i^{n_i}\}(i = 1, \cdots, N)$ be the components of the exceptional divisor S; then there exist positive integers $\{m_i\}(i = 1, \cdots, N)$ such that the canonical map from $q^*(V)$ to $q^*(E)$ maps $q^*(V)$ isomorphically onto a proper subbundle of $q^*(E) \otimes \mathcal{O}(-m_i P_i^{n_i})$.

We first prove 1), i.e.,

$$\deg(V) = \frac{\sqrt{-1}}{2\pi} \int_X tr(F_{K_1|_V}) \wedge *\omega.$$

Note that for any choice of a metric γ on $q^*(V)$ we have

$$\deg(V) = \frac{\sqrt{-1}}{2\pi} \int_{\widetilde{X}} tr(F_\gamma) \wedge *q^*(\omega).$$

Now consider the metric on $q^*(V)$ obtained from the metric on $q^*(E) \otimes \mathcal{O}(-m_i P_i^{n_i})$, where we use on $q^*(E)$ pullback of K_1 and on each $\mathcal{O}(P_i^{n_i})$ some metric. Let $|S_i|$ be the norm of the canonical section of $\mathcal{O}(P_i^{n_i})$ with respect to the metric. we have

$$\begin{aligned}
\deg V &= \frac{\sqrt{-1}}{2\pi}\int_{\tilde{X}} tr(F_\gamma)\wedge *q^*(\omega)\\
&= \frac{\sqrt{-1}}{2\pi}\int_{\tilde{X}\backslash S} tr(q^*(F_{K_1|V}))\wedge *q^*(\omega)\\
&+ \frac{\sqrt{-1}}{2\pi}\int_{\tilde{X}\backslash S} -\frac{m_i}{2}\overline{\partial}\partial\log|S_i|^2\wedge *q^*(\omega)\\
&= \frac{\sqrt{-1}}{2\pi}\int_{X} tr(F_{K_1|V})\wedge *\omega + \frac{\sqrt{-1}}{2\pi}\int_{\tilde{X}\backslash S} -\frac{m_i}{2}\overline{\partial}\partial\log|S_i|^2\wedge *q^*(\omega)\\
&= \frac{\sqrt{-1}}{2\pi}\int_{X} tr(F_{K_1|V})\wedge *\omega - \frac{m_i}{2}\int_{\tilde{X}} C_1(\mathcal{O}(P_i^{n_i}))\wedge *q^*(\omega)\\
&= \frac{\sqrt{-1}}{2\pi}\int_{X} tr(F_{K_1|V})\wedge *\omega
\end{aligned}$$

where $C_1(\mathcal{O}(P_i^{n_i}))$ is the first Chern form of $\mathcal{O}(P_i^{n_i})$.

From now on, we suppose that K_1 is the metric of E defined in this section.

It is clear that

$$trF_{K_0|_V} = trF_{K_1|_V} + \overline{\partial}\partial\log\det((K_1|_V)^{-1}(K_0|_V))$$

Using 1) we have

$$d_0(V, K_0) = \deg V + \frac{\sqrt{-1}}{2\pi}\int_X \overline{\partial}\partial\log\det((K_1|_V)^{-1}(K_0|_V))\wedge *\omega$$

We will use blow up as above to calculate the second term on the right hand side of the last identity, which equals

$$\frac{\sqrt{-1}}{2\pi}\int_{\tilde{X}\backslash q^{-1}(D)} \overline{\partial}\partial\log\det((q^*(K_1)|_{q^*(V)})^{-1}(q^*(K_0)|_{q^*(V)}))\wedge *q^*(\omega)$$

We denote by D_i^* the proper transform of D_i, we may assume that $q^{-1}(D) = \sum_{i=1}^m D_i^* + S$ is a divisor with normal crossings.

Suppose that the flag of $V|_{D_i}$ by coherent subsheaves is

$$V|_{D_i} = F_1^i V \supset \cdots \supset F_{n_i}^i V \supset \{0\} = F_{n_i+1}^i V,$$

the weight attached to the flag is $\alpha_1^i(V), \cdots, \alpha_{n_i}^i(V)$ (see Section 2). Let $a = rankV$.

Put $\delta_l^i = \alpha_j^i$ if $a - r_j^i < l \le a - r_{j+1}^i$ where $r_j^i = rankF_j^iV$, $j = 1, \cdots, n_i$. Set

$$A^i = \begin{pmatrix} |\sigma_i^*|^{2\delta_1^i} & & \\ & \ddots & \\ & & |\sigma_i^*|^{2\delta_a^i} \end{pmatrix}$$

$$B^i = \begin{pmatrix} \prod_{l=1}^N |S_l|^{2\delta_1^i} & & \\ & \ddots & \\ & & \prod_{l=1}^N |S_l|^{2\delta_a^i} \end{pmatrix}$$

where σ_i^* is the canonical section of D_i^*.

We have

$$\begin{aligned} &\frac{\sqrt{-1}}{2\pi}\int_{\widetilde{X}\backslash q^{-1}(D)} \bar{\partial}\partial \log\det((q^*(K_1)|_{q^*(V)})^{-1}(q^*(K_0)|_{q^*(V)})) \wedge *q^*(\omega) \\ &= \frac{\sqrt{-1}}{2\pi}\int_{\widetilde{X}\backslash q^{-1}(D)} \bar{\partial}\partial \log\det\left(\prod_i (A^iB^i)^{-1}(q^*(K_1)|_{q^*(V)})^{-1}(q^*(K_0)|_{q^*(V)})\right) \\ &\qquad \wedge *q^*(\omega) + \frac{\sqrt{-1}}{2\pi}\int_{\widetilde{X}\backslash q^{-1}(D)} \bar{\partial}\partial \log\det\left(\prod_i (A^iB^i)\right) \wedge *q^*(\omega) \end{aligned}$$

Let $p \in q^{-1}(D)$.Assume that $D_1^*, \cdots, D_k^*, P_1^{n_1}, \cdots, P_b^{n_b}$ pass through p. Suppose that $f_1, \cdots, f_a$ is a holomorphic frame of $q^*(V)$ in a neighborhood U_p of p and that f is a holomorphic frame of $\mathcal{O}(-m_iP_i^{n_i})$, assume that $f_i = g_{ij}q^*(e_j) \otimes f$ where $e_1, \cdots, e_r$ is a holomorphic basis of E around $q(p)$ preserving the flags, $g = (g_{ij})$ is a matrix with constant rank a. Furthermore, we may assume that $f_i = q^*(e_{k_i}) \otimes f$ on $D_p^* = \cup_{j=1}^k D_j^*, i = 1, \cdots, a$, that is $g_{ij}|_{D_p^*} = 0$ if $j \neq k_i$ and $g_{ik_i}|_{D_p^*} = 1$. We have

$$\langle f_i, f_j \rangle_{H'} = g_{il}\overline{g}_{jl} \tag{5}$$

$$\langle f_i, f_j \rangle_{q^*(H)} = \prod_{t=1}^k |\sigma_t^*|^{2\beta_l^t} \prod_{t_1=1}^b |S_{t_1}|^{2\beta_l^t} g_{il}\overline{g}_{jl} \tag{6}$$

Here H is the local metric on E' defined in Section 3, H' is the local metric on E constructed in this section, β_l^t is defined in Section 3.

Let $(z_1, \cdots, z_n)$ be a local coordinate system around p and $P_i^{n_i}$ is defined by $z_i = 0, i = 1, \cdots, b$. Then

$$q^*(\omega) = \sum_{l,m}\left(\prod_{i=1,i\neq l}^b z_i \prod_{j=1,j\neq m}^b \overline{z}_j\omega_{l\overline{m}}dz_l \wedge d\overline{z}_m\right) \tag{7}$$

By (5),(6) and (7) we can see that

$$\int_{\widetilde{X}\setminus q^{-1}(D)} \overline{\partial}\partial \log\det\left(\prod_i (A^iB^i)^{-1}(q^*(K_1)|_{q^*(V)})^{-1}(q^*(K_0)|_{q^*(V)})\right)\wedge *q^*(\omega)$$
$$=0$$

We therefore have

$$\frac{\sqrt{-1}}{2\pi}\int_{\widetilde{X}\setminus q^{-1}(D)} \overline{\partial}\partial \log\det((q^*(K_1)|_{q^*(V)})^{-1}(q^*(K_0)|_{q^*(V)}))\wedge *q^*(\omega)$$
$$=\sum_{i=1}^m\sum_{j=1}^a \delta_j^i\left(\frac{\sqrt{-1}}{2\pi}\int_{\widetilde{X}\setminus q^{-1}(D)}\overline{\partial}\partial\log|\sigma_i^*|^2\wedge *q^*(\omega)\right)$$
$$+\sum_{i=1}^m\sum_{j=1}^a \delta_j^i\sum_{l=1}^N\frac{\sqrt{-1}}{2\pi}\int_{\widetilde{X}\setminus q^{-1}(D)}\overline{\partial}\partial\log|S_l|^2\wedge *q^*(\omega)$$
$$=\sum_{i=1}^m\sum_{j=1}^a \delta_j^i\left(\frac{\sqrt{-1}}{2\pi}\int_{\widetilde{X}\setminus D}\overline{\partial}\partial\log|\sigma_i^*|^2\wedge *q^*(\omega)\right)$$
$$=\sum_{i=1}^m\sum_{j=1}^a \delta_j^i\deg[D_i]$$

So

$$d_0(V,K_0)=par\deg V$$

This completes the proof of the proposition.

Similarly we also have

Proposition 5.6. *Let V be a proper coherent subsheaf of E with quotient torsion free.*

$$d_\alpha(V,K_0)=C_\alpha^{n-1}par\deg V$$

Proposition 5.7. *Suppose that K_0 is the metric constructed in Section 3. If K and K_0 are compatible with respect to ω_α, then $d_\alpha(V,K)=d_\alpha(V,K_0)$.*

Proof. Let $h=K_0^{-1}K$. We adopt the notations in the proof of Proposition 5.5. In particular $q:\widetilde{X}\in\overline{X}$ is the blow up. We shall prove

$$\int_{\widetilde{X}\setminus q^{-1}(D)} \overline{\partial}\partial \log\det\left(\prod_i (A^iB^i)^{-1}(q^*(K_1)|_{q^*(V)})^{-1}(q^*(K)|_{q^*(V)})\right)\wedge *q^*(\omega_\alpha)$$
$$=0$$

For any point $p \in q^{-1}(D)$, around it we use the holomorphic frame $e_i^{(1,\cdots,k)}$ defined in Section 3.

Suppose that

$$q^*(h)q^*(e_i^{(1,\cdots,k)}) = h_{ij}q^*(e_j^{(1,\cdots,k)}),$$

and

$$f_i = g'_{ij}q^*(e_j^{(1,\cdots,k)}) \otimes f$$

It is clear that

$$g_{ij} = g'_{ij}q^*\left(\prod_{l=1}^{k} \sigma_l^{-\alpha^l_{k_l}}\right) \text{ if } rk(F^l_{k_l+1}) < j \leq rk(F^l_{k_l})$$

where $k_l = 1, \cdots, m_l$.

We have

$$\langle q^*(h)f_i, f_j\rangle_H = g'_{il}h_{l\delta}\overline{g'_{j\delta}}\langle f, f\rangle$$

Here H is the local metric on E' defined in Section 3.

Since $q^*\sigma_i = \sigma_i^* \prod_{t_1=1}^{b} S_{t_1}$, $|h|_{K_0} \in L^\infty(X)$, $|\overline{\partial} h|_{K_0,\omega_\alpha} \in L^2(X,\omega_\alpha)$, using (6) we can show the claim, which yields the proposition.

Finally, we consider the equivalence between parabolic stability and analytic stability.

The following lemma is a corollary of a theorem of Siu ([Siu], Theorem 4.5).

Lemma 5.8. *Suppose that A is a thick set in $P^*(k_0, n-1)(0 \leq k_0 \leq n-1)$, assume that $\mathcal{G}$ is a coherent analytic sheaf on $P^*(k_0, n-1) \times \triangle$ and that $\mathcal{F}$ is a coherent analytic subsheaf of $\mathcal{G}$ with quotient torsion free on $P^*(k_0, n-1) \times \triangle^*$, where $\triangle = \{|z| < 1\}, \triangle^* = \triangle \setminus \{0\}$. If for every point $p \in A, \mathcal{F}|_{\{p\}\times\triangle^*}$ can be extended to $\{p\} \times \triangle$ as a coherent analytic subsheaf of $\mathcal{G}|_{\{p\}\times\triangle}$, then $\mathcal{F}$ can be extended uniquely to a coherent analytic subsheaf of $\mathcal{G}$ on $P^*(k_0, n-1) \times \triangle$.*

Li-Narasimhan showed in [L-N] (Lemma 6.2) that the extension of a coherent subsheaf of $E' = E|_X$ is a local problem. So applying Lemma 10.6 in [S1] and using Lemma 5.8 at most n times, we can prove the following proposition.

Proposition 5.9. *Suppose that K_0 is the metric on $E' = E|_X$ constructed in Section 3. If V is a proper coherent subsheaf of E' with quotient torsion free and $|\overline{\partial}\pi_V|_{K_0} \in L^2(X)$, then it extends to a coherent subsheaf of E.*

Proposition 5.10. *Suppose that E is a parabolic bundle, assume that K_0 is the metric constructed in Section 3. Then E is parabolic stable if and only if (E, K_0) is analytic stable with respect to ω_α.*

Proof. Using (4) we can obtain

$$\int_X |\overline{\partial}\pi_V|_{K_0} dV \leq C_\alpha \int_X |\overline{\partial}\pi_V|_{K_0,\omega_\alpha} dV_\alpha.$$

So the proposition follows from Proposition 5.4, Proposition 5.6 and Proposition 5.9.

6. The existence of H-E metrics .

In this section we prove one of our main theorems in this paper.

Definition 6.1. A Hermitian metric H on $E' = E|_X$ is called Hermitian-Einstein with respect to ω_α, if $\Lambda_\alpha F_H^\perp = 0$ where $F_H^\perp = F_H - \frac{trF_H}{rankE} I$ is the trace free part of the curvature F_H, I is the identity endomorphism of E'.

Definition 6.2. Suppose that E is a parabolic bundle, K is a Hermitian metric on $E' = E|_X$, we say that it is compatible with the parabolic structure with respect to ω_α if K and K_0 are compatible with respect to ω_α, where K_0 is defined in Section 3.

We set

$$\gamma_0 = \min\{ \alpha_l^i - \alpha_{l-1}^i,\ 1 - (\alpha_l^i - \alpha_{l-1}^i) \mid l = 2, \cdots, m_i + 1,\ i = 1, \cdots, m \}.$$

Theorem 6.3. *Let $\overline{X}$ be a compact Kähler manifold of complex dimension n and D a divisor of $\overline{X}$ with normal crossings. Let E be a holomorphic vector bundle with a parabolic structure along D. If E is parabolic stable there exists a Hermitian-Einstein metric with respect to ω_α for any $2(1-\gamma_0) \leq \alpha < 2$ on E' compatible with the parabolic structure with respect to ω_α. Conversely, if E is indecomposable and E' admits a Hermitian-Einstein metric with respect to ω_α $(0 \leq \alpha < 2)$ compatible with the parabolic structure with respect to ω_α, then E is parabolic stable.*

Proof. If E is parabolic stable, by Proposition 5.10 we know that (E, K_0) is analytic stable with respect to ω_α for any $0 \leq \alpha < 2$. According to (3) we know that we can choose any $2(1-\gamma_0) \leq \alpha < 2$ such that $|\Lambda_\alpha F_{K_0}|_{K_0} \in L^\infty(X)$. Theorem 1 in [S1] yields that there is a H-E metric on E' compatible with the parabolic structure with respect to ω_α.

Conversely, suppose that H is a H-E metric compatible with the parabolic structure with respect to ω_α, we have $par \deg E = C_\alpha^{1-n} d_\alpha(E, H)$. Suppose that V is a proper coherent subsheaf of E with quotient torsion free, by Proposition 5.6 and Proposition 5.7 we have $par \deg V = C_\alpha^{1-n} d_\alpha(V, H)$. Then by an argument similar to the one used in the proof of Theorem 7.3 in [L-N] we can show that E is parabolic stable if E is indecomposable.

7. Chern number inequality(I).

Suppose that H is a Hermitian-Einstein metric on E' compatible with the parabolic structure with respect to ω_α $(2(1-\gamma_0) \leq \alpha < 2)$, which is obtained in Theorem 6.3. It was proved in [S1] (Proposition 3.4) that

$$\left(2C_2(E,H) - \frac{r-1}{r} C_1(E,H)^2\right) [\omega_\alpha]^{n-2} \geq 0 \tag{8}$$

where $r = rankE$.

$$C_1(E,H) = \frac{\sqrt{-1}}{2\pi} trF_H,$$
$$C_2(E,H) = -\frac{1}{8\pi^2}(trF_H \wedge trF_H - trF_H \wedge F_H)$$

Since $\det H = \det K_0$, we have $C_1(E,H) = C_1(E,K_0)$.

Lemma 7.1.

$$\int_X C_2(E,H) \wedge \omega_\alpha^{n-2} \leq \int_X C_2(E,K_0) \wedge \omega_\alpha^{n-2}$$

Proof. It suffices to show that

$$\int_X tr(F_H \wedge F_H) \wedge \omega_\alpha^{n-2} \leq \int_X tr(F_{K_0} \wedge F_{K_0}) \wedge \omega_\alpha^{n-2}$$

Suppose that f is a compactly suppored function on X, and we set $v = -4\pi\sqrt{-1}\partial\bar{\partial} f$. Simpson [S1] showed that

$$\int_X f(tr(F_{K_0} \wedge F_{K_0}) - tr(F_H \wedge F_H)) \wedge \omega_\alpha^{n-2}$$
$$= \frac{\sqrt{-1}}{2\pi} \int_X tr(sF_{K_0}) v \wedge \omega_\alpha^{n-2} - \frac{\sqrt{-1}}{2\pi} \int_X tr(\Psi(s)(\bar{\partial} s)\partial_{K_0} s) v \wedge \omega_\alpha^{n-2}$$

where $e^s = K_0^{-1}H$, $\Psi(s)$ is constructed as in the definition of Donaldson's functional (see [S1] Section 5).

We choose

$$f_\beta = \max\left\{0, 1 + \frac{\log|\sigma|^2}{\beta}\right\}$$

Set $X_\beta = \{x \in X | \log|\sigma|^2 > -\beta\}$. We now recall how one gets H from K_0 (see [S1] Section 6 and Section 7).

One solves the heat equation

$$\begin{cases} H^{-1}\dfrac{dH}{dt} = -\sqrt{-1}\Lambda_\alpha F_H^\perp \\ H\big|_{t=0} = K_0 \\ \det H = \det K_0 \end{cases}$$

on X_β with Dirichlet boundary condition $H|_{\partial X_\beta} = K_0$. If the solution is denoted by $H_\beta(t)$, one shows that $H_\beta(t) \to H(t)$ in $C^{1,0}$ over compact sets in X, as $\beta \to \infty$. $H(t)$ is a solution of the heat equation, and there exists a subsequence $t_i \to \infty$ such that $H(t_i) \to H$ weakly in $L^p_{2,loc}$.

Set $e^{s_\beta} = h_\beta = K_0^{-1}H_\beta(t)$ in X_β and $s_\beta = 0$ outside X_β. Since ([S1], Lemma 3.1 (c))

$$\Delta_\alpha trh_\beta = -\sqrt{-1}tr(h_\beta(\Lambda_\alpha F_{H_\beta}^\perp - \Lambda_\alpha F_{K_0}^\perp)) + \left|(\bar\partial h_\beta)h_\beta^{-\frac{1}{2}}\right|^2_{K_0,\omega_\alpha}$$

and $\frac{\partial}{\partial n}trh_\beta|_{\partial X_\beta} \le 0$, because $trh_\beta \ge r = trh_\beta|_{\partial X_\beta}$, where Δ_α is the Laplace operator with respect to ω_α, we have

$$\int_X \left|(\bar\partial h_\beta)h_\beta^{-\frac{1}{2}}\right|^2_{K_0,\omega_\alpha} dV_\alpha \le C.$$

Here C is a positive constant independent of β.

By Proposition 5.3 and Lemma 7.1 in [S1] we can see that $|h_\beta|_{K_0}$ is bounded on both side. So

$$\int_X |\bar\partial s_\beta|^2_{K_0,\omega_\alpha} dV_\alpha \le C.$$

We have

$$\int_{X_\beta} f_\beta tr(F_{H_\beta} \wedge F_{H_\beta}) \wedge \omega_\alpha^{n-2}$$
$$= \int_{X_\beta} f_\beta tr(F_{K_0} \wedge F_{K_0}) \wedge \omega_\alpha^{n-2} - \frac{\sqrt{-1}}{2\pi} \int_{X_\beta} tr(s_\beta F_{K_0}) v \wedge \omega_\alpha^{n-2}$$
$$+ \frac{\sqrt{-1}}{2\pi} \int_{X_\beta} tr(\Psi(s_\beta)(\bar{\partial} s_\beta) \partial_{K_0} s_\beta) v \wedge \omega_\alpha^{n-2}$$

Note that $s_\beta|_{\partial X_\beta} = 0$, $f_\beta|_{\partial X_\beta} = 0$, we have

$$tr(s_\beta F_{K_0}) v|_{\partial X_\beta} = 0$$

and

$$tr(\Psi(s_\beta)(\bar{\partial} s_\beta)\partial_{K_0} s_\beta) v|_{\partial X_\beta} = 0$$

In X_β,

$$v = -4\pi\sqrt{-1}\partial\bar{\partial} f_\beta = -\frac{4\pi\sqrt{-1}}{\beta}\partial\bar{\partial}\log|\sigma|^2$$

By Poincaré-Lelong formula, we have

$$\int_{X_\beta} f_\beta tr(F_{H_\beta} \wedge F_{H_\beta}) \wedge \omega_\alpha^{n-2}$$
$$\leq \int_{X_\beta} f_\beta tr(F_{K_0} \wedge F_{K_0}) \wedge \omega_\alpha^{n-2}$$
$$+ \frac{C}{\beta} \int_{X_\beta} |tr\Lambda_\alpha F_{K_0}| dV_\alpha + \frac{C}{\beta} \int_{X_\beta} |\bar{\partial} s_\beta|^2_{K_0,\omega_\alpha} dV_\alpha$$

By the Riemman bilinear relations, one gets

$$tr(F_{H_\beta} \wedge F_{H_\beta}) \wedge \omega_\alpha^{n-2} \geq -C|\Lambda_\alpha F_{H_\beta}|^2 \omega_\alpha^n$$

Since

$$\sup_{X_\beta} |\Lambda_\alpha F_{H_\beta}^\perp| \leq \sup_{X_\beta} |\Lambda_\alpha F_{K_0}^\perp| \leq C$$

and $trF_{H_\beta} = trF_{K_0}$, we have

$$tr(F_{H_\beta} \wedge F_{H_\beta}) \wedge \omega_\alpha^{n-2} \geq -C\omega_\alpha^n$$

Letting $\beta \to \infty$, using Fatou's lemma we obtain

$$\int_X tr(F_{H(t)} \wedge F_{H(t)}) \wedge \omega_\alpha^{n-2} \leq \int_X tr(F_{K_0} \wedge F_{K_0}) \wedge \omega_\alpha^{n-2}$$

Applying Fatou's lemma again, we get

$$\int_X tr(F_H \wedge F_H) \wedge \omega_\alpha^{n-2} \leq \int_X tr(F_{K_0} \wedge F_{K_0}) \wedge \omega_\alpha^{n-2}$$

Proposition 7.2. *Let $\overline{X}$ be a compact Kähler manifold of complex dimension n and D a divisor of $\overline{X}$ with normal crossings. Let E be a holomorphic vector bundle over $\overline{X}$ with a parabolic structure along D. Let K_0 be the metric of E' constructed in Section 3. If E is parabolic stable, for any $2(1-\gamma_0) \leq \alpha < 2$ such that*

$$\left(2C_2(E, K_0) - \frac{r-1}{r} C_1(E, K_0)^2\right) [\omega_\alpha]^{n-2} \geq 0$$

where $r = rankE$.

The proposition follows from Lemma 7.1 and (8).

Lemma 7.3.

$$\begin{aligned} C_\alpha^{2-n} \int_X & C_1(E, K_0) \wedge C_1(E, K_0) \wedge \omega_\alpha^{n-2} \\ &= \int_X C_1(E) \wedge C_1(E) \wedge \omega^{n-2} \\ &\quad + 2\sum_{i=1}^{m}\sum_{l=1}^{m_i} \alpha_l^i rank(F_l^i/F_{l+1}^i) \deg(E|_{D_i}) \\ &\quad + \sum_{i,j=1}^{m} \left(\left(\sum_{l=1}^{m_i} \alpha_l^i rank(F_l^i/F_{l+1}^i)\right) \left(\sum_{l=1}^{m_j} \alpha_l^j rank(F_l^j/F_{l+1}^j)\right) D_i \cdot D_j \right) \end{aligned}$$

where $D_i \cdot D_j = \int_X C_1([D_i]) \wedge C_1([D_j]) \wedge \omega^{n-2}$ is the intersection number of D_i and $D_j (i, j = 1, \cdots, m)$.

Proof. Suppose that K_1 is the metric constructed in Section 5. Set $h = K_1^{-1} K_0$, then $trF_{K_0} = trF_{K_1} + \overline{\partial}\partial \log\det h$. So,

$$\begin{aligned} C_1(E, K_0) & \wedge C_1(E, K_0) \\ &= \left(\frac{\sqrt{-1}}{2\pi}\right)^2 (trF_{K_1} + \overline{\partial}\partial \log\det h) \wedge (trF_{K_1} + \overline{\partial}\partial \log\det h) \\ &= \left(\frac{\sqrt{-1}}{2\pi}\right)^2 ((trF_{K_1})^2 + 2trF_{K_1} \wedge \overline{\partial}\partial \log\det h \\ &\quad + \overline{\partial}\partial \log\det h \wedge \overline{\partial}\partial \log\det h) \end{aligned}$$

We adopt the same notations as that in the proof of Proposition 5.3, we have

$$\begin{aligned}
&\int_X C_1(E,K_0)\wedge C_1(E,K_0)\wedge\omega_\alpha^{n-2}\\
&=\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_X (trF_{K_1})^2\wedge\omega_\alpha^{n-2}\\
&\quad+2\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_X trF_{K_1}\wedge\overline{\partial}\partial\log\det(\Pi_i(S^i)^2)\wedge\omega_\alpha^{n-2}\\
&\quad+2\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_X trF_{K_1}\wedge\overline{\partial}\partial\log\det(\Pi_i(S^i)^{-2}h)\wedge\omega_\alpha^{n-2}\\
&\quad+\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_X \overline{\partial}\partial\log\det(\Pi_i(S^i)^2)\wedge\overline{\partial}\partial\log\det(\Pi_i(S^i)^2)\wedge\omega_\alpha^{n-2}\\
&\quad+\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_X \overline{\partial}\partial\log\det(\Pi_i(S^i)^{-2}h)\wedge\overline{\partial}\partial\log\det(\Pi_i(S^i)^{-2}h)\wedge\omega_\alpha^{n-2}\\
&\quad+2\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_X \overline{\partial}\partial\log\det(\Pi_i(S^i)^2)\wedge\overline{\partial}\partial\log\det(\Pi_i(S^i)^{-2}h)\wedge\omega_\alpha^{n-2}
\end{aligned}$$

By the construction of the metrics K_1 and K_0, it is not difficult to see that $\log\det(\Pi_i(S^i)^{-2}h)$ can be extended smoothly to $\overline{X}$, so

$$\begin{aligned}
&\int_X trF_{K_1}\wedge\overline{\partial}\partial\log\det(\Pi_i(S^i)^{-2}h)\wedge\omega_\alpha^{n-2}\\
&\qquad=\int_X \overline{\partial}\partial\log\det(\Pi_i(S^i)^{-2}h)\wedge\overline{\partial}\partial\log\det(\Pi_i(S^i)^{-2}h)\wedge\omega_\alpha^{n-2}\\
&\qquad=\int_X \overline{\partial}\partial\log\det(\Pi_i(S^i)^2)\wedge\overline{\partial}\partial\log\det(\Pi_i(S^i)^{-2}h)\wedge\omega_\alpha^{n-2}\\
&\qquad=0
\end{aligned}$$

Thus the Poincaré-Lelong formula yields the lemma.

Lemma 7.4.

$$\begin{aligned}
C_\alpha^{2-n}\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_X tr(F_{K_0}\wedge F_{K_0})\wedge\omega_\alpha^{n-2}
&= \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_X tr(F_{K_1}\wedge F_{K_1})\wedge\omega^{n-2} \\
&+ 2\sum_{i=1}^{m}\sum_{l=1}^{m_i}\alpha_l^i \deg(F_l^i/F_{l+1}^i) \\
&+ \sum_{i=1}^{m}\sum_{l=1}^{m_i}(\alpha_l^i)^2 rank(F_l^i/F_{l+1}^i) D_i^2
\end{aligned}$$

where $D_i^2 = \int_X C_1[D_i]\wedge C_1[D_i]\wedge\omega^{n-2}$ *is the self-intersection number of* $D_i, (i=1,\cdots,m)$.

Proof. It is clear that

$$F_{K_0} = F_{K_1} + \overline{\partial}(h^{-1}\partial_{K_1}h)$$

where $h = K_1^{-1}K_0$. So,

$$F_{K_0}\wedge F_{K_0} = F_{K_1}\wedge F_{K_1} + 2F_{K_1}\wedge\overline{\partial}(h^{-1}\partial_{K_1}h) + \overline{\partial}(h^{-1}\partial_{K_1}h)\wedge\overline{\partial}(h^{-1}\partial_{K_1}h)$$

Note that

$$\overline{\partial}(h^{-1}\partial_{K_1}h) = \overline{\partial}((Sh)^{-1}\partial_{K_1}(Sh)) - \overline{\partial}(h^{-1}(S^{-1}\partial_{K_1}S)h)$$

where $S = \Pi_i(S^i)^{-2}$. By the construction of the metrics K_1 and K_0 we know that (Sh) can be seen as an endomorphism of E, we have

$$\begin{aligned}
&\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_X tr(F_{K_0}\wedge F_{K_0})\wedge\omega_\alpha^{n-2} \\
&= \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_X tr(F_{K_1}\wedge F_{K_1})\wedge\omega_\alpha^{n-2} \\
&\quad - 2\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_X tr(F_{K_1}\wedge\overline{\partial}(h^{-1}(S^{-1}\partial_{K_1}S)h))\wedge\omega_\alpha^{n-2} \\
&\quad + \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_X tr(\overline{\partial}(h^{-1}(S^{-1}\partial_{K_1}S)h)\wedge\overline{\partial}(h^{-1}(S^{-1}\partial_{K_1}S)h))\wedge\omega_\alpha^{n-2}
\end{aligned}$$

A simple calculation shows that

$$C_\alpha^{2-n}\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_X tr(F_{K_1}\wedge\overline{\partial}(h^{-1}(S^{-1}\partial_{K_1}S)h))\wedge\omega_\alpha^{n-2}$$
$$=\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_X tr(F_{K_1}\wedge\overline{\partial}(S^{-1}\partial_{K_1}S))\wedge\omega^{n-2}$$
$$=-\sum_{i=1}^m\sum_{l=1}^{m_i}\alpha_l^i\deg(F_l^i/F_{l+1}^i)$$

Similarly, we have

$$C_\alpha^{2-n}\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_X tr(\overline{\partial}(h^{-1}(S^{-1}\partial_{K_1}S)h)\wedge\overline{\partial}(h^{-1}(S^{-1}\partial_{K_1}S)h))\wedge\omega_\alpha^{n-2}$$
$$=\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_X tr(\overline{\partial}(S^{-1}\partial_{K_1}S)\wedge\overline{\partial}(S^{-1}\partial_{K_1}S))\wedge\omega^{n-2}$$
$$=\sum_{i=1}^m\sum_{l=1}^{m_i}(\alpha_l^i)^2 rank(F_l^i/F_{l+1}^i)D_i^2$$

This completes the proof of the lemma.

Theorem 7.5. *Let $\overline{X}$ be a compact Kähler manifold of complex dimension n and D a divisor of $\overline{X}$ with normal crossings. Let E be a rank r holomorphic vector bundle over $\overline{X}$ with a parabolic structure along D. If E is parabolic stable, the following Chern number inequality holds.*

$$(C_1^2-2C_2)+2\sum_{i=1}^m\sum_{l=1}^{m_i}\alpha_l^i\deg(F_l^i/F_{l+1}^i)+\sum_{i=1}^m\sum_{l=1}^{m_i}(\alpha_l^i)^2 rank(F_l^i/F_{l+1}^i)D_i^2$$
$$\leq\frac{1}{r}\left(C_1^2+2\sum_{i=1}^m\sum_{l=1}^{m_i}\alpha_l^i rank(F_l^i/F_{l+1}^i)\deg(E|_{D_i})\right.$$
$$\left.+\sum_{i,j=1}^m\left(\sum_{l=1}^{m_i}\alpha_l^i rank(F_l^i/F_{l+1}^i\right)\right)\left(\sum_{l=1}^{m_j}\alpha_l^j rank(F_l^j/F_{l+1}^j))D_i\cdot D_j\right)$$

where $D_i\cdot D_j$ is the intersection number of D_i and D_j $(i,j=1,\cdots,m)$, D_i^2 is the self-intersection number of D_i, $C_2=\int_{\overline{X}}C_2(E)\wedge\omega^{n-2}$, $C_1^2=\int_{\overline{X}}C_1(E)\wedge C_1(E)\wedge\omega^{n-2}$.

Proof. By Proposition 7.2, we have

$$2\int_X C_2(E,K_0)\wedge\omega_\alpha^{n-2} \geq \frac{r-1}{r}\int_X C_1(E,K_0)\wedge C_1(E,K_0)\wedge\omega_\alpha^{n-2}$$

for any $2(1-\gamma_0)\leq\alpha<2$. that is,

$$-\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_X tr(F_{K_0}\wedge F_{K_0})\wedge\omega_\alpha^{n-2} \geq -\frac{1}{r}\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_X trF_{K_0}\wedge trF_{K_0}\wedge\omega_\alpha^{n-2}$$

so

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_X tr(F_{K_0}\wedge F_{K_0})\wedge\omega_\alpha^{n-2} \leq \frac{1}{r}\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_X trF_{K_0}\wedge trF_{K_0}\wedge\omega_\alpha^{n-2}$$

Then the theorem follows from Lemma 7.3 and Lemma 7.4.

8. Chern number inequality (II).

In this section, we assume that D is a divisor in $\overline{X}$ and that $D=\sum_{i=1}^m D_i$ where the irreducible components D_i of D are smooth, we do not assume that D_i meat transversely. Let E be a holomorphic vector bundle over $\overline{X}$, we shall define the notion of parabolic structure of E along D and the notion of parabolic stability for a parabolic bundle, we shall derive at last a Chern number inequality for a stable parabolic bundle.

Definition 8.1. A parabolic structure on E with respect to D consists of

a) flags of $E|_{D_i}(i=1,\cdots,m)$,

$$E|_{D_i}=F_1^i\supset F_2^i\supset\cdots\supset F_{m_i}^i\supset\{0\}=F_{m_i+1}^i$$

where F_{l+1}^i is a proper subbundle of $F_l^i(l=1,\cdots,m_i-1)$.

b) weights $\alpha_1^i,\cdots,\alpha_{m_i}^i$ attached to $F_1^i,\cdots,F_{m_i}^i$ satisfying $0\leq\alpha_1^i<\cdots<\alpha_{m_i}^i<1$.

A holomorphic vector bundle E with a parabolic structure is called parabolic bundle.

We define the parabolic degree of a parabolic bundle E by

$$par \deg E = \deg E + \sum_{i=1}^{m} \sum_{l=1}^{m_i} \alpha_l^i rank(F_l^i/F_{l+1}^i) \deg[D_i]$$

Suppose that V is a proper coherent subsheaf of E with quotient torsion free. There is a natural flag of $V|_{D_i}$ by coherent subsheaves

$$V|_{D_i} = F_1^i V \supset \cdots \supset F_{n_i}^i V \supset \{0\} = F_{n_i+1}^i$$

induced by $F_1^i \cap V \supseteq \cdots \supseteq F_{m_i}^i \cap V \supset \{0\}$. We define the weights attached to the flag by $\alpha_l^i(V) =$ the largest α_k^i such that $F_l^i V \subseteq F_k^i \cap V, l = 1, \cdots, n_i$. We define the parabolic degree of V by

$$par \deg V = \deg V + \sum_{i=1}^{m} \sum_{l=1}^{n_i} \alpha_l^i(V) rank(F_l^i V/F_{l+1}^i V) \deg[D_i]$$

Definition 8.2. We say that a parabolic bundle E is parabolic stable if for every proper coherent subsheaf V of E with quotient torsion free we have

$$\frac{par \deg V}{rank V} < \frac{par \deg E}{rank E}$$

In this section we mainly prove the following Chern number inequality for a parabolic stable bundle.

Theorem 8.3. *Let $\overline{X}$ be a compact Kähler manifold of complex dimension n. Let $D = \sum_{i=1}^m D_i$ be a divisor in $\overline{X}$ where the irreducible components D_i of D are smooth. Let E be a rank r holomorphic vector bundle over $\overline{X}$ with a parabolic structure along D. If E is parabolic stable,*

$$\begin{aligned}(C_1^2 - 2C_2) &+ 2\sum_{i=1}^{m}\sum_{l=1}^{m_i} \alpha_l^i \deg(F_l^i/F_{l+1}^i) + \sum_{i=1}^{m}\sum_{l=1}^{m_i} (\alpha_l^i)^2 rank(F_l^i/F_{l+1}^i) D_i^2 \\ &\leq \frac{1}{r}\left(C_1^2 + 2\sum_{i=1}^{m}\sum_{l=1}^{m_i} \alpha_l^i rank(F_l^i/F_{l+1}^i) \deg(E|_D)\right. \\ &\left. + \sum_{i,j=1}^{m} \left(\sum_{l=1}^{m_i} \alpha_l^i rank(F_l^i/F_{l+1}^i\right)\right)\left(\sum_{l=1}^{m_j} \alpha_l^j rank(F_l^j/F_{l+1}^j)) D_i \cdot D_j\right)\end{aligned}$$

where $D_i \cdot D_j$ is the intersection number of D_i and D_j $(i,j = 1,\cdots,m)$, D_i^2 is the self-intersection number of D_i, $C_1^2 = \int_{\overline{X}} C_1(E) \wedge C_1(E) \wedge \omega^{n-2}$, $C_2 = \int_{\overline{X}} C_2(E) \wedge \omega^{n-2}$.

Proof. We use the theorem of Hironaka ([H],p.145, Corollary 2) again. By successively blowing up complex submanifolds, we can find a complex manifold $\widetilde{X}$ such that the following holds. Let $q : \widetilde{X} \to X$ be the canonical map and let $\{P_i^{n_i}\}(i = 1, \cdots, N)$ be the components of the exceptional divisor S. Let D_i^* be the proper transform of D_i $(i = 1, \cdots, m)$. We may assume that D_i^* do not meet each other, and that $q^{-1}(D) = \sum_{i=1}^m D_i^* + S$ forms a divisor with normal crossings.

Note that $q^*\omega$ is a Kähler metric on $\widetilde{X} \setminus S$, but it is not a metric on S. Suppose that $\widetilde{\omega}$ is a Kähler metric on $\widetilde{X}$, then for any $\varepsilon > 0, \omega_\varepsilon = q^*\omega + \varepsilon\widetilde{\omega}$ is a Kähler metric on $\widetilde{X}$.

The parabolic structure of E along D induces a parabolic structure of q^*E along $D^* = \sum_{i=1}^m D_i^*$ which consists of

a') flags of $q^*E|_{D_i^*}(i = 1, \cdots, m)$

$$q^*E|_{D_i^*} = q^*F_1^i|_{D_i^*} \supset q^*F_2^i|_{D_i^*} \supset \cdots q^*F_{m_i}^i|_{D_i^*} \supset \{0\} = q^*F_{m_i+1}^i|_{D_i^*}$$

b') weights $\alpha_1^i, \cdots, \alpha_{m_i}^i$ attached to the flags.

Set

$$\begin{aligned} par \deg q^*E = &\deg_{q^*\omega} q^*E \\ &+ \sum_{i=1}^{m}\sum_{l=1}^{m_i} \alpha_l^i rank(q^*F_l^i|_{D_i^*}/q^*F_{l+1}^i|_{D_i^*}) \deg_{q^*\omega}[D_i^*] \end{aligned}$$

where

$$\begin{aligned} \deg_{q^*\omega} q^*E &= \int_{\widetilde{X}} C_1(q^*E) \wedge *q^*\omega \\ &= \int_{\overline{X}} C_1(E) \wedge \omega = \deg E \\ \deg_{q^*\omega}[D_i^*] &= \int_{\widetilde{X}} C_1([D_i^*]) \wedge *q^*\omega = \deg[D_i] \end{aligned}$$

so

$$par \deg q^*E = par \deg E.$$

Put

$$par\deg_\varepsilon q^*E = \deg_{\omega_\varepsilon} q^*E + \sum_{i=1}^{m}\sum_{l=1}^{m_i} \alpha_l^i rank(q^*F_l^i|_{D_i^*}/q^*F_{l+1}^i|_{D_i^*}) \deg_{\omega_\varepsilon}[D_i^*]$$

where

$$\deg_{\omega_\varepsilon} q^*E = \int_{\widetilde{X}} C_1(q^*E)\wedge *(q^*\omega+\varepsilon\widetilde{\omega}) = \deg E + \delta_0(\varepsilon)$$
$$\deg_{\omega_\varepsilon}[D_i^*] = \int_{\widetilde{X}} C_1([D_i^*])\wedge *(q^*\omega+\varepsilon\widetilde{\omega}) = \deg[D_i] + \delta_i(\varepsilon)$$

and $\delta_i(\varepsilon)\to 0$ as $\varepsilon\to 0$ $(i=0,\cdots,m)$.

So $par\deg_\varepsilon q^*E = par\deg E + \delta(\varepsilon)$ where $\delta(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Let V^* be a proper coherent subsheaf of q^*E with quotient torsion free. $(q^{-1})^*V^*|_{q^{-1}(\widetilde{X}\backslash S)}$ can be extended to $\overline{X}$ as a coherent subsheaf of E, we denote it by V. Similarly, we can define $par\deg V^*$ and $par\deg_\varepsilon V^*$ by

$$par\deg V^* = \deg_{q^*\omega} V^* + \sum_{i=1}^{m}\sum_{l=1}^{n_i} \alpha_l^i(V^*) rank(q^*F_l^i|_{D_i^*}(V^*)/q^*F_{l+1}^i|_{D_i^*}(V^*)) \deg_{q^*\omega}[D_i^*]$$

and

$$par\deg_\varepsilon V^* = \deg_{\omega_\varepsilon} V^* + \sum_{i=1}^{m}\sum_{l=1}^{n_i} \alpha_l^i(V^*) rank(q^*F_l^i|_{D_i^*}(V^*)/q^*F_{l+1}^i|_{D_i^*}(V^*)) \deg_{\omega_\varepsilon}[D_i^*]$$

It is clear that $par\deg V^* = par\deg V$ and $par\deg_\varepsilon V^* = par\deg V + \eta(\varepsilon)$, where $\eta(\varepsilon)\to 0$ as $\varepsilon\to 0$.

The following lemma is obvious.

Lemma 8.4. *E is parabolic stable if and only if q^*E is parabolic stable with respect to $q^*\omega$.*

Furthermore, we have

Lemma 8.5. *Suppose that E is parabolic stable. Then q^*E is parabolic stable with respect to ω_ε for sufficiently small ε.*

Proof. If q^*E were not parabolic stable with respect to ω_ε, there would exist a proper coherent subsheaf V^* of q^*E with quotient torsion free such that

$$\frac{par\deg_\varepsilon V^*}{rankV^*} \geq \frac{par\deg_\varepsilon q^*E}{rankq^*E} \tag{9}$$

On the other hand, since q^*E is parabolic stable with respect to $q^*\omega$ (Lemma 8.4), we have

$$\frac{par\deg V^*}{rankV^*} < \frac{par\deg q^*E}{rankq^*E} \tag{10}$$

We consider $\deg_{q^*\varepsilon}((V^*)^*\otimes q^*E)$.

By (9), we have

$$\begin{aligned}
(11)\quad & \deg_{q^*\omega}((V^*)^*\otimes q^*E)\\
&= rankV^*\deg_{q^*\omega} q^*E - rankq^*E\deg_{q^*\omega} V^*\\
&= rankV^*rankq^*E\left(\frac{\deg_{q^*\omega} q^*E}{rankq^*E} - \frac{\deg_{q^*\omega} V^*}{rankV^*}\right)\\
&\leq rankV^*rankq^*E\left(\frac{1}{rankV^*}\sum_{i=1}^{m}\sum_{l=1}^{n_i}(\alpha_l^i(V^*)\right.\\
&\quad\cdot rank(q^*F_l^i|_{D_i^*}(V^*)/q^*F_{l+1}^i|_{D_i^*}(V^*)))\deg_{q^*\omega}[D_i^*]\\
&\quad\left.-\frac{1}{rankq^*E}\sum_{i=1}^{m}\sum_{l=1}^{m_i}\alpha_l^i rank(q^*F_l^i|_{D_i^*}/q^*F_{l+1}^i|_{D_i^*})\deg_{q^*\omega}[D_i^*]\right)\\
&\quad + C(\varepsilon)
\end{aligned}$$

where $C(\varepsilon)\to 0$ as $\varepsilon\to 0$.

By (10), we have

$$\begin{aligned}
(12)\quad & \deg_{q^*\omega}((V^*)^*\otimes q^*E)\\
&> rankV^*rankq^*E\left(\frac{1}{rankV^*}\sum_{i=1}^{m}\sum_{l=1}^{n_i}(\alpha_l^i(V^*)\right.\\
&\quad\cdot rank(q^*F_l^i|_{D_i^*}(V^*)/q^*F_{l+1}^i|_{D_i^*}(V^*)))\deg_{q^*\omega}[D_i^*]\\
&\quad\left.-\frac{1}{rankq^*E}\sum_{i=1}^{m}\sum_{l=1}^{m_i}\alpha_l^i rank(q^*F_l^i|_{D_i^*}/q^*F_{l+1}^i|_{D_i^*})\deg_{q^*\omega}[D_i^*]\right)
\end{aligned}$$

(11) contradicts (12) when ε is sufficiently small, because $\deg_{q^*\omega}((V^*)^*\otimes q^*E) = \deg(V^*\otimes E)$ is an integer. This completes the proof of the lemma.

Now we can finish the proof of Theorem 8.3.

Since $(\widetilde{X}, \omega_\varepsilon)$ is a compact Kähler manifold, $D^* = \sum_{i=1}^m D_i^*$ is a divisor in $\widetilde{X}$, and D_i^* $(i = 1, \cdots, m)$ do not meet each other. Since E is parabolic stable, q^*E is parabolic stable with respect to ω_ε for sufficiently small ε. By Theorem 7.5 we have

$$\begin{aligned}
&(C_1^{2,\varepsilon} - 2C_2^\varepsilon) + 2\sum_{i=1}^{m}\sum_{l=1}^{m_i} \alpha_l^i \deg_{\omega_\varepsilon}(q^*F_l^i|_{D_i^*}/q^*F_{l+1}^i|_{D_i^*}) \\
&+ \sum_{i=1}^{m}\sum_{l=1}^{m_i} (\alpha_l^i)^2 rank(q^*F_l^i|_{D_i^*}/q^*F_{l+1}^i|_{D_i^*}) \\
&\cdot \int_{\widetilde{X}} C_1([D_i^*]) \wedge C_1([D_i^*]) \wedge \omega_\varepsilon^{n-2} \\
&\qquad \leq \frac{1}{r}\Bigg(C_1^{2,\varepsilon} + 2\sum_{i=1}^{m}\sum_{l=1}^{m_i} \alpha_l^i rank(q^*F_l^i|_{D_i^*}/q^*F_{l+1}^i|_{D_i^*}) \deg_{\omega_\varepsilon}(E|_{D_i^*}) \\
&\qquad + \sum_{i,j=1}^{m} \left(\sum_{l=1}^{m_i} \alpha_l^i rank(q^*F_l^i|_{D_i^*}/q^*F_{l+1}^i|_{D_i^*}) \right) \\
&\qquad \cdot \left(\sum_{l=1}^{m_j} \alpha_l^j rank(q^*F_l^j|_{D_j^*}/q^*F_{l+1}^j|_{D_j^*}) \right) \\
&\qquad \cdot \int_{\widetilde{X}} C_1([D_i^*]) \wedge C_1([D_j^*]) \wedge \omega_\varepsilon^{n-2} \Bigg)
\end{aligned}$$

where

$$\begin{aligned}
C_1^{2,\varepsilon} &= \int_{\widetilde{X}} C_1(q^*E) \wedge C_1(q^*E) \wedge \omega_\varepsilon^{n-2} \\
C_2^{\varepsilon} &= \int_{\widetilde{X}} C_2(q^*E) \wedge \omega_\varepsilon^{n-2}
\end{aligned}$$

Letting $\varepsilon \to 0$, we get the desired inequality. This completes the proof of the theorem.

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