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Quasi-convergence of the Ricci flow

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We study a collection of Riemannian metrics which collapse under the Ricci flow, and show that the quasi-convergence equivalence class of an arbitrary metric in this collection contains a 1-parameter family of locally homogeneous metrics.

1. Introduction and statement of main theorem.

In [1], Hamilton and Isenberg studied the Ricci flow of a family of solv-geometry metrics on twisted torus bundles. This family contains no Einstein metrics, so the (normalized) Ricci flow cannot converge. Hamilton–Isenberg introduced the concept of *quasi-convergence* to describe its behavior, writing

> *"...the Ricci flow of all metrics in this family asymptotically approaches the flow of a sub-family of locally homogeneous metrics..."*

The intent of this paper is to make that statement more precise. In so doing, we answer a question of Hamilton, who asked whether an arbitrary metric in this class would converge to a unique locally homogeneous limit or would exhibit a more nuanced behavior.

Definition 1.1. If g, h are evolving Riemannian metrics on a manifold $\mathcal{M}^n$, we say g *quasi-converges* to h if for any $\varepsilon > 0$ there is a time t_ε such that

$$\sup_{\mathcal{M}^n \times [t_\varepsilon, \infty)} |g - h|_h < \varepsilon.$$

Quasi-convergence is an equivalence relation. Indeed, the standard fact that $|U(V,V)| \leq |U|_h |V|_h^2$ for any symmetric 2-tensor U and vector field V implies that g quasi-converges to h if and only if for all $t \geq t_\varepsilon$,

$$(1-\varepsilon)\, h(V,V) \leq g(V,V) \leq (1+\varepsilon)\, h(V,V).$$

We now state our result, using notation defined in [1] and to be reviewed in §2 below.

Theorem 1.2. *If g is any* solv-*Gowdy metric on a twisted torus bundle M^3_Λ, there is a locally homogeneous metric h in its quasi-convergence equivalence class $[g]$. Moreover, if h corresponds to the data $(\alpha(\theta), \Omega, F)$, the locally homogeneous metrics in $[g]$ are exactly those with the data $(\ell + \alpha(\theta), \Omega, F)$, $\ell \in \mathbb{R}$.*

Remark 1.3. Similar quasi-convergence of the Ricci flow to a 1-parameter family was conjectured for a class of $\mathcal{T}^3$ metrics studied in [2].

The paper is organized as follows. §2 describes the bundles $\mathcal{T}^2 \to \mathcal{M}^3_\Lambda \to \mathcal{S}^1$ and the solv-Gowdy metrics under study. It turns out that at large times, an arbitrary solv-Gowdy metric g behaves much like locally homogeneous metrics. §3 quantifies this observation and explicitly constructs a family h_ε of locally homogeneous metrics existing for all $t \geq 0$ which approximate g for times $t \geq t_\varepsilon$. In §4, we show that this family enjoys a certain compactness property which allows us to prove the existence part of the main theorem. The heuristic here is that g resembles a single locally homogeneous metric closely enough that the metrics h_ε are not too far apart at $t = 0$. §5 completes the main theorem by explaining the very special sort of non-uniqueness which can occur: distinct locally homogeneous metrics define distinct equivalence classes unless they differ only by a dilation of the base circle.

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2. Review of solv-Gowdy geometries.

We begin by briefly recalling some notation and results of [1]. Readers familiar with that paper may skip this section.

To construct an arbitrary solv-Gowdy metric g, take $\Lambda \in \mathrm{SL}(2, \mathbb{Z})$ with eigenvalues $\lambda_+ > 1 > \lambda_-$. In coordinates θ, x, y on $\mathbb{R}^3$, chosen so that the x, y axes coincide with the eigenvectors of Λ, define

$$g \doteqdot e^{2A}\, d\theta \otimes d\theta + e^{F+W}\, dx \otimes dx + e^{F-W}\, dy \otimes dy, \tag{2.1}$$

where F is constant and A, W depend only on θ. Clearly, g descends to a metric on the product of the line and the torus $\mathcal{T}^2$. Let Λ act on $\mathbb{R} \times \mathcal{T}^2$ by $(\theta, x, y) \mapsto (\theta + 2\pi, \lambda_- x, \lambda_+ y)$. If

$$A(\theta + 2\pi) = A(\theta) \tag{2.2}$$

and

$$W(\theta + 2\pi) = W(\theta) + 2\log\lambda_+, \tag{2.3}$$

then Λ is an isometry, and g becomes a well defined metric on the mapping torus $\mathcal{M}^3_\Lambda$, regarded as a twisted $\mathcal{T}^2$ bundle over $\mathcal{S}^1$. Notice that A governs the length of the base circle, while F and W respectively describe the scale and skew of the fibers. We denote arc length by

$$s(\theta) \doteqdot \int_0^\theta e^{A(u)}\,du \tag{2.4}$$

and set

$$Z \doteqdot \frac{\partial}{\partial s}W. \tag{2.5}$$

Then we can write the Ricci tensor as

$$\mathrm{Rc} = -\frac{1}{2}e^{2A}Z^2\,d\theta\otimes d\theta - \frac{1}{2}e^{F+W}\frac{\partial Z}{\partial s}\,dx\otimes dx + \frac{1}{2}e^{F-W}\frac{\partial Z}{\partial s}\,dy\otimes dy. \tag{2.6}$$

The locally homogeneous solv-Gowdy metrics are easily characterized.

Lemma 2.1. *A* solv-*Gowdy metric g is locally homogeneous if and only if W depends linearly on arc length.*

Proof. If g is locally homogeneous, then $R = -\frac{1}{2}Z^2$ is constant in space. Since Z is continuous, it follows that $\partial^2 W/\partial s^2 = 0$.

If Z is constant in space, let $P_0 = (\theta_0, x_0, y_0)$, $P_1 = (\theta_1, x_1, y_1)$ be points in $\mathcal{M}^3_\Lambda$. It will suffice to construct a diffeomorphism $\Phi : \mathcal{U}_0 \to \mathcal{U}_1$, where $\mathcal{U}_0$, $\mathcal{U}_1$ are neighborhoods of P_0, P_1 respectively, such that $\Phi(P_0) = P_1$ and $\Phi^* g = g$. If Φ is given in coordinates (θ, x, y) by

$$\Phi(\theta,x,y) = (\tau(\theta,x,y),\, \xi(\theta,x,y),\, \eta(\theta,x,y)),$$

the pullback condition $\Phi^* g = g$ is equivalent to the system

$$e^{2A(\theta)} = \left(\frac{\partial\tau}{\partial\theta}\right)^2 e^{2A(\tau)} + \left(\frac{\partial\xi}{\partial\theta}\right)^2 e^{F+W(\tau)} + \left(\frac{\partial\eta}{\partial\theta}\right)^2 e^{F-W(\tau)} \tag{2.7a}$$

$$e^{F+W(\theta)} = \left(\frac{\partial\tau}{\partial x}\right)^2 e^{2A(\tau)} + \left(\frac{\partial\xi}{\partial x}\right)^2 e^{F+W(\tau)} + \left(\frac{\partial\eta}{\partial x}\right)^2 e^{F-W(\tau)} \tag{2.7b}$$

$$e^{F-W(\theta)} = \left(\frac{\partial\tau}{\partial y}\right)^2 e^{2A(\tau)} + \left(\frac{\partial\xi}{\partial y}\right)^2 e^{F+W(\tau)} + \left(\frac{\partial\eta}{\partial y}\right)^2 e^{F-W(\tau)}. \tag{2.7c}$$

Note that $s(\theta)$ is invertible, because $\partial s/\partial\theta = e^{A(\theta)} > 0$, and define

$$\tau(\theta, x, y) = s^{-1}\left(s(\theta) + s(\theta_1) - s(\theta_0)\right)$$
$$\xi(\theta, x, y) = x_1 + e^{-\frac{Z}{2}(s(\theta_1)-s(\theta_0))}(x - x_0)$$
$$\eta(\theta, x, y) = y_1 + e^{\frac{Z}{2}(s(\theta_1)-s(\theta_0))}(y - y_0).$$

Clearly, $\Phi : P_0 \mapsto P_1$. Equation (2.7a) is satisfied, because

$$\frac{\partial\tau}{\partial\theta} = \frac{\partial\theta}{\partial s}(\tau) \cdot \frac{\partial s}{\partial\theta}(\theta) = e^{-A(\tau)+A(\theta)}.$$

To see that (2.7b) is satisfied, let ω denote W regarded as a linear function of arc length, so that $W(\theta) = \omega(s(\theta))$. Then we can write

$$\log\left(\left(\frac{\partial\xi}{\partial x}\right)^2 e^{W(\tau)}\right) = -Z \cdot (s(\theta_1) - s(\theta_0)) + \omega(s(\theta) + s(\theta_1) - s(\theta_0))$$
$$= \omega(s(\theta)) = W(\theta).$$

Equation (2.7c) is verified in a similar fashion. □

Remark 2.2. When studying a single locally homogeneous solv-Gowdy metric, one can always make A constant in space by a reparameterization of $\mathcal{S}^1$; but it will not be convenient for us to do so.

If an arbitrary solv-Gowdy metric g evolves by the Ricci flow

$$\frac{\partial}{\partial t}g = -2\,\mathrm{Rc}, \tag{2.8}$$

we shall abuse notation and allow the quantities introduced above to depend also on time. We find that g remains a solv-Gowdy metric and that (2.8) is equivalent to the system

$$\frac{\partial}{\partial t}A = \frac{1}{2}Z^2 \tag{2.9a}$$
$$\frac{\partial}{\partial t}W = \frac{\partial}{\partial s}Z \tag{2.9b}$$
$$\frac{\partial}{\partial t}F = 0, \tag{2.9c}$$

whose solution exists for all $t \geq 0$. It is most convenient to study Z and recover A and W by integration. Z evolves by

$$\frac{\partial}{\partial t}Z = \frac{\partial^2}{\partial s^2}Z - \frac{1}{2}Z^3, \tag{2.10}$$

where the operator $\partial^2/\partial s^2$ plays the role of the Laplacian and evolves according to the commutator

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = -\frac{1}{2} Z^2 \frac{\partial}{\partial s}. \tag{2.11}$$

For all $t \geq 0$, we identify $\mathcal{S}^1$ with the circle $x = 0$, $y = 0$ and denote its length by

$$L(t) \doteqdot \int_{\mathcal{S}^1} ds = \int_0^{2\pi} e^{A(\theta,t)} d\theta. \tag{2.12}$$

Notice that (2.3) implies the important *integral condition*

$$\int_{\mathcal{S}^1} Z\, ds = 2 \log \lambda_+, \tag{2.13}$$

which is preserved by the flow.

If an evolving solv-Gowdy metric is locally homogeneous at $t = 0$, it remains so under the Ricci flow. For such metrics, Z is the function of time alone

$$Z(t) = \frac{1}{\sqrt{t + 1/\zeta^2}}, \tag{2.14}$$

where $\zeta \doteqdot Z(0)$ is positive by (2.13). The sub-family of locally homogeneous solv-Gowdy metrics can thus be indexed by $(\alpha(\theta), \Omega, F)$, where

$$\alpha(\theta) \doteqdot A(\theta, 0) \tag{2.15a}$$

$$\Omega \doteqdot W(0,0). \tag{2.15b}$$

We now summarize the estimates we shall use from [1]. Let g be a solution to the Ricci flow whose initial data $g(\cdot, 0)$ is a C^2 solv-Gowdy metric. Hamilton–Isenberg organize the proof of their main theorem into four steps. In *Step 1*, they show there is $C > 0$ depending on $Z(\cdot, 0)$ such that for all $t > 0$,

$$|Z(\cdot, t)| \leq \frac{1}{\sqrt{t + C}} < \frac{1}{\sqrt{t}}. \tag{2.16}$$

By *Step 2*, there is a time $T > 0$ and constants $m \doteqdot Z_{\min}(T)$, $M \doteqdot Z_{\max}(T)$ depending on $L(0)$, $Z(\cdot, 0)$ and satisfying $0 < m \leq M < 1/\sqrt{T}$ such that for all $t \geq T$,

$$\frac{1}{\sqrt{t - T + 1/m^2}} \leq Z(\cdot, t) \leq \frac{1}{\sqrt{t - T + 1/M^2}}. \tag{2.17}$$

By *Step 1* again, there are $C, C' > 0$ depending on $L(0)$, $Z(\cdot, 0)$ such that for all $t \geq T+1$,

$$C\sqrt{t-T} \leq L(t) \leq C'\sqrt{t-T}. \tag{2.18}$$

By *Step 4*, there is $C > 0$ depending on $L(0)$, $Z(\cdot, 0)$ such that for all $t \geq T$,

$$\left|\frac{\partial}{\partial s} Z(\cdot, t)\right| \leq \frac{C}{\left(1 + m^2 (t-T)\right)^2}. \tag{2.19}$$

3. Construction of approximating metrics.

As a first step in proving the existence part (Theorem 4.1) of our main theorem, we find times t_ε and construct locally homogeneous metrics h_ε with the following properties: h_ε is in a sense the average of g at t_ε; h_ε remains ε-close to g for all times $t \geq t_\varepsilon$; and most importantly, h_ε exists for all $t \geq 0$.

Proposition 3.1. *For any $\varepsilon > 0$, there is a time $t_\varepsilon > 0$ and a locally homogeneous* solv-*Gowdy metric h_ε evolving by the Ricci flow for $0 \leq t < \infty$ such that*

$$\sup_{\mathcal{M}^3_\Lambda \times [t_\varepsilon, \infty)} |g - h_\varepsilon|_{h_\varepsilon} < \varepsilon.$$

Before proving this, we collect some technical observations.

Lemma 3.2. *For any $\varepsilon > 0$, there is $t_\varepsilon > 0$ such that Z satisfies the pinching estimate*

$$Z_{\max}(t) - Z_{\min}(t) \leq \frac{\varepsilon}{L(t)}, \tag{3.1}$$

and the decay estimate

$$\frac{1}{\sqrt{t - t_\varepsilon + 1/m_\varepsilon^2}} \leq Z(\cdot, t) \leq \frac{1}{\sqrt{t - t_\varepsilon + 1/M_\varepsilon^2}}, \tag{3.2}$$

for all $t \geq t_\varepsilon$, where m_ε, M_ε are defined by

$$0 < m_\varepsilon \doteqdot Z_{\min}(t_\varepsilon) \leq Z_{\max}(t_\varepsilon) \doteqdot M_\varepsilon < \infty \tag{3.3}$$

and satisfy

$$m_\varepsilon \leq M_\varepsilon \leq m_\varepsilon + \varepsilon \qquad \textit{and} \qquad M_\varepsilon^2 \leq (1+\varepsilon)\, m_\varepsilon^2. \tag{3.4}$$

Moreover, we can choose t_ε so that

$$\int_{t_\varepsilon}^{\infty}\left|\frac{\partial Z}{\partial s}\right| dt \leq \varepsilon.$$

Proof. Let T, m, M be as in (2.17) and let C be the constant in (2.19). Let $t_* \doteqdot \max\left\{T + C/\left(m^4\varepsilon\right), T+1\right\}$ and suppose $t \geq t_*$. Then (2.19) implies

$$\int_{t_*}^{\infty}\left|\frac{\partial Z}{\partial s}\right| dt \leq \int_0^{\infty}\frac{C}{m^4\left(t+t_*-T\right)^2}\,dt = \frac{C}{m^4\left(t_*-T\right)} \leq \varepsilon,$$

and (2.18) implies there is $C' > 0$ such that

$$L(t) \leq C'\sqrt{t-T}.$$

Hence for such times

$$Z_{\max}(t) - Z_{\min}(t) \leq \int_{S^1}\left|\frac{\partial Z}{\partial s}\right| ds \leq CC'\frac{\sqrt{t-T}}{\left(1+m^2\left(t-T\right)\right)^2}.$$

Choose $t_\varepsilon \geq t_*$ large enough that (3.1) holds for $t \geq t_\varepsilon$, and that (3.4) holds for m_ε, M_ε defined by (3.3). This is possible, because

$$\left(\frac{Z_{\max}(t)}{Z_{\min}(t)}\right)^2 \leq \frac{t-T+1/m^2}{t-T+1/M^2} \leq 1 + \frac{1}{m^2\left(t-T\right)}.$$

Then since $\frac{\partial}{\partial t}Z = \frac{\partial^2}{\partial s^2}Z - \frac{1}{2}Z^3$, we observe that

$$\frac{d}{dt}Z_{\min} \geq -\frac{1}{2}Z_{\min}^3 \qquad \text{and} \qquad \frac{d}{dt}Z_{\max} \leq -\frac{1}{2}Z_{\max}^3.$$

A routine use of the maximum principle (proved in [3]) now establishes (3.2) for all $t \geq t_\varepsilon$. □

Remark 3.3. The proof shows that for $t \geq T+1$,

$$Z_{\max} - Z_{\min} = O\left(t-T\right)^{-3/2},$$

a result which also follows directly from (2.17).

Lemma 3.4. *Let $\varepsilon > 0$ be given and let t_ε, m_ε, M_ε be as in Lemma 3.2. Then there is a locally homogeneous* solv-*Gowdy metric*

$$h_\varepsilon = e^{2A_\varepsilon}\, d\theta \otimes d\theta + e^{F_\varepsilon + W_\varepsilon}\, dx \otimes dx + e^{F_\varepsilon - W_\varepsilon}\, dy \otimes dy$$

evolving by the Ricci flow for $0 \leq t < \infty$ so that for $t \geq t_\varepsilon$,

$$\frac{1}{\sqrt{t - t_\varepsilon + 1/m_\varepsilon^2}} \leq Z_\varepsilon(t) \leq \frac{1}{\sqrt{t - t_\varepsilon + 1/M_\varepsilon^2}},$$

where $Z_\varepsilon = \frac{\partial W_\varepsilon}{\partial s_\varepsilon} = e^{-A_\varepsilon} \frac{\partial W_\varepsilon}{\partial \theta}$. Moreover, h_ε is constructed so that for all $\theta \in \mathcal{S}^1$, $A_\varepsilon(\theta, t_\varepsilon) = A(\theta, t_\varepsilon)$ and $|W(\theta, t_\varepsilon) - W_\varepsilon(\theta, t_\varepsilon)| \leq \varepsilon$.

Proof. Define

$$Z_\varepsilon(t) \doteqdot \frac{1}{\sqrt{t + (1/\zeta_\varepsilon^2 - t_\varepsilon)}}, \tag{3.5}$$

where

$$\zeta_\varepsilon \doteqdot \int_{\mathcal{S}^1} Z\, ds \Big/ \int_{\mathcal{S}^1} ds, \tag{3.6}$$

with the RHS evaluated at t_ε. Observe that Z_ε is well defined for all $t \geq 0$, because $|Z(t)| < 1/\sqrt{t}$ by (2.16), whence

$$1/\zeta_\varepsilon^2 - t_\varepsilon \geq 1/Z_{\max}^2(t_\varepsilon) - t_\varepsilon > 0.$$

Now recall that locally homogeneous solv-Gowdy metrics form a 3-parameter family and define

$$\alpha_\varepsilon(\theta) \doteqdot A(\theta, t_\varepsilon) - \frac{1}{2}\int_0^{t_\varepsilon} Z_\varepsilon^2\, dt \tag{3.7a}$$

$$\Omega_\varepsilon \doteqdot W(0, t_\varepsilon) \tag{3.7b}$$

$$F_\varepsilon \doteqdot F. \tag{3.7c}$$

Notice that h_ε is well defined; indeed, the identities

$$2\log\lambda_+ = \int_{\mathcal{S}^1} Z\, ds = \zeta_\varepsilon \int_{\mathcal{S}^1} ds = \int_{\mathcal{S}^1} \zeta_\varepsilon e^{A_\varepsilon}\, d\theta = \int_{\mathcal{S}^1} Z_\varepsilon\, ds_\varepsilon$$

show that the integral condition (2.13) is satisfied at t_ε, hence for all time.

The first assertion of the lemma is verified by the elementary observation

$$m_\varepsilon = Z_{\min}(t_\varepsilon) \le \zeta_\varepsilon \le Z_{\max}(t_\varepsilon) = M_\varepsilon,$$

which follows from (3.6). The second assertion is trivial; to prove the third, simply notice that

$$|W(\theta, t_\varepsilon) - W_\varepsilon(\theta, t_\varepsilon)| \le \int_{S^1} |Z - \zeta_\varepsilon|\, ds \le (Z_{\max} - Z_{\min})(t_\varepsilon) \cdot L(t_\varepsilon) \le \varepsilon.$$

□

Proof of Proposition 3.1. Without loss of generality, assume $0 < \varepsilon \le 1/6$.
Let $t \ge t_\varepsilon$ and observe that

$$\begin{aligned} |(A - A_\varepsilon)(\theta, t)| &= \frac{1}{2}\left|\int_{t_\varepsilon}^{t} \left(Z^2 - Z_\varepsilon^2\right)(\theta, \tau)\, d\tau\right| \\ &\le \frac{1}{2}\int_{t_\varepsilon}^{t} \left(\frac{1}{\tau - t_\varepsilon + 1/M_\varepsilon^2} - \frac{1}{\tau - t_\varepsilon + 1/m_\varepsilon^2}\right) d\tau \\ &= \log\sqrt{\frac{1 + M_\varepsilon^2(t - t_\varepsilon)}{1 + m_\varepsilon^2(t - t_\varepsilon)}}. \end{aligned}$$

Then since $|e^u - 1| \le e^U - 1$ when $|u| \le U$, we have

$$\left|\left(e^{2A} - e^{2A_\varepsilon}\right)(\theta, t)\right| = e^{2A_\varepsilon}\left|e^{2(A - A_\varepsilon)} - 1\right| \le e^{2A_\varepsilon}\frac{M_\varepsilon^2 - m_\varepsilon^2}{m_\varepsilon^2}$$

and hence

$$\left((h_\varepsilon)^{\theta\theta}\right)^2 \left(g_{\theta\theta} - (h_\varepsilon)_{\theta\theta}\right)^2 \le \varepsilon^2.$$

Because W_ε is constant in time, we have

$$\begin{aligned} |(W - W_\varepsilon)(\theta, t)| &\le |W(\theta, t) - W(\theta, t_\varepsilon)| + |W(\theta, t_\varepsilon) - W_\varepsilon(\theta, t_\varepsilon)| \\ &\le \left|\int_{t_\varepsilon}^{t} \frac{\partial Z}{\partial s}\, d\tau\right| + \varepsilon \\ &\le 2\varepsilon, \end{aligned}$$

whence substituting $\delta = 2\varepsilon \le 1/3$ in the crude estimate $e^\delta \le 1 + \delta + \frac{e}{2}\delta^2$ (which holds for $0 \le \delta \le 1$) gives

$$\left|\left(e^{F+W} - e^{F_\varepsilon + W_\varepsilon}\right)(\theta, t)\right| = e^{F_\varepsilon + W_\varepsilon}\left|e^{(W - W_\varepsilon)} - 1\right| \le 3\varepsilon e^{F_\varepsilon + W_\varepsilon}$$

and thus

$$((h_\varepsilon)^{xx})^2 (g_{xx} - (h_\varepsilon)_{xx})^2 \leq 9\varepsilon^2.$$

The estimate for $((h_\varepsilon)^{yy})^2 \left(g_{yy} - (h_\varepsilon)_{yy}\right)^2$ is entirely analogous. We have shown that

$$|g - h_\varepsilon|^2_{h_\varepsilon} = (h_\varepsilon)^{ac} (h_\varepsilon)^{bd} (g_{ab} - (h_\varepsilon)_{ab}) (g_{cd} - (h_\varepsilon)_{cd}) \leq 19\varepsilon^2$$

for $t \geq t_\varepsilon$, which is clearly equivalent to the desired result. □

4. Existence.

We have seen that for any $\varepsilon > 0$, there is a natural choice h_ε of locally homogeneous metric approximating g for times $t \geq t_\varepsilon$. In view of our non-uniqueness result (Theorem 5.1), it is remarkable that these choices are close enough to one another that we can prove the existence of a locally homogeneous metric in $[g]$.

Theorem 4.1. *There is a locally homogeneous* solv-*Gowdy metric* h_∞ *evolving by the Ricci flow for* $0 \leq t < \infty$ *such that for any* $\varepsilon > 0$ *there is a time* $t_\varepsilon > 0$ *with*

$$\sup_{\mathcal{M}^3_\Lambda \times [t_\varepsilon, \infty)} |g - h_\infty|_{h_\infty} < \varepsilon.$$

Again, we first obtain some preliminary results.

Lemma 4.2. *Let* $\{\varepsilon_j\}$ *be a sequence with* $\varepsilon_j \searrow 0$. *For each* j, *let* h_j *denote the metric* h_{ε_j} *given by Proposition* 3.1. *Then there is a subsequence* j_k *and a locally homogeneous metric* h_∞ *with data* $(\alpha_\infty(\theta), \Omega_\infty, F_\infty)$ *such that*

$$(\alpha_{j_k}(\theta), \Omega_{j_k}, F_{j_k}) \to (\alpha_\infty(\theta), \Omega_\infty, F_\infty)$$

uniformly in θ. *(Here, and throughout the proof, a subscript such as* j *denotes quantities corresponding to the metric* $h_j \equiv h_{\varepsilon_j}$.*)*

Proof. The argument is constructed from four claims, as follows: Claim 4.3 bounds $\frac{\partial}{\partial\theta} A(\cdot, t_j)$, hence $\frac{\partial}{\partial\theta} A_j(\cdot, t_j)$ by construction, hence $\frac{\partial}{\partial\theta} A_j(\cdot, 0)$ by (4.1) and the local homogeneity of h_j. Combining this with Claim 4.4 proves $\{A_j(\cdot, 0)\}$ is bounded and equicontinuous. Since Claim 4.5 bounds

$\frac{\partial}{\partial s_j} W_j(\cdot, 0)$, this lets us bound $\frac{\partial}{\partial \theta} W_j(\cdot, 0)$. Combining this with Claim 4.6 then proves $\{W_j(\cdot, 0)\}$ is bounded and equicontinuous. Because $F_j \equiv F$ by construction, this lets us extract a subsequence of the h_j whose initial data converge uniformly to the data of a locally homogeneous metric h_∞ existing for all $t \geq 0$.

Notice that if $j < k$, we may (and shall) assume $t_j \leq t_k$.

Claim 4.3. *There is $C < \infty$ such that*

$$\sup_{\mathcal{M}^3_\Lambda \times [T,\infty)} \left| \frac{\partial A}{\partial \theta} \right| < C.$$

Compute

$$\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\left(\frac{1}{2}Z^2\right) = e^A Z \frac{\partial Z}{\partial s}. \tag{4.1}$$

Since by (2.17),

$$\frac{\partial}{\partial t} A \leq \frac{1}{2} \cdot \frac{1}{t - T + 1/M^2}$$

for $t \geq T$, there is $C' > 0$ such that

$$A(\cdot, t) \leq \log C' + \log \sqrt{t - T + 1/M^2}$$

for $t \geq T$. Then by (2.19), we have

$$\begin{aligned} \left| \frac{\partial}{\partial t}\left(\frac{\partial A}{\partial \theta}\right) \right| &\leq C' \sqrt{t - T + 1/M^2} \frac{1}{\sqrt{t - T + 1/M^2}} \cdot \frac{C''}{(1 + m^2 (t - T))^2} \\ &\leq \frac{C' C''}{1 + m^4 (t - T)^2} \end{aligned}$$

for all $t \geq T$. Since there is $B > 0$ depending only on the initial data such that $-B \leq \partial A / \partial \theta \leq B$ at $t = T$, the claim follows.

Claim 4.4. *The sequence $\{\alpha_j(\theta)\}$ is bounded for each $\theta \in S^1$.*

Let $\theta \in S^1$ be arbitrary. For $j < k$, consider

$$\begin{aligned} \alpha_j(\theta) - \alpha_k(\theta) &= A(\theta, t_j) - \frac{1}{2} \int_0^{t_j} Z_j^2 \, dt - A(\theta, t_k) + \frac{1}{2} \int_0^{t_k} Z_k^2 \, dt \\ &= \frac{1}{2} \int_{t_j}^{t_k} \left(Z_k^2 - Z^2\right) dt + \frac{1}{2} \int_0^{t_j} \left(Z_k^2 - Z_j^2\right) dt. \end{aligned}$$

Since $1/\zeta_k^2 - t_k \geq 1/M_j^2 - t_j$, we obtain a familiar estimate for the first integral:

$$\frac{1}{2}\left|\int_{t_j}^{t_k} \left(Z_k^2 - Z^2\right)\, dt\right| \leq \log\sqrt{\frac{1+M_j^2\,(t_k - t_j)}{1+m_j^2\,(t_k-t_j)}} \leq \log\sqrt{1+\varepsilon_j}.$$

Write the second integral as

$$\begin{aligned}\frac{1}{2}\int_0^{t_j} \left(Z_k^2 - Z_j^2\right)\, dt &= \log\sqrt{\frac{1/\zeta_j^2 - t_j}{1/\zeta_k^2 - t_k}} + \log\sqrt{\frac{t_j + (1/\zeta_k^2 - t_k)}{1/\zeta_j^2}} \\ &= \log\sqrt{P_{jk}},\end{aligned}$$

where

$$P_{jk} \doteqdot \left(1 - \zeta_j^2 t_j\right)\left(1 + \frac{t_j}{1/\zeta_k^2 - t_k}\right) > 0. \tag{4.2}$$

Since

$$\frac{1/M^2 - T}{t_j + 1/M^2 - T} \leq 1 - \zeta_j^2 t_j \leq \frac{1/m^2 - T}{t_j + 1/m^2 - T}$$

and

$$\frac{t_j + 1/m^2 - T}{1/m^2 - T} \leq 1 + \frac{t_j}{1/\zeta_k^2 - t_k} \leq \frac{t_j + 1/M^2 - T}{1/M^2 - T},$$

we conclude that

$$\frac{1/M^2 - T}{1/m^2 - T} \leq P_{jk} \leq \frac{1/m^2 - T}{1/M^2 - T}.$$

Claim 4.5. *There are* $0 < Z_* \leq Z^* < \infty$ *such that* $Z_j\,(0) \in [Z_*, Z^*]$ *for all* j.

Note how

$$1/Z_j^2\,(0) = 1/\zeta_j^2 - t_j \geq 1/Z_{\max}^2\,(t_j) - t_j \geq 1/M^2 - T > 0$$

by (2.16) and (2.17), and similarly

$$1/Z_j^2\,(0) = 1/\zeta_j^2 - t_j \leq 1/Z_{\min}^2\,(t_j) - t_j \leq 1/m^2 - T < \infty.$$

Claim 4.6. *There are* $\Omega_* \leq \Omega^*$ *such that* $\Omega_j \in [\Omega_*, \Omega^*]$ *for all* j.

Suppose $j < k$. Then since $\Omega_j \doteqdot W(0, t_j)$, we have

$$|\Omega_k - \Omega_j| = |W(0, t_k) - W(0, t_j)| \leq \int_{t_j}^{t_k} \left|\frac{\partial W}{\partial t}\right| dt = \int_{t_j}^{t_k} \left|\frac{\partial Z}{\partial s}\right| dt \leq \varepsilon_j.$$

□

Lemma 4.7. *If h_∞ is a locally homogeneous metric with data*

$$(\alpha_\infty(\theta), \Omega_\infty, F)$$

and $\{h_j\}$ is a sequence of locally homogeneous metrics with data

$$(\alpha_j(\theta), \Omega_j, F)$$

converging to $(\alpha_\infty(\theta), \Omega_\infty, F)$ uniformly in θ, then for any $\varepsilon > 0$ there is J_ε such that for each $j \geq J_\varepsilon$

$$\sup_{\mathcal{M}^3_\Lambda \times [0,\infty)} |h_j - h_\infty|_{h_\infty} < \varepsilon.$$

Proof. The integral condition

$$\int_{S^1} Z_\infty(0)\, e^{\alpha_\infty(\theta)}\, d\theta = 2\log\lambda_+ = \int_{S^1} Z_j(0)\, e^{\alpha_j(\theta)}\, d\theta$$

shows that $Z_j(0) \to Z_\infty(0)$. For $\delta > 0$ to be determined, choose J_ε large enough that

$$\sup_{\theta\in S^1} |\alpha_\infty(\theta) - \alpha_j(\theta)| \leq \delta \qquad \text{and} \qquad \left|\frac{Z^2_\infty(0)}{Z^2_j(0)} - 1\right| \leq \delta$$

for all $j \geq J_\varepsilon$, and consider

$$(A_\infty - A_j)(\theta, t) = (\alpha_\infty - \alpha_j)(\theta) + \frac{1}{2}\int_0^t (Z^2_\infty - Z^2_j)\, dt.$$

For any $\lambda, \mu > 0$ we have the now-familiar inequality

$$\log\left(1 - \frac{|\mu-\lambda|}{\lambda}\right) \leq \int_0^t \left(\frac{1}{t+\lambda} - \frac{1}{t+\mu}\right) dt \leq \log\left(1 + \frac{|\mu-\lambda|}{\lambda}\right).$$

Since

$$\frac{1}{2}\int_0^t (Z^2_\infty - Z^2_j)\, dt = \frac{1}{2}\int_0^t \left(\frac{1}{t+1/Z^2_\infty(0)} - \frac{1}{t+1/Z^2_j(0)}\right) dt$$

and

$$\frac{\left|1/Z_j^2(0) - 1/Z_\infty^2(0)\right|}{1/Z_\infty^2(0)} \leq \delta,$$

we get our first estimate:

$$|(A_\infty - A_j)(\theta, t)| \leq \delta + \log\sqrt{1+\delta}.$$

Next observe that when $0 < \delta \leq \log 2$ we have $e^\delta \leq 1 + 2\delta$ and thus obtain our second estimate:

$$\begin{aligned}
|(W_\infty - W_j)(\theta, t)| &= |W_\infty(\theta, 0) - W_j(\theta, 0)| \\
&= \left|\int_0^\theta Z_\infty(0) \cdot e^{\alpha_\infty(u)} du - \int_0^\theta Z_j(0) \cdot e^{\alpha_j(u)} du\right| \\
&\leq \int_0^\theta Z_\infty(0) \cdot e^{\alpha_\infty(u)} \left|1 - e^{\alpha_j(u) - \alpha_\infty(u)}\right| du \\
&+ \int_0^\theta Z_j(0) \cdot e^{\alpha_j(u)} \left|\frac{Z_\infty(0)}{Z_j(0)} - 1\right| du \\
&\leq 3\delta\,(2\log\lambda_+)\,.
\end{aligned}$$

As in the proof of Theorem 3.1, it follows that we can make $|h_\infty - h_j|_{h_\infty}$ as small as desired by choosing $\delta = \delta(\varepsilon)$ appropriately. □

Proof of Theorem 4.1. Note that $|g - h_\infty|_{h_\infty}$ will be small if both $|g - h_j|_{h_j}$ and $|h_j - h_\infty|_{h_\infty}$ are. So take the subsequence of metrics h_{j_k} and times t_{j_k} given by Lemma 4.2 and pass to a further subsequence according to Lemma 4.7. □

5. Uniqueness.

Distinct locally homogeneous solv-Gowdy metrics belong to the same equivalence class if and only if they differ merely by a dilation of arc length. In that case, we shall see that they approach one another at the rate C/t, where the constant depends on the initial difference in length of the base circle.

Theorem 5.1. *Let h and h_* be locally homogeneous metrics corresponding to the data $(\alpha(\theta), \Omega, F)$ and $(\alpha_*(\theta), \Omega_*, F_*)$ respectively. If for some* *constant ℓ we have $\alpha_* \equiv \alpha + \ell$ and $\Omega_* = \Omega$ and $F_* = F$, then h and h_* quasi-converge with*

$$|h_* - h|_h = O\left(\frac{1}{t}\right).$$

In all other cases, there are $\delta > 0$ and $\theta \in \mathcal{S}^1$ such that

$$|h_* - h|_h (\theta, t) \geq \delta$$

for all $t > 0$, so h and h_* do not quasi-converge.

Proof. We consider three cases.

Case 5.2. $\alpha_* \equiv \alpha + \ell,\ \Omega_* = \Omega,\ F_* = F.$

Writing

$$Z(t) = \frac{1}{\sqrt{t + 1/\zeta^2}} \qquad \text{and} \qquad Z_*(t) = \frac{1}{\sqrt{t + 1/\zeta_*^2}},$$

we observe that $\ell = \log(\zeta/\zeta_*)$, since by the integral condition (2.13) we have

$$\frac{\zeta}{\zeta_*} = \frac{\int_{\mathcal{S}^1} e^{\alpha_*(\theta)}\, d\theta}{\int_{\mathcal{S}^1} e^{\alpha(\theta)}\, d\theta} = e^{\ell}. \tag{5.1}$$

It follows that the function

$$\omega(\theta) \doteqdot \int_0^\theta \left(\zeta_* e^{\alpha_*(u)} - \zeta e^{\alpha(u)}\right) du \tag{5.2}$$

is identically zero. So for all $\theta \in \mathcal{S}^1$ and $t \geq 0$ we have

$$(W_* - W)(\theta, t) = (W_* - W)(\theta, 0) = \Omega_* - \Omega + \omega(\theta) = 0.$$

Now notice that

$$(A_* - A)(\theta, t) = (\alpha_* - \alpha)(\theta) + \frac{1}{2}\int_0^t \left(Z_*^2(\tau) - Z^2(\tau)\right) d\tau = \ell + \phi(t),$$

where

$$\phi(t) \doteqdot \frac{1}{2}\log\frac{1 + \zeta_*^2 t}{1 + \zeta^2 t}. \tag{5.3}$$

It is clear by (5.1) that $A_* - A \to 0$ uniformly in θ as $t \to \infty$. In fact, this identifies the critical rate at which distinct locally homogeneous metrics h, h_* approach each other, because

$$\left(e^{2A_*} - e^{2A}\right)(\theta, t) = e^{2A(\theta,t)}\left(e^{2(\ell+\phi(t))} - 1\right)$$

and hence

$$|h_* - h|_h = \left|h^{\theta\theta}\left(h_* - h\right)_{\theta\theta}\right| = \left|e^{2(\ell+\phi(t))} - 1\right| = \frac{\left|1/\zeta_*^2 - 1/\zeta^2\right|}{t + 1/\zeta^2}.$$

Case 5.3. $\alpha_* \equiv \alpha + \ell,\ \Omega_* = \Omega,\ F_* \neq F.$

Notice that $W_* - W \equiv 0$ and $A_* - A \to 0$ as above. Without loss of generality, suppose $F_* - F = \delta > 0$. Then for all $\theta \in \mathcal{S}^1$ and $t \geq 0$ we have

$$e^{F_*+W_*} - e^{F+W} = e^{F+W}\left(e^{F_*-F} - 1\right) > \delta e^{F+W}$$

and hence

$$|h_* - h|_h \geq \left|h^{xx}\left(h_* - h\right)_{xx}\right| > \delta > 0.$$

Case 5.4. *Either* $\alpha_* \not\equiv \alpha + \ell$ *or* $\Omega_* \neq \Omega$.

Observe that we can always find θ with

$$\left(W_* - W\right)(\theta, 0) = \Omega_* - \Omega + \omega(\theta) \neq 0,$$

since ω cannot be identically zero if $\alpha_* \not\equiv \alpha + \ell$. Without loss of generality, assume $\left(W_* - W\right)(\theta, 0) = \delta > 0$. Then if $F_* \geq F$, we have

$$e^{F_*+W_*(\theta,t)} - e^{F+W(\theta,t)} = e^{F+W(\theta,t)}\left(e^{F_*-F}e^{\delta} - 1\right) \geq e^{F+W(\theta,t)}\left(e^{\delta} - 1\right)$$

for all $t \geq 0$ and hence

$$|h_* - h|_h(\theta, t) \geq \left|h^{xx}\left(h_* - h\right)_{xx}\right|(\theta, t) > \delta > 0.$$

On the other hand, if $F \geq F_*$ we obtain

$$e^{F_*-W_*(\theta,t)} - e^{F-W(\theta,t)} = e^{F-W(\theta,t)}\left(e^{F_*-F}e^{-\delta} - 1\right) \leq e^{F-W(\theta,t)}\left(e^{-\delta} - 1\right)$$

for all $t \geq 0$ and thus

$$|h_* - h|_h(\theta, t) \geq \left|h^{yy}\left(h_* - h\right)_{yy}\right|(\theta, t) > \frac{\delta}{1+\delta} > 0.$$

□

References.

[1] R. Hamilton and J. Isenberg, *Quasi-convergence of Ricci flow for a class of metrics*, Comm. Anal. Geom. **1**:4 (1993), 543–559.

[2] M. Carfora, J. Isenberg, and M. Jackson, *Convergence of the Ricci flow for metrics with indefinite curvature*, J. Differential Geom. **31** (1990), 249–263.

[3] R. Hamilton, *Four-manifolds with positive curvature operator*, J. Differential Geom. **24** (1986), 153–179.

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