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An application of the h-principle to C^1-isometric immersions in contact manifolds.

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We prove here a version of the Nash C^1-isometric immersion theorem for contact manifolds equipped with Carnot-Caratheodory metrics.

0. Introduction and motivation.

Let us first recall the logical structure of the C^1-isometric theory for the Riemannian manifolds *without* contact structure. Here one starts with a general smooth (not at all isometric) immersion f_0 of a Riemannian manifold $V^n = (V^n, g)$ into $\mathbf{R}^q$. If V^n is compact, then by an obvious scaling one can make such $f_0 : (V^n, g) \to \mathbf{R}^q$ *strictly short.* This means that the Riemannian metric induced by f_0 from $\mathbf{R}^q$ is strictly smaller than g, i.e. the difference $g - g_0$ is positive definite on V^n. The key idea of the C^1-immersion theory of Nash and Kuiper is as follows. One "stretches" a given strictly short immersion f_0 to an isometric C^1-immersion f_1, i.e. such that the form g_1 induced by f_1 equals g. This remarkable stretching was performed in the celebrated 1954 paper [13] by Nash under the assumption $q \geq n + 2$ and then Kuiper (1955) improved this result by showing that it is true when $q = n + 1$. (Clearly, this is impossible, in general, for $q = n$.)

To complete the construction what remained was to have at one's disposal the starting immersion $f_0 : V^n \to \mathbf{R}^q$. For this one could invoke the classical result by Whitney which claims that such an f_0 always exists for $q \geq 2n$. In fact, a generic C^∞-map $f : V \to \mathbf{R}^q$ is an immersion. Another possibility is offered by the Smale-Hirsch immersion theory which provides smooth immersions $V^n \to \mathbf{R}^q$ for a given $q > n$, provided the manifold V^n satisfies the necessary topological restrictions. For example, every parallelizable (e.g. contractible) manifold V^n can be smoothly immersed to

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$\mathbf{R}^{n+1}$ according to the Hirsch theorem (which was unavailable to Nash and Kuiper as it was proven in 1959, see [9]).

Now we turn to the subject matter of the present paper where we are given a *contact* structure on a manifold V, i.e. a "maximally non integrable" subbundle $S \subset T(V)$ (see §1 for the definitions) as well as a metric on S, i.e. a positive definite quadratic form g on S. We want to *isometrically immerse our contact manifold* V to another such manifold $W = (W, T \subset T(W), h)$. This means we seek an immersion $f : V \to W$ which is both, *contact*, i.e. sends S to T, and *isometric*, i.e. induces g from h via the differential Df of f restricted to S (and ranging in T by the contactness assumption on f). The basic example of W, playing the role of $\mathbf{R}^q$, is the *Heisenberg group* H^{2m+1} (defined in 1.4) which is, topologically speaking, a Euclidean space $\mathbf{R}^{2m+1}$ fibered over $\mathbf{R}^{2m}$ thought of as $\mathbf{C}^m$. This H^{2m+1} carries a (rather natural) contact structure $T \subset T\ (H^{2m+1})$ and a metric h on T, such that the differential of the fibration (map) $H^{2m+1} \to \mathbf{C}^m$ is an isometry on T for the standard metric on $\mathbf{C}^m$. Here, in order to construct a contact isometric C^1-immersion $V \to H^{2m+1}$, one may begin with a non-isometric but yet contact immersion $(V, S) \to (H^{2m+1}, T)$ which can be easily (at least for compact V, see 1.4) scaled to a contact strictly short immersion $f_0 : (V, S) \to (H^{2m+1}, T)$. Then one can start stretching this f_0, but this does not seem work in general. In fact, in order to perform a stretching in the same style as Nash, one needs an additional regularity assumption on f_0 (called (h, ω) regularity in 1.3) which automatically implies the inequality $m \geq rankS$, i.e. $q = 2m + 1 = \dim(H^{2m+1}) \geq 2n - 1$, $n = \dim V$. This assumption, for $W = H^{2m+1}$ amounts to the projected map $V \to \mathbf{C}^m$ being *totally real on* S. This means in our context that the differential of $f : V \to W = H^{2m+1}$ maps each fiber S_v to an $(n-1)$-dimensional subspace in $\mathbf{C}^m$ containing no $\mathbf{C}$-line (cf 1.3, 1.7). Actually, Nash's stretching construction in the contact case (similarly to the Riemannian one) needs extra directions in the ambient space which further restricts the dimension to $q \geq 2n + 3$. This stretching procedure was carried out in [1] thus providing a contact version of Nash's C^1-isometric immersion theorem (while it is still unknown if the stretching obtained by Kuiper under the weakened condition $q = n+1$ can be rendered contact). What remained unfinished in [1] is the Smale-Hirsch aspect of the immersion theory which is what we complete in this paper.

Namely we prove here a *Hirsch type theorem for regular contact immersions* $(V, S) \to (W, T, h)$ (where the key example is $W = H^{2m+1}$). We do this by a rather direct application of general theory from [7]. (The reader who is interested in the modern developements in contact geometry is referred to [4]). What is actually done here is checking that the general conditions required by [7] are satisfied provided $q \geq 2n+1$. We prove, under this restriction, the *h-principle for regular contact immersions* (see 1.8) which, in combination with [1] yields a similar result for *contact regular isometric* C^1*-immersions* $V^n \to H^{2m+1}$ *for* $q \geq 2n+3$ (see 1.5).

1. Definitions and the statement of the main theorem..

A 1-form β on a $(2m+1)$ -dimensional manifold W is called *contact* if $\beta \wedge (d\beta)^m$ does not vanish on W. Equivalently, one can say that β is contact if the restriction of $d\beta$ on the $2m$-dimensional tangent subbundle $T = \{\beta = 0\}$ is non degenerate or, as we also say, symplectic. A codimension 1 tangent subbundle T on W is called *contact* if it can be defined (at least locally) by an equation $\beta = 0$ for a contact form β. Notice that β is unique modulo multiplication by a nowhere vanishing function on W. The pair (W, T) in this case is called a *contact* manifold or a manifold W with a contact structure T. If a contact structure T is defined by a 1-form β, then the restriction $d\beta|_T$ of $d\beta$ to the subbundle T defines a *symplectic structure* on it i.e. a field of symplectic forms of the fibers $T_w \subset T, w \in W$. Denote $d\beta|_T$ by $d'\beta$. The symplectic form $d'\beta$ can be naturally identified (see, e.g. [2], p.96) with the 'curvature' $\omega = \omega_T : \Lambda^2 T \to T(W)/T$ of the subbundle $T \subset T(W)$ measuring nonintegrability of T itself. Note, that according to the Frobenius theorem the contact distribution is maximally non integrable: its integral submanifolds have dimension $\leq m$. To simplify our notation, in the sequel we shall use ω to denote the form $d'\beta$ on T.

1.1 In what follows we study almost complex structures, i.e. endomorphisms $J : T \to T$ with $J^2 = -Id$ on contact subbundles and associated quadratic forms on T.

1.1.A. Definition. An *almost complex structure* $J : T \to T$ is called *compatible* (with the contact structure) if $\omega(J\tau_1, J\tau_2) = \omega(\tau_1, \tau_2)$ for the form $\omega = d'\beta$ on T.

Notice that this definition does not depend on the choice of β defining T since ω is unique up to multiplication by a non-vanishing function. Namely, if β and β_1 define the same T and so $\beta_1 = \lambda\beta$, then $d\beta_1 = \lambda\beta + d\lambda \wedge \beta$ and so $d\beta_1|_T = \lambda d\beta|_T$ as $d\lambda \wedge \beta|_T = 0$, since $\beta|_T = 0$.

Let J be a compatible almost complex structure on (T, ω) and, for $\tau_1, \tau_2 \in T_w \subset T, w \in W$, put

$$h(\tau_1, \tau_2) = \omega(\tau_1, J\tau_2) \quad (\textit{or } h(J\tau_1, \tau_2) = \omega(\tau_1, \tau_2)). \tag{1}$$

Then it follows easily that

$$(2) \qquad h(\tau_1,\tau_2)=h(\tau_2,\tau_1), \qquad h(J\tau_1,J\tau_2)=h(\tau_1,\tau_2)$$

and the symmetric bilinear form h is non-degenerate.

1.1.A_1. Definition. A *form* h defined by (1) via some compatible J is called *ω-compatible* .

Notice that this definition depends on a choice of β.

1.1.A_1'. We say that h is *T-compatible* if it is $\lambda\omega$-compatible for some non-vanishing function λ on W, where, recall, T is defined as $ker\beta$.

1.1.A_2. Remark. If the above h is positive definite, the pair (h,ω) defines a *Hermitian* structure on the subbundle $T\subset T(W)$. That is, every fiber $T_w\subset T, w\in W$, can be equipped with a Hermitian metric $\mathbf{h}$ such that its real part equals h and the imaginary part equals ω.

1.2. In this paper we consider manifolds V and W with contact structures $S\subset T(V)$ and $T\subset T(W)$ where we assume that S and T are globally represented as $S=ker\alpha$ and $T=ker\beta$ for some 1-forms α and β defined on V and W respectively. (Recall, that the global existence of a contact form is equivalent to the coorientability of the implied contact subbundle).

We shall be considering on $S\subset T(V)$ and $T\subset T(W)$ positive definite quadratic forms called g and h respectively, where the form g on S is arbitrary and the form h on T is assumed to be T-compatible.

1.2.A. Definition. Call a manifold V a *contact-Carnot-Caratheodory* manifold (or, briefly, *contact C-C manifold*) if V is endowed with a contact structure $S\subset T(V)$ together with a positive definite quadratic form (Riemannian metric) g on S.

An immersion $f:(V,S)\to(W,T)$ is called a *contact immersion* if the differential Df sends S to T.

Let $V=(V,S,g)$ and $W=(W,T,h)$ be contact C-C manifolds. We say a *contact immersion* $f:(V,S,g)\to(W,T,h)$ is *isometric* if its differential Df is an isometric homomorphism $(S,g)\to(T,h)$.

1.2.B. Remark. A contact immersion $f:(V,S)\to(W,T)$ sends the curvature form $\omega=d\beta$ of T to the curvature $d\alpha$ of S, but only up to a multiplicative scalar. For example, if (h,ω) make a quasi-hermitian pair on S, the differential $df:S\to T$ is not necessarily a complex linear map for associated complex structures in S and in T. It would be so if $df(\omega)=d\alpha$, but $df(\omega)=\lambda\alpha$ is not sufficient. In fact, we can not handle at all maps

$f : V \to W$ with complex df on S and our definition of (h, ω)-regularity helps to avoid any trace of complex linearity. (Thus our result is useless for inducing CR-structures despite a formal similarity between two questions).

1.3. Regular linear subspaces and regular contact immersions. Let $L = (L, h, \omega)$ be a linear space endowed with a symmetric quadratic form h and an antisymmetric form ω.

We have the following

1.3.A. Definition. *A subspace* $L_0 \subset L = (L, h, \omega)$ *is called* (h, ω)-*regular if the intersection of the orthogonal complements* $(L \ominus_h L_0) \bigcap (L \ominus_\omega L_0)$ *has the correct codimension, namely* $2 \dim L_0$ (see 1.7 for a clarification of this notion).

We shall call this intersection (h, ω)-orthogonal complement and denote it by $L \ominus_{h,\omega} L_0$.

Now, let (V, S) be a contact manifold and let $W = (W, T, h)$ be a contact C-C manifold.

1.3.A_1. Definition. A *contact immersion* $f : (V, S) \to (W, T)$ is called (h, ω)-*regular* if, for all $v \in V$, the (image) subspace $Df(S_V) \subset T_W$ of $S_v \subset T_v(V)$, $w = f(v)$ is (h, ω)- regular. An injective *bundle homomorphism* $\phi : T(V) \to T(W)$ is called (h, ω)- *regular* if the image subspace $\phi_v(S_v) \subset T_w(W), w = f(v)$ is an (h, ω)-regular subspace for all $v \in V$.

1.3.B. Remark. This notion of regularity is a rather technical one and begs for justification. The situation here is similar to C^∞-isometric immersions of Riemannian manifolds, $f : V \to \mathbf{R}^q$ where the relevant regularity condition is *freedom* of f i.e. linear independence of the $n + \frac{n(n+1)}{2}$ vectors of the first and second partial derivatives of f (see [7],p.8).

The present techniques do not allow to approach general maps without suitable regularity conditions and so we are forced to add "freedom", "(h, ω)-regularity", etc.. But this does not limit our existence result as thus we get our contact isometric immersion with (h, ω)-regularity as a bonus.

1.4. The Heisenberg group example. The basic example of a contact manifold is the *Heisenberg group* H^{2m+1} which is a nilpotent Lie group with a left invariant contact structure. This H^{2m+1} is, by definition, simply connected and so is uniquely determined by its Lie Algebra $L(H^{2m+1})$, where this Lie Algebra has the following presentation: there is a basis $x_1, x_2, \ldots\ldots x_{2m}, \; y_1, y_2, \ldots\ldots y_{2m}, z,$ of vectors in $\mathbf{R}^{2m+1} = L(H^{2m+1})$ satisfying the following properties:

$$[x_i, x_j] = 0 \quad \text{for} \quad i, j = 1, \dots, m$$
$$[y_i, y_j] = 0 \quad \text{for} \quad i, j = 1, \dots, m$$
$$[x_i, y_j] = \delta_{ij}\, z \quad \text{for} \quad 1 \le i \le j \le m, \text{ where } \delta_{ij} \text{ is the Kronecker symbol}$$
$$[x_i, z] = [y_i, z] = 0.$$

Thus the vector z generates the *center* of $L(H^{2m+1})$ and the quotient $L(H^{2m+1})/center$ is abelian. Consequentely, H^{2m+1} has one-dimensional *center*, say, $C \in H^{2m+1}$ and $H^{2m+1}/C = \mathbf{R}^{2m}$. These properties $\dim(center) = 1$ and $H^{2m+1}/center$ being abelian, uniquely characterize H^{2m+1} as a simply connected Lie group. Then one shows, that the left translates of the $spanT_0 = span(x_i, y_j) \subset T_{Id}(H^{2m+1}) = L(H^{2m+1})$ form a maximally non-integrable hyperplane field T on H^{2m+1}. This field obviously is horizontal for the fibration

$$H^{2m+1} \to \mathbf{R}^{2m} = H^{2m+1}/C$$

and the pull back to T, that we denote by h of the Euclidean metric $\sum x_i^2 + \sum y_i^2$ on $\mathbf{R}^{2m}$ (for $\mathbf{R}^{2m}$ identified with the quotient algebra $L(H^{2m+1})/span\{z\} = span(x_i, y_i)$) is compatible with T.

The basic property on H^{2m+1} with this geometry, and the only one we use below, is the existence of a 1-parameter group of *self-similarities*:

$$A_t : H^{2m+1} \to H^{2m+1}$$

induced by the following automorphisms a_t of $L(H^{2m+1})$, $t \in (-\infty, +\infty)$

$$a_t(x_i) = e^t x_i$$
$$a_t(y_i) = e^t y_i$$
$$a_t(z) = e^{2t} z.$$

These A_t are obviously *contact*, i.e. their differential DA_t send $T \subset T(H^{2m+1})$ to T and moreover they *scale* the metric h

$$A_t^*(h) = e^{2t} h,$$

as is seen by looking at the action of a_t on $L(H^{2m+1})$. Note, that this H^{2m+1} plays the same distinguished role in Carnot-Caratheodory geometry (see (A) in Remark 1.6 below) as the Euclidean space in the Riemannian geometry. (The reader may consult [8] for more details on this).

1.5. Our main existence result is the following

1.5.*A*. h-principle for contact isometric immersions. *Let (V, S, g) and (W, T, h) be contact C-C manifolds such that there exists a contact (h, ω)-regular homomorphism $T(V) \to T(W)$, (for ω being the curvature of the contact structure T on W), pulling back ω to the curvature $d'\alpha$ of $S \subset T(V)$ (where $d'\alpha = d\alpha|_S$). Then V admits a contact isometric C^1-immersion into W if the following three conditions (i)-(iii) are satisfied:*

(i) $\dim W \geq 2 \dim V + 3$;

(ii) *V is compact;*

(iii) *W is the Heisenberg group.*

1.5.A_1. Remarks.

1. The general notion of h-principle (h-for homotopy) is explained in §2.
2. The condition (ii) can be dropped while (iii) can be replaced by T-compatibility of h, but the proof is more difficult in the general case.

1.5.A_2. Example. Suppose V is a compact contractible manifold: e.g. V is homeomorphic to the n-ball. Then the required homomorphism $T(V) \to T(W)$ does exist whenever $\dim W \geq 2 \dim V - 1$. In fact these homomorphisms can be seen as sections of the fibration $X^{(1)} \to V$ where the fiber $X_v^{(1)}, v \in V$, consists of the injective linear maps $T_v(V) \to T_w(W) \subset T(W), w \in W$, which send the subspace $S_v \subset T_v(V)$ to $T \subset T(W)$ such that the resulting map $S_v \to T_w \subset T_w(W) \subset T(W)$ is (h, ω)- regular and sends the curvature of S at v to that of T at w. Such a map $S_v \to T_w$ exists iff $rank T_w \geq 2\, rank S_v$ and then the fiber of the fibration $X^{(1)} \to V$ is non-empty and so it does admit a section whenever V is contractible.

Thus, our h-principle implies *the existence of contact isometric C^1-immersions $V \to W = H^{2m+1}$ provided V is compact contractible, and* $\dim V \leq m - 1$.

1.6. Remarks and comments.

(A) Every contact Carnot-Caratheodory manifold (V, S, g) carries a natural metric, called *Carnot-Caratheodory metric* where the $dist(v_1, v_2)$ is defined as the infimum of the g-lengths of curves joining v_1 and v_2 and which are everywhere tangent to S.

Our contact isometric immersions $V \to W$ clearly are isometric for the respective C-C metrics in V and W in the sense that they preserve the C-C lengths of smooth curves. Thus our Theorem 1.5.A. can be thought of as the C-C counterpart of the *h-principle* for isometric C^1-immersions of Riemannian manifolds to $\mathbf{R}^q$. (See [7]). Notice that this h-principle directly follows from the Hirsch-Smale immersion theory and the Nash-Kuiper isometric C^1-immersion theorem ([10, 13]). In fact, our h-principle for (h, ω)-regular contact immersions (see 1.8.A. below) is a counterpart of the Smale-Hirsch theory, while Nash's ideas (explained in detail in [1]) are extended to the contact category by the following

(A') **Short Approximation Theorem.** (see [1].) *Let* (V, S) *be a contact manifold and let* (W, T, h) *be a contact C-C manifold. If the metric* h *on* W *is* T*-compatible, then every strictly short* (h, ω)*-regular immersion* $f_0 : V \to W$ *can be* C^0*- approximated by contact isometric* C^1*-immersions provided* $\dim W \geq \max(2\dim V + 3, 3\dim V - 2)$.

Recall, that "strictly short" means that the pull-back h^* of the metric h to S by Df_0 is strictly smaller than g, i.e. the difference $g - h^*$ is positive definite on S.

(A") The above Short approximation theorem was combined in [1] with a Whitney-type contact immersion theorem from [7] and the conclusion was the following

Isometric Immersion Theorem. *Let* (V, S) *be a contact manifold, let* (W, T) *be a contact C-C manifold and assume* V *is compact. If the metric* h *on* $T \subset T(W)$ *is* T*-compatible and if* $\dim W \geq \max(2\dim V + 3, 3\dim V - 2)$ *then there exists a* C^1*-immersion* $f : V \to W$ *which is contact and isometric.*

Notice, that here W is not assumed to be Heisenberg, but our dimension assumption is more restrictive than in the above h-principle for immersions 1.5.A. On the other hand, under the present (stronger) dimension assumptions it is easy to see that $T(V)$ admits the homomorphism to $T(W)$ required by the h-principle if h is T-compatible. Thus our h-principle implies the isometric immersion theorem for $W = H^{2m+1}$. What is new is the weaker dimension assumption in the case when V is contractible.

1.7. On the algebraic meaning of (h, ω)-regularity. To have a better understanding of the condition of (h, ω)-regularity, let's supplement our Definition 1.3.A. of regular linear subspace by the following

1.7.A. A subspace $L_0 \subset L = (L, h, \omega)$ is (h, ω)*-regular* if one of the two following equivalent conditions (I) -(I') is satisfied:

(I) The linear system in $x \in L$

$$(3) \qquad h(x, l_i) = 0, \quad \omega(x, l_i) = 0,$$

where the vectors $l_1, l_2, \dots l_m$ form a basis in L_0, is *non singular.*

(I') The $2m$ covectors $x \to h(x, l_i)$ and $x \to \omega(x, l_i)$, $i = 1, \dots, m$ are linearly independent.

1.7.A_1. Remarks.

1. The space L_1 of solutions of (3) (obviously) equals the (h, ω)-orthogonal complement $L \ominus_{h,\omega} L_0$ (see 1.3.A.).
2. Observe that, if the quadratic form h is non singular (e.g. is positive definite) then $codim(L \ominus_h L_0) = \dim L_0$ and the same is true for non singular ω, but our (h, ω)-regularity condition is stronger than the joint regularity of h and ω.
3. If a subspace $L_0 \subset L$ is regular, then obviously every subspace $L_{00} \subset L_0$ is regular. Furthermore, if the forms h and ω are non-singular, then every 1-dimensional subspace of the linear space L is regular.

1.7.B. Definition. *We call the pair (h, ω) quasi-Hermitian if the space $L = (L, h, \omega)$ admits a complex structure $J : L \to L$ and a Hermitian form* h *such that its real part equals h and the imaginary part $\omega_{\mathrm{h}}(x, y) = h(x, Jy)$, is proportional to ω, i.e. $\omega_{\mathrm{h}} = \lambda\omega$, $\lambda \neq 0$.*

1.7.B_1. Remark. This is equivalent in our framework (compare with Def. 1.1.A_1'.) to the form h being T-compatible. It is also clear that J, if it exists, is uniquely determined by h and ω via the condition $\omega_{\mathrm{h}}(x, y) = \lambda h(x, Jy)$ as ω and h are non singular. One sees immediately that, when $\lambda = 1$, $\mathbf{h}$ is the usual hermitian metric of the complex vector space (L, J). This motivates the name 'quasi-Hermitian' which is introduced here only for terminological reasons).

1.7.C. Another useful characterization of regularity is related to the notion of *totally-real subspaces.*

Recall, that a $\mathbf{R}$-linear subspace $E \subset \mathbf{C}^q$ is said *totally real* if one of the following equivalent conditions (i)-(iv) is satisfied:

(i) $E \cap \sqrt{-1}E = \{0\}$;

(ii) The real dimension of $span_{\mathbf{R}}(E, \sqrt{-1}E)$ equals $2 \dim E$;

(iii) If $t_1, t_2, \dots, t_n$ is a real basis in E, then the vectors $t_1, t_2, \dots, t_n$ are $\mathbf{C}$- independent in $\mathbf{C}^q$;

(iv) If $t_1, t_2, \dots, t_n$, is a real basis in E, then the vectors $t_1, t_2, \dots, t_n$, $\sqrt{-1}t_1, \sqrt{-1}t_2, \dots, \sqrt{-1}t_n$, are $\mathbf{R}$-independent .

In what follows $L = (L, h, \omega)$ is a quasi-hermitian space, with the implied complex structure denoted by J. Such L is isomorphic to some $\mathbf{C}^q, q = \frac{1}{2}\dim_{\mathbf{R}} L$, with $J \leftrightarrow \sqrt{-1}$.

1.7.C_1. Lemma. *Let $L_0 \subset L = (L, h, \omega)$ be a linear subspace. Then the space L_1 of solutions of the system (3) equals the Hermitian orthogonal complement $L \ominus_h L_0$. In particular, L_1 is complex linear, i.e. $JL_1 = L_1$.*

The proof is immediate. Just observe that Hermitian orthogonality and the orthogonality with respect to h "together with" ω expressed by the system of equations (3) are equivalent.

1.7.C_1'. Corollary. *A subspace $L_0 \subset L$ is regular iff it is totally real.*

1.7.C_1''. Corollary. *The restriction $(h, \omega)|_{L_1}$ to the space L_1 remains quasi-Hermitian and, in particular, both forms h and ω are non-singular on L_1.*

It should be remarked that if $L_0 \subset L$ is a *complex* subspace, then it is *highly* non-regular, as $codim_{\mathbf{R}}(L \ominus_{h,\omega} L_0) = \dim_{\mathbf{R}} L_0$, instead of the required $2\dim L_0$.

We end this section with a few other properties of regular subspaces contained in the following

1.7.D. Lemma. *Let $L_0 \subset L$ be a regular subspace in (L, h, ω) and let $l' \in L$ be a non zero vector (h, ω)-orthogonal to L_0. Then the linear space $L_0' = span(L_0, l')$ is regular.*

(This follows from $L \ominus_{\mathrm{h}} L_0' == L_1 \ominus_{\mathrm{h}} l'$).

1.7.D_1. Corollary. *Let l_1, l_2 be two independent vectors (h, ω)- orthogonal to $L_0 \subset L$ and also mutually (h, ω)-orthogonal. Then the space $L_0'' = span(L_0, l_1, l_2)$ is regular, provided the subspace L_0 was regular.*

1.8. Derivation of the h-principle for isometric immersions from the h-principle for (h, ω)-regular contact immersions. According to the Short Approximation Theorem that we stated in 1.6.(A')., all we need for the existence of a contact isometric immersion is a strictly short contact immersion $V \to W$ and therefore our h-principle 1.5.A. for contact isometric immersions follows from the following

1.8.A. h-principle for (h,ω)-regular contact immersions. *Let $(V,S,g),(W,T,h)$ be contact C-C-manifolds where h is T-compatible, and suppose there exists a contact fiberwise injective homomorphism $T(V)\to T(W)$, pulling back the (curvature) form $\omega(\equiv d'\beta)$ on T to a non zero multiple of the curvature form $d'\alpha$ of S,(where $d'\alpha=(d\alpha|_s)$) which is (h,ω)-regular on S.*

Then we have the following (I) *and* (II)

(I) *There exists a continuous (h,ω)-regular immersion $f:V\to W$, provided* $\dim W\geq 2\dim V+1$;

(II) *If V is compact and $W=H^{2m+1}$ for $m\geq\dim V$, then the above f can be chosen strictly short besides being contact and (h,ω)-regular.*

1.8.A_1. Remarks.

1. The first statement does not need the metric g on S.

2. The second claim follows from the first with the help of the self-similarities $A_t:H^{2m+1}\to H^{2m+1}$. These A_t, for $t<<-const$, strongly contract the C-C metric in H^{2m+1} and thus every contact immersion $f_0:V\to H^{2m+1}$ becomes strictly short when composed with A_t provided V is compact and t is small enough. On the other hand, if f_0 is contact and (h,ω)-regular then so is the composition $f=A_t\circ f_0$ since A_t is a homothety. Thus (I) $\Rightarrow$ (II) as we claimed.

   The remaining part of the paper is dedicated to proving the first statement in the above h-principle. This will be done in the framework of continuous sheaves of [7] which is explained in the following §2.

3. The relation : $(Df)^*(d\beta)|_S=\lambda d\alpha$ does not depend on the choice of α and β. In fact if we change β by a non-vanishing function $\beta\mapsto\lambda\beta$, then $d\beta|_T$ becomes replaced by $\lambda d\beta|_T$ as was explained earlier.

2. Partial Differential Relations and Sheaves.

2.1. Let $p:X\to V$ be a C^∞-smooth fibration and let $X^{(r)}\to V$ be the corresponding space of r-jets fibered over V. In our case $X=V\times W\to V$ and $X^{(r)}$ consists of the space of r-jets of maps $V\to W$ at all points $v\in V$ with the projection $p^{(r)}:j^r_v\mapsto v\in V$.

2.1.A. Definition. *A differential relation of order r is a subset $\mathsf{R}\subset X^{(r)}$.*

2.1.B. Example. Let (V, S) and (W, T) be contact manifolds and let us describe the relation R_{con} distinguishing contact immersions $V \to W$. Our X here is $V \times W$ fibered over V via the projection $p : V \times W \to V, r = 1$, and $\mathsf{R}_{con} \subset X^{(1)} \to V$, where $X^{(1)}$ consists of the linear maps $T_v(V) \to T_w(W)$ for all $(v, w) \in V \times W$. This R_{con} consists of those *injective* linear maps which send $S_v \subset T_v(V)$ to $T_v \subset T_w(W)$, for all $(v, w) \in V \times W$. This means that a smooth map $f : V \to W$ is a contact immersion, iff its jet $j^1_f : V \to X^{(1)}$ lands in $\mathsf{R}_{con} \subset X^{(1)}$. However this needs some further refinement. If we write S and T locally as the kernels of 1-forms α on V and β on W, then the inclusion property $Df(S) \subset T$, written as $(Df)^*(\beta) = \lambda\alpha$ (which is always possible) implies that $(Df)^*(\beta) = \lambda d\alpha$ on S.

Thus we arrive at a smaller condition $\mathsf{R}'_{con} \subset \mathsf{R}_{con}$ consisting of linear maps $T_v(V) \to T_w(W)$ sending $S_v \to T_w$ and pulling back $d\beta|_{T_w}$ to a multiple of $d\alpha$ on S_v. This motivates the introduction of our basic relation $\mathsf{R}_{cr} \subset X^{(1)}$ defining *contact (h, ω)-regular immersions* . This R_{cr} consists of the linear maps $\delta : T_v(V) \to T_w(W)$ satisfying the following four conditions.

1. the maps δ are injective (which makes our maps $V \to W$ immersions);
2. $\delta(S) \subset T$ (contact condition);
3. $\delta^*(d\beta)|_S = \lambda d\alpha$ for non-vanishing λ (as motivated above);
4. $\delta(S) \subset T$ is (h, ω)-regular for $\omega = d'\beta$ (where, according to the notation introduced in §1, $d'\beta$ denotes the restriction of $d\beta$ to T).

Clearly, if $f : V \to W$ has $j^1_f(V)$ in R_{cr}, then it is a contact regular immersion, but a contact regular immersion may have $j^1_f(V)$ outside of this R_{cr} as the above 3. may be violated. (Say, the implied λ may vanish). But if we strengthen the contact condition $Df(S) \subset T$ by insisting that $(Df)^*(\beta) = \lambda\alpha$ for a non vanishing function λ, then we see that such contact immersion which we call *strict* (adopting the terminology of [3]), have their 1-jets in R_{cr}.

Finally, to have a perfect agreement between our relation and the relevant class of maps, we take $\mathsf{R}_{scr} \subset \mathsf{R}_{cr}$ where we insist that not only $\delta(S_v) = T_w$, but $\delta^{-1}(T_w) = S_v$. Then we arrive at the desired 'agreement'.

Maps $f : V \to W$ satisfying R_{scr} (i.e. with $j^1_f(V) \subset \mathsf{R}_{scr}$) are exactly *strict contact (h, ω)-regular immersions $V \to W$.*

2.1.B_1. Remarks.

1. Our construction of (h, ω)-regular immersions via the sheaf theory automatically provides *strict* such immersions but this is rather irrelevant for our ultimate purpose of constructing contact isometric immersions.

2. Notice, that R_{scr} is contained in R_{con} and $\mathsf{R}_{scr} \subset \mathsf{R}_{con}$ is an *open* subset in R_{con}. This will be crucial in the reduction of our h-principle (see §3) to that in [7].

2.2. h-principle. This principle says that every continuous section $\phi_0 : V \to \mathsf{R} \subset X^{(r)}$ can be homotoped in R to another section, say $\phi_1 : V \to \mathsf{R}$ which comes from a *solution* of R . In other words there exists a section $f : V \to X$, such that the r-jet of f, $j^r_f : V \to X^{(r)}$ sends V into R and this jet is homotopic to ϕ_0 in R . Notice that those sections of $X^{(r)}$ which appear as jets of sections of X constitute a rather small subspace in the space of sections of $X^{(r)}$. For example, if $X = V \times W$ and $r = 1$, then the section of $X^{(1)}$ are just continuous homomorphisms $T(V) \to T(W)$ while jets of sections $V \to X$ are *differentials* of maps $V \to W$. Notice that every homomorphism $T(V) \to T(W)$ defines the underlying maps $V \to W$, but not at all equals the differential of such a map.

Having this in mind, we say that a section $V \to X^{(r)}$ is *holonomic* if it is of the form $\phi = j^r(f)$ for some $f : V \to X$. Notice, that such an f, if exists, is unique and equal to the composition of f with the tautological projection $X^{(r)} \to X$.

Thus the *h-principle* can be expressed by saying that *every continuous section $V \to \mathsf{R}$ is homotopic in R to a holonomic section $V \to \mathsf{R}$ by a continuous homotopy of sections $V \to \mathsf{R}$.*

2.2.A. Comments and Remarks.

1. In our case the h-principle says that *every strictly contact (h, ω)-regular homomorphism $T(V) \to T(W)$ is homotopic to a differential of a strict contact (h, ω)-regular map.*

2. The h-principle is a very strong claim allowing us to solve a differential relation R. It does not hold in general, but by the work of Gromov it holds under a variety of assumptions on R. In fact, we shall see that these assumptions are met by R_{scr} if $\dim W \geq 2 \dim V + 1$.

3. Actually, our ultimate purpose is the solvability of another relation whose solutions are contact isometric maps, call it R_{is} . This R_{is} also lies in $X^{(1)}$ and we solve it by deforming holonomic sections $V \to \mathsf{R}_{cr} \supset \mathsf{R}_{scr}$ to holonomic sections of $\mathsf{R}_{is} \bigcap \mathsf{R}_{cr}$.

2.3. Local h-principle. The global solution of an R should be preceeded by local solutions. Namely, we may ask if, at a given point $v \in V$, there exists a germ of a solution of R, say f_v, defined on a small neighbourhood $U_v \subset V$. If R is an open subset, then the germ f_v representing the r-jet $y \in \mathsf{R} \subset X^{(r)}$, necessarily satisfies R in a small enough neighbourhood $U_v \subset V$ as the inclusion $j^r(f_v)(v) \in \mathsf{R}$ implies $j^r(f_v)(v') \in \mathsf{R}$ for all v' close to v. But for general non-open R the existence of local solutions at a point v does not necessarily follow from the existence of a single point y in R over v.

2.3.*A*. The local h-principle at a point v is the claim that every germ of a section $U_v \to \mathsf{R}$ can be deformed to a holonomic germ. Notice that in the course of such a deformation the implied neighbourhood U_v (which is the domain of the definition of the germ) may vary. Yet, there exists a smaller neighbourhood, say, U'_v where all germs making up the definition are simultaneously defined.

Every solution f of R gives rise to local solutions at all points $v \in V$. This suggests looking at families $f_v : U_v \to X$ of solutions also denoted $f_v(v')$, for families of neighbourhoods $U_v \subset V$ for all $v \in V$. Here a family of U_v' s signifies a neighbourhood $U_\nabla \subset V \times V$ of the diagonal $\nabla = V \subset V \times V$ and our family f_v is a continuous section of the fibration $X \to V \times V$ induced from X by the projection on the first factor V. This section, called $f_v(v')$, $(v, v') \in V \times V$, is supposed to be continuous in v and smooth in v', where moreover we assume the r-jet of $f_v(v')$ in v' to be continuous in v. And, if these v'-jets lie in R for all $v \in V$, we speak of families of local solutions of R, where the parameter of the family is v.

Notice, that every such a family of local solutions $f_v(v')$ gives rise to a section (jet) $\phi : V \to X^{(r)}$ sending V to R, namely $v \mapsto j^r_{v'}(f_v(v'))$ at $v' = v$. Sections $V \to \mathsf{R}$ of this nature are called *extendable* jets (this is much weaker than holonomic) and then we make the following

2.3.*B*. Definition. *We say that the relation* R *satisfies the local h- principle over V if every section $V \to \mathsf{R}$ can be deformed to an extendable jet $V \to \mathsf{R}$*

.

In other words, this h-principle claims that every section (jet) after a deformation comes from a family of local solutions.

2.3.*B*$_1$. Important Remark. Every open R satisfies the above h- principle. (To see this requires just going through the above definitions).

2.4. Continuous sheaves. We shall review now some of the basic notions and results from the theory of topological sheaves which are needed to prove the h-principle. We shall follow the same approach and use the same terminology as in [7][2].

The solutions of a relation $\mathsf{R} \subset X^{(r)}$ form what is called a *continuous sheaf*, namely for every $U \subset V$ we have a space of solutions of R over U, i.e. the space of sections $f : U \to X|_U$ with $j_r(f)(U) \subset \mathsf{R}$, or equivalently, the space of holonomic sections $U \to \mathsf{R}$. These spaces form a sheaf in the usual sense (see [5]).

We recall from [6], pp.74-75, that an *abstract sheaf* Φ over a topological space V is, by definition, the assignement of a set $\Phi(U)$ to each open subset $U \subset V$ and of a map $\Phi(I) : \Phi(U) \to \Phi(U')$ to each inclusion $I : U' \subset U$, such that the following three axioms are satisfied.

(1) If $I' : U'' \subset U'$ and $I : U' \subset U$, then the value of Φ at the inclusion $I \circ I' : U'' \subset U$ satisfies the property: $\Phi(I \circ I') = \Phi(I') \circ \Phi(I)$.

One also requires that $(Id : U \subset U) = Id$ for all $U \subset V$ and $\Phi(\emptyset) = \emptyset$. The elements $\phi \in \Phi$ are called sections of Φ over U and the value of Φ on the inclusion $I : U' \subset U$ is denoted $\phi|_{U'}$ instead of $\Phi(I)$.

(2) If two sections ϕ_1 and ϕ_2 of Φ over U are *locally equal*, then they are equal. Here the local equality means that for every point $u \in U'$ there exists a neighbourhood $U' \subset U$, such that $\phi_1|_{U'} = \phi_2|_{U'}$.

(2') Let open subsets $U_\mu \subset U, \mu \in M$, cover U. If the sections $\phi_\mu \in \Phi(U_\mu)$ satisfy: $\phi_\mu|_{U_\mu \cap U_{\mu'}} = \phi_{\mu'}|_{U_\mu \cap U_{\mu'}}$ for all μ and μ' in M, then there exists a section $\phi \in \Phi(U)$ (which is unique by (2)), such that $\phi|_{U_\mu} = \phi_\mu$ for all $\mu \in M$.

The conditions (2) and (2') show that every sheaf Φ is uniquely defined by $\Phi(U_v)$ for any *base* of open subsets $U_v \subset V$.

In order to extend Φ to non open-subsets $C \subset V$ we set $\Phi(C) = \Phi(OpC)$ (see Remark 2.4.A below) where $\Phi(C)$ denotes the inductive (direct) limit of $\Phi(U)$ over all neighborhoods $U \subset V$ of C. In particular, we define the *stalk* $\Phi(v) = \Phi(Opv)$ for all $v \in V$ and we write $\phi(v) \in \Phi(v)$ instead of $\phi|_{Opv}$. This allows us to restrict our Φ to a sheaf over C, called $\Phi|_C$ and defined by $(\Phi|_C)(D) = \Phi(OpD)$ for all open subsets $D \subset C$ and for $OpD \subset V$. Thus, the (restricted) sheaf $\Phi|_C$ has the same stalks over the points $c \in C$ as Φ.

2.4.A. Remark. Following Sect. 1.4.1. in [7] we introduce OpC (*opening of* C *in* V) as "an arbitrarily small but non-specified neighboorhood of C",

[2]The book [7] is the main reference for all the sheaf theoretic material used in this paper. Some of the definitions which are quoted in this section 2.5. are, for the convenience of the reader, reproduced almost literally from Sect 2.2 in [7].

which may become even smaller in the course of an argument when we need it.

The space of C^k- sections $OpC \to X$, by definition, is the direct (inductive) limit of the spaces of C^k- sections $U \to X$ over all open neighboorhoods $U \subset V$ of C. This space does not have a useful natural topology; however, it may be given a weaker structure, called *quasi-topology*, which nicely behaves under direct limits (see Def. 2.5.A. below).

2.4.A_1. A sheaf Φ is called *continuous* (or quasi-topological) if every set $\Phi(U), U \subset V$, is endowed with a quasi-topology such that the map $\Phi(I)$ is continuous in the ordinary sense for all inclusions $I : U' \subset U$. In this case the space $\Phi(C)$ is equipped with the inductive limit quasi-topology for all subsets $C \subset V$.

2.4.A_2. Let Φ and Ψ be two continuous sheaves over V. A *homomorphism* $\alpha : \Phi \to \Psi$ is a family of continuous maps $\alpha_U : \Phi(U) \to \Psi(U)$ which, for all open $U \subset V$, satisfy to the property: $\alpha_{U'} \circ \Phi(I) = \Psi(I) \circ \alpha_U$ for all $I : U' \subset U$. (i.e. the maps α_U commute with the restrictions of sections). Finally, a *subsheaf* $\Phi' \subset \Phi$ is defined by giving a subspace $\Phi'(U) \subset \Phi(U)$ for all $U \subset V$, such that Φ' satisfies the above (2) and (2').

2.5. Let $\Phi = \Phi_{\mathsf{R}}$ be our sheaf of solutions and observe that each space $\Phi(U), U \subset V$, (i.e. the space of solutions of R over U) has a natural topology, coming from the C^0-topology on the space of sections $U \to X^{(r)}|_U$. It will be useful to extend this to $\Phi(K)$ for all compact subsets $K \subset V$, where $\Phi(K)$ is the direct (inductive) limit of $\Phi(U)$ for all neighborhoods $U \supset K$. Thus an element $\phi \in \Phi(K)$ is given by a section $\Phi(U)$ for a "small" neighboorhood $U \subset V$ of K. The space $\Phi(K)$ being an inductive limit has no natural suitable topologies but it has a good quasi-topology in the following sense (cfr. [7], p.36).

2.5.A. Definition. ([14]) We shall say that the set A is endowed with a *quasi-topological structure* if for each topological space P the set of all maps $P \to A$ has a distinguished subset of maps that we temporarily call "continuous", which satisfy the following formal properties of maps which are continuous in the ordinary sense.

(i) If $\mu : P \to A$ is "continuous" and if $\phi : Q \to P$ is an ordinary continuous map, then the composed map $\mu \circ \phi : Q \to A$ is "continuous".

(ii) If a map $\mu : P \to A$ is locally "continuous", then it is "continuous" where the local "continuity" means that there exists a neighboorhood

$U \subset P$ of every point in P such that the map $\mu|_U : U \to A$ is "continuous".

(iii) Let P be covered by two *closed* subsets P_1 and P_2 in P. If a map μ is "continuous" on P_1 and on P_2, then it is "continuous" on all of P. Therefore, if $\bigcup_{i=1}^k P_i = P$ is a covering of P by finitely many closed subsets, then a map $\mu : P \to A$ is "continuous" if and only if $\mu|_{P_i} : P_i \to A$ is "continuous" for all $i = 1, \dots, k$.

A map between quasi-topological spaces, say, $\alpha : A \to B$, is called continuous if $\alpha \circ \mu : P \to B$ is "continuous" for all continuous maps $\mu : P \to A$ and for all topological spaces P.

From now on we will write continuous instead of "continuous".

The space of C^k-sections $OpC \to X$ (compare with what we said in Remark 2.4.A. above) is equipped with the quasi-topology which is the direct limit of the quasi-topologies associated to the C^k-topologies in the spaces of C^k-sections $U \to X$ for all neighborhoods $U \subset V$ of C. Thus, the notion of continuity for maps $\mu : p \mapsto f_p : OpC \to X, p \in P$, agrees with the C^k-continuity of families described before.

Notice that the main notions of homotopy theory make sense for quasi-topological spaces. For example, the definition of *weak homotopy equivalence* as well as the definitions of *Serre-fibration* and also of *microflexibility* which we shall be using in the sequel obviously generalize to quasi-topological spaces.

2.5.B. Definition.

(I) Let A and A' be topological spaces. A continuous map say $\mu : A \to A'$, is called a *weak homotopy equivalence* if either of the two following equivalent conditions is satisfied.

- (i) The map μ is *bijective* on the homotopy groups, $\mu_i : \pi_i(A) \to \pi_i(A'), i = 0, 1, \dots;$
- (ii) For an arbitrarily given cell complex P, let $P_0 \subset P$ be a subcomplex and let $\alpha_0 : P_0 \to A$ be an arbitrary continuous map. Then the map α_0 extends to a continuous map $\alpha : P_0 \to A$ if and only if $\alpha'_0 = \mu \circ \alpha_0$ extends to a continuous map $\alpha' : P_0 \to A'$.

(II) Let $\alpha : A \to A'$ be a continuous map between quasi-topological spaces. Consider a compact polyhedron P and a continuous map $\phi : P \to A$. Let $\Phi' : P \times [0,1] \to A'$ satisfy $\Phi'|_{P \times 0} = \phi'$ for $\phi' = \alpha \circ \phi : P \to A'$.

The map a is called a *Serre fibration* if Φ' lifts to a map $\Phi : P \times [0,1] \to A$ such that $\Phi|_{P\times 0} = \phi$ and $\alpha \circ \Phi = \Phi'$, for all polyhedra P, maps $\phi : P \to A$ and homotopies Φ' of ϕ. (In other words, the lifting Φ of Φ' can be extended over $P \times [0,1]$ in such a way that one still has a commutative diagram).

(III) We call α a *micro-fibration* if for all P, ϕ and Φ', there exists a positive $\epsilon \leq 1$ and a map $\Phi : P \times [0,\epsilon] \to A$ (where ϵ may depend on P , ϕ and Φ') such that $\Phi|_{P\times 0} = \phi$ and $\alpha \circ \Phi = \Phi'|_{P\times[0,\epsilon]}$.

2.5.B_1. Example. A submersion between smooth manifolds, $\alpha : A \to A'$, is a microfibration. If such a map α is also proper, then it is necessarily a fibration. Another condition which insures the Serre fibration property of a submersion α is the contractibility of the fiber $\alpha^{-1}(a) \subset A$ for all $a \in A'$ where $\alpha^{-1}(a)$ is assumed to be non-contractible in case it is empty. (See [7], p.307 for further discussion related to this example).

2.5.B_2.

A continuous sheaf Φ is said to be *flexible* iff, for each pair of compact subsets $C' \subset C$ in V the restriction map $\Phi(C) \to \Phi(C')$ is a Serre fibration.

One calls the sheaf Φ *microflexible* if all its restriction maps $\Phi(C) \to \Phi(C')$ are microfibrations.

2.5.C. Comments and Remarks. It is not hard to see ([7], pag.41) that the sheaf corresponding to any open relation R is microflexible. In those cases when the pertinent differential relation is not open (the case of contact immersions is one of these) one needs an extra argument to show that the relevant sheaves are microflexible.

One of the more significant results proven by Gromov in this connection, concerns sheaves which may be defined as the solution sheaf Φ_D of some partial differential operator D (for example, the sheaf of contact immersions has this form). We recall from [7] (see Section 2.3.1.), that D is an operator which goes from the space say, χ, of C^r-sections of some fibration $X \to V$ to the space G of C^s-sections of some vector bundle $G \to V$, and the sheaf Φ_D is defined by setting $\Phi_D(U)$ equal to the set of solutions of $Df = 0$ over $U \subset V$. By an application of the *Nash's implicit function theorem for infinitesimally invertible* differential operators (see [7], section 2.3.2.) it follows that the sheaf Φ_D is microflexible for these D.

In the case of contact immersions, the needed implicit function theorem reduces to elementary O.D.E. and the microflexibility of the corresponding sheaf is established ([7], Lemma (A), p.339) by using an appropriate version of Gray-Moser ([6],[11]) stability theorem. (Originally, J. Gray showed in [6] that a small perturbation of a contact structure is isomorphic to the original one by a small diffeomorphism to the underlying manifold. Then Moser proved a similar result for symplectic manifolds and his method was streamlined by Weinstein and presented in a form suitable for symplectic immersions on p. 337 of [7], while the contact case was worked out in [3]).

Here is the basic relation between flexibility and the h-principle.

2.5.D. Theorem. (see [7]). *Let* $\mathsf{R} \subset X^{(r)}$ *be a differential relation such that:*

1. *The sheaf of solutions of* R *is flexible;*
2. *The relation* R *satisfies the local h-principle over V.*

Then R *satisfies the h-principle.*

2.5.D_1. Comments. The flexibility is a strong global property of R which plays the crucial role for proving the h-principle. On the other hand the local h-principle is essentially a local analytic fact which may be difficult but not geometrically significant.

2.5.D_2. Idea of the proof of Theorem 2.5.D. The local h-principle allows a deformation of a section $V \to \mathsf{R}$ to a family of local solutions $(f_v)(v') : U_v \to X|_{U_v}, v \in V$ and if $\Phi = \Phi_{\mathsf{R}}$ is flexible these can be simultaneously deformed to 'new' $(f_v)(v')$ so that these new local solutions will agree on different neighbourhoods whenever these overlap, i.e. $f_{v_1}^{new}(v') = f_{v_2}^{new}(v')$ on the intersection $U_{v_1} \bigcap U_{v_2}$ for all v_1 and v_2 in V. Such a coherent family of solutions, $f_{v_1}^{new}(v')$ form an actual solution $f^{new}(v')$ over V required by the h- principle.

Thus the problem of proving the h-principle is reduced to establish the local h-principle and the flexibility of Φ_{R}.

2.5.E. Proving the local h-principle for the relation R_{scr}. It is shown in [7], Sect. 3.4.3. that the sheaf Φ_{con} of contact immersions $V \to W$ satisfy the local h-principle for each *contact* structure T on W. Then we observe the following:

If R_0 satisfies the local h-principle and $\mathsf{R}_0 \subset \mathsf{R}$ is an open subset then R also satisfies the local h-principle. This is an obvious generalization of the above remark 2.3.B_1. on the local h-principle for open relations $\mathsf{R} \subset X^{(r)}$.

2.6. Parametric h-principle. The notion of the h-principle can be strenghtened by extending it to families of solutions of R parametrized by an arbitrary polyhedron P, say $f_p : V \to X, p \in P$.

Here is the precise definition.

2.6.*A*. Definition. Let $\Phi = \Phi_{\mathsf{R}}$ denote the sheaf of solutions of R, and $\Psi = \Psi_{\mathsf{R}}$ the sheaf of sections $U \to \mathsf{R} \subset X^{(r)}$ for all open $U \subset V$. (This means $\Psi(U)$ equals the space of sections $U \to X^{(r)}|_U$ sending U to R). We have the natural sheaf homomorphism $J^r : \Phi \to \Psi$ sending each solution $f : U \to X$ of R from $\Phi(U)$ to its r-jet $f \mapsto j_r(f) \in \Phi(U)$. (Equivalently, one could identify Φ with the subsheaf of *holonomic* sections of Ψ and then instead of the jet homomorphism $\Phi \to \Psi$ one could just take the inclusion of Φ to Ψ).

We say that the relation R *satisfies the parametric h-principle if this homomorphism* $J^r : \Phi \to \Psi$ *is a weak homotopy equivalence,* which means $J^r(U) : \Phi(U) \to \Psi(U)$ is a w.h.e. for every open U.

2.6.A_1. Remarks.

1. Since the notion of w.h.e. extends to *quasi-topological* spaces, (compare 2.5.*A*.) one may speak of the map $\Phi(K) \to \Psi(K)$ being (or not being) a w.h.e. for compact $K \subset V$, e.g. for points $v \in V$. One can show easily that if a sheaf homomorphism is a w.h.e. on open subsets, then it is also a w.h.e. on all compact subsets in V.

2. The parametric h-principle trivially yields the ordinary h-principle, which is our prime goal. It also allows one to decide when two (h, ω)-regular contact immersions are isotopic by a (h, ω)-regular contact isotopy and so we state our final theorem 3.2.*E*. in the parametric form.

2.7. Some Remarks about restriction and extensions. We shall later on meet a situation where we need to extend certain maps from V to $V \times \mathbf{R}$ for V embedded in $V \times \mathbf{R}$ as $V \times 0$. In fact, we shall not need sections on all of $V \times \mathbf{R}$ but only near $V \times 0 \subset V \times \mathbf{R}$. Here is the necessary sheaf theoretic terminology which we present in a full generality.

Let Ψ be a sheaf over a topological space A and $A_0 \subset A$ be a closed subset (where A plays the role of $V \times \mathbf{R}$ above and A_0 stands for $V \times 0 \subset V \times \mathbf{R}$). Then Ψ *restricts* to a sheaf on A_0 say, Ψ_U , where $\Psi_U(U_0)$ equals the inductive limit of $\Psi(U)$ for all open U in A containing U_0.

Next, suppose Ψ is a (sub)-sheaf of maps from A to some space B. Then we can restrict maps from A to A_0 thus passing from Ψ_0 to a sheaf Φ_0 over A_0 of maps $A_0 \to B$. This new restriction is a sheaf homomorphism over A_0, denoted $\tau : \Psi_0 \to \Phi_0$ and called *restriction homomorphism.* (Notice that Ψ and Φ_0 are defined over different spaces and one can *not* speak of homomorphisms from Ψ to Φ_0, although one may restrict Ψ to Φ_0).

Coming back to $V = V \times 0 \subset V \times \mathbf{R}$ our problem will be local extendibility of certain maps $V \to W$ to maps $V \times \mathbf{R} \to W$ which will be expressed in the language of the corresponding restriction homomorphism (see Sect. 3.2).

3. The proof of flexibility and thus of the parametric h-principle for the relation R_{scr}.

3.1. Microflexibility of Φ_{scr}. Let again V and W be contact manifolds with the implied structures called $S \subset T(V)$ and $T \subset T(W)$ as in §1.

3.1.A. Lemma. *The sheaf Φ_{con} of contact immersions $V \to W$* (where, we recall, "contact" for an $f : V \to W$ means $Df(S) \subset T$), is *microflexible.* (see (A) on p.339 in [7] for the proof).

3.1.A_1. Remark. Observe that the subbundle $S \subset T(V)$ does not have to be contact here, it may be an arbitrary (codimension 1)- subbundle in $T(V)$ since the regularity required in [6] to prove the statement in the Lemma holds whenever $T \subset T(W)$ is contact, as the property contact makes the curvature form ω of the subbundle T non singular. Actually, we need microflexibility of the sheaf $\Phi^{\mathbf{R}}_{con}$ (see below) where the corresponding subbundle is not contact. (Once again we remind the reader that, according to our previous notation , the form ω is the 2-form $d'\beta = d\beta|_T$ for β being the 1-form on W defining the contact structure T by $\ker\beta = T$).

Next, we introduce a Riemannian metric h on T and let $\Phi_{scr} \subset \Phi_{con}$ be the subsheaf of strict (h,ω)-regular contact immersions. Clearly, this is an *open* subsheaf and thus *it is also microflexible.*

Here $\Phi' \subset \Phi$ is called open if $\Phi'(U)$ is open in $\Phi(U)$ for all open U for the topology of $\Phi(U)$, and the quasi-topological version of openness (which we do not need here) is left to the reader. In particular, an open $\Phi' \subset \Phi$ appears as $\Phi_{\mathsf{R}'} \subset \Phi_{\mathsf{R}}$ for open subsets $\mathsf{R}' \subset \mathsf{R}$.

We eventually want to show that Φ_{scr} is actually flexible and satisfies the h- principle, provided $rankT \geq 2(rankS + 1)$. To do this,we take the sheaf $\Phi^{\mathbf{R}}_{con}$ of contact immersions $V \times \mathbf{R} \to W$ and its subsheaf $\Phi^{\mathbf{R}}_{scr} \subset \Phi^{\mathbf{R}}_{con}$ consisting of (h,ω)-regular strict contact immersions $V \times \mathbf{R} \to W$. We shall also consider the restrictions to $V = V \times 0 \subset V \times \mathbf{R}$ of $\Phi^{\mathbf{R}}_{con}$ and $\Phi^{\mathbf{R}}_{scr}$ and shall denote these restrictions by $\tilde{\Phi}_{con}$ and $\tilde{\Phi}_{scr}$ respectively.

Notice, that here the term "contact" refers to the subbundle $S \times \mathbf{R} \subset T(V \times \mathbf{R})$ whose rank is equal to $rankS + 1$. (In other words, this subbundle equals the pull- back of S under the differential of the projection $V \times \mathbf{R} \to W$).

3.2. Flexibility of $\tilde{\Phi}_{scr}$. The microflexibility result established in 3.1. applies to $\Phi^{\mathbf{R}}_{scr}$ and so it is microflexible as well as $\Phi^{\mathbf{R}}_{con}$. Furthermore, the sheaf of contact immersions $V \times \mathbf{R} \to W$ as well as its natural subsheaves (such as $\Phi^{\mathbf{R}}_{scr}$) is acted upon by the group of diffeomorphisms of $V \times \mathbf{R}$ preserving the $\mathbf{R}$-fibers of the projection $V \times \mathbf{R} \to V$. Thus the microflexibility of $\Phi^{\mathbf{R}}_{con}$ implies the flexibility of $\tilde{\Phi}_{con}$ as claimed in the Main flexibility theorem on p.78 in [7] and this equally applies to the sheaf $\Phi^{\mathbf{R}}_{scr}$ and yields flexibility of $\tilde{\Phi}_{scr}$. Next, once we know that local sections of Φ_{scr} extend to those from $\tilde{\Phi}_{scr}$, we expect the flexibility of $\tilde{\Phi}_{scr}$ to imply that of Φ_{scr}. Indeed, this can be done but we need for this purpose a certain property (described in general terms in the definition 1 below) of the restriction homomorphism $\rho : \tilde{\Phi}_{scr} \to \Phi_{scr}$ which better expresses the idea of consistent extensions of contact (and related) immersions from V to $V \times \mathbf{R} \supset V$.

Consider two sheaves Φ and $\tilde{\Phi}$ over V and a homomorphism $\rho : \tilde{\Phi} \to \Phi$ over V. Take two sets Z and $Z' \subset Z$ in V and call two sections $\phi \in \Phi(Z)$ and $\tilde{\phi} \in \tilde{\Phi}(Z')$ *coherent* if the restriction $\phi|_Z \in \Phi(Z)$ equals to $\rho(\tilde{\phi}) \in \Phi(Z')$. The space of all coherent pairs $(\phi, \tilde{\phi}) \in \Phi(Z) \times \tilde{\Phi}(Z')$ is denoted by $\Omega = \Omega(Z, Z')$ and the natural map $\tilde{\Phi}(Z) \to \Omega$ is denoted by $\eta = \eta(Z, Z')$.

3.2.*A*. Definition. A homomorphism ρ is called *microextension* if it is surjective and if for every pair of compact subsets Z and Z', $Z' \subset Z$ in V the map $\eta : \tilde{\Phi}(Z) \to \Omega$ is a microfibration.

The usefulness of this property of the homomorphism $\rho : \tilde{\Phi} \to \Phi$ comes from the following :

3.2.*B*. Microextension Theorem (see [7], p.85). *If $\tilde{\Phi}$ is a flexible sheaf and $\rho : \tilde{\Phi} \to \Phi$ is a microextension then the sheaf Φ is also flexible. In other words, if Φ admits a flexible microextension then Φ itself is a flexible sheaf.*

Now, as we mentioned earlier, the sheaf $\Phi^{\mathbf{R}}_{scr}$ is flexible and so the flexibility of Φ_{scr} would follow from the microextension property of the restriction homomorphism $\rho : \tilde{\Phi}_{scr} \to \Phi_{scr}$.

So the next fact to be proven is the following:

3.2.*C*. Theorem. *If $\dim W \geq 2\dim V + 1$, then the restriction homomorphism $\rho : \tilde{\Phi}_{scr} \to \Phi_{scr}$ is a microextension.*

Proof. Let us first recall how this theorem is proven for the restriction homomorphism in the case of contact immersions without extra conditions of strictness and regularity, namely the fact that the restiction homomorphism $\tilde{\Phi}_{con} \to \Phi_{con}$ is a microextension. The essential point here is extension of germs of contact immersions $V \to W$ to germs of contact immersions $V \times \mathbf{R} \to W$ for $V = V \times 0 \subset V \times \mathbf{R}$ which is equivalent to show that $\tilde{\Phi}_{con} \to \Phi_{con}$ is surjective. Following Gromov, such extensions can be viewed as solutions of the Cauchy problem for contact immersions for which the Nash-Gromov implicit function theorem applies and yields the extendibility in a sufficiently strong form to give microextension, where 'strong' means sufficient continuity in the parameter as required by the microextension property. All this is done in a full generality in [7], but unfortunately tracking down our specific example there appears an unsurmontable task. On the other hand, fortunately the contact case can be handled directly without the use of Nash-Gromov implicit function theorem as indicated on p. 339 of [7] and elaborated in [3] for strict contact immersions. In fact, an appropriate extension scheme is indicated in §2 of [1], where contact extensions from V to $V \times \mathbf{R}$ are achieved with suitable contact vector fields in $W \supset V$. Such a field starts at V and its orbit gives $V \times \mathbf{R} \subset W$ with the contact structure equal the one coming from the projection $V \times \mathbf{R} \to V$. Moreover, the fields in [1] were chosen in such a way as to respect the (h, ω)-regularity of the immersion theorem implying the surjectivity of the restriction homomorphism $\tilde{\Phi}_{cr} \to \Phi_{cr}$ for $rankT \geq 2(rankS + 1)$. Furthermore, this process automatically preserves the strictness property of maps: i.e. if some contact immersion $V \to W$ is strict, then so is every its extension to $V \times \mathbf{R}$ if we remain close to $V = V \times 0 \subset V \times \mathbf{R}$. Thus we see that $\tilde{\Phi}_{scr} \to \Phi_{scr}$ is surjective. However, the "microfibration" aspect of the microextension property needs to be verified. This can be done in the following two ways a) and b).

a) Observe that the sheaves $\tilde{\Phi}_{scr} \subset \tilde{\Phi}_{con}$ and $\Phi_{scr} \subset \Phi_{con}$ are *open* subsheaves and thus, as we said before, their microfibration property reduces to that of $\rho : \tilde{\Phi}_{con} \to \Phi_{con}$ which is established (but not explicitely stated) in [7].

b) Check that the extension obtained by using contact fields satisfies all the microextension requirements.

To see how it works, one has to look at another example. Namely, one takes the sheaf, say, Φ_{all} of all smooth maps $V \to W$, and studies the corresponding extension problem for the homomorphism $\tilde{\Phi}_{all} \to \Phi_{all}$. Or, one may look at the homomorphism $\tilde{\Phi}_{imm} \to \Phi_{imm}$ for immersions $V \to \mathbf{R}^q$ for $\dim V < q$. The major problem here is to interpret adequately the meaning of microextension. This is explained for $\Phi = \Phi_{imm}$ in the example immediately after the definition of microextension on p. 85 in [7], where the microfexibility property of the map η follows from the openness of the immersion condition.

Now, in our case the sheaf $\Phi = \Phi_{con}$ (or $\Phi = \Phi_{scr}$). The space $\tilde{\Phi}(Z)$ consists of contact (strict contact) immersions of $Op(Z) \subset V \times \mathbf{R}$ to W and the space $\Omega(Z, Z')$ for $Z' \subset Z$ consists of the maps of the union $Op(Z') \cup (V \cap Op(Z))$, $Op(Z') \subset V \times \mathbf{R}$, to W, such that the restrictions of these maps to $Op(Z')$ and to $V \cap Op(Z)$ are contact (strict contact) immersions: $Op(Z') \to W$ and $V \cap Op(Z) \to W$ respectively. Here the map $\eta : \tilde{\Phi}(Z) \to \Omega$ amounts to the restrictions of contact (strict contact) immersions from $Op(Z)$ to the union of $Op(Z')$ and $V \cap Op(Z)$.

Since the microfibration property of η is known for $\Phi = \Phi_{con}$ it follows for $\Phi_{scr} \subset \Phi_{con}$ as it is an open subsheaf.

This gives a realization of the above a). To work out b) one has to look closer into the construction of the contact fields in [1] and observe that they are continous with respect to all parameters involved which is essentially equivalent to being a microextension in the sheaf theoretic language. Once that the meaning of microextension is clear for the above examples, then the contact extension becomes clear with the extension Lemma 2.3.A. on p.114 in [1].

3.2.C_1. Remarks.

1. The microextension property for the sheaf $\Phi = \Phi_{sc}$ is proven in [3] and so we could repeat the above with the sheaf $\Phi_{sc} \supset \Phi_{scr}$ instead of $\Phi_{con} \supset \Phi_{scr}$, thus deriving our microextension proof from [3] instead [7].

2. Both proofs of microextension in [7] and [3] follow the same route, essentially the one indicated in b) and briefly explained above by involving "continuity" of contact fields on parameters. So if we start from scratch the b)-approach is better as one avoids references to [7] or [3] but then one has to look into the matter more closely.

3.2.D. Let us now summarize the logic of the proof of the h-principle for the relation $\mathsf{R}_{scr} \subset X^{(1)}$ for $X = V \times W \to V$ by dividing it into the following 7 steps:

Step 1. Introduce the sheaf of strict contact (h, ω)-regular immersions Φ_{scr} consisting of solutions of the relation R_{scr} (see Example 2.1.B.) and the sheaf $\Phi^{\mathbf{R}}_{scr}$ of strict contact (h, ω)- regular immersions of $V \times \mathbf{R} \to V$ (which are solutions of the corresponding relation over $V \times \mathbf{R}$).

Step 2. Invoke the microflexibility of $\Phi^{\mathbf{R}}_{scr}$ which is a direct consequence of the microflexibility of $\Phi^{\mathbf{R}}_{con}$. (See Sect. 3.4.3(A) in [7]). Alternatively, one can derive microflexibility of Φ_{scr} from that for $\Phi_{sc} \supset \Phi_{scr}$ with the refering to [3] instead of [7].

Step 3. Use the fact that the sheaf $\Phi^{\mathbf{R}}_{scr}$ is invariant under special diffeomorphisms of $V \times \mathbf{R}$ and then (by applying the Main flexibility theorem in [7]) conclude to the flexibility of $\tilde{\Phi}_{scr}$ which is the restriction to $V = V \times 0 \subset V \times \mathbf{R}$ of the sheaf $\Phi^{\mathbf{R}}_{scr}$.

Step 4. Prove that the restriction homomorphism $\rho : \tilde{\Phi}_{scr} \to \Phi_{scr}$ is a microextension provided $\dim W \geq 2 \dim V + 1$.

Step 5. Apply the microextension theorem 3.2.B. and show that the sheaf Φ_{scr} is flexible.

Step 6. Show that the relation R_{scr} satisfies the local h-principle.

Step 7. Combine the previous steps 5 and 6 according to the pattern presented in [7], p.119 and arrive to the conclusion that Φ_{scr} satisfies the parametric h-principle and hence the (ordinary) h-principle.

This parametric h-principle is our main result in this article which we state here once more.

3.2.E. Theorem. *Denote by Ψ_{scr} the sheaf of injective bundle homomorphisms $\psi : T(V) \to T(W)$ with the following properties:*

1. *ψ pulls back T to S;*
2. *ψ is (h, ω)-regular, i.e. $\psi_v(S_v) \subset T_w$, is (h, ω)-regular for all $v \in V$.*

Then the natural homomorphism from the sheaf Φ_{scr} to the sheaf Ψ_{scr} (assigning $\psi = Df$ to each $f \in \Phi_{scr}$) is a weak homotopy equivalence, provided W is quasi Hermitian and $\dim W \geq 2 \dim V + 1$.

This theorem implies our h-principle for strict contact regular immersions 1.8.A. and thus the h-principle for contact isometric immersions stated in 1.5.A. □

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