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A Correction to "The Dirichlet Problem for Complex Monge-Ampère Equations and Regularity of the Pluri-Complex Green Function"

BO GUAN

I recently learned that the proof of Lemma 3.1 in [1] contains an error. As a result, the proof of Theorem 1.3 in [1] is incomplete. It also affects the proof of Theorem 1.2 which states that the pluri-complex Green function g for a strongly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with a logarithmic pole at a point $\zeta \in \Omega$ belongs to $C^{1,\alpha}(\overline{\Omega} - \{\zeta\})$ for any $0 < \alpha < 1$. Here we present a proof of this result independent of Lemma 3.1 of [1].

Theorem 1. *Let Ω be a smooth bounded strongly pseudoconvex domain in $\mathbb{C}^n$ and $\zeta \in \Omega$. Let g be the pluri-complex Green function for Ω with a logarithmic pole at ζ. Then $g \in C^{1,\alpha}(\overline{\Omega} - \{\zeta\})$ for any* $0 < \alpha < 1$.

Proof. It is known that the pluri-complex Green function g is the unique week solution of the problem

$$(42) \qquad \begin{cases} u \text{ is pluri-subharmonic} & \text{in } \Omega - \{\zeta\} \\ \det(u_{z_j \bar{z}_k}) = 0 & \text{in } \Omega - \{\zeta\} \\ u = 0 & \text{on } \partial\Omega \\ u(z) = \log|z - \zeta| + O(1) & \text{as } z \to \zeta. \end{cases}$$

We will show that the solution to (42) is in $C^{1,1}(\overline{\Omega} - \{\zeta\})$. Without loss of generality, we may assume $\zeta = 0$ and $B_1 = B_1(0) \subset \Omega$.

In [1] we proved that, for each positive $\varepsilon \leq \varepsilon_0$ (for some fixed $\varepsilon_0 \leq \frac{1}{2}$ so that $B_{2\varepsilon_0} \subset \Omega$), there exists a unique strictly pluri-subharmonic solution $u^\varepsilon \in C^\infty(\overline{\Omega}_\varepsilon)$ to the Dirichlet problem

$$(43) \qquad \det(u_{j\bar{k}}) = \varepsilon \ \text{ in } \Omega_\varepsilon \equiv \Omega - \overline{B}_\varepsilon, \quad u = \underline{u} \ \text{ on } \partial\Omega_\varepsilon$$

where $\underline{u} \equiv v + \log|z| \in C^\infty(\overline{\Omega} - \{0\})$ and v is the unique strictly pluri-subharmonic solution in $C^\infty(\overline{\Omega})$ of the Dirichlet problem

$$\det(v_{j\bar{k}}) = 1 \ \text{ in } \Omega, \quad v = -\log|z| \ \text{ on } \partial\Omega.$$

By the maximum principle,

$$\log|z| - C_0 \le \underline{u} \le u^\varepsilon \le u^{\varepsilon'} \le \log|z| \text{ in } \Omega_\varepsilon \text{ if } \varepsilon' \le \varepsilon. \tag{44}$$

Thus the limit

$$u(z) \equiv \lim_{\varepsilon\to 0} u^\varepsilon(z)$$

exists for all $z \in \overline{\Omega} - \{0\}$. We need to show that $u \in C^{1,\alpha}(\overline{\Omega} - \{0\})$.

Lemma 2. *There exists a constant C_1 independent of ε such that*

$$|\nabla u^\varepsilon| \le C_1 \text{ on } \partial\Omega \text{ and } |\nabla u^\varepsilon| \le \frac{C_1}{\varepsilon} \text{ on } \partial B_\varepsilon. \tag{45}$$

Proof. Since $\underline{u} \le u^\varepsilon \le 0$ in Ω_ε and $u^\varepsilon = \underline{u} = 0$ on $\partial\Omega_\varepsilon$, we have

$$|\nabla u^\varepsilon| = u^\varepsilon_\nu \le \underline{u}_\nu = |\nabla \underline{u}| \text{ on } \partial\Omega.$$

This proves the first inequality in (45). To prove the second one, let $\tilde{u}(z) = u^\varepsilon(\varepsilon z) - \log\varepsilon$ and $\underline{\tilde{u}}(z) = \underline{u}(\varepsilon z) - \log\varepsilon = v(\varepsilon z) + \log|z|$ for $z \in \bar{B}_2 - B_1$. Note that

$$\det(\tilde{u}_{j\bar{k}}) = \varepsilon^{2n}\det(u_{j\bar{k}}) = \varepsilon^{2n+1}. \tag{46}$$

Let $\tilde{h}$ be the harmonic function on $\bar{B}_2 - B_1$ with $\tilde{h} = \log 2$ on ∂B_2 and $\tilde{h}(z) = v(\varepsilon z)$ on ∂B_1. Then $\underline{\tilde{u}} \le \tilde{u} \le \tilde{h}$ on $\bar{B}_2 - B_1$ by the maximum principle, since $\Delta\tilde{u} \ge 0$ in $B_2 - \bar{B}_1$, $\tilde{u} \le \tilde{h}$ on ∂B_2 and $\tilde{u} = \tilde{h}$ on ∂B_1. Consequently,

$$|\nabla\tilde{u}| \le C_1 \text{ on } \partial B_1.$$

This implies the seocnd ineqality in (45) as $\nabla u^\varepsilon(z) = \frac{1}{\varepsilon}\nabla\tilde{u}(\frac{z}{\varepsilon})$. □

Lemma 3. *Let u be a strictly pluri-subharmonic C^3 function with*

$$\det u_{j\bar{k}} = \textit{constant}.$$

Let $\{u^{j\bar{k}}\} = \{u_{j\bar{k}}\}^{-1}$. Then, for any constant $a \ge 0$,

$$u^{i\bar{j}}(e^{au}|\nabla u|^2)_{i\bar{j}} \ge 0.$$

Proof. We verify this by direct calculation. First,

$$(|\nabla u|^2)_i = \sum_k (u_k u_{\bar{k}})_i = \sum_k (u_{ki} u_{\bar{k}} + u_k u_{\bar{k}i})$$

$$(|\nabla u|^2)_{i\bar{j}} = \sum_k (u_{ki} u_{\bar{k}} + u_k u_{\bar{k}i})_{\bar{j}} = \sum_k (u_{ki\bar{j}} u_{\bar{k}} + u_k u_{\bar{k}i\bar{j}} + u_{ki} u_{\bar{k}\bar{j}} + u_{k\bar{j}} u_{\bar{k}i}).$$

Since $\det u_{j\bar{k}}$ is constant, we see that

$$u^{i\bar{j}}(u_{ki\bar{j}} u_{\bar{k}} + u_k u_{\bar{k}i\bar{j}}) = 0$$

and therefore

$$u^{i\bar{j}}(|\nabla u|^2)_{i\bar{j}} = u^{i\bar{j}} u_{ki} u_{\bar{k}\bar{j}} + \sum_k u_{k\bar{k}}.$$

We also note that

$$u^{i\bar{j}}(|\nabla u|^2)_i u_{\bar{j}} = |\nabla u|^2 + u^{i\bar{j}} u_{ki} u_{\bar{k}} u_{\bar{j}}.$$

By Cauchy-Schwarz inequality,

$$2a|\operatorname{Re}\{u^{i\bar{j}} u_{ki} u_{\bar{k}} u_{\bar{j}}\}| \leq a^2 |\nabla u|^2 u^{i\bar{j}} u_i u_{\bar{j}} + u^{i\bar{j}} u_{ki} u_{\bar{k}\bar{j}}.$$

Finally,

$$\begin{aligned} e^{-au} u^{i\bar{j}}(e^{au}|\nabla u|^2)_{i\bar{j}} =& |\nabla u|^2 u^{i\bar{j}}(a u_{i\bar{j}} + a^2 u_i u_{\bar{j}}) \\ &+ 2a \operatorname{Re}\{u^{i\bar{j}}(|\nabla u|^2)_i u_{\bar{j}}\} + u^{i\bar{j}}(|\nabla u|^2)_{i\bar{j}} \\ =& a(n+2)|\nabla u|^2 + a^2|\nabla u|^2 u^{i\bar{j}} u_i u_{\bar{j}} \\ &+ 2a \operatorname{Re}\{u^{i\bar{j}} u_{ki} u_{\bar{k}} u_{\bar{j}}\} + u^{i\bar{j}} u_{ki} u_{\bar{k}\bar{j}} + \sum_k u_{k\bar{k}} \\ \geq& a(n+2)|\nabla u|^2 + \sum_k u_{k\bar{k}} \geq 0. \end{aligned}$$

This proves Lemma 3. □

It follows from Lemma 2 and Lemma 3 by the maximum principle that

(47) $$|\nabla u^\varepsilon| \leq C_1 e^{-u^\varepsilon} \quad \text{on } \overline{\Omega_\varepsilon}.$$

Lemma 4. *There exists a constant C_2 independent of ε such that*

(48) $$|\nabla^2 u^\varepsilon| \leq C_2 \ \text{ on } \partial\Omega \text{ and } |\nabla^2 u^\varepsilon| \leq \frac{C_2}{\varepsilon^2} \ \text{ on } \partial B_\varepsilon.$$

Proof. The first estimate in (48) may be proved as in [1]. We only prove the second one here. Let $\tilde{u}$, $\underline{\tilde{u}}$ and $\tilde{h}$ be as in the proof of Lemma 2. It suffices to show that

$$|\nabla^2 \tilde{u}| \le C_2 \ \text{ on } \partial B_1. \tag{49}$$

For a fixed point $z^0 \in \partial B_1$, we may assume $z^0 = (0, \dots, 1)$, i.e., the coordinates of z^0 are $x_j = y_j = 0$, $1 \le j \le n-1$, $x_n = 1$ and $y_n = 0$. Since $\tilde{u}(z) = v(\varepsilon z)$ on ∂B_1 and $|\nabla \tilde{u}| \le C_1$, it is trivial to obtain a bound for the pure tangential seocnd order derivatives at z^0

$$|\tilde{u}_{x_i x_k}|,\ |\tilde{u}_{x_i y_j}|,\ |\tilde{u}_{y_j y_l}| \le C, \ \ 1 \le i, k \le n-1, 1 \le j, l \le n. \tag{50}$$

To estimate the mixed tangential normal derivatives we need the following analogue of Lemma 2.1 of [1].

Lemma 5. *Let $U_\delta = (B_2 - B_1) \cap B_\delta(z^0)$ and $w = (\tilde{u} - \underline{\tilde{u}}) + t(\tilde{h} - \underline{\tilde{u}}) - Nd^2$, where d is the distance function from ∂B_1, and t, N are positive constants. For N sufficiently large and t, δ sufficiently small, we have*

$$\tilde{u}^{j\bar{k}} w_{j\bar{k}} \le -\frac{1}{64}\left(1 + \sum \tilde{u}^{k\bar{k}}\right) \ \ \text{in } U_\delta, \ \ v \ge 0 \ \text{ on } \partial U_\delta. \tag{51}$$

Proof. We first note that this does not follow from Lemma 2.1 of [1] as $\{\underline{\tilde{u}}_{j\bar{k}}\}$ is not uniformly positive definite in ε. In order to prove (51) we have to make use of a special property of $\underline{\tilde{u}}$. Since $\underline{\tilde{u}}(z) = v(\varepsilon z) + \log|z|$ and v is plurisubharmonic, we see that

$$\tilde{u}^{j\bar{k}} \underline{\tilde{u}}_{j\bar{k}} \ge \tilde{u}^{j\bar{k}} (\log|z|)_{j\bar{k}} = \frac{1}{2|z|^2} \tilde{u}^{j\bar{k}} \left(\delta_{jk} - \frac{\bar{z}_j z_k}{|z|^2}\right) \ge \frac{1}{16} \sum_{k=1}^{n-1} \tilde{u}^{k\bar{k}} \ \ \text{in } U_\delta$$

for δ sufficiently small. It follows that

$$\tilde{u}^{j\bar{k}} (\tilde{u} - \underline{\tilde{u}})_{j\bar{k}} \le n - \frac{1}{16} \sum_{k=1}^{n-1} \tilde{u}^{k\bar{k}} \ \ \text{in } U_\delta$$

when δ is sufficiently small. The rest of the proof is similar to that of Lemma 2.1 in [1] and therefore omitted. □

Returning to the proof of Lemma 4, as in [1] we may derive a bound for the mixed tangential normal derivatives at z^0 with the aid of Lemma 5

$$|\tilde{u}_{x_k x_n}|,\ |\tilde{u}_{x_n y_j}| \le C, \ \ 1 \le k \le n-1, 1 \le j \le n. \tag{52}$$

It remains to establish an estimate for the pure normal second order derivative

$$|\tilde{u}_{x_n x_n}(z^0)| \le C. \tag{53}$$

Because of (50) and (52) it suffices to prove

$$|\tilde{u}_{n\bar{n}}(z^0)| \le C. \tag{54}$$

Since $\tilde{u} - \underline{\tilde{u}} = 0$ on ∂B_1,

$$\tilde{u}_{j\bar{k}}(z^0) = \underline{\tilde{u}}_{j\bar{k}}(z^0) + \frac{1}{2}(\tilde{u} - \underline{\tilde{u}})_{x_n}(z^0)\delta_{jk}$$

and therefore

$$\sum_{j,\bar{k}<n} \tilde{u}_{j\bar{k}}(z^0)\xi_j\bar{\xi}_k \ge \sum_{j,\bar{k}<n} \underline{\tilde{u}}_{j\bar{k}}(z^0)\xi_j\bar{\xi}_k = |\xi|^2 \tag{55}$$

for any $\xi = (\xi_1, \dots, \xi_{n-1}) \in \mathbb{C}^{n-1}$. Finally, solving equation (46) for $\tilde{u}_{n\bar{n}}$ we see that (54) follows from (50), (52) and (55). This completes the proof of (49) and therfore that of Lemma 4. □

Lemma 6. *There exists a constant C_3 independent of ε such that*

$$|u^\varepsilon_{j\bar{k}}| \le C_3 e^{-2u^\varepsilon} \quad in\ \overline{\Omega}_\varepsilon. \tag{56}$$

Proof. It suffice to derive an upper bound

$$M \equiv \max_{z\in\overline{\Omega}_\varepsilon} \max_{|\xi|=1, \xi\in\mathbb{C}^n} e^{2u^\varepsilon} \sum u^\varepsilon_{j\bar{k}}(z)\xi_j\bar{\xi}_{\bar{k}} \le C \quad \text{independent of } \varepsilon. \tag{57}$$

We claim that M is achieved on $\partial\Omega_\varepsilon$. Suppose M is achieved at an interior point z^0 for some $\xi \in \mathbb{C}^n$. We may assume $\xi = (1, 0, \dots, 0)$ and $\{u^\varepsilon_{j\bar{k}}(z^0)\}$ is diagonal. Thus the function $\varphi \equiv 2u^\varepsilon + \log u^\varepsilon_{1\bar{1}}$ attains a maximum value at z^0 where, therefore

$$\sum \frac{\varphi_{k\bar{k}}}{u^\varepsilon_{k\bar{k}}} \le 0.$$

On the other hand, differentiating equation (42) twice, we obtain

$$\sum \frac{u^\varepsilon_{1\bar{1}k\bar{k}}}{u^\varepsilon_{k\bar{k}}} - \sum \frac{u^\varepsilon_{1\bar{j}k} u^\varepsilon_{\bar{1}j\bar{k}}}{u^\varepsilon_{j\bar{j}} u^\varepsilon_{k\bar{k}}} = 0 \tag{58}$$

and hence

$$\sum \frac{\varphi_{k\bar{k}}}{u^\varepsilon_{k\bar{k}}} = 2n + \frac{1}{u^\varepsilon_{1\bar{1}}} \sum \left(\frac{u^\varepsilon_{1\bar{1}k\bar{k}}}{u^\varepsilon_{k\bar{k}}} - \frac{u^\varepsilon_{1\bar{1}k} u^\varepsilon_{1\bar{1}\bar{k}}}{u^\varepsilon_{1\bar{1}} u^\varepsilon_{k\bar{k}}} \right) \geq 2n.$$

This contradiction shows that M is achieved on $\partial\Omega_\varepsilon$. By Lemma 4, we obtain (57). □

We are now in a position to finish the proof of Theorem 1. Let K be a compact subset of $\overline{\Omega} - \{0\}$. We show that, for ε sufficiently small so that $K \subset \overline{\Omega}_\varepsilon$,

$$\|u^\varepsilon\|_{C^{1,\alpha}(K)} \leq C = C(K) \quad \text{independent of } \varepsilon. \tag{59}$$

First, by (44) and (47) we have

$$\|u^\varepsilon\|_{C^1(K)} \leq C = C(K) \quad \text{independent of } \varepsilon. \tag{60}$$

Next, from Lemma 6 we see that

$$\Delta u^\varepsilon \equiv \sum_{j=1}^n (u^\varepsilon_{x_j x_j} + u^\varepsilon_{y_j y_j}) \equiv 4 \sum_{j=1}^n u^\varepsilon_{j\bar{j}} \leq C \quad \text{in } K \text{ independent of } \varepsilon. \tag{61}$$

Now, (59) follows from (60) and (61) with the aid of the standard regularity theory. This proves that $u \in C^{1,\alpha}(\overline{\Omega} - \{0\})$ and therefore completes the proof of Theorem 1. □

References.

[1] B. Guan, *The Dirichlet problem for complex Monge-Ampère equations and regularity of the pluri-complex Green function*, Commun. in Analysis and Geometry, **6** (1998), 687–703.

University of Tennessee,
Knoxville, TN 37996
E-mail address: guan@math.utk.edu