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Pseudo-holomorphic curves and the Weinstein conjecture

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1. Introduction.

In this paper we will illustrate how certain non-vanishing theorems of Gromov-Witten invariants can be used to prove the Weinstein conjecture.

Let $S \subset (N, \omega)$ be a hypersurface in a symplectic manifold. The characteristic distribution $\mathcal{L}_S$ on S consists of the tangent vectors $v \in TS$ such that $i(v)\omega_S = 0$, where ω_S is the pull back of ω to S. The flow lines generated by a vector field in $\mathcal{L}_S$ are called characteristics. $S \subset (N, \omega)$ is said to be of contact type if there is a 1-form α on S such that $d\alpha = \omega_S$ and $\alpha(v) \neq 0$ for any $v \neq 0$ in $\mathcal{L}_S$. The 1-form α is called a contact form on S, which induces a contact structure $\xi = \{\alpha = 0\}$ on S (see [W]).

Conjecture 1.1 (Weinstein [W]). *If S in (N, ω) is a compact hypersurface of contact type with $H^1(S, \mathbb{R}) = 0$, then S has at least one closed characteristic.*

We will focus on dimension 4. Our main result is

Theorem 1.2. *Let $S \subset (X, \omega)$ be a compact hypersurface of contact type in a closed symplectic 4-manifold with $b_2^+(X) > 1$. Let α be an induced contact form and $\xi = \{\alpha = 0\}$ be the corresponding contact structure on S. If we orient S by $\alpha \wedge d\alpha$, then*

1. *$c_1(\xi)$ is Poincare dual to a finite union of closed characteristics in S, each of which is oriented by $-\alpha$. In particular, S has at least one closed characteristic if $c_1(\xi) \neq 0$.*

2. *If S bounds a submanifold $W \subset X$ such that $c_1(W) \neq 0$ and ω is exact on W, then S has at least one closed characteristic.*

As a corollary of Theorem 1.2, we prove the Weinstein conjecture for compact hypersurfaces of contact type in a 2-dimensional Stein manifold under a mild restriction.

Let (V, J) be a Stein surface with a strictly plurisubharmonic function φ. Then $\omega_\varphi = -d(J^*(d\varphi))$ is a Kahler form on (V, J) and (V, ω_φ) is a symplectic manifold ([EG]). Suppose that S is a connected orientable compact hypersurface in V. Then S bounds a compact submanifold $W \subset V$ since $H_3(V) = 0$.

Corollary 1.3. *If $S \subset (V, \omega_\varphi)$ is of contact type and $c_1(W) \neq 0$, then S has at least one closed characteristic. Moreover, suppose α is an induced contact form on S and S is oriented by $\alpha \wedge d\alpha$, then the first Chern class of the induced contact structure on S is Poincare dual to a finite union of closed characteristics in S, each of which is oriented by $-\alpha$.*

Proof. Pick a regular value $r > \max(\varphi|_S)$ of φ so that $S \subset \overline{V_r} = \{\varphi \leq r\}$ which is compact. Note that $H_3(\overline{V_r}) = 0$ because $\overline{V_r}$ is homotopic to a 2-complex. It then follows that $\overline{V_r} \setminus S$ is disconnected, since otherwise S would have intersected a 1-dimensional cycle geometrically once which contradicts the fact that the fundamental class of S (note that S is orientable) in $\overline{V_r}$ is zero. Now S must bound a compact submanifold $W \subset V_r$ since $\partial\overline{V_r}$ is connected.

On the other hand, by Theorem 3.2 in [LM], $V_r = \{\varphi < r\}$ admits a holomorphic embedding into a Kahler surface X with $b_2^+(X) > 1$ such that the pull back of the Kahler form equals ω_φ. It is easy to see that Corollary 1.3 follows from Theorem 1.2. □

The Weinstein conjecture has a version for contact manifolds. Let (M, α) be a closed contact manifold with contact form α. The Reeb vector field v on M associated to α is defined by $i(v)d\alpha = 0$ and $\alpha(v) = 1$. One can regard $M = \{0\} \times M \subset \mathbb{R} \times M$ as a compact hypersurface of contact type in the symplectization $(\mathbb{R} \times M, d(e^t\alpha))$ with an induced contact form α. The induced characteristics in M are the orbits of the Reeb vector field on M. The Weinstein conjecture for a contact manifold (M, α) states that M has at least one closed Reeb orbit if $H^1(M, \mathbb{R}) = 0$.

Let (Y, α) be a closed contact 3-manifold. Hofer proved in [H] that Y has at least one closed Reeb orbit if $Y = S^3$ or Y is covered by S^3; or $\pi_2(Y) \neq 0$; or α induces an overtwisted contact structure on S. The remaining case is that $\pi_2(Y) = 0$ and α is tight.

Basic examples of (W, S) in Corollary 1.3 are the Stein surfaces with boundary ([Go]), which have tight induced contact structures on the boundary. So our result is somehow complement to Hofer's. Moreover, the closed

Reeb orbits obtained by Hofer in [H] are all contractible while we can produce closed Reeb orbits of non-trivial homology class when the first Chern class of the contact structure is non-zero. However, whether any connected tight contact 3-manifold (Y, α) is fillable, especially Stein fillable, is still a basic open question in contact geometry, although one can construct many concrete examples of that sort ([Go]).

In the proof of Theorem 1.2 we make use of the following deep theorem of Taubes.

Theorem 1.4 (Taubes [T]). *Let (X, ω) be a closed symplectic 4-manifold with $b_2^+(X) > 1$ and a non-trivial canonical bundle K. Then for a generic ω-compatible almost complex structure J, the Poincare dual to $c_1(K)$ is represented by the fundamental class of an embedded J-holomorphic curve Σ (may have several components) in X.*

The proof starts with the observation that the induced contact structure ξ as a complex line bundle on S is isomorphic to $K^{-1}|_S$ where K is the canonical bundle of (X, ω). By Theorem 1.4, $c_1(K)$ is Poincare dual to an embedded J-holomorphic curve Σ in X for a generic almost complex structure J. It is easy to see that $c_1(\xi)$ is Poincare dual to $\Sigma \bigcap S$ in S. The upshot is that $\Sigma \bigcap S$ converges to a union of closed characteristics in S through a deformation of almost complex structures in a neighborhood of S. The technical part of the proof is a combinatorial argument which produces convergent sequences of annuli in $\mathbb{R} \times S$ from the pseudo-holomorphic curves Σ such that the S-component of each limiting annulus lies in a Reeb orbit (see Lemmas 2.8, 2.9). The fact that each component of the pseudo-holomorphic curves has a uniformly bounded genus plays an essential role in the proof. The assumption $c_1(W) \neq 0$ is used to ensure that each pseudo-holomorphic curve goes through W, and the assumption that ω is exact on W is needed to get a non-trivial closed Reeb orbit.

We end this introduction with two remarks. First, analogous versions of Theorem 1.2 and Corollary 1.3 hold for $b_2^+ = 1$ case or higher dimensions provided that an analogous non-vanishing theorem of Gromov-Witten invariants exists. In fact, after this paper was finished, we learned that some relevant results (for any dimensions), especially the stabilized version of the Weinstein conjecture, had been proved recently by G. Liu and G. Tian in [LT] by a different argument. Second, the first Chern class of the contact structure being Poincare dual to a union of closed Reeb orbits is the analogue of Taubes' theorem (Theorem 1.4) for contact 3-manifolds which can be realized as a hypersurface of contact type in a closed symplectic 4-manifold with $b_2^+ > 1$; it would be interesting to know whether this is true for more general contact 3-manifolds, as well as what contact 3-manifolds can be realized as a hypersurface of contact type in a closed symplectic 4-manifold with $b_2^+ > 1$.

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2. The proof.

We first recall some basic facts about pseudo-holomorphic curves (or maps) in (or into) a hermitian manifold. Let (N, J) be an almost complex manifold. A C^1 map f from a Riemann surface (Σ, j) into (N, J) is said to be pseudo-holomorphic if the equation $J \circ df = df \circ j$ holds. The image of f is called a pseudo-holomorphic curve in (N, J). If we choose a Kahler metric μ on (Σ, j) and a hermitian metric h on (N, J), the energy of a map $f : \Sigma \to N$ is

$$E(f) = \frac{1}{2}\int_\Sigma |df|^2.$$

Note that the energy $E(f)$ depends only on the complex structure j of Σ and the metric h on N, while the integrand $|df|^2$ also depends on the choice of the metric μ. For a pseudo-holomorphic map $f : (\Sigma, j) \to (N, J, h)$, the area of the image of f measured with the metric h is equal to the energy of f:

$$Area_h(f) = E(f) = \frac{1}{2}\int_\Sigma |df|^2.$$

Let (N, ω) be a symplectic manifold of dimension $2n$. Since $U(n)$ is a deformation retract of $Sp(2n)$ — the group of $2n$-dimensional symplectic matrices, the tangent bundle of (N, ω) admits an almost complex structure and any two such structures are homotopic. An almost complex structure J on (N, ω) is said to be ω-compatible if $\omega(\cdot, J\cdot)$ is a hermitian metric on (N, J). It is a well-known fact that the space of all ω-compatible almost complex structures is nonempty and contractible. In a symplectic manifold (N, ω), for any ω-compatible almost complex structure J and the associated hermitian metric $h = \omega(\cdot, J\cdot)$, we have

$$Area_h(f) = E(f) = \frac{1}{2}\int_\Sigma |df|^2 = \int_\Sigma f^*\omega$$

for a pseudo-holomorphic map $f : \Sigma \to N$. In particular, the pseudo-holomorphic maps (curves) representing a fixed homology class in N have uniformly bounded energy (area). In fact, they are absolutely area minimizing in the homology class, and the images are minimal surfaces. Any C^1 pseudo-holomorphic map into an almost complex manifold (N, J) is smooth, and a bounded sequence of pseudo-holomorphic maps $f_n : (\Sigma, j, \mu) \to (N, J, h)$ with a uniform C^0 upper bound on the gradient df_n (in fact only a uniform L^p bound for $p > 2$ is needed) has a C^∞ convergent subsequence on any compact subset of Σ.

We particularly need the following well-known local properties of pseudo-holomorphic maps (curves) into (in) a hermitian manifold (see, for example, [PW] or [Ye]).

Lemma 2.1 (Energy estimate). *Let (Σ, j, μ) be a compact Riemann surface and (N, J, h) be a hermitian manifold with bounded geometry. Suppose $U \subset \Sigma$ is an open subset such that $U \bigcap \partial\Sigma = \emptyset$. Then there exists a constant $\varepsilon > 0$ such that for any pseudo-holomorphic map $f : \Sigma \to N$ with*

$$\int_{B(z,2r)} |df|^2 < \varepsilon$$

on a geodesic disc $B(z, 2r)$ in U centered at $z \in U$ with radius $2r$, the following estimate holds for a constant $C > 0$:

$$\sup_{B(z,r)} |df|^2 \leq \frac{C}{r^2}\int_{B(z,2r)} |df|^2.$$

Lemma 2.2 (Monotonicity). *Let (N, J, h) be a hermitian manifold with bounded geometry. Then there exist constants $C_0, r_0 > 0$ with the following property: let (Σ, j) be a compact Riemann surface with boundary, for any pseudo-holomorphic map $f : (\Sigma, j) \to (N, J)$, if $f(\partial\Sigma)$ lies outside of a closed r-ball $B(f(p), r)$ in N for some $p \in \Sigma \setminus \partial\Sigma$ and $r \leq r_0$, the following inequality holds:*

$$Area_h\left(f(\Sigma) \bigcap B(f(p), r)\right) \geq C_0 r^2.$$

The following lemma is a well-known fact of which we give a short proof here for completeness.

Lemma 2.3. *$S \subset (X,\omega)$ is of contact type with a contact form α if and only if a neighborhood of S in X is symplectomorphic to $(-\delta,\delta)\times S$ with the symplectic form $d(e^t\alpha)$.*

Proof. The "if" part is obvious; we give a proof for the "only if" part.

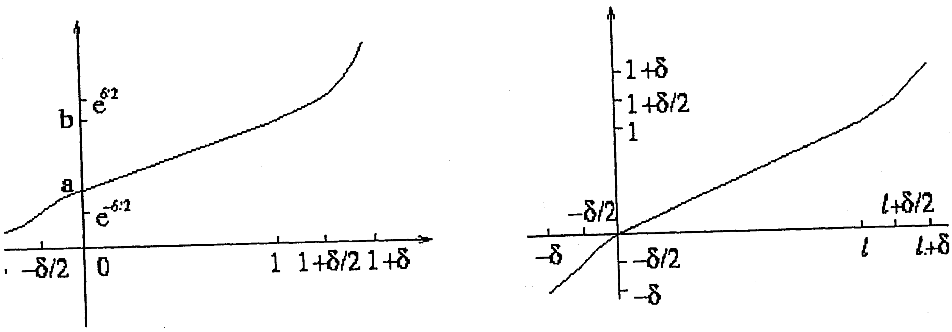
According to [W], the contact form α on S can be extended to a 1-form $\tilde{\alpha}$ in a neighborhood of $S \subset (X,\omega)$ in which $d\tilde{\alpha} = \omega$. Define the Liouville vector field Θ in the neighborhood by $i(\Theta)d\tilde{\alpha} = \tilde{\alpha}$. Then Θ is nowhere zero on S and transversal to S since α is a contact form on S. Let ψ_t be the flow generated by Θ, then $\psi_t(S)$, $t \in (-\delta,\delta)$, parameterizes a neighborhood $(-\delta,\delta)\times S$ of S in X for some $\delta > 0$. On the other hand, $L_\Theta\tilde{\alpha} = d(i(\Theta)\tilde{\alpha}) + i(\Theta)d\tilde{\alpha} = \tilde{\alpha}$, from which it follows that $\psi_t^*(\tilde{\alpha}) = e^t\tilde{\alpha}$. So $\omega = d\tilde{\alpha} = d(e^t\alpha)$, observing that $\tilde{\alpha} = \alpha$ on S since Θ is transversal to S and $\tilde{\alpha}(\Theta) = d\tilde{\alpha}(\Theta,\Theta) = 0$. □

The Proof of Theorem 1.2. The proof consists of three steps.

Step 1: Let $S \subset (X,\omega)$ be a compact hypersurface of contact type and α be an induced contact form on S. By Lemma 2.3, $\omega = d(e^t\alpha)$ in a collar neighborhood $(-\delta,\delta)\times S$ of S in (X,ω). Let X_1 be the 4-manifold obtained by cutting X open along S and then inserting the cylinder $[0,1]\times S$ into it. Then X_1 is also a symplectic manifold with the symplectic form ω_1 given by

$$\omega_1 = \begin{cases} \omega & \text{on } X \setminus (-\delta,\delta)\times S \\ d(\rho(t)\alpha) & \text{on } (-\delta, 1+\delta)\times S. \end{cases}$$

Here $\rho(t)$ is a smooth function on $(-\delta,1+\delta)$ such that $\rho'(t) > 0$ and $\rho(t) = e^t$ on $(-\delta,-\frac{1}{2}\delta)$ and $\rho(t) = e^{t-1}$ on $(1+\frac{1}{2}\delta, 1+\delta)$, and $\rho(0) = a$, $\rho(1) = b$ with $\rho(t)$ being linear on $[0,1]$ of slope ε_0 where a, b are in $(e^{-\frac{1}{2}\delta}, e^{\frac{1}{2}\delta})$ with $b - a = \varepsilon_0 > 0$ (see Figure 1, (a)). Let $\sigma_l : (-\delta, l+\delta) \to (-\delta, 1+\delta)$ be a strictly increasing smooth function which equals to t on $(-\delta,-\frac{1}{2}\delta)$ and $t-l+1$ on $(l+\frac{1}{2}\delta, l+\delta)$, and maps $[0,l]$ linearly onto $[0,1]$ (see Figure 1, (b)). Let X_l be the manifold obtained by cutting X open along S and inserting the cylinder $[0,l]\times S$ into it. Define a diffeomorphism $g_l : X_l \to X_1$ such that $g_l = (\sigma_l, id)$ on the neck and g_l equals to identity on the rest. Then X_l is a symplectic manifold with the pull back symplectic form $\omega_l = g_l^*\omega_1$. Note that the canonical class $c_1(K_l)$ of (X_l,ω_l) is the pull back of the canonical class $c_1(K_1)$ of (X_1,ω_1) via g_l.



(a) the graph of ρ (t)

(b) the graph of σ_l

Recall that the Reeb vector field v on S associated to the contact form α is defined by $i(v)d\alpha = 0$ and $\alpha(v) = 1$. We define an almost complex structure J_0 on $\mathbb{R} \times S$ as follows: J_0 equals to a fixed almost complex structure on $\xi = \{\alpha = 0\}$ which is compatible to the symplectic form $d\alpha|_\xi$, and $J_0(\frac{\partial}{\partial t}) = v$, $J_0(v) = -\frac{\partial}{\partial t}$.

We will use a slightly extended version of Theorem 1.4 which was communicated to the author by Taubes (see [McS]). In fact, in Taubes' theorem, the generic compatible almost complex structures can be chosen freely in a region as long as no pseudo-holomorphic curve is supported in that region. On each symplectic manifold (X_l, ω_l), the neck $(-\delta, \delta + l) \times S$ does not support any pseudo-holomorphic curve since the symplectic form ω_l is exact on it. Therefore by the extended version of Taubes' theorem, for each l there is a generic ω_l-compatible almost complex structure J_l on (X_l, ω_l) which equals to J_0 on the neck $[0, l] \times S$, and as $l \to \infty$, $\{J_l\}$ converges over $X_l \setminus (-\delta, \delta + l) \times S$ to a fixed ω-compatible almost complex structure J on X, and there is an embedded J_l-holomorphic curve Σ_l in (X_l, ω_l) (may have several components) such that $c_1(K_l)$ is Poincare dual to $[\Sigma_l]$.

Lemma 2.4. *The genus of each component of Σ_l is uniformly bounded from above by $c_1(K_1)^2 + b_2^-(X_1) + 1$.*

Proof. Suppose that $\{\Sigma_l^i\}$ are the components of Σ_l and $[\Sigma_l^i]$ is Poincare dual to $e_i \in H^2(X_l, \mathbb{Z})$. Then $c_1(K_l) = \sum e_i$. By the adjunction equality,

$$2g_i - 2 = e_i^2 + c_1(K_l) \cdot e_i = 2e_i^2,$$

where g_i is the genus of Σ_l^i. One easily sees that if $e_i^2 < 0$, then $e_i^2 = -1$ and $g_i = 0$. Therefore

$$\sum e_i^2 = c_1(K_l)^2 + s_l \leq c_1(K_l)^2 + b_2^-(X_l) = c_1(K_1)^2 + b_2^-(X_1)$$

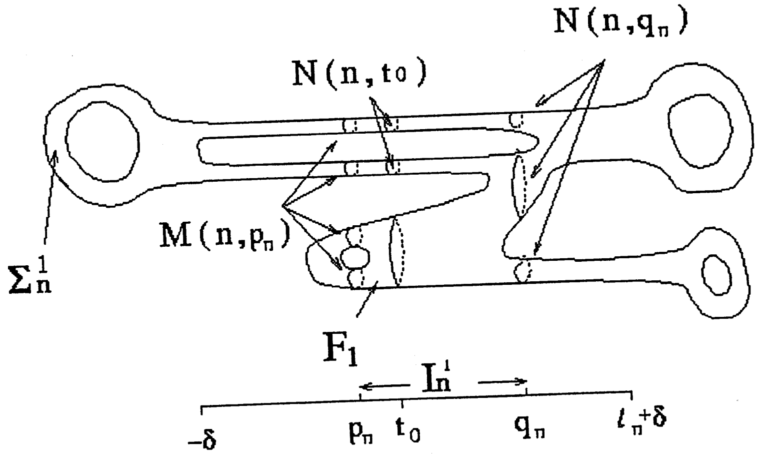
where the summation is taken over all the $e_i's$ with $e_i^2 \geq 0$, and s_l is the number of -1-sphere components of Σ_l. Now it is easy to see that $g_i = 1+e_i^2$ is bounded from above by $c_1(K_1)^2 + b_2^-(X_1) + 1$, which is uniform in l. □

Step 2: Take a sequence of $l_n \to \infty$. For simplicity, the subscript l_n is replaced by n in the notation. Let h_n be the hermitian metric on (X_n, ω_n, J_n) defined by $h_n = \omega_n(\cdot, J_n\cdot)$, and h_0 be the hermitian metric on $\mathbb{R} \times S$ defined by $h_0 = \omega_0(\cdot, J_0\cdot)$ where $\omega_0 = dt \wedge \alpha + d\alpha$. Observe that the area of Σ_n in (X_n, J_n, h_n) is uniformly bounded from above by a constant $c(= \int_{\Sigma_1} \omega_1)$. Therefore, on each neck $[0, l_n] \times S$, there is a sub-neck $I_n \times S$ (we can assume that the endpoints of I_n are regular values of the function $t'_n = t|_{\Sigma_n}$) with the following property: $|I_n| = \delta_0$ for a small $\delta_0 > 0$ and $Area_{h_n}(\Sigma_n \bigcap (I_n \times S))$ is bounded from above by $\frac{1}{2} l_n^{-1} \varepsilon_0 \varepsilon$ (recall that ε_0 is involved in the definition of $\rho(t)$ in Step 1). Here ε is chosen as in Lemma 2.1 for the hermitian metric h_0 on $(\mathbb{R} \times S, J_0)$ and the annulus $[0,1] \times S^1$ (S^1 has unit length) with the standard complex structure and metric. Note that $Area_{h_0}(\Sigma_n \bigcap (I_n \times S))$ is bounded from above by $\frac{1}{2}\varepsilon$ for large enough n. This is because on each neck $[0, l_n] \times S$, $h_0 \leq (l_n \varepsilon_0^{-1}) h_n$ for large enough n. For each n, choose a sub-interval I_n^1 of I_n such that the endpoints of I_n^1 are regular values of t'_n, and $|I_n^1| = \frac{1}{2}|I_n| = \frac{1}{2}\delta_0$ and $dist(\partial I_n, \partial I_n^1) \geq \frac{1}{8}\delta_0$.

We observe that the contact structure $\xi = \{\alpha = 0\}$ as a complex line bundle on S is isomorphic to $K^{-1}|_S$ where K is the canonical bundle of (X, ω). So for any regular value $t_0 \in I_n^1$ of the function $t'_n = t|_{\Sigma_n}$, $c_1(\xi)$ is Poincare dual to $\Sigma_n \bigcap (\{t_0\} \times S)$. Each component of $\Sigma_n \bigcap (\{t_0\} \times S)$ is oriented by $-\alpha$ if we orient S by $\alpha \wedge d\alpha$ (note that t_0 is a regular value of t'_n so that the pull back of α to $\Sigma_n \bigcap (\{t_0\} \times S)$ is nowhere zero).

Lemma 2.5. *For any regular value $t_0 \in I_n^1$, each component of $\Sigma_n \bigcap (\{t_0\} \times S)$ is connected through $\Sigma_n \bigcap ([t_0, l_n + \delta) \times S)$ to the inside of $X_n \setminus (-\delta, l_n + \delta) \times S$ (see Figure 2).*

Proof. If not, then it is easy to see that there are some components $\{\gamma_i\}$ of $\Sigma_n \bigcap (\{t_0\} \times S)$ which bound a pseudo-holomorphic curve $F \subset \Sigma_n \bigcap ([t_0, l_n + \delta) \times S)$. If we orient $\{\gamma_i\}$ canonically as the boundary of the oriented surface F, then each integral $\int_{\gamma_i} \rho_n(t_0)\alpha$ is negative, where $\rho_n(t) = \rho \circ \sigma_n(t)$ (see Step 1). The reason is that t_0 is a regular value of $t'_n = t|_{\Sigma_n}$ and $J_n(\frac{\partial}{\partial t}) = v$ so that the pull back of α to γ_i is a negative multiple of the volume form on γ_i. On the other hand, $0 < \int_F \omega_n = \sum_i \int_{\gamma_i} \rho_n(t_0)\alpha$ by the Stokes' theorem. This contradiction proves the lemma. □



Now by Lemma 2.5, the components of $\Sigma_n \bigcap(\{t_0\} \times S)$ are divided into two groups:

1. A component of $\Sigma_n \bigcap(\{t_0\} \times S)$ belongs to Group **I** if it is connected to the inside of $X_n \setminus (-\delta, l_n + \delta) \times S$ through both $\Sigma_n \bigcap((-\delta, t_0] \times S)$ and $\Sigma_n \bigcap([t_0, l_n + \delta) \times S)$.

2. The rest of $\Sigma_n \bigcap(\{t_0\} \times S)$ belongs to Group **II** which bounds a pseudo-holomorphic curve $F_1 \subset \Sigma_n \bigcap((-\delta, t_0] \times S)$ (may have several components). The homology class of the union of these components (each of which is oriented by $-\alpha$) is zero in S (see Figure 2).

Lemma 2.6. *For any n and regular value $t_0 \in I_n^1$ of t'_n, there is a subset $N(n, t_0)$ of $\Sigma_n \bigcap(\{t_0\} \times S)$ such that $c_1(\xi)$ is Poincare dual to $N(n, t_0)$ and $\#N(n, t_0) \leq N$ for a constant N independent of n and $t_0 \in I_n^1$.*

Proof. We define $N(n, t_0)$ to be the set of components of $\Sigma_n \bigcap(\{t_0\} \times S)$ which belong to Group **I** (see Figure 2). It remains to prove that there is an N independent of n and $t_0 \in I_n^1$ such that $\#N(n, t_0) \leq N$. But this follows from the following two reasons:

- the h_n-area of any component of $\Sigma_n \bigcap(X_n \setminus (-\delta, l_n + \delta) \times S)$ is bounded from below by a constant $c_1 > 0$ by Lemma 2.2 (Monotonicity), so the number of components of $\Sigma_n \bigcap(X_n \setminus (-\delta, l_n + \delta) \times S)$ is uniformly bounded from above;

- the genus of each component of Σ_n is uniformly bounded from above.

□

Let p_n and q_n be the left and right endpoints of I_n^1. We define a subset $M(n,p_n)$ of $\Sigma_n \bigcap(\{p_n\} \times S)$ as follows: a component γ of $\Sigma_n \bigcap(\{p_n\} \times S)$ is in $M(n,p_n)$ if there is a smooth path Γ in $\Sigma_n \bigcap([-\delta, q_n] \times S)$ joining $N(n,q_n)$ to $\Sigma_n \bigcap(\{-\delta\} \times S)$ which intersects $\Sigma_n \bigcap(\{p_n\} \times S)$ first time at γ geometrically once. (See Figure 2.)

Lemma 2.7. *There is an M independent of n such that $\#M(n,p_n) \leq M$.*

Proof. It is easy to see from Lemma 2.5 that any element in $N(n,p_n)$ is connected to $N(n,q_n)$ through $\Sigma_n \bigcap([p_n, q_n] \times S)$. So $N(n,p_n) \subset M(n,p_n)$. It suffices to prove that $\#(M(n,p_n) \setminus N(n,p_n))$ has a uniform upper bound.

Let γ be in $M(n,p_n) \setminus N(n,p_n)$. By the definition of $M(n,p_n)$, there is a smooth path Γ in $\Sigma_n \bigcap([-\delta, q_n] \times S)$ joining $N(n,q_n)$ to $\Sigma_n \bigcap(\{-\delta\} \times S)$ which intersects $\Sigma_n \bigcap(\{p_n\} \times S)$ first time at γ geometrically once. Since γ is not in $N(n,p_n)$, Γ must intersect $\Sigma_n \bigcap(\{p_n\} \times S)$ at another component γ'. By Lemma 2.5, γ' is connected to $X_n \setminus (-\delta, l_n + \delta) \times S$ through a path in $\Sigma_n \bigcap([p_n, l_n + \delta) \times S)$ which must intersect $\Sigma_n \bigcap(\{q_n\} \times S)$ at a component in $N(n,q_n)$. So there is a path Γ' on Σ_n with two ends in $N(n,q_n)$ which intersects $\Sigma_n \bigcap(\{p_n\} \times S)$ only at γ and γ'. The relative homology class $[\Gamma']$ is non-zero in $H_1(\Sigma_n, N(n,q_n))$ since Γ' intersects γ geometrically once. Now it follows from Lemmas 2.4 and 2.6 that $\#(M(n,p_n) \setminus N(n,p_n))$ has a uniform upper bound. □

Now we pick a Morse function t_n on $\Sigma_n \bigcap(I_n \times S)$ such that $\|t_n - t'_n\|_{C^k} < e^{-l_n}$ (recall that $t'_n = t|_{\Sigma_n}$) and any two different critical points of t_n have different values. We further require that $t_n = t'_n$ at the endpoints of both I_n and I_n^1.

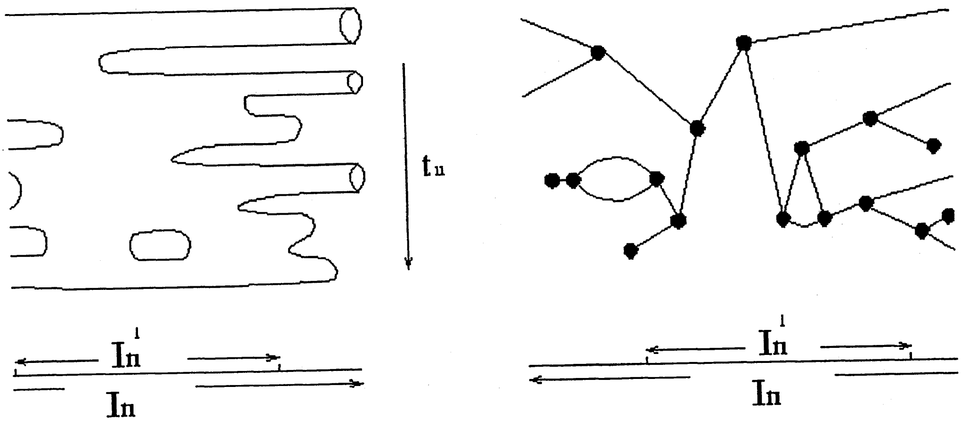
We associate a graph G_n to each curve $\Sigma_n \bigcap(I_n \times S)$ by the Morse function t_n (see Figure 3). The graph G_n has the following properties:

(a) each edge of G_n corresponds to an open annulus in $\Sigma_n \bigcap(I_n \times S)$ on which t_n is regular and each vertex corresponds to a critical point of t_n;

(b) there is a projection $\pi_n : G_n \to I_n$ such that $\pi_n = t_n$ under the correspondence between G_n and $\Sigma_n \bigcap(I_n \times S)$ (π_n is one to one on each edge and maps the vertices to the critical values of t_n);

(c) vertices corresponding to critical points of index 1 have three edges (one or two on the left) and vertices corresponding to critical points of index 0 (or 2) have only one edge which is on the right (or left).

Let G_n^1 be the sub-graph of G_n which corresponds to $\Sigma_n \bigcap (I_n^1 \times S)$, i.e. $G_n^1 = \pi_n^{-1}(I_n^1)$.



the curve Σn (b) the graph Gn

Let Λ_n be the sub-graph of G_n^1 which is the union of all the paths in G_n^1 joining the points corresponding to $M(n, p_n)$ with the points corresponding to $N(n, q_n)$ on G_n^1. Clearly the number of paths in Λ_n has a uniform upper bound since $\#M(n, p_n) \leq M$, $\#N(n, q_n) \leq N$ and the genus of each component of Σ_n has a uniform upper bound.

Lemma 2.8. *There exists a $\delta_1 > 0$ such that for large n, Λ_n has the following property:*

- *there are annuli $A_{n,i}$ in $\Sigma_n \bigcap (I_n \times S)$ such that $\partial A_{n,i}$ are regular level circles of t_n' with regular values x_n, y_n (the same for all i) satisfying $y_n - x_n = \delta_1$;*
- *$N(n, x_n)$ consists of the left boundary components of $A_{n,i}$, in particular, $\#A_{n,i} \leq N$.*

Proof. Let Γ be a path in G_n^1 such that the projection $\pi_n|_\Gamma : \Gamma \to I_n^1$ is surjective. We put the vertices on Γ (critical points of t_n of Morse index 1) into four groups as follows:

1. Group (1) consists of those vertices whose only edge that is not on the path Γ is part of a closed cycle in G_n involving part of Γ. Each of such vertices is associated with a closed cycle in G_n such that the vertex

is the most left or the most right one amongst the vertices on both Γ and the closed cycle. It is easy to see that the number of vertices in Group (1) is at most twice of the number of the associated closed cycles.

2. Group (2) consists of those vertices which start a sub-graph of G_n (never connect to Γ through another vertex) such that there is a closed cycle in this sub-graph.

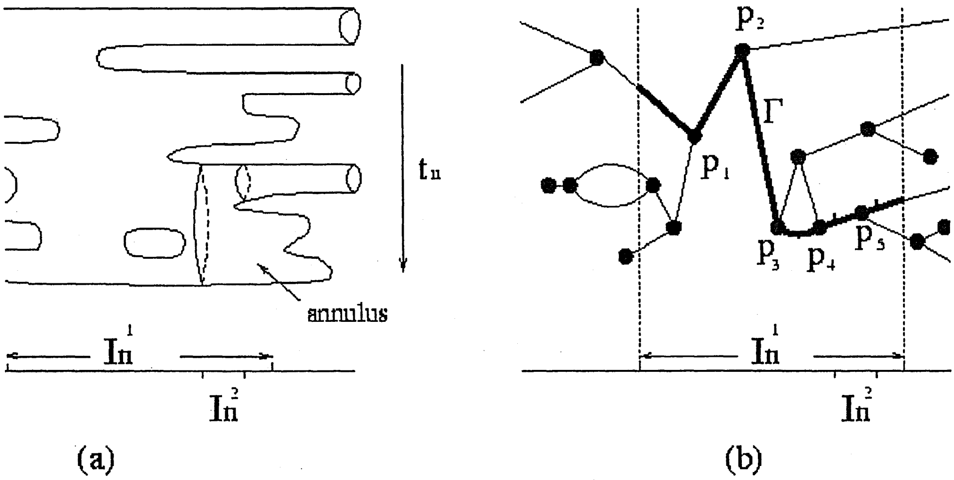
3. Group (3) consists of those vertices which start a tree that goes all the way to one of the two ends of G_n (i.e. $\pi_n^{-1}(\partial I_n)$).

4. Group (4) consists of those vertices which start a tree whose projection under π_n is within I_n.

(In Figure 4, (b), the path Γ is thickened. The vertex p_1 is in Group (2), p_2 in Group (3), p_3, p_4 in Group (1) and p_5 in Group (4).)

Since the genus of each component of Σ_n is uniformly bounded from above, the number of elements in Groups (1), (2) has an upper bound independent of n or the path Γ. The number of elements in Group (3) is also uniformly bounded from above. The reason is that each component of the pseudo-holomorphic curve in $(I_n \setminus intI_n^1) \times S$ with boundary on both of the boundaries of $I_n \times S$ and $I_n^1 \times S$ has h_0-area bounded from below by a constant $c_2 > 0$ (independent of n or Γ) by Lemma 2.2 (Monotonicity). Therefore the number of vertices on Γ which belong to Groups (1), (2) or (3) is uniformly bounded from above.

Since the number of paths in Λ_n is uniformly bounded from above, the total number of vertices on Λ_n which belong to Groups (1), (2) or (3) for some path in Λ_n is also uniformly bounded from above. Let M' be such an upper bound, and set $\delta_1 = (4M'+8)^{-1}\delta_0$. Then there is an interval I_n^2 in I_n^1 whose endpoints are regular values of t_n such that $|I_n^2| = (M'+2)^{-1}|I_n^1| = (2M'+4)^{-1}\delta_0 = 2\delta_1$ and I_n^2 does not contain $\pi_n(p)$ for any vertex p on Λ_n which belongs to Groups (1), (2) or (3) for some path in Λ_n (but I_n^2 may contain $\pi_n(p)$ for some p in Group (4)). Now look at $\pi_n^{-1}(\partial I_n^2) \bigcap \Lambda_n$. It is easy to see that the corresponding circles on $\Sigma_n \bigcap (I_n^1 \times S)$ bound a set of annuli in $\Sigma_n \bigcap (I_n \times S)$, since the tree started by any vertex in Group (4) (for example, the vertex p_5 in Figure 4, (*b*)) corresponds to a topological disc in $\Sigma_n \bigcap (I_n \times S)$. (The case when Λ_n consists of only one path is shown in Figure 4).



Pick regular values x_n, y_n ($x_n < y_n$) of t'_n in I_n^2 such that $|x_n - y_n| = \delta_1 (= \frac{1}{2}|I_n^2|)$ and $\min(dist(x_n, \partial I_n^2), dist(y_n, \partial I_n^2)) > \frac{1}{8}\delta_1$. By the construction of Λ_n, $N(n, x_n)$ must lie in the part of Σ_n that corresponds to Λ_n, and for large n, must lie in the annuli constructed in the previous paragraph (since $\|t_n - t'_n\|_{C^k} < e^{-l_n}$). Each annulus contains at most one component of $N(n, x_n)$ which has non-zero homology in that annulus. Collect all the annuli that contain a component of $N(n, x_n)$. Each of these annuli also contains one component of $N(n, y_n)$ so that it has a sub-annulus which is bounded by a component of $N(n, x_n)$ on the left and a component of $N(n, y_n)$ on the right. These sub-annuli are the claimed annuli $A_{n,i}$. □

Lemma 2.9. *There exist sequences of pseudo-holomorphic maps* $f_{n,i} : [0, L] \times S^1 \to I_n \times S$ *for an* $L \in (0, 1]$ *such that the image of each* $f_{n,i}$ *lies in the annulus* $A_{n,i}$ *(constructed in Lemma 2.8) and* $f_{n,i}(\{0\} \times S^1)$ *is the left boundary component of* $A_{n,i}$*. After applying a translation to* $f_{n,i}$ *if necessary,* $\{f_{n,i}\}$ *has a subsequence which is* C^∞ *convergent on* $[\frac{L}{4}, \frac{3L}{4}] \times S^1$*. Furthermore, the* S*-component of the image of each limiting pseudo-holomorphic map* $f_i : [\frac{L}{4}, \frac{3L}{4}] \times S^1 \to \mathbb{R} \times S$ *lies in a Reeb orbit.*

Proof. First recall a fact from the classical theory of Riemann surfaces. Let A be an annulus equipped with a smooth Riemannian metric μ. The complex structure determined by μ has a uniformization. More precisely, A is holomorphically equivalent to the product $[0, L] \times S^1$ with the standard complex structure and metric (S^1 has unit length), where L is the modulus of A, and is given by

$$L^{-1} = \inf \int_A |du|^2$$

over smooth real valued functions u on A which take the value 0 on one boundary component and 1 on the other (see [P]).

For our purpose here, we need to prove that the modulus $L_{n,i}$ of each annulus $A_{n,i}$ has a lower bound independent of n and i. This goes as follows. Recall that $t'_n = x_n$ on the left boundary component of each $A_{n,i}$ and $t'_n = y_n$ on the right boundary component. Define a smooth function $u_{n,i}$ on $A_{n,i}$ by $u_{n,i} = \delta_1^{-1}(t'_n - x_n)$ $(\delta_1 = y_n - x_n)$. Then we have

$$L_{n,i}^{-1} \leq \int_{A_{n,i}} |du_{n,i}|^2 \leq \delta_1^{-2} \int_{A_{n,i}} |dt'_n|^2 \leq 2\delta_1^{-2} Area_{h_0}(A_{n,i}) \leq \delta_1^{-2}\varepsilon.$$

So $L_{n,i} \geq \varepsilon^{-1}\delta_1^2$. Now it is easy to see that we have sequences of pseudo-holomorphic maps $f_{n,i} : [0, L] \times S^1 \to I_n \times S$ with $L = \min(\varepsilon^{-1}\delta_1^2, 1)$ such that the image of $f_{n,i}$ lies in $A_{n,i}$ and $f_{n,i}(\{0\} \times S^1)$ is the left boundary component of $A_{n,i}$. Equip $I_n \times S$ with the hermitian metric h_0. Then the energy $E(f_{n,i})$ is bounded from above by $\frac{1}{2}\varepsilon$. By Lemma 2.1, the C^0 norm of $df_{n,i}$ over $[\frac{L}{4}, \frac{3L}{4}] \times S^1$ is uniformly bounded from above. By the regularity theory of pseudo-holomorphic maps, after applying a translation to $f_{n,i}$ if necessary, $\{f_{n,i}\}$ has a subsequence which is C^∞ convergent on $[\frac{L}{4}, \frac{3L}{4}] \times S^1$. Let $f_i : [\frac{L}{4}, \frac{3L}{4}] \times S^1 \to \mathbb{R} \times S$ denote the limit (here $\mathbb{R} \times S$ is equipped with the almost complex structure J_0, see Step 1). Then the S-component of the image of f_i lies in a Reeb orbit. This is because

$$\int_{[\frac{L}{4}, \frac{3L}{4}] \times S^1} f_{n,i}^*(d\alpha) \leq e^\delta Area_{h_n}(A_{n,i}) \leq e^\delta l_n^{-1}\varepsilon_0\varepsilon$$

which goes to 0 as $n \to \infty$. □

Recall that $N(n, x_n) = \{f_{n,i}(\{0\} \times S)\}$, which is Poincare dual to $c_1(\xi)$, is oriented by $-\alpha$, where α is the contact form on S and $\xi = \{\alpha = 0\}$ is the contact structure. We orient $\{\frac{L}{2}\} \times S^1$ as the negative of the oriented boundary component of oriented surface $[0, \frac{L}{2}] \times S^1$ so that $\bigcup_i f_{n,i}(\{\frac{L}{2}\} \times S^1)$ is homologous to $N(n, x_n)$. Let $\tilde{f}_i$ be the projection of f_i into S. Then $c_1(\xi)$ is Poincare dual to $\bigcup_i \tilde{f}_i(\{\frac{L}{2}\} \times S^1)$ with each $\tilde{f}_i(\{\frac{L}{2}\} \times S^1)$ lying in a Reeb orbit. We can throw away any $\tilde{f}_i(\{\frac{L}{2}\} \times S^1)$ which is not a closed orbit, since its homology class is zero. So to finish the proof of the first assertion in Theorem 1.2, it suffices to show that for each i, $\int_{\tilde{f}_i(\{\frac{L}{2}\} \times S^1)} \alpha \leq 0$. But this follows from

$$\int_{f_{n,i}(\{\frac{L}{2}\}\times S^1)} (\rho\circ\sigma_n)\alpha = \int_{f_{n,i}(\{0\}\times S^1)} (\rho\circ\sigma_n)\alpha - \int_{f_{n,i}([0,\frac{L}{2}]\times S^1)} d((\rho\circ\sigma_n)\alpha) < 0.$$

(Recall from Step 1 that $\omega_n = d((\rho\circ\sigma_n)\alpha)$ on the neck $[0, l_n]\times S$.)

Step 3: Suppose that S bounds a submanifold $W \subset (X,\omega)$ such that $c_1(W)\neq 0$ and ω is exact on W. To finish the proof we need to show that S has at least one closed characteristic.

Let $\omega = d\lambda$ on W. Then one can extend λ to a neighborhood of $W\subset X$ in which $\omega = d\lambda$ still holds ([W]). We can assume that $(-\delta,\delta)\times S$ is contained in this neighborhood. For each $l_n \geq 1$, let $W_n = W\bigcup([0, l_n+\delta)\times S)$ in (X_n,ω_n). Then there is a 1-form λ_n on W_n such that $\omega_n = d\lambda_n$. The proof is as follows: first let $\lambda_1 = g^*\lambda$ on W_1 where $g : X_1 \to X$ is a diffeomorphism which is given by $(t,x)\to(\ln\rho(t),x)$ on the neck, then let $\lambda_n = g_n^*\lambda_1$ where the diffeomorphism $g_n : X_n \to X_1$ is defined in Step 1. Without loss of generality, we assume that the normal vector field $\frac{\partial}{\partial t}$ is outward with respect to W.

Note that there is at least one component of each Σ_n which intersects both of $W\subset X_n$ and $X_n\backslash W_n$ since $c_1(W)\neq 0$ and ω_n is exact on $W_n\subset X_n$. In particular, $N(n,t_0)$ is not empty for any n and regular value $t_0\in I_n^1$ of $t'_n = t|_{\Sigma_n}$.

We recall from Step 2 that $t'_n = x_n$ on the left boundary component of each annulus $A_{n,i}$ (for all i). By the definition of $N(n,x_n)$, the union of the left boundary components of annuli $A_{n,i}$ (which is $N(n,x_n)$) bounds a pseudo-holomorphic curve Σ_n^1 in W_n which goes through the inside of W (see Figure 2). By Lemma 2.2, the h_n-area of Σ_n^1 is bounded from below by a constant $c_3>0$. So by Stokes' theorem and the fact that $\omega_n = d\lambda_n$ in W_n, we have $\max_{\gamma\in\{\gamma_{n,i}\}}\int_\gamma\lambda_n \geq N^{-1}c_3 > 0$, where $\{\gamma_{n,i}\}$ are the left boundary components of $A_{n,i}$, each of which is oriented by α.

Pick an annulus $A_n\in\{A_{n,i}\}$ whose left boundary component $\gamma_n\in\{\gamma_{n,i}\}$ satisfies

$$\int_{\gamma_n}\lambda_n \geq N^{-1}c_3 > 0.$$

Let $f_n : [0,L]\times S^1 \to I_n\times S$ be the sequence of pseudo-holomorphic maps associated to A_n, and $f : [\frac{L}{4},\frac{3L}{4}]\times S^1\to\mathbb{R}\times S$ be the limit of f_n (see Lemma 2.9). Then the S-component of the image of f must lie in a closed Reeb orbit. The proof is as follows. Suppose it does not lie in a closed orbit. Let $\gamma = \{\frac{L}{2}\}\times S^1$ be the circle in $[0,L]\times S^1$ canonically oriented as a boundary component of $[0,\frac{L}{2}]\times S^1$, and s be the time coordinate of the Reeb flow, then $\int_\gamma \tilde{f}^*\alpha = \int_\gamma \tilde{f}^* ds = \int_\gamma d\tilde{f} = 0$ where $\tilde{f}$ is the projection of f into S. On the other hand, $\int_\gamma f_n^*\lambda_n = \int_\gamma f_n^*((\rho \circ \sigma_n)\alpha)$ for large n, since the homology class of $\tilde{f}(\gamma)$ is zero in S and the difference $\lambda_n - (\rho \circ \sigma_n)\alpha$ is closed on the neck $[0, l_n] \times S$. This leads to a contradiction as follows: on the one hand, $\int_\gamma f_n^*\lambda_n > \int_{\gamma_n} \lambda_n \geq N^{-1}c_3 > 0$, on the other hand, let $T_n(1) = \min(t|_{f_n(\gamma)})$ and $T_n(2) = \max(t|_{f_n(\gamma)})$, and $\tilde{f_n}$ be the S-component of f_n, then

$$\left|\int_\gamma f_n^*((\rho\circ\sigma_n)\alpha)\right| \leq \int_\gamma \left|\max(\rho\circ\sigma_n)'\right| |T_n(2) - T_n(1)| \left|\tilde{f_n}^*\alpha\right| + \left|\int_\gamma \rho\circ\sigma_n(T_n(1))\tilde{f_n}^*\alpha\right|$$

which goes to zero as $n \to \infty$, since $\int_\gamma \tilde{f_n}^*\alpha$ and $(\rho\circ\sigma_n)' = \varepsilon_0 l_n^{-1}$ go to zero and $T_n(2) - T_n(1)$, $|\tilde{f_n}^*\alpha|$ and $\rho \circ \sigma_n(T_n(1))$ remain bounded as $n \to \infty$. Therefore Theorem 1.2 is proved. □

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