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Vanishing theorems for L^2-cohomology on infinite coverings of compact Kähler manifolds and applications in algebraic geometry

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1. Introduction.

In geometry, various notions of hyperbolicity have been introduced, and the appellation "hyperbolic" is intended to signify that a space shares some of the geometric properties that distinguish the standard model $SO(n,1)/SO(n)$ from the Euclidean space. Thus, typical examples are manifolds that have negative curvature in a suitable sense.

The starting point for the present investigation is Gromov's notion of Kähler hyperbolicity [G1]. Let (X, ω) be a compact Kähler manifold with Kähler form ω. It is called Kähler hyperbolic if the lifting $\tilde{\omega}$ of ω to some covering $\tilde{X} \to X$ is of the form $\tilde{\omega} = d\beta$ with a 1-form β that is bounded w.r.t. the metric on $\tilde{X}$ induced by the Kähler form $\tilde{\omega}$. Of course, this condition is satisfied on the Poincaré hyperbolic disk, but not on Euclidean space. More generally, the typical examples of Kähler hyperbolic manifolds are locally Hermitian symmetric spaces of noncompact type. Gromov showed that for a Kähler hyperbolic manifold X with a covering $\tilde{X}$ as above, the L^2 cohomology of $\tilde{X}$ vanishes except in the middle dimension $\dim_C X$. In the case of a Kähler manifold with negatively pinched sectional curvature, this was independently shown by M. Stern [St].

One of the points of the present note is that in contrast to what one might expect from Gromov's work, this vanishing theorem does not distinguish negatively curved spaces from flat ones. More precisely, we wish to introduce a condition that is weaker than Gromov's and includes flat spaces but that still allows one to deduce such vanishing theorems. Thus, in a geometric sense, the line of distinction will be drawn not between negatively and nonpositively curved spaces, but rather between nonpositively and positively curved ones.

In this note we demonstrate a Gromov type vanishing theorem for the cup product between L^2-cohomology on some infinite coverings of X and the so called $\tilde{d}$(linear growth) cohomology classes on X (see the definition below). This theorem has some interesting applications in algebraic geometry. For example, it easily implies the Green–Lazarsfeld vanishing theorem for cohomology groups of generic flat line bundles over X. Also some new theorems are proved. We obtain a generalization of the Green–Lazarsfeld type vanishing theorem in the non abelian case (Theorem 2') using the Busemann function technique due to Philippe Eyssidieux (see below) and theorems of harmonic maps into Bruhat–Tits buildings and Higgs bundles, and we also verify Kollár's conjecture about $\chi(K_X)$ for large $\pi_1(X)$ in the representation case (Theorem 4).

Also, we are able to verify the Hopf-Singer conjecture on the sign of the Euler characteristic of a compact Riemannian manifold of nonpositive curvature in the Kähler case (see Cor. 1).

Remark 1. Recently, Philippe Eyssidieux [E2] also proved a similar type vanishing theorem for L^2-cohomology and derived the Green–Lazarsfeld vanishing theorem independently. In his thesis [E1], (June 1994, Orsay) he proved also this kind of statement for large variations of Hodge structures. In fact, the existence of a Kähler form of $\tilde{d}$(linear growth) in the general case is based partly on his theorem. We thank him for pointing out an error in the first version of our paper. We also thank the referee for bringing some inaccuracies in the paper to our attention.

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2. Synopsis.

In order to prepare our definition, let (X, g) be a compact Riemannian manifold. A closed differential form α on X with coefficients in a metrized local system (V, h) is called $\tilde{d}$(linear growth) if for some covering $\tilde{X} \to X$, the lifting $\tilde{\alpha}$ is of the form $\tilde{d}\beta$ with

$$\|\beta(x)\|_{\tilde{g},\tilde{h}} \leq c \cdot \mathrm{dist}_{\tilde{g}}(x, x_0) + c',$$

where c and c' are constants (that may depend on β and x_0, but not on x), where $\tilde{d}$ is the exterior derivative on $\tilde{X}$, $\tilde{g}$ is the lift of the Riemannian metric g to $\tilde{X}$, and x_0 is an arbitrary point in $\tilde{X}$.

In order to familiarize ourselves with this notion, we present the following example of H. Whitney: For a differential q-form ω on a Riemannian manifold (X, g), one puts:

$$\|\omega\|_\infty := \mathrm{Sup}\{\omega(e_1, \dots, e_q)\},$$

where $e_1, \dots, e_q$ are unit tangent vectors at some $x \in X$.

For a cohomology class $\alpha \in H^q(X)$, one puts

$$\|\alpha\|^* := \mathrm{Inf}_{\omega\in\alpha} \|\omega\|_\infty.$$

This infimum is always achieved on a compact X, i.e. there exists some $\omega_0 \in \alpha$ with $\|\omega_0\|_\infty = \|\alpha\|^*$, but this ω_0 need not to be unique.

Let X be compact, ω_0 be a closed 1-form and let $\pi : \tilde{X} \to X$ be the Galois covering corresponding to the homomorphism $\pi_1(X) \to H_1(X, Z)$. Then the pull back satisfies $\pi^*(\omega_0) = \tilde{d}\tilde{f}$, where $\tilde{f} : \tilde{X} \to R$ is a Lipschitz function with Lipschitz constant $\|\alpha\|^*$. This shows

Proposition 1. *Any closed 1-form on a compact Riemannian manifold is $\tilde{d}$ (linear growth).*

Of course, this may also be verified by integration along geodesic paths, but the preceding construction yields the optimal constant for the growth condition.

Remark. The fundamental group strongly influences the growth of primitive forms of the pulled back forms of degree ≥ 2. Some examples in [G2] show that they may even have exponential growth.

Definition. A compact Kähler manifold (X, ω) is called *Kähler nonelliptic* if ω is $\tilde{d}$(linear growth).

In order to make this notion compatible with morphisms between Kähler manifolds we also introduce the following

Definition. A compact Kähler manifold (X, ω) is called *singular Kähler nonelliptic* if there exists a closed 2-form ω' on X of $(1,1)$-type that is positive definite on a nonempty Zariski open subset $X_0 \subset X$, and ω' is $\tilde{d}$(linear growth).

A compact Kähler manifold (X, ω) is called *semi Kähler nonelliptic* if there exists a nontrivial closed 2-form ω' on X of $(1,1)$-type that is positive semidefinite and $\tilde{d}$(linear growth).

Examples of Kähler nonelliptic manifolds.

1) Kähler hyperbolic implies Kähler nonelliptic.

2) If X has nonpositive sectional curvature, then X is Kähler nonelliptic.

3) If X admits a holomorphic immersion into a torus T, then X with the pull back of the Euclidean metric of T is Kähler nonelliptic.

   If X admits a generically finite holomorphic map into T, then X with the pull back metric is singular Kähler nonelliptic.

   If X admits a holomorphic map into T, then X with the pull back metric is semi Kähler nonelliptic.

4) If some covering of X admits a pluriharmonic map into some symmetric space or Bruhat–Tits building, then the Higgs structure or the multivalued holomorphic 1-forms via this map define a semi Kähler nonelliptic structure on X. Consequently if X has a generically large reductive representation $\rho : \pi_1(X) \to GL_n$, then X is singular Kähler nonelliptic.

Question. All examples above have always to do with the curvature on X, or some pluriharmonic maps on X. It would be very interesting to find such examples only via some properties of the fundamental group. For example, X with $\pi_1(X)$ of subexponential growth (see Mok's recent work).

With these notions, one may extend Gromov's vanishing theorem

Theorem 1. *Let (X, g) be a compact Riemannian manifold, and let $\bar{H}^j_{(2)}(\tilde{X})$ denote the j-th reduced L^2-de Rham cohomology group on $\tilde{X}$ with respect to the metric $\tilde{g}$ and $H^i(X, V)$ denote the i-th de Rham cohomology group on X valued in a metrized local system (V, h). If $\alpha \in H^i(X, V)$ is $\tilde{d}$(linear growth) then for any $\eta \in \bar{H}^j_{(2)}(\tilde{X})$, $\tilde{\alpha} \wedge \eta$ is a L^2-form valued in the metrized local system $(\tilde{V}, \tilde{h})$ and*

$$\tilde{\alpha} \wedge \eta = 0 \quad \text{in} \quad \bar{H}^{i+j}_{(2)}(\tilde{X}, \tilde{V})$$

Remark. Most of the present note extends to noncompact manifolds with complete Riemannian metrics, but here we shall not explore this point.

Let (X, ω) be a compact Kähler manifold, and let $\mathcal{H}^*_{(2)}(\tilde{X})$ denote the space of L^2-harmonic forms on $\tilde{X}$. The above vanishing theorem becomes particularly useful if combined with the Hodge decomposition

$$\mathcal{H}^i_{(2)}(\tilde{X}) = \bigoplus_{p+q=i} \mathcal{H}^{p,q}_{(2)}(\tilde{X}).$$

Our first applications are vanishing theorems: Green–Lazarsfeld type vanishing theorem for L^2-cohomology:

Theorem 2. *Let $\alpha_1, \ldots, \alpha_l$ be holomorphic 1-forms on X which are linearly independent at generic points of X. Suppose that $\tilde{X} \to X$ is a covering such that the liftings $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_l$ are exact. Then*

i) $\mathcal{H}^{p,0}_{(2)}(\tilde{X}) = 0$ *for* $p < l$.

ii) *If* $\mathcal{H}^{l,0}_{(2)}(\tilde{X}) \neq 0$, *then there exists a proper holomorphic map* $\tilde{f} : \tilde{X} \to \tilde{Y}$ *such that* $\mathcal{H}^{l,0}_{(2)}(\tilde{X})$ *factors through* $\tilde{f}$.

Remark. If p, $q > 0$ then an example in [GL] implies in fact that $\mathcal{H}^{p,q}_{(2)}(\tilde{X})$ does not need to vanish for $p + q < l$.

We have the following extension of the Green–Lazarsfeld type vanishing theorem in the non abelian case

Theorem 2'. *Suppose X is a projective variety and (E, θ) is a Higgs bundle coming from a reductive linear representation of $\pi_1(X)$. Then*

$$H^0_{(2)}(\tilde{X}, \Omega^i) = 0 \quad \textit{for} \quad i < \operatorname{rank} \theta.$$

These theorems follows from the Gromov type vanishing theorem:

Theorem 3. i) *Let (X, ω) be a Kähler nonelliptic manifold. Then*

$$\mathcal{H}^{p,q}_{(2)}(\tilde{X}) = 0 \quad \textit{for} \quad p + q \neq \dim_C X.$$

ii) *Suppose (X, ω) is a singular Kähler nonelliptic manifold. Then*

$$\mathcal{H}^{p,0}_{(2)}(\tilde{X}) = 0 \quad \textit{for} \quad p < \dim_C X.$$

Remark. Again the example in [GL] shows that for singular Kähler nonelliptic manifolds $\mathcal{H}^{p,q}_{(2)}(\tilde{X}) = 0$ does not need to vanish for $p, q > 0$ and $p+q < \dim_C X$.

Extending a conjecture of Hopf, it has been conjectured by Singer that the Euler characteristic $\chi(X)$ of all aspherical (or at least all nonpositively curved) compact manifolds X of dimension $2n$ satisfies

either

$$\chi(X) = 0,$$

or

$$\operatorname{sign} \chi(X) = (-1)^n$$

Here, we have a positive answer for the Kähler manifolds of nonpositive sectional curvature:

Corollary 1. (Hopf-Singer conjecture on the sign of the Euler characteristic for Kähler manifolds of nonpositive curvature) *Suppose that X is an n-dimensional compact Kähler manifold of nonpositive sectional curvature. Then all L^2-reduced cohomology groups of $\tilde{X}$ vanish with the possible exception of degree n. In particular, the above conjecture is true.*

In order to obtain vanishing theorems in the sense of Green–Lazarsfeld for compact Kähler manifolds we must combine the preceding with Kazhdan's theorem [K] on the growth of Betti numbers of coverings.

Consider a sequence of finite index subgroups of $\pi_1(X)$

$$\cdots \subset \Gamma_n \subset \Gamma_{n-1} \subset \cdots \subset \Gamma_1 \subset \pi_1(X).$$

This corresponds to a sequence of finite coverings of X

$$\cdots \to \tilde{X}_n \to \tilde{X}_{n-1} \to \cdots \to \tilde{X}_1 \to X.$$

The projective limit of this sequence

$$\varprojlim \tilde{X}_l =: \tilde{X}_\infty \to X$$

is the covering of X with

$$\pi_1(\tilde{X}_\infty) = \bigcap \Gamma_l \subset \pi_1(X).$$

Putting

$$d_l := \text{degree of the covering map} \quad \pi_l : \tilde{X}_l \to X,$$

the *normalized Betti numbers* of $\tilde{X}_l$ are defined as

$$\tilde{b}^i(\tilde{X}_l) := b^i(\tilde{X}_l)/d_l,$$

and the *normalized Hodge numbers* of $\tilde{X}_l$ are defined as

$$\tilde{h}^{p,q}(\tilde{X}_l) = h^{p,q}(\tilde{X}_l)/d_l.$$

Kazhdan's theorem [K] says that if

$$\lim_{l\to\infty} \sup \tilde{h}^{p,q}(\tilde{X}_l) > 0,$$

then

$$h^{p,q}_{(2)}(\tilde{X}_\infty) > 0.$$

On the other hand, some cohomology class on the initial manifold is going to be exact on $\tilde{X}_\infty$. So, sometimes Theorem 1 gives obstructions to the existence of L^2-cohomology on $\tilde{X}_\infty$. See more details in Cor.2.

A further application of our vanishing theorem is concerned with the signature of the holomorphic Euler characteristic of the canonical line bundle of Shafarevich varieties.

Conjecture (Kollár). *If $\pi_1(X)$ is generically large, then*

$$\chi(K_X) \geq 0.$$

Roughly speaking, suppose $\pi_1(X)$ is residually finite. Then $\pi_1(X)$ is called generically large if $\pi_1(X)$ does not factor through any rational surjective map $f : X \to Y$ with $\dim X > \dim Y$. The precise definition of generically large $\pi_1(X)$ has been given in [Ko, Def. 4.6]. Here we verify his conjecture in the representation case:

Theorem 4. *Suppose that $\rho : \pi_1(X) \to GL_n$ is a reductive and generically large representation. Then*

$$\chi(K_X) \geq 0.$$

3. A vanishing theorem for cup products.

We start by reviewing L^2-de Rham cohomology groups of an oriented complete Riemannian manifold (X, g). All the formal properties are the same as in the compact case. Let $*$ denote the Hodge operator with respect to g.

We consider the Hilbert space $A^i_{(2)}$ of the completion of square-integrable i-forms α. Thus, $\alpha \in A^i_{(2)}$ has to satisfy

$$\int_X \alpha \wedge *\alpha < \infty.$$

One defines two subspaces

$$Z^i_{(2)}(X) = \{\alpha \in A^i_{(2)}(X) \mid d\alpha = 0\}$$
$$B^i_{(2)}(X) = d(A^{i-1}_{(2)}(X)) \cap A^i_{(2)}(X).$$

Then $Z^i_{(2)}(X)$ is a closed subspace of $A^i_{(2)}(X)$ and contains the closure $\bar{B}^i_{(2)}(X)$ of $B^i_{(2)}(X)$ in $A^i_{(2)}(X)$. The (reduced) L^2-de Rham cohomology groups of (X, g) are defined by

$$\bar{H}^i_{(2)}(X) = Z^i_{(2)}(X)/\bar{B}^i_{(2)}(X).$$

Let $\triangle = d^*d + dd^*$ be the Laplacian operating on $A^i_{(2)}(X)$,

$$\mathcal{H}^i_{(2)}(X) = \{\alpha \in A^i_{(2)}(X) \mid \triangle(\alpha) = 0\}$$

the space of harmonic L^2-forms, and

$$\mathcal{B}^i_{(2)}(X) := d^* \left(A^{i+1}_{(2)}(X)\right) \cap A^i_{(2)}(X).$$

Theorem (Hodge decomposition [de Rham]). *Let (X, g) be an oriented complete Riemannian manifold. The following orthogonal sum decompositions hold:*

$$\Omega^i_{(2)}(X) = \mathcal{H}^i_{(2)}(X) \oplus \bar{B}^i_{(2)}(X) \oplus \bar{\mathcal{B}}^i_{(2)}(X),$$
$$Z^i_{(2)}(X) = \mathcal{H}^i_{(2)}(X) \oplus \bar{B}^i_{(2)}(X).$$

Now suppose that (X, ω) is a complete Kähler manifold. The Laplacian preserves the Hodge decomposition

$$A^i_{(2)}(X) = \bigoplus_{p+q=i} A^{p,q}_{(2)}(X)$$

Thus, we obtain the decomposition of Hodge type

$$\mathcal{H}^i_{(2)}(X) = \bigoplus_{p+q=i} \mathcal{H}^i_{(2)}(X) \cap A^{p,q}_{(2)}(X) =: \bigoplus_{p+q=i} \mathcal{H}^{p,q}_{(2)}(X).$$

We notice that the space $\mathcal{H}^{i,0}_{(2)}(X)$ is nothing but the space of L^2-holomorphic i-forms $H^0_{(2)}(X,\Omega^i)$ on X.

As the Kähler form ω is parallel with respect to the Riemannian connection on (X,ω), the operator $\omega^k\wedge : A^i_{(2)}(X) \to A^{i+2k}_{(2)}(X)$ sends harmonic forms to harmonic forms. We have the following strong L^2-Lefschetz theorem for complete Kähler manifolds (for example, see [G1]).

Theorem (Lefschetz). *The map* $\omega^k\wedge : \mathcal{H}^i_{(2)}(X) \to \mathcal{H}^{i+2k}_{(2)}(X)$ *is injective for* $i+k \leq \dim_C X$ *and surjective for* $i+k \geq \dim_C X$.

We return to an oriented complete Riemannian manifold (Y,h). Let α be a closed i-differential form on Y which is g-bounded. Recall that α is d(linear growth) if α satifies the following condition:

$$\alpha = d(\beta) \quad \text{and} \quad \|\beta(x)\|_g \leq c \cdot \mathrm{dist}_g(x,x_0) + c'$$

for constants c, c' and some $x_0 \in Y$. Typically, we apply this to a Riemannian covering $(\tilde{X},\tilde{g})$ of an oriented compact Riemannian manifold (X,g) and α the lifting of a closed form on X, that is exact on $\tilde{X}$. We have shown in Prop.1 that any closed 1-form is $\tilde{d}$(linear growth). Suppose now X is a compact Kähler manifold, whose Albanese map

$$\mathrm{alb} : X \to \mathrm{Alb}(X)$$

is an immersion. Then the lifting of the Euclidean Kähler form to the covering $\tilde{X}' \to X$ corresponding to the abelian fundamental group of X is d(linear growth). Hence, the lifting to any covering $\tilde{X} \to \tilde{X}'$ is also d(linear growth) by Prop.1.

As a slight generalization we have

Proposition 2. *Let* (X,g) *be an oriented compact Riemannian manifold of nonpositive sectional curvature. Then the lifting of any closed form to the universal covering* $(\tilde{X},\tilde{g})$ *is* $\tilde{d}$*(linear growth).*

Proof. Let $x_0 \in \tilde{X}$, S a compact submanifold of $\tilde{X}$ of dimension m. Then the $(m+1)$-dimensional volume of the geodesic cone over S with vertex x_0 satisfies

$$\mathrm{Vol}_{m+1}(\mathrm{Cone} \quad S) \leq c_m \quad \max_{x\in S} \quad \mathrm{dist}(x,x_0) \quad \mathrm{Vol}_m(S)$$

for some constant c_m depending only on m. Therefore, the conclusion can be derived as in [G1; 0.1.B]. □

Theorem 1 (Vanishing theorem for cup products). *Let (X, g) be an oriented complete Riemannian manifold, and let α be a closed differential i-form on X valued in a metrized local system (V, h) which is (g, h)-bounded. Suppose that α is d(linear growth).*

Then for any $\eta \in \bar{H}^j_{(2)}(X)$ we have $\alpha \wedge \eta = 0 \in \bar{H}^{i+j}_{(2)}(X, V)$, the L^2-de Rham cohomology valued in the metrized local system (V, h).

Proof. We only prove this for the constant local system case. The argument for the general case is the same. We write $\alpha = d(\beta)$ with $\|\beta\|_g \leq c \cdot \mathrm{dist}_g(x, x_0) + c'$. Let B_r denote the ball in X with center x_0 and radius r with respect to g. We may find a smooth function $\chi_r : X \to R^+$ with $0 \leq \chi_r(x) \leq 1$ for all $x \in X$, $\chi_r(x) = 1$ for $x \in B_r$, $\chi_r(x) = 0$ for $x \in X \setminus B_{2r}$ and $\|d\chi_r(x)\|_g \leq \mathrm{Constant}/\mathrm{dist}_g(x, x_0)$ for $x \in B_{2r} \setminus B_r$. Since $d(\chi_r \beta \wedge \eta)$ has compact support, $d(\chi_r \beta \wedge \eta) \in B^{i+j}_{(2)}(X)$. We want to show that $d(\chi_r \beta \wedge \eta)$ L^2-converges to $\alpha \wedge \eta$ as $r \to \infty$.

We consider

$$d(\chi_r \beta \wedge \eta) = d\chi_r \wedge \beta \wedge \eta + \chi_r \alpha \wedge \eta.$$

Since α is bounded, $\alpha \wedge \eta$ is in L^2, and

$$\int_X \|\alpha \wedge \eta\|^2 = \lim_{r \to \infty} \int_{B_r} \|\alpha \wedge \eta\|^2,$$

and

$$\lim_{r \to \infty} \int_X |\chi_r|^2 \|\alpha \wedge \eta\|^2 = \lim_{r \to \infty} \int_{B_r} \|\alpha \wedge \eta\|^2 + \lim_{r \to \infty} \int_{B_{2r} \setminus B_r} |\chi_r|^2 \|\alpha \wedge \eta\|^2.$$

Since

$$\int_{B_{2r} \setminus B_r} |\chi_r|^2 \|\alpha \wedge \eta\|^2 \leq \int_{B_{2r} \setminus B_r} \|\alpha \wedge \eta\|^2 \to 0$$

for $r \to \infty$ since $\alpha \wedge \eta \in L^2$, we conclude that $\chi_r \alpha \wedge \eta$ converges to $\alpha \wedge \eta$ in L^2 for $r \to \infty$.

Next

$$\int_X \|d\chi_r \wedge \beta \wedge \eta\|^2 \leq \mathrm{const} \int_{B_{2r} \setminus B_r} \|\eta\|^2$$

by the growth properties of $d\chi_r$ and β, and this expression again converges to 0 as $r \to \infty$, since $\eta \in L^2$. The preceding estimates imply that $d(\chi_r \beta \wedge \eta)$ converges to $\alpha \wedge \eta$ in L^2 for $r \to \infty$. □

4. Vanishing theorems for L^2-cohomology.

The next result derives consequences from the existence of holomorphic 1-forms and $\tilde{d}$(linear growth)-semi Kähler forms on Kähler manifolds:

Theorem 2 (Green–Lazarsfeld type vanishing theorem). *Let X be a compact Kähler manifold, and $\alpha_1, \dots, \alpha_l$ be holomorphic 1-forms on X that are linearly independent at generic points of X. Suppose that $\tilde{X} \to X$ is a covering for which the liftings $\tilde{\alpha}_1, \dots, \tilde{\alpha}_l$ are exact. Then*

i) $\mathcal{H}^{p,0}_{(2)}(\tilde{X}) = 0$ *for* $p < l$.

ii) *If* $\mathcal{H}^{l,0}_{(2)}(\tilde{X}) \neq 0$, *then there exists a proper rational map* $\tilde{f} : \tilde{X} \to \tilde{Y}$ *with* $\dim \tilde{Y} = l$ *such that* $\mathcal{H}^{l,0}_{(2)}(\tilde{X})$ *factors through* $\tilde{f}$.

Proof.

i) Let $\eta \in \mathcal{H}^{p,0}_{(2)}(\tilde{X})$. So, η is an L^2-holomorphic p-form on $\tilde{X}$. Since $\tilde{\alpha}_i$ is bounded, $\tilde{\alpha}_i \wedge \eta$ is an L^2- holomorphic $(p+1)$-form, hence in $\mathcal{H}^{p+1,0}_{(2)}(\tilde{X})$. By Theorem 1 we obtain $\tilde{\alpha}_i \wedge \eta = 0$ as products of differential forms for $1 \leq i \leq l$. Since $l > p$ and $\alpha_1, \dots, \alpha_l$ are linearly independent, it follows from elementary linear algebra (c.f. [GL]) that $\eta = 0$.

ii) First we show that all sections from $\mathcal{H}^{l,0}_{(2)}(\tilde{X})$ generate a rank-1 coherent subsheaf $\tilde{L} \subset \Omega^l_{\tilde{X}}$. It follows for the same reason as in i) that any section from $\mathcal{H}^{l,0}_{(2)}(\tilde{X})$ can be written as $f\tilde{\alpha}_1 \wedge \dots \wedge \tilde{\alpha}_l$. It is clear that $\tilde{L}$ is invariant under the action of the deck transformation group Γ, and all sections from $\mathcal{H}^{l,0}_{(2)}(\tilde{X})$ are L^2-holomorphic sections of $\tilde{L}$. It is known classically (c.f. [G1], [K] and [Ko, Chapter 5]) that given an L^2-holomorphic section s one can construct a Poincare series of weight k

$$P(s^k(x)) = \sum_{\gamma \in \Gamma} s^k(\gamma x).$$

If $k \geq 2$ then $P(s^{\otimes k})$ is convergent and defines a Γ-invariant holomorphic section of $\tilde{L}^k$, hence a holomorphic section of the corresponding line bundle L on X. Further, let

$$R_k = \left\langle \prod_i P(s^{k_i}) \,\middle|\, \sum k_i = k \right\rangle$$

be the subspace of $H^0(\tilde{X}, \tilde{L}^k)$ generated by the Poincare series $P(s^{k_i})$. If $k >> 0$, then R_k defines a proper rational map $\tilde{f} : \tilde{X} \to \tilde{Y}$. This statement is due to Gromov [G1], page 285. It is also discussed in detail in [Ko], Page 143-147, 13.9 Theorem, Statement 13.9.3. Since the generic fibre of $\tilde{f}$ is compact and $\tilde{\alpha}_i$ is exact, the pull back of $\tilde{\alpha}_i$ to the generic fibre is zero. So, all $\tilde{\alpha}_i$ factor through $\tilde{f}$. This implies that $\dim \tilde{Y} \geq l$. On the other hand, we consider the quotient map $f : X \to Y$. By the Bogomolov-Sommese Theorem ([EV], Page 58) the Kodaira–dimension of the rank-1 subsheaf $L \subset \Omega^l_X$ cannot be bigger than l. This shows that $\dim Y \leq \kappa(X, L) \leq l$. So, we get $\dim Y = l$. It is easy to see that $\tilde{L}$ descends to the canonical line bundle on $\tilde{Y}$ in the orbifold sense. Hence all elements of $\mathcal{H}^{l,0}_{(2)}(\tilde{X})$ are pulled back from $\tilde{Y}$. □

We next wish to derive an extension of Green–Lazarsfeld's theorem to the non abelian case: Let (V, h) be a metrized local system, such that h is a harmonic metric. This is equivalent to saying that V comes from a reductive linear representation of $\pi_1(X)$ [S1]. The harmonic metric h, equivalently an equivariant pluriharmonic map $u : \tilde{X} \to N$ to the corresponding symmetric space N, gives rise to a new holomorphic structure E on the vector bundle V and the (1,0)–part of the differential $d'u$ is a holomorphic section $\theta \in H^0(X, \mathrm{End}\, E \otimes \Omega^1_X)$, satisfying $\theta \wedge \theta = 0$. The pair (E, θ) is called the Higgs bundle corresponding to V. Suppose V is an abelian local system. Then θ is nothing new but a collection of holomorphic 1-forms corresponding to V. So, we are in the Green–Lazarsfeld situation. And by Prop.1 the pull back of the euclidian metric on the torus via the Albanese map defined by those forms gives rise to a $\tilde{d}$(linear growth)-semi Kähler form on X.

Another extreme case is a local system arising from a variation of Hodge structures. This kind of local system corresponds to holomorphic maps into symmetric Hermitian spaces in the special case and horizontal holomorphic maps into Griffiths period domains (see [GS] for details). The work of Gromov [G1] for the Hermitian case and the recent work of Eyssidieux [E1] for the period domain case show that X does admit a $\tilde{d}$(bounded) semi Kähler form.

In general we have the following proposition:

Proposition 3. *Suppose that there exists a Higgs bundle (E, θ) coming from a reductive local system. Then X admits a nonelliptic semi Kähler form ω'. And the null spaces of ω' coincide with the tangent spaces of the* *foliation defined by* θ.

The proof of Prop.3 is a combination of Eyssidieux's construction for a $\tilde{d}$(bounded) semi Kähler form via Busemann functions on symmetric spaces and pluriharmonic maps into Bruhat–Tits buildings. We postpone it to Section 6.

The following generalization of the Green–Lazarsfeld type vanishing theorem in the non abelian case is derived from a Gromov type vanishing theorem, Theorem 3 below.

Theorem 2'. *Suppose* X *is a projective variety and* (E,θ) *is a Higgs bundle coming from a reductive linear representation of* $\pi_1(X)$. *Then*

$$H^0_{(2)}(\tilde{X},\Omega^i)=0 \quad \textit{for} \quad i<\operatorname{rank}\theta,$$

where $\operatorname{rank}\theta$ *is defined as the rank of the map* $\theta:\operatorname{End}E\to\Omega^1$.

The *proof* follows directly from Prop.3 and iii) in Theorem 3 below. □

Our second type vanishing theorem is applicable for so called *Kähler nonelliptic manifolds* (also including singular ones), i.e. there exists a covering $(\tilde{X},\tilde{\omega})\to(X,\omega)$ such that $\tilde{\omega}=d$(linear growth).

The main examples of such manifolds are: manifolds of nonpositive sectional curvature, generically finite maps into holomorphic tori, harmonic maps with maximal rank at generic points into symmetric spaces and into Bruhat–Tits buildings (see Section 6).

Theorem 3 (Gromov type vanishing theorem). i) *Suppose that* (X,ω) *is a compact Kähler nonelliptic manifold. Then* $\bar{H}^i_{(2)}(\tilde{X})$ *vanishes except possibly for* $i=\dim_C X$.

ii) *Suppose that* (X,ω) *is compact Kähler manifold, and there exists a singular Kähler nonelliptic form* ω' *on* X, *i.e.* ω' *is a closed 2-form on* X *of (1,1)-type that is positive definite on a nonempty Zariski open subset* $X_0\subset X$ *and* ω' *is* $\tilde{d}$*(linear growth). Then* $H^0_{(2)}(\tilde{X},\Omega^i)$ *vanishes except possibly for* $i=\dim_C X$.

iii) *Suppose that the following two conditions hold:*

(a) Let X *be a projective algebraic manifold, and suppose there exists a semi Kähler nonelliptic form* ω' *on* X. *i.e.* ω' *is*

a closed 2-form on X of (1,1)-type that is positive semidefinite on a nonempty Zariski open subset $X_0 \subset X$ and ω' is $\tilde{d}$(linear growth).

(b) $\pi_1(\tilde{X}) \subset \pi_1(X)$ *is a normal subgroup and* $\pi_1(X)/\pi_1(\tilde{X})$ *is residually finite, i.e. there exists a sequence of normal subgroups of finite index*

$$\cdots \subset \Gamma_n \subset \ldots \Gamma_1 \subset \pi_1(X)/\pi_1(\tilde{X})$$

with $\cap_i \Gamma_i = \{1\}$.

Then $H^0_{(2)}(\tilde{X}, \Omega^i)$ *vanishes for* $i < \operatorname{rk} \omega'$, *where* $\operatorname{rk} \omega' := \operatorname{rk} \omega'(x)$ *for generic points* $x \in X$.

Remark. We believe that the assumption that X is a projective algebraic manifold in Condition a) and Condition b) in the statement iii) is only a technical condition, but we do not know how to get rid of it.

Proof of i). Straightforward: Let $\tilde{\omega} = d(\beta)$ such that β has at most linear growth with respect to the pull back Kähler metric. So, for any $k \in N$ we have $\wedge^k \tilde{\omega} = d(\beta \wedge \wedge^{k-1} \tilde{\omega})$ and $\beta \wedge \wedge^{k-1} \tilde{\omega}$ has again at most linear growth. Applying Theorem 1 we obtain $\wedge^k \omega \wedge \eta = 0$ for $k \geq 1$ and $\eta \in \bar{H}^*_{(2)}(\tilde{X})$. Hence, the hard Lefschetz theorem implies that $\bar{H}^i_{(2)}(\tilde{X}) = 0$ for $i \neq \dim X$. i) is proved.

Proof of ii). The proof is also quite standard. Let η be an L^2-holomorphic i-form on $\tilde{X}$ with $i < \dim_C X =: n$. We consider the differential form

$$\eta \wedge \bar{\eta} \wedge \tilde{\omega'}^{n-i}$$

of degree $2n$, where $\bar{\eta}$ is the complex conjugation of η. Since ω'^{n-i} is $\tilde{d}$(linear growth), by Theorem 1 $\eta \wedge \tilde{\omega'}^{n-i}$ is in the L^2-closure $\overline{d(L^2) \cap L^2}$. Since $\bar{\eta}$ is closed and in L^2, $\eta \wedge \bar{\eta} \wedge \tilde{\omega'}^{n-i}$ is in the L^1-closure $\overline{d(L^1) \cap L^1}$. By Gromov's L^1-Lemma ([G1], 1.1.A.) any $2n$-form ζ that is in $d(L^1) \cap L^1$ has

$$\int_{\tilde{X}} \zeta = 0.$$

This implies that any $2n$-form ζ that is in the L^1-closure $\overline{d(L^1) \cap L^1}$ has also

$$\int_{\tilde{X}} \zeta = 0.$$

Applying this fact to $\eta \wedge \bar{\eta} \wedge \tilde{\omega'}^{n-i}$, we obtain

$$\int_{\tilde{X}} \eta \wedge \bar{\eta} \wedge \tilde{\omega'}^{n-i} = 0.$$

On the other hand, let $\tilde{X}_0$ be the open subset of $\tilde{X}$, where $\tilde{\omega}'$ is positive definite. Since $\tilde{X} \setminus \tilde{X}_0$ is a zero measure subset,

$$\int_{\tilde{X}_0} \eta \wedge \bar{\eta} \wedge \tilde{\omega'}^{n-i} = 0.$$

The following argument can be found in [GH], page 110. Let $\phi_1, \ldots, \phi_n$ be the local holomorphic unitary coframe w.r.t. $\tilde{\omega}'$; if

$$\eta = \sum_I \eta_I \phi_I,$$

then

$$\eta \wedge \bar{\eta} = \sum_{I,J} \eta_I \bar{\eta}_J \phi_I \wedge \bar{\phi}_J.$$

Now

$$\tilde{\omega}' = \frac{\sqrt{-1}}{2} \sum \phi_i \wedge \bar{\phi}_i,$$

so

$$\tilde{\omega'}^{n-i} = C_i (n-i)! \sum_{|K|=n-i} \phi_K \wedge \bar{\phi}_K;$$

for suitable $C_i \neq 0$, thus

$$\eta \wedge \bar{\eta} \wedge \tilde{\omega'}^{n-i} = C_i \sum_I |\eta_I|^2 \cdot \Phi,$$

where Φ is the volume form of $\tilde{\omega}'$ on $\tilde{X}_0$. So, the vanishing of the preceding integral implies that all $\eta_I = 0$. ii) is done.

Proof of iii). If $\mathrm{rk}\,\omega' = \dim X$ then (X, ω, ω') is singular Kähler nonelliptic. So, iii) follows from ii). In general, suppose $\mathrm{rk}\,\omega' = r > 0$.

Since under Condition b) $\pi_1(X)/\pi_1(\tilde{X})$ is residually finite, there exists a sequence of normal subgroups of finite indices

$$\{1\} \subset \cdots \subset \Gamma_n \subset \cdots \subset \Gamma_1 \subset \pi_1(X)/\pi_1(\tilde{X})$$

with $\cap_i \Gamma_i = \{1\}$. Letting Γ'_n denote the preimage of $\Gamma_n \subset \pi_1(X)/\pi_1(\tilde{X})$ in $\pi_1(X)$, we obtain a sequence of normal subgroups of finite indices

$$\pi_1(\tilde{X}) \subset \cdots \subset \Gamma'_n \subset \cdots \subset \Gamma'_1 \subset \pi_1(X)$$

with $\cap_n \Gamma'_n = \pi_1(\tilde{X})$. It corresponds to a sequence of coverings of finite degrees

$$\tilde{X} \to \cdots \to \tilde{X}_n \to \cdots \to \tilde{X}_1 \to X$$

with $\pi_1(\tilde{X}_n) = \Gamma'_n$. Let d_n denote the degree of the covering $\tilde{X}_n \to X$ and $h^{i,0}(\tilde{X}_n)$ denote the dimension of the space of holomorphic i-forms on $\tilde{X}_n$. Since $\cap_n \pi_1(\tilde{X}_n) = \pi_1(\tilde{X})$, by Kazhdan's and Lück's theorems [K], [L]

$$\lim_{n\to\infty} \sup \frac{h^{i,0}(\tilde{X}_n)}{d_n} = h^{i,0}_{(2)}(\tilde{X}),$$

where $h^{i,0}_{(2)}(\tilde{X})$ is the von Neumann-dimension (w.r.t. the group $\pi_1(X)/\pi_1(\tilde{X})$) of the space of L^2-holomorphic i-forms on $\tilde{X}$.

Claim 1.

$$\lim_{n\to\infty} \sup \frac{h^{i,0}(\tilde{X}_n)}{d_n} = 0.$$

Proof of Claim 1. Since X is a projective algebraic manifold, there exists a linear system $|D|$ for some very ample divisor D. An element from $|D|$ is a hypersurface in X, and a generic one is a smooth hypersurface. If we take l generic smooth hypersurfaces $D_1, \dots, D_l$ from $|D|$, then the intersection

$$D_1 \cap \cdots \cap D_l =: Y$$

is a smooth projective submanifold in X of dimension $\dim X - l$. Since Y is the intersection of ample divisors, by the Lefschetz hyperplane theorem for fundamental groups, the homomorphism $i_* : \pi_1(Y) \to \pi_1(X)$ is surjective.

For $l = \dim X - \mathrm{rk}\,\omega'$, we may choose such a Y so that the pull back $i^*(\omega')$ via the inclusion $i : Y \hookrightarrow X$ is positive definite in a nonempty Zariski open subset of Y. So, $i^*(\omega')$ is a singular Kähler nonelliptic form on Y and the pull back $\widetilde{i^*(\omega')}$ on the covering $\tilde{Y}$ is d(linear growth). Applying ii) we get $H^0_{(2)}(\tilde{Y}, \Omega^i) = 0$ for $0 \le i \le \dim Y - 1 = r - 1$.

Now let $\tilde{Y}_n \subset \tilde{X}_n$ denote the preimage of $Y \subset X$, and we consider the sequence of coverings

$$\tilde{Y} \to \cdots \to \tilde{Y}_n \to \cdots \to \tilde{Y}_1 \to Y$$

with $\cap_n \pi_1(\tilde{Y}_n) = \pi_1(\tilde{Y})$ and of the same covering degrees d_n as before. Applying Kazhdan's and Lück's theorems and the vanishing for L^2-holomorphic i-forms on $\tilde{Y}$ we get

$$\lim_{n\to\infty} \sup \frac{h^{i,0}(\tilde{Y}_n)}{d_n} = 0, \quad 0 \le i \le r-1.$$

So, in order to prove Claim 1 it is enough to show that for any n we have

$$h^{i,0}(\tilde{Y}_n) = h^{i,0}(\tilde{X}_n) \quad 0 \le i \le \dim Y - 1 = r-1.$$

Since $\tilde{Y}_n = \tilde{D}_1 \cap \cdots \cap \tilde{D}_l$ is the intersection of the ample divisors $\tilde{D}_1, \ldots, \tilde{D}_l$ on $\tilde{X}_n$, by applying the Lefschetz hyperplane theroem for cohomology groups successively we get the the equality. Claim 1 is proved. Hence, iii) is complete.

Theorem 3 is done. □

Remark. In the preceding argument, we have reduced the situation to finite coverings of X so that the L^2 condition is trivially satisfied. In general, of course, the restriction of an L^2-form to a subvariety need no longer be of class L^2, but if we are dealing with the universal covering of a projective algebraic manifold X, for a given L^2-form $\eta \in H^0_{(2)}(\tilde{\Omega}^i), i < r$, we consider projective subvarieties $j : Z \to X$ of dimension r in generic position, with $j^*\eta \neq 0$ and $rkj^*\omega' = r$. In that case, we have enough flexibility in the choice of Z, by taking a generic pencil and moving the base locus around, to be able to assume by Fubini's theorem that $j^*\eta$ is of class L^2. This remark may be useful for establishing other vanishing theorems in the spirit of iii) of Thm. 3.

Theorem 3 has the following

Corollary 2. *Let (X, ω) be a compact Kähler manifold, and let $u : X \to N$ be a pluriharmonic map into some Riemannian manifold N that is of maximal rank $\dim_{\mathbb{R}} X$ at generic points. Suppose that on the universal cover $\tilde{N}$ of N, there exists a strictly convex function φ with gradient of at most linear growth, and with bounded Hessian. Then $H^0_{(2)}(\tilde{X}, \Omega^i) = 0$ except possibly for $i = \dim_{\mathbb{C}} X$.*

Proof. We consider

$$\omega' := \partial\bar{\partial}(\varphi \circ \tilde{u}),$$

where $\tilde{u} : \tilde{X} \to \tilde{N}$ is the lift of u.

Since the composition of a convex function with a pluriharmonic map is plurisubharmonic, ω' is nonnegative. In fact, it is positive definite at generic points since u is assumed to be generically of maximal rank, and φ is strictly convex. Since φ has gradient of linear growth, and the derivative of u is bounded as X is compact, ω' is d (linear growth). Altogether, (X, ω, ω') is singular Kähler nonelliptic and Theorem 3, ii) applies. □

5. Vanishing theorems in algebraic geometry.

Our first application here is to reprove some vanishing or nonvanishing theorems in algebraic geometry. The general idea is simple. On one hand, we use Kazhdan's theorem [K] to produce some L^2-cohomology class on some infinite covering $\tilde{X}_\infty \to X$ via algebraic cohomology classes on sequences of algebraic coverings that converge to $\tilde{X}$, provided the growth of algebraic cohomology groups is proportional to the growth of degrees of the coverings. On the other hand, some cohomology class on the initial manifold is going to be exact $\tilde{X}$. We may apply this idea to the following interesting problem, the so called generic vanishing theorem for cohomology groups of local systems in algebraic geometry. In this paper we only consider the rank-1 case, which has been studied by Green and Lazarsfeld [GL].

We consider the Picard variety $\mathrm{Pic}^0(X)$ of X. It is the moduli space of rank-1 unitary local systems (flat line bundles) on X. Let $S^p_u \subset \mathrm{Pic}^0(X)$ denote the subset defined by

$$S^p_u = \{L \in \mathrm{Pic}^0(X) \,|\, H^0(X, \Omega^p \otimes L) \neq 0\}.$$

It is well known that S^p_u is a subvariety (for example see [GL])

Corollary 3. (Generic vanishing theorem of Green–Lazarsfeld) *Let X be a compact Kähler manifold. Suppose that X has l holomorphic 1-forms, which are linearly independent at generic points. Then S^p_u is a proper subvariety of* $\mathrm{Pic}^0(X)$, *for $p < l$. (That means also that $H^0(X, \Omega^p \otimes L) = 0$ for a generic $L \in \mathrm{Pic}^0(X)$ and $p < l$.)*

Proof. If the statement were not true, then there would exist some $p < l$ with $H^0(X, \Omega^p \otimes L) \neq 0$ for all $L \in \mathrm{Pic}^0(X)$. The main point here is to construct a sequence

$$\cdots \to X_i \to \cdots \to X_1 \to X$$

of coverings of finite covering degree d_i, such that

$$h^{p,0}(\tilde{X}_i)/d_i \geq \text{constant}.$$

Let T_i denote the subgroup of 2^i-torsion points in $\mathrm{Pic}^0(X)$. (Here in place of 2^i-torsion points one can also take p^i-torsion points for any prime number p.) The sequence of groups

$$\cdots \supset T_i \supset T_{i-1} \supset \cdots \supset \{1\}$$

corresponds to a sequence of abelian coverings

$$\cdots \to X_i \to \cdots \to X_1 \to X$$

such that

$$\mathrm{Gal}(\tilde{X}_i/X) \simeq T_i \quad \text{and} \quad \pi_* \mathcal{O}_{\tilde{X}_i} = \bigoplus_{L \in T_i} L.$$

Denote

$$\Gamma_i = \mathrm{kernel}(\pi_1(X) \to \mathrm{Gal}(\tilde{X}_i/X)),$$

and let $\pi : \tilde{X}_\infty \to X$ be the covering such that $\pi_1(\tilde{X}_\infty) = \cap_i \Gamma_i$. One sees easily that

$$\mathrm{Im}\{\pi_* : H_1(\tilde{X}_\infty, Z) \to H_1(X, Z)\} = 0.$$

This means that the liftings of all holomorphic 1-forms from X to $\tilde{X}$ are exact.

It is straightforward to see that the finite covering map $\pi_i : \tilde{X}_i \to X$ induces an isomorphism

$$H^0(\tilde{X}_i, \Omega^p_{\tilde{X}_i}) \simeq H^0(X, \pi_{i*}\Omega^p_{\tilde{X}_i}) = H^0(X, \Omega^p_X \otimes \pi_{i*}\mathcal{O}_{\tilde{X}_i}) = \bigoplus_{L \in T_i} H^0(X, \Omega^p_X \otimes L)$$

As $h^{p,0}(X, L) \geq 1$ for $L \in T_i$, we obtain

$$h^{p,0}(\tilde{X}_i) = \sum_{L \in T_i} h^{p,0}(X, L) \geq |T_i| = d_i.$$

So, by Kazhdan's theorem ([K], Theorem 2) there exists a non zero L^2-holomorphic p-form on $\tilde{X}$. On the other hand, since the liftings of all holomorphic 1-forms from X to $\tilde{X}$ are exact and l of them are linearly independent at the generic point of X, hence, by Theorem 2 all L_2-holomorphic p-forms on $\tilde{X}_\infty$ for $p < l$ must be vanishing. A contradiction. □

Using the same argument we also reprove a Nakano–type generic vanishing theorem, which is again due to Green-Lazarsfeld. The statement here is weaker than the original one (c.f.[GL]).

Corollary 4. (Nakano-type generic vanishing theorem of Green-Lazarsfeld) *Suppose that X is a compact Kähler manifold that admits an immersion $X \to T$ into a holomorphic torus. Then $h^{p,q}(X, L) = 0$ for generic $L \in \mathrm{Pic}^0(X)$ and $p + q < \dim X$.*

Proof. Let $\omega = \sum_i dz_i \wedge d\bar{z}_i$ be the flat Kähler metric on T. The pull back of ω to X via the immersion is a Kähler metric on X. Further, let $\cdots \to \tilde{X}_i \to \tilde{X}_{i-1} \to \cdots \to \tilde{X}_1 \to X$ be the sequence of coverings constructed in the proof of Cor.3. One checks easily that the lifting $\tilde{\omega}$ is $\tilde{d}$(linear growth). Hence, from Theorem 3 $\bar{H}^i_{(2)}(\tilde{X})$ is zero for $i < \dim X$.

If there were some $p + q < \dim X$ such that $h^{p,q}(X, L) \geq 1$ for all $L \in \mathrm{Pic}^0(X)$, then by applying Kazhdan's theorem as above, we would get a non zero class in $\mathcal{H}^{p,q}_{(2)}(\tilde{X}_\infty)$. A contradiction. □

In fact, Arapura [Ar] and Simpson [S2] have the following description of the subvariety

$$S^i(X) := \{L \in \{1\text{–dim. local systems}| H^1(X, L) \neq 0\};$$

they have shown that $S^i(X)$ is a union of translates of subtori of $C^{*b_1(X)}$. Simpson shows further that these translates of subtori are translations by torsion points. As a consequence, they reproved a theorem of Green–Lazarsfeld, namely that the subvariety

$$S^i_u(X) = \{L \in \mathrm{Pic}^0(X) \,|\, H^i(X, L) \neq 0\,\};$$

is a union of translates of subtori of $\mathrm{Pic}^0(X)$.

Combining those descriptions and the same argument as above, we may also reprove the general form of Green–Lazarsfeld's vanishing theorem. We omit the details.

6. Further examples of singular Kähler nonelliptic manifolds and $\chi(K_X)$ of Shafarevich varieties.

A special and important class of Shafarevich varieties are locally symmetric Hermitian spaces $X = \tilde{X}/\Gamma$. In this case the Shavarevich map is the identity map. Gromov [G1] showed that the invariant Kähler form is $\tilde{d}$(bounded). And using the metric he proved a vanishing theorem for L^2-cohomology except in the middle dimension $\dim_C X$.

An important generalization along this direction was obtained by P. Eyssidieux [E1]. He considered the Griffiths–period map $f : X \to D/\Gamma$ corresponding to a variation of Hodge–structures. Suppose that f is an immersion (this means that the corresponding representation $\pi_1(X) \to \Gamma$ is large). Then X admits a Kähler hyperbolic metric. We will discuss his argument below.

However, in the most general case there is no hope to get a Kähler hyperbolic metric. A typical example is in the Green–Lazarsfeld vanishing–theorem where we have holomorphic maps into tori. We can only get some Kähler nonelliptic metrics. So, we must replace Kähler hyperbolic metrics by Kähler nonelliptic ones in the general case. We have seen in Section 4 that the existence of this kind of metrics is strong enough to deduce vanishing theorems.

We start with a compact Kähler manifold (X, ω) that admits a reductive representation $\rho : \pi_1(X) \to GL_n(C)$. This means that the Zariski closure $\overline{\rho\pi_1(X)}$ is a reductive algebraic group and decomposes into an almost direct product of a torus G_0 and some almost simple groups G_i, $i \geq 1$,

$$\overline{\rho\pi_1(X)} = G_0 \times \prod_i G_i.$$

By [C] there exists an equivariant pluriharmonic map

$$u : \tilde{X} \to N$$

corresponding to ρ into the symmetric space N of noncompact type with isometry group $GL_n(\mathbb{C})$. Let $T^c_{u(x)}(N)$ denote the complexified tangent space of N. Then by [Sa] the image $d'u(T_x(\tilde{X})) \subset T^c_{u(x)}(N)$ is contained in an abelian subspace $W \subset T^c_{u(x)}(N)$ of maximal dimension. W is decomposed into a direct sum of the nilpotent and semisimple subspaces

$$W = W_n + W_s.$$

The semisimple part W_s is just the complexified tangent space of a flat $A \subset N$ of maximal dimension passing through $u(x)$.

The main idea here is that we shall construct a Kähler form as a sum of a Kähler form that is non degenerate in the W_n-direction, and a Kähler form that is non degenerate in the W_s-direction.

For the nilpotent part, we have the argument of Eyssidieux [E1]:

We consider the equivariant pluriharmonic map

$$u : \tilde{X} \to N$$

into the symmetric space corresponding to ρ, and we would like to utilize u in order to construct a semi Kähler form on $\tilde{X}$ that is $\tilde{d}$ (linear growth). We start with the construction of Eyssidieux [E1]. The idea is to pull back a suitable convex function from N, and the natural starting point here is the distance function $d(\cdot, p)$ from some point $p \in N$ (or rather $\sqrt{1+d(p,\cdot)^2}$, in order to have a smooth function). The problem is that the Hessian of $d(\cdot, p)$ does not have a positive lower bound in those directions that correspond to flats in N. If the map is holomorphic, then the image of $T_x\tilde{X}$ under du cannot be contained in the tangent space to a flat, unless $du(x) = 0$, because these flats are totally real. However, there exist so-called singular directions in $T\tilde{X}$ that are contained in more than one maximal flat going through p. Eyssidieux overcomes this problem by considering in place of the distance function from an interior point the normalized distance function from a suitable point q at infinity, the so-called Busemann function of q.

One fixes a point $p \in N$, and q then can be considered as the tangent direction of a geodesic ray $\gamma(t)$ starting at p. The Busemann function $\varphi_q : N \to \mathbb{R}$ then is defined as

$$\varphi_q(x) := \lim_{t\to\infty} (d(x, \gamma(t)) - d(p, \gamma(t))).$$

By a proper choice of q, each tangent vector in N is contained in a unique maximal flat going through q. Consequently, pulling back the Hessian of such a Busemann function under a holomorphic map yields the desired form on $\tilde{X}$.

However, if the map is only pluriharmonic instead of holomorphic, then $\partial\bar{\partial}(\varphi_q \circ u)$ does not necessarily have a positive lower bound anymore as in the third inequality in Prop. 4.5.1 of [E1]. It is still nonnegative, because the composition of a convex function with a pluriharmonic map is plurisubharmonic, and it is not identically zero, because even if the image of u should be contained in a flat, we may choose q not contained in this flat. (One might also take sums of Busemann functions.) Since we have a stronger vanishing theorem at our disposal (see Thm. 1 and Thm. 3 (ii) and Cor. 2) than [E1], we may get by below with those weaker properties.

We return to the general case.

Let $\rho_i : \pi_1(X) \to G_i$ denote the representation induced by the i-th projection. It is clear that for the abelian representation $\rho_0 : \pi_1(X) \to G_0$, we have a factor map

$$\pi_1(X) \to C^{*b_1(X)} \to G_0,$$

where $\rho_0' : \pi_1(X) \to C^{*b_1(X)}$ is the standard discrete representation. By enlarging the representation we may simply assume that $\rho_0 = \rho_0'$. This makes the new representation even bigger.

Now we consider $\rho_i : \pi_1(X) \to G_i$, $i \geq 1$, a Zariski dense representation into an almost simple algebraic group G_i, $1 \leq i$.

We now summarize some general properties of such representations for which details and references can be found in [JZ2] Page 499 and [Z] Pages 142-143. Since $\pi_1(X)$ is finitely presented, the moduli space of representations of $\pi_1(X) \to G_i$ is defined over some number field.

If ρ_i is rigid, then ρ_i is defined over some number field K (see [S1] for details). By Simpson's theorem on VHS [S1] ρ_i is a complex component of a Q-variation of Hodge–structures τ_i. We have two possibilities for this kind of representations.

1) ρ_i is bounded w.r.t. every prime ideal $p \subset \mathcal{O}_K$, the ring of algebraic integers of K. By taking the diagonal embedding into the direct product of different field embeddings $K \hookrightarrow C$, we see that τ_i is a complex component of a Z-variation of Hodge–structures (see [Z]). By enlarging the representation we may assume that τ_i itself is a Z-VHS.

2) ρ_i is unbounded w.r.t. some prime ideal $p \subset \mathcal{O}_K$.

If ρ_i is nonrigid, then ρ_i can be deformed to a $\rho_{i,t}$ such that $\rho_{i,t}$ is defined over some number field L and is unbounded w.r.t. a prime ideal $p \subset \mathcal{O}_L$.

Summing up the above discussion we obtain a new representation

$$\rho' := \rho_0 \times \prod_{i \geq 1} \tau_i \times \prod_{i \geq 1} \rho_{i,t} : \pi_1(X) \to C^{*b_1(X)} \times \prod_{i \geq 1} H_i \times \prod_{i \geq 1} G_i,$$

such that

$$\rho_0 : \pi_1(X) \to C^{*b_1(X)}$$

is the standard representation,

$$\tau_i : \pi_1(X) \to H_i, \, i \geq 1,$$

is a Z-VHS and

$$\rho_i, t : \pi_1(X) \to G_i(K_p), \, i \geq 1$$

is a Zariski dense and p-adically unbounded representation into an almost simple p-adic algebraic group $G_i(K_p)$ over some p-adic number field K_p.

Definition. (see [Ko, Chapter 4.1] for the general case)
Let $\tau : \pi_1(X) \to GL_n$ be a representation, and $\tilde{X}_\tau \to X$ be the covering such that $\pi_1(\tilde{X}_\tau) = \mathrm{Ker}\,\tau$. We call τ large if all compact and positive dimensional subvarieties in $\tilde{X}_\tau$ are contained in a proper subvariety of $\tilde{X}_\tau$.

Campana and Kollár ([C], [Ko, Chapter 3]) have proven that if τ is not large then there exists a surjective rational map, the so called Shafarevich map

$$sh_\tau : X \to Sh(X)$$

such that τ factors through sh_τ.

We come back to our situation. If the original ρ is large then ρ' is again large. Consider the family of representations $\{\rho'_t\}_{t\in T}$ such that $\rho'_0 = \rho = \rho_0 \times \Pi_{i\geq 1}\tau_i \times \Pi_{i\geq 1}\rho_1, \rho'_t = \rho_0 \times \Pi_{i\geq 1}\tau_i \times \Pi_{i\geq 1}\rho_{i,t}$ and ρ'_0 is large by the assumption. We claim that ρ'_t is again large for generic $t \in T$. Otherwise we would get a family of the Shafarevich maps $\{sh_t : X \to Sh_t(x)\}_{t\in T}$. With positive fibre dimension for the family $\{\rho'_t\}_{t\in T_0}$ where $T_0 \subset T$ is a Zariski open dense subset. Since the generic fibres $sh_t^{-1}(y), y \in sh_t(X)$ are homotopic to each other and $\rho'_{t\in}|_{sh_t^{-1}(y)} = Id$, $t \in T_0$ we may fix some fibre $sh_{t_0}^{-1}(y)$ such that $\rho_t|_{sh_{t_0}^{-1}(y)} = Id$ for $t \in T_0$. Consider the holomorphic map

$$T_\varphi \xrightarrow{\pi} Hom(\Pi_1(sh_{t_0}^{-1}(y), G)/G$$

$$t \mapsto \rho'_t|_{sh_{t_0}^{-1}(y)}.$$

Since $\pi(T_0) = Id$ and T_0 is Zariski open dense, $\rho'_0 = \pi(0) = Id$. By applying the exact sequence of the homotopy groups

$$\to \pi_1(sh_{t_0}^{-1}(y)) \to \pi_1(X) \to \pi_1(Sh_{t_0}(X)) \to 1,$$

we show that ρ'_0 factors through $sh_{f_0} : X \to sh_{t_0}(X)$. A contradiction to ρ'_0 being large. In general we may use the Shafarevich map

$$sh_\rho : X \to Sh(X).$$

The representation ρ factors through sh_ρ, and is large on $Sh(X)$. So, we may work on $Sh(X)$ to obtain such a representation ρ'.

For the standard representation $\rho_0 : \pi_1(X) \to C^{*b_1(X)}$, let $f_0 : X \to Y_0$ be the Stein-factorisation of the Albanese map of X. Then ρ_0 factors through f_0 up to some etale covering of X.

Similarly, for the Z-VHS τ_i we consider the corresponding Griffiths period map

$$\tilde{g}_i : \tilde{X} \to D_i.$$

Since the action of $\tau_i\pi_1(X)$ on the period domain D_i is discrete, we obtain a quotient map (take again the Stein-factorisation)

$$g_i : X \to Z_i \subset D_i/\tau_i\pi_1(X).$$

As before, τ_i factors through g_i up to some etale covering of X.

Finally, for the p-unbounded Zariski dense representation $\rho_{i,t} : \pi_1(X) \to G(K_p)$ one may construct an equivariant pluriharmonic map

$$u_i : \tilde{X} \to \triangle_i$$

into the Bruhat–Tits building of $G(K_p)$ (see [GSch]). Since $\rho_{i,t}$ is p-unbounded, u_i is not constant. Considering the complexified differential

$$du_i^c = (du_i^c)^{1,0} + (du_i^c)^{0,1},$$

the $(1,0)$-part $(du_i^c)^{1,0}$ is a collection of holomorphic 1-forms on $\tilde{X}$. By Theorem 2 in [JZ1] these 1-forms define a surjective morphism $f_i : X \to Y_i$ of connected fibres, such that dim Y is equal to the dimension of the linear space spanned by $(du^c)^{1,0}$ at the generic points, and $\rho_{i,t}$ factors through f_i.

Taking the product of all the preceding morphisms

$$h := f_0 \times \prod_{i\geq 1} g_i \times \prod_{i\geq 1} f_i : X \to Y_0 \times \prod_{i\geq 1} Z_i \times \prod_{i\geq 1} Y_i,$$

ρ' factors through h. Now we want to construct a positive semidefinite Kähler metric ω' on X, such that ω' is $\tilde{d}$(linear growth) and the null spaces of ω' coincide with the tangent spaces of the fibres of h.

Let $\beta_1, \ldots, \beta_i, \ldots$ be the holomorphic 1-forms on X defining the map f_0 and $\partial\bar{\partial}\phi_1, \ldots, \partial\bar{\partial}\phi_i, \ldots$ be the Hessian of Busemann functions on the symmetric spaces corresponding to the VHS $g_1, \ldots, g_i, \ldots$ as explained above. Then from Prop.1 and the Prop. 4.5.1 of Eyssidieux [E1] the sum

$$\sum_i \beta_i \wedge \bar{\beta}_i + \sum_i \partial\bar{\partial}\phi_i$$

is a positive semi definite Kähler form on X and is $\tilde{d}$(linear growth).

Further, let $ds_{\triangle_i}$ be the Euclidean metric on the building and let $(u_i^* ds_{\triangle_i})^{1,1}$ be the $(1,1)$-part of the pull back metric. Then it is positive semidefinite, and Kähler, since u_i is pluriharmonic. The null spaces of $(u_i^* ds_{\triangle_i})^{1,1}$ at the generic points are vertical tangent vectors along the fibres of f_i.

The semi Kähler metric $\tilde{\omega}'$ on $\tilde{X}$ is now defined as

$$\tilde{\omega}' = \sum_i \beta_i \wedge \bar{\beta}_i + \sum_i \partial\bar{\partial}\phi_i + \sum_i (u_i^* ds_{\triangle_i})^{1,1}.$$

We have to check that $\tilde{\omega}'$ is positive definite on some open subset. From the above discussion we see that the null spaces of $\tilde{\omega}'$ at the generic points are the vertical tangent vectors along fibres of the map h. Since h is generically finite, $\tilde{\omega}'$ is positive definite on a non empty Zariski open subset $\tilde{X}_0 \subset \tilde{X}$. It is clear that $\tilde{\omega}'$ is $\pi_1(X)$-invariant, and we conclude that it descends to a singular Kähler metric ω' on X.

Lemma 1. *ω' is $\tilde{d}$(linear growth).*

Proof. Recalling Prop. 1 and observing that $\partial\bar{\partial}\varphi_i$ is even $\tilde{d}$ (bounded) as the gradient of a Busemann function is bounded, it only needs to be checked that $\sum_i (u_i^* ds_{\triangle_i})^{1,1}$ is $\tilde{d}$(linear growth). We start by reviewing the pluriharmonic map u into the building

$$u : \tilde{X} \to \triangle.$$

The general properties of buildings that will be used below can be found in [B]. The properties of harmonic maps into buildings and forms on buildings can be found in [GSch] and [JZ1] and [JZ2], Page 491.

The building $\triangle$ is covered by apartments

$$\triangle = \bigcup A.$$

On each apartment A there is a natural choice of a collection of real linear functions $\{x_1, \ldots, x_l\}_A$ (they are essentially the root system on R^r of the group G up to some translations on R^r), such that on the intersection of two apartments A and A' the two collections of differentials coincide

$$\{dx_1, \ldots, dx_l\}_A|_{A\cap A'} = \{dx_1, \ldots, dx_l\}_{A'}|_{A\cap A'},$$

the elements will be permuted by the Weyl group W of G.

Consider now the complexified pull back 1-forms

$$u^{*c}dx_i = \alpha_i + \bar{\alpha}_i.$$

The harmonicity of u implies that α_i, $1 \le i \le l$ are holomorphic. The sum

$$\sum_{i=1}^{l} \alpha_i \wedge \bar{\alpha}_i$$

is then W-invariant. Hence, they piece together and give rise to a 2–form on $\tilde{X}$, the exterior 2–form corresponding to the semi Kähler form $(u_i^* ds_{\triangle_i})^{1,1}$.

Claim. *$\sum_{i=1}^l \alpha_i \wedge \bar{\alpha}_i$ is $\tilde{d}$(linear growth).*

Proof. It is clear that on the preimage $u^{-1}(A)$ of each apartment A we may write

$$\left(\sum_{i=1}^{l} \alpha_i \wedge \bar{\alpha}_i\right)_{u^{-1}(A)} = \left(\sum_{i=1}^{l} (\alpha_i + \bar{\alpha}_i) \wedge \bar{\alpha}_i\right)_{u^{-1}(A)}$$
$$= d\left(\left(\sum_{i=1}^{l} u^*(x_i) \wedge \bar{\alpha}_i\right)_{u^{-1}(A)}\right).$$

The main point here is that we can piece all of these primitive forms on $u^{-1}(A)$ together to obtain a global primitive form on $\tilde{X}$

$$\sum_{i=1}^{l} \alpha_i \wedge \bar{\alpha}_i = d(\beta).$$

The following discussion of affine coordinates can be found in [B]. Fixing a vertex $p_0 \in \triangle$, then for any point $p \in \triangle$ we may find an apartment A containing p_0 and p (not unique). We choose those linear functions $\{x_1, \ldots, x_l\}_A$ on A that vanish on p_0, and define β restricted on the preimage of A as

$$\beta|_{u^{-1}(A)} := \left(\sum_{i=1}^{l} u^*(x_i) \wedge \bar{\alpha}_i\right)_{u^{-1}(A)}$$

β is well defined on $\tilde{X}$, which can be seen as follows: On the intersection of two apartments A and A' containing p_0 and p the two collections of linear functions coincide

$$\{x_1, \ldots, x_l\}_A|_{A \cap A'} = \{x_1, \ldots, x_l\}_{A'}|_{A \cap A'},$$

since all x_i vanish at p_0, and the Weyl group W just permutes them. So, the above sum does not depend on which A we have chosen. Thus, β is well defined.

We now show that β has linear growth. It is clear that all linear functions x_i have linear growth w.r.t. the metric on $\triangle$. Since u is equivariant and Lipschitz, all u^*x_i are linear growth w.r.t. a fixed chosen Kähler metric $\tilde{\omega}$ pulled back from X.

We still need to show that all $\{\bar{\alpha}_1, \ldots, \bar{\alpha}_l\}$ are bounded w.r.t $\tilde{\omega}$. This is equivalent to showing that the sum of their norms

$$\sum_{i=1}^{l} \|\bar{\alpha}_i\|_{\tilde{\omega}}$$

is bounded. Since this function is well defined, continuous on $\tilde{X}$, and is $\pi_1(X)$-invariant, it is bounded. Thus, we show that β is linear growth. The claim is proved. Hence, Lemma 1 is proved. □

Lemma 1 implies Prop.3 in Section 4. In particular, if ρ is large then ω' is a singular nonelliptic Kähler form on X. Hence, from ii) in Theorem 3 we get

Corollary 5. *Let X be a compact Kähler manifold, suppose that X admits a generically large and reductive linear representation $\rho : \pi_1(X) \to GL_n$. Then*

$$H^0_{(2)}(\tilde{X}, \Omega^i) = 0 \quad for \quad i < \dim_C X.$$

Combining this corollary with Atiyah's L^2-index theorem [A] we verify Kollár's conjecture in the representation case

Theorem 4. *Let X be a compact Kähler manifold, suppose that X admits a generically large and reductive linear representation $\rho : \pi_1(X) \to GL_n$. Then*

$$\chi(K_X) \geq 0.$$

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