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Morse-Bott functions and the Witten Laplacian

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Given a compact Riemannian manifold (N, g), a flat vector bundle V over N, and a Morse-Bott function h, Witten considered the following one-parameter deformation of the differential d in the de Rham complex of V-valued differential forms on N:

$$d(\alpha) : \omega \mapsto e^{-\alpha h} d e^{\alpha h}.$$

This paper studies the asymptotic as $\alpha \to \infty$ of the discrete spectrum of the Witten Laplacian

$$L(\alpha) = d(\alpha)d^*(\alpha) + d^*(\alpha)d(\alpha).$$

Suppose g is a metric on N, associated to a Morse-Bott function h. Suppose M is the critical manifold of h. The main result of the paper states that as $\alpha \to \infty$ the eigenvalues of $L(\alpha)$, which stay bounded, converge to eigenvalues of the Laplacian $\triangle$ on M, twisted by the orientation bundle of the negative directions in the normal bundle to M in N. All the eigenvalues of $\triangle$ are limits of the eigenvalues of $L(\alpha)$. The paper provides the estimates on the rate of convergence as $\alpha \to \infty$ of the bounded eigenvalues of $L(\alpha)$. The main idea of the proof is to use the adiabatic limit technique of Mazzeo-Melrose and Forman to analyze the spectrum of the Witten Laplacian on the tubular neighborhood of M .
As an application a new Hodge theoretic proof of the Thom isomorphism and of the degenerate inequalities of Morse is given.

0. Introduction.

Summary. Suppose N is a smooth compact manifold without boundary. Let $h : N \to \mathbb{R}$ be a smooth function. We assume that critical points of h form a (disconnected) submanifold M of N and that the Hessian D^2h of h

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is a non-degenerate quadratic form on the normal bundle to M in N. In this case the function h is called a *Morse-Bott* function.

In [Wi1] E. Witten considered the following one-parameter deformation of the differential in the de Rham complex of N:

$$d(\alpha) : \omega \mapsto e^{-\alpha h} d e^{\alpha h} \omega = d\omega + \alpha dh \wedge \omega, \quad \alpha \in \mathbb{R}_{(\geq)}, \ \omega \in \Omega^\bullet(N). \tag{0.1}$$

In this paper we study the asymptotic as $\alpha \to \infty$ of the (discrete) spectrum of the *Witten Laplacian*

$$L(\alpha) = d(\alpha)d^*(\alpha) + d^*(\alpha)d(\alpha). \tag{0.2}$$

Here $d^*(\alpha)$ denotes the operator adjoint to $d(\alpha)$ with respect to a fixed Riemannian metric g on N.

Suppose g is a metric on N, associated to a Morse-Bott function h (see Section 1). Then the spectrum of $L(\alpha)$ consists of the eigenvalues which stay bounded as $\alpha \to \infty$, and the large eigenvalues which grow faster than $C\alpha$ for some constant $C > 0$. The main result of the paper is Theorem 8.6 which states that the bounded eigenvalues of $L(\alpha)$ approach all eigenvalues of the (twisted) Laplacian $\triangle$ on M as $\alpha \to \infty$. The theorem also includes the estimates on the rate of convergence of the bounded eigenvalues of $L(\alpha)$.

The main idea of the proof is to use the adiabatic limit technique of Mazzeo-Melrose and Forman ([Ma-Me], [Fo]) to analyze the spectrum and the eigenspaces of the Witten Laplacian on the tubular neighborhood of the critical submanifold M of h. This analysis is accomplished in sections 2 through 7.

In Section 6 we give a new Hodge theoretic proof of the Thom isomorphism. In Section 9 we use our results about the spectrum of $L(\alpha)$ to prove the degenerate inequalities of Morse.

We now give a more detailed description of the results of the paper.

A localization of the Witten Laplacian to a neighborhood of the critical submanifold. For a Morse-Bott function $h : N \to \mathbb{R}$ let $M_1, ..., M_\Lambda$ denote the connected components of the critical submanifold M of h. According to the Generalized Morse Lemma (Lemma 8.3) each M_i has a *tubular neighborhood* E_i, which is diffeomorphic to the normal bundle to M_i in N. An easy calculation shows that

$$L(\alpha) = \Box + \alpha^2 |dh|^2 + \alpha A, \tag{0.3}$$

where $\Box$ is the usual Laplacian associated with the metric g and A is a bounded zeroth order operator. As $\alpha \to \infty$ the term $\alpha^2|dh|^2$ becomes very large, except in the neighborhood of the critical submanifolds M_i, where $dh = 0$. Therefore, the eigenforms of $L(\alpha)$ corresponding to bounded eigenvalues are, for large α, concentrate near the critical submanifolds of h.

In Section 8 (Theorem 8.5) we show that the asymptotics of the bounded eigenvalues of $L(\alpha)$ can be calculated by restricting $L(\alpha)$ to a tubular neighborhood of M. We call this restriction $\square(\alpha)$. Thus we are led to study the spectrum of $\square(\alpha)$ on each $E_i \to M_i$.

If for some i the manifold M_i is a point, then the tubular neighborhood around M_i is diffeomorphic to $\mathbb{R}^{\dim N}$. Then the study of $\square(\alpha)$, restricted to such neighborhood, reduces to the study of the standard harmonic oscillator.

Zero eigenvalues of the Witten Laplacian on the tubular neighborhood. We begin our study of the spectrum of $\square(\alpha)$ with the examination of zero eigenvalues. Let $E \to M$ be a tubular neighborhood of a single connected component M ($\dim M = m > 0$) of the critical submanifold of h. By a slight abuse of notation we denote by the same letter h the pull-back of h from N to E under the diffeomorphism between E and the normal bundle to M in N. Since the Hessian of h is non-degenerate, the bundle E splits into the Whitney sum of two subbundles E^+ and E^-, such that the Hessian is strictly positive on E^+ and strictly negative on E^-. The dimension n^- of the bundle E^- is called *the Morse index of* M (as a critical submanifold of h). Moreover,

$$h(y) = |y^+|^2 - |y^-|^2, \tag{0.4}$$

where $y = (y^+, y^-) \in E^+ \oplus E^-$ is the coordinate in the fiber.

We study the spectrum of $\square(\alpha)$ on E in a more general situation. Namely, we assume in addition that we are given a flat vector bundle $V \to E$. Then $\square(\alpha)$ is defined on the space $\Omega^\bullet(E, V)$ of smooth differential forms on E with values in V.

In sections 1 through 5 we study the kernel of $\square(\alpha)$. Since the square of $d(\alpha)$ is zero, for each α we can define the de Rham cohomology $H^\bullet(E, V, \alpha)$, associated to a differential complex $(\Omega^\bullet(E, V, \alpha), d(\alpha))$. In Section 1 we show that the spectrum of $\square(\alpha)$ is discrete. Then by Hodge theory for $d(\alpha)$,

$$H^p(E, V, \alpha) \cong \ker \square^p(\alpha), \quad p = 0, 1, \ldots \dim E. \tag{0.5}$$

As an essential step in his analytic proof of Morse-Bott inequalities, J. M. Bismut proved the following theorem about $H^\bullet(E, \alpha)$ (without considering an additional bundle V):

Theorem A ([Bis, Section 2(h)]). *For all large enough α and any p*

$$\dim H^p(E,\alpha) = \dim H^{p-n^-}(M, o(E^-)), \tag{0.6}$$

where $o(E^-)$ denotes the orientation bundle of E^-.

In his proof Bismut uses the existence of a Thom form on E^- and the retraction of E^+ on M to construct local isomorphisms between $H^p(E_{|U},\alpha)$ and $H^{m-n^-}(U)$ over open sets U. Then the Mayer-Vietoris argument finishes the proof.

M. Braverman and M. Farber in [Bra-Far, Section 3] proved Theorem B, which is the generalization of Theorem A, where in addition we have a flat vector bundle $V \to E$.

Theorem B (Theorem 5.3). *For all large enough α and any p*

$$\dim H^p(E,V,\alpha) = \dim H^{p-n^-}(M, V\otimes o(E^-)). \tag{0.7}$$

Their proof is different from the proof of Bismut. They used the existence of a Thom form of the bundle E^- to construct homotopy equivalences between the complexes $(\Omega^\bullet(E,V,\alpha), d(\alpha))$ and $(\Omega^\bullet(M, V\otimes o(E^-)), d)$.

In this paper we give a proof of Theorem B based on a new approach. This approach is motivated by the notion of the "adiabatic limit", introduced in this sort of mathematical context by Witten in [Wi2]. Moreover, we use our method to study the eigenvalues of the Witten deformation of the Laplacian $\square(\alpha)$ as the metric on E is deformed.

The adiabatic limit. Suppose we chose a metric g_E on E, compatible in the sense of Section 1 with the $(E^+\oplus E^-, h)$-structure on E. We now describe the general set up for our approach. Let A be a smooth distribution of k-planes, $A\subset TE$. Let B be the orthogonal complement of A in TE. Writing $g_E = g_A \oplus g_B$, for $0<\delta\le 1$ we define a 1-parameter family of metrics on E by setting

$$g_\delta = g_A \oplus \delta^{-2} g_B. \tag{0.8}$$

In addition, let $V\to E$ be a flat vector bundle. Then the limit of (E, g_δ) as $\delta\to 0$ is known as the *adiabatic limit.*

The adiabatic limit was introduced in this form by Witten in [Wi2]. He considered the distribution A consisting of the vertical vectors of a fibration

$$\mathcal{F} \hookrightarrow E \to M, \tag{0.9}$$

where $\mathcal{F}$ is *compact*, $M = S^1$, and the metric g makes (0.9) a Riemannian submersion. Witten investigated the limit of the eta-invariant of E as $\delta \to 0$. We also refer to [Bis-Fr] and [Ch]. In [Bis-Ch] and [Dai] this investigation was extended to general base spaces M.

In [Ma-Me], R. Mazzeo and R. Melrose study the behavior of the space of harmonic forms on a *compact* manifold E for the fibration (0.9) as $\delta \to 0$. They show that modulo a change of coordinates, the space of harmonic p-forms approaches a finite dimensional space, which can be identified from the Taylor series analysis. They use Melrose's calculus of pseudodifferential operators on manifolds with corners to construct a parametrix for $\square_\delta^p$, where $\square_\delta^p$ denotes the Laplacian induced by the metric g_δ acting on p-forms on E with values in V. This parametrix has a uniform extension to the closed interval $[0, 1]$. This implies that in the case of a fibration (0.9) the eigenvalues and eigenvectors have well-defined asymptotics as $\delta \to 0$.

This paper owes much to the ideas and techniques in [Fo]. In [Fo] R. Forman considers a more general situation than in [Ma-Me]. In particular, he does not require that the distribution $A \subset TE$ arises from a fibration. He investigates a spectral sequence for the cohomology of E, associated with A and B. This spectral sequence arises naturally from a Taylor series analysis of the eigenvalues of $\square_\delta^p$ near $\delta = 0$. Moreover, in Section 5 of [Fo] he shows that the leading order asymptotics of the small eigenvalues of $\square_\delta^p$ and the corresponding eigenspaces are determined by the information contained in the spectral sequence.

The setting for the adiabatic limit arises naturally in our situation (see Section 7.1) after we change coordinates in fibers and let $\delta = \alpha^{-1/2}$. However we point out that, as opposed to the setting in [Ma-Me] or in [Fo], the fibers of the fibration $E \to M$ we study are not compact. As a result it is crucial in the analysis of Sections 4 through 7 that for all large enough α the eigenforms of $\square(\alpha)$ have a rapid decay at ∞. The proof of this result (Appendix 2) uses the notion of the *Bismut connection on* E, introduced in [Bis]. This result is one of the main technical difficulties of the paper.

Many results proved in Sections 1 through 7 of this paper are similar to those proved in [Fo], if instead of the standard Laplacian on E one considers its Witten deformation.

The Thom isomorphism. In Section 7 we prove the Thom isomorphism [Bott-Tu, Theorem 7.10] as an application of Theorem B. For $E \to M$ a rank n smooth vector bundle over a compact connected manifold M, we denote as $H_c^p(E, V)$ the compactly supported de Rham cohomology of E with values in V. Then our version of the Thom isomorphism is

Theorem C (Theorem 6.1). *For any p*

$$\dim H_c^p(E,V) = \dim H^{p-n}(M, V\otimes o(E)), \tag{0.10}$$

where $o(E)$ is the orientation bundle of E.

To prove Theorem C we choose function $h(y) = -|y|^2$ as a Morse-Bott function on E. In this case $E = E^-$. Then Theorem C follows from Theorem B and the following equality, which is the main result of Section 6,

$$\dim H_c^p(E,V) = \dim\ker \Box^p(\alpha), \tag{0.11}$$

This equality holds for all large enough α and all p. Equality (0.11) can be considered as a generalization of the following well-known Hodge-theoretic equality:

$$\dim H^p(N,V) = \dim\ker \Box^p, \tag{0.12}$$

where N is compact and $\Box$ is the Laplace-Beltrami operator.

A simple example. Before we proceed further, let us consider a simple example which illustrates our results. In this example we know explicitly the eigenvalues and eigenforms of the deformed Laplacian. Let

$$E = M\times\left(\mathbb{R}^{n^+}\oplus\mathbb{R}^{n^-}\right),$$

and

$$g = g_M \oplus g_{\mathbb{R}^n},$$

where $g_{\mathbb{R}^n}$ is the standard Euclidean metric on $\mathbb{R}^n$. Let $\{y_i\}$ be the coordinates on $\mathbb{R}^n$, then

$$h(y) = \sum_{i=1}^{n^+} y_i^2 - \sum_{i=(n^+)+1}^{n} y_i^2.$$

We find

$$\Box(\alpha) = H(\alpha) + \triangle_M,$$

where $\triangle_M$ is the Laplacian on M and $H(\alpha) = \oplus_{k=1}^n H_k(\alpha)$ is the direct sum of harmonic oscillators. For any $\phi dy_{i_1}\wedge dy_{i_2}\wedge\cdots\wedge dy_{i_p}\in\Omega^p(\mathbb{R}^n)$,

$$\begin{aligned} H_k(\phi dy_{i_1}\wedge dy_{i_2}\wedge\cdots\wedge dy_{i_p}) &= \left(-\frac{d^2\phi}{dy_k^2} + 4\alpha^2 y_k^2\phi\right) dy_{i_1}\wedge dy_{i_2}\wedge\cdots\wedge dy_{i_p} \\ &\quad + \alpha\frac{\partial^2 h}{\partial y_k^2}\phi B_k(dy_{i_1}\wedge dy_{i_2}\wedge\cdots\wedge dy_{i_p}). \end{aligned} \tag{0.13}$$

The operator B_k is a zeroth order operator defined by

$$B_k(dy_{i_1} \wedge dy_{i_2} \wedge \cdots \wedge dy_{i_p}) = \pm dy_{i_1} \wedge dy_{i_2} \wedge \ldots \wedge dy_{i_p},$$

where we have $(+)$ if $k \in \{i_1, i_2, \ldots, i_p\}$ and we have $(-)$ otherwise.

We can compute the eigenvalues and eigenforms of $\square(\alpha)$ by separating variables. Let $\omega \in \ker \square^p(\alpha)$ then $\omega = \gamma \otimes \beta$, where $\gamma \in \ker H_k(\alpha)$ for all k and $\beta \in \ker \triangle_M$. It follows from (0.13) that

$$\gamma = \gamma^{n^-} = e^{-\alpha|y|^2} dy_{(n^+)+1} \wedge \cdots \wedge dy_n. \tag{0.14}$$

Therefore, $\beta \in \ker \triangle^{p-n^-}$. Since $\dim H^p(E, \alpha) = \dim \ker \square^p(\alpha)$ and $\dim H^p(M) = \dim \ker \triangle_M^p$, we have demonstrated (for this example) the following theorem (see Theorem C and Corollary 5.4):

Theorem D. *For any p and all α*

$$\dim H^p(E, \alpha) = \dim H^{p-n^-}(M).$$

We also have

Theorem E. *If $\lambda_i^p(\alpha)$ is an eigenvalue of $\square^p(\alpha)$ which is bounded as $\alpha \to \infty$, then*

$$\lambda_i^p(\alpha) = \mu_i^{(p-n^-)},$$

where $\mu_i^{p-n^-}$ is an eigenvalue of $\triangle_M^{(p-n^-)}$.

This theorem should be compared with Theorem 7.23.

A more detailed description of the content of Sections 1 through 7. In Section 1 we put a natural metric g on E, compatible with the given decomposition $E = E^+ \oplus E^-$. Then we define a one parameter family of deformations $\square(\alpha)$ of the Laplacian and describe the Hodge theory for $d(\alpha)$. We also define $\Omega_s^\bullet(E, V)$, the space of rapidly decreasing or Schwartz forms on E, in terms of the Bismut connection. Finally, we observe that the cohomology $H^\bullet(E, V, \alpha)$ of the complex $(\Omega^\bullet(E, V), d(\alpha))$ is the same as the cohomology $H_s^\bullet(E, V, \alpha)$ of the complex $(\Omega_s^\bullet(E, V), d(\alpha))$.

In Section 2 we consider a manifold E as a vector bundle over E^-. For reasons which will be clear in Section 4, we have to consider the bundle $E \to M$ as a filtration $E \to E^- \to M$. We denote as $A \subset TE$ the set of all vectors tangent to fibers of $E \to E^-$, and B denotes the orthogonal complement of A in TE in the metric g. Then the family of metrics g_δ is defined as in (0.8). We also define a rescaling map ρ_δ, which is an isometry

$$\rho_\delta : (\Omega^p(E,V), g_\delta) \to (\Omega(E,V), g).$$

Fixing $\alpha \geq 0$ large enough,

$$\hat{d}_\delta = \rho_\delta d(\alpha)\rho_\delta^{-1}, \quad \hat{d}_\delta^* = \rho_\delta d_{g_\delta}^*(\alpha)\rho_\delta^{-1}.$$

Then we show that $\dim\ker \Box^p(\alpha) = \dim\ker \hat{\Box}^p(\delta)$, where

$$\hat{\Box}(\delta) = \hat{d}_\delta \hat{d}_\delta^* + \hat{d}_\delta^* \hat{d}_\delta. \tag{0.15}$$

In Sections 3 and 4 we study the behavior of the space of the $\hat{d}_\delta$-harmonic forms on M as $\delta \to 0$.

In Section 3 we define a nested sequence of spaces

$$E_0^p \supseteq E_1^p \supseteq E_2^p \supseteq \ldots$$

by

$$\begin{aligned} E_k^p = \{\omega \in \Omega_s^p(E,V) |\ &\exists \omega_1, \ldots, \omega_{k-1}, \\ &\hat{d}_\delta(\omega + \delta\omega_1 + \ldots + \delta^{k-1}\omega_{k-1}) \in 0(\delta^k), \\ &\hat{d}_\delta^*(\omega + \delta\omega_1 + \ldots + \delta^{k-1}\omega_{k-1}) \in 0(\delta^k)\}, \end{aligned}$$

By explicitly computing the spaces E_1^p and E_2^p, we show that E_2^p is isomorphic to $H_s^p(E^-, V, \alpha)$, which denotes the cohomology of the complex $(\Omega_s^p(E^-, V), d(\alpha))$, where a differential $d(\alpha)$ on $\Omega_s^p(E^-, V)$ is defined by $d(\alpha) = d - \alpha dh^- \wedge$.

In Section 4 we show that the nested sequence define in Section 3 stabilizes at E_2^p:

$$E_2^p = E_3^p = \ldots = E_\infty^p.$$

Then we describe an isomorphism between $H_s^p(E, V, \alpha)$ and E_∞^p. We note that we can prove this fact directly for the fibration $E \to E^-$, but not for the fibration $E \to M$. Together with the results of Section 2, we have an isomorphism

$$H_s^p(E, V, \alpha) \cong H_s^p(E^-, V, \alpha). \tag{0.16}$$

In Section 5 we use the Hodge theoretic $*$-operator as a convenient tool to deduce the following equality

$$\dim H_s^p(E^-, V, \alpha) = \dim H^{m+n^- - p}(M, V \otimes o(E^-)),$$

Together with (0.16) and the Poincare duality on M this equality proves Theorem B of Braverman and Farber. In Section 6 we present a Hodge theoretic version of the de Rham cohomology of E and prove the Thom isomorphism.

In Section 7 we study the asymptotics of the bounded eigenvalues of $\square(\alpha)$ as $\alpha \to \infty$. Observe that if we put

$$\delta = \alpha^{-1/2},$$

then the operators $\square(\alpha)$ and $\delta^{-2}\hat{\square}(\delta)$ will be isospectral (we assume that in the definition of $\hat{\square}(\delta)$ the parameter α equals to one). This implies that if $\lambda_j^p(\alpha)$ denotes the j-th eigenvalues of $\square^p(\alpha)$ and $\lambda_j^p(\delta)$ of $\hat{\square}^p(\delta)$, then for any j $\alpha \geq 0$, and $p = 1, 2, \dots \dim E$, we have

$$\lambda_j^p(\alpha) = \delta^{-2}\lambda_j^p(\delta).$$

To investigate the asymptotic of the small spectrum of $\hat{\square}(\delta)$, which corresponds to the bounded spectrum of $\square(\alpha)$, we use the Taylor analysis of the eigenspaces of the Witten Laplacian in the spirit of Section 5 of [Fo]. The main result of Section 7 is Theorem 7.23, which states that all bounded eigenvalues of $\square(\alpha)$ converge to the eigenvalues of the Laplacian

$$\triangle : \Omega^\bullet(M, V \otimes o(E^-)) \to \Omega^\bullet(M, V \otimes o(E^-)).$$

In addition we show (Section 7.5) that modulo a change of coordinates the eigenspaces which correspond to small eigenvalues of $\hat{\square}(\delta)$ approach fixed spaces as $\delta \to 0$.

The main result. In Section 8 we observe that the bounded eigenvalues of $L(\alpha) : \Omega^\bullet(N, V) \to \Omega^\bullet(N, V)$ can be calculated by restricting $L(\alpha)$ to tubular neighborhoods E_i of connected components M_i of the critical submanifold M and then applying Theorem 7.23. We prove Theorem 8.6, which is the main result of the paper. This theorem states that all the bounded eigenvalues of $L(\alpha)$ on N converge to the eigenvalues of the standard Laplacian $\triangle$ on M, twisted by orientation bundles $o(E^-)$ as $\alpha \to \infty$. Moreover, all the eigenvalues of $\triangle$ are limits of the bounded eigenvalues of $L(\alpha)$. The theorem also contains estimates on the rate of convergence of the eigenvalues of the Witten Laplacian $L(\alpha)$ on N.

The Morse-Bott inequalities. In 1954, R. Bott [Bott1] generalized the Morse inequalities to the case when the critical points of the function h

form a submanifold M, satisfying some conditions of non-degeneration (see Section 8). Let $M = \oplus_{i=1}^{\Lambda} M_i$, where M_i denotes the connected component of M of index n_i^-. For each i, consider the *twisted Poincare polynomial* of M_i

$$P_i^-(t) = \sum t^p \dim H^p(M_i, V_{|M_i} \otimes o(E_i^-)),$$

and the following *Poincare polynomial* of N

$$P(t) = \sum t^p \dim H^p(N, V).$$

Then the Morse-Bott inequalities say that there exists a polynomial $Q(t)$ given by $Q(t) = Q_0 + Q_1 t + \dots$ with with all non-negative coefficients such that

$$\sum_{i=1}^{\Lambda} t^{n_i^-} P_i^-(t) - P(t) = Q(t)(1+t).$$

The idea of applying the Witten deformation to prove the Morse-Bott inequalities was suggested by Witten in [Wi1].

J.-M. Bismut [Bis] introduced a slight modification of the Witten deformation (using two parameters), such that the study of the corresponding family of operators leads to a family of operators on $E \to M$. Bismut applied a probabilistic technique to study the eigenvalues of the deformed Laplacian on $E \to M$ which approach 0.

M. Braverman and M. Farber [Bra-Far] used essentially the same modification of the Witten deformation as Bismut. However, they excluded probability considerations and used instead an explicit estimate of the number of the eigenvalues of the deformed Laplacian approaching 0. They proved the existence of the spectral gap which separates the eigenvalues that approach 0 from the rest of the spectrum. Moreover, Braverman and Farber also proved the twisted degenerate Novikov inequalities.

In [H-S] Helffer and Sjöstrand gave an analytic proof of the Morse-Bott inequalities. Although they also used the ideas of [Wi1], their method is completely different from the method in [Bis] and [Bra-Far].

In our case the existence of spectral gap and the Morse-Bott inequalities easily follow from Theorem 8.6. In particular, the twisted degenerate Novikov inequalities of Braverman and Farber [Bra-Far] also can be easily recovered from this theorem.

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1. The Witten Laplacian.

1.0. Introduction. In this section we put a natural metric on E, compatible with the given decomposition of E into positive and negative bundles E^+ and E^- for a Morse-Bott function h. Then we define the family

$$\Box(\alpha) = d(\alpha)d^*(\alpha) + d^*(\alpha)d(\alpha),$$

of the Witten deformations of the Laplacian and describe its properties. We prove the Hodge decomposition for $\Box(\alpha)$, and we define the space of rapidly decreasing forms.

1.1. Description of the data. The following set up naturally arises when we consider tubular neighborhoods of connected components of the critical submanifold for a Morse-Bott function (see Section 0.2 and Section 8). Let $E = E^+ \oplus E^-$ be the $\mathbb{Z}_2$ graded finite-dimensional vector bundle of rank n, not necessarily orientable, over a compact connected Riemannian manifold (M, g_M) of dimension m. Let $p : E \to M$ be the projection. Dimensions of E^+ and E^- are denoted as n^+ and n^-, where $n = n^+ + n^-$.

Suppose that in addition we are given a complex vector bundle V over E with a choice of a flat connection. Let $\Omega^p(E, V)$ denote the space of smooth differential p-forms on E with values in V. Then $d : \Omega^p(E, V) \to \Omega^{p+1}(E, V)$ is the standard differential on $\Omega^\bullet(E, V)$, corresponding to our choice of a flat connection on V, $d \circ d = 0$.

1.2. The Morse-Bott function on E. We choose Euclidean metrics on fibers of E^+ and E^-. We define the metric on E to be the direct sum of the metrics on E^+ and E^-.

For a vector $y \in E^+ \oplus E^-$ we denote by $|y|$ its Euclidean norm with respect to the metric on E:

$$|y|^2 = |y^+|^2 + |y^-|^2. \tag{1.1}$$

Let $h : E \to \mathbb{R}$ be the function defined on a vector

$$y = (y^+, y^-) \in E^+ \oplus E^-$$

by the formula

$$h(y) = |y^+|^2 - |y^-|^2 = h^+(y) - h^-(y). \tag{1.2}$$

Then $h(y)$ is a Morse-Bott function on E, having M as its critical submanifold. Moreover, the Hessian of h is positive on E^+ and is negative on E^-. Thus the index of M is n^-.

In order to study the Witten Laplacian we need metrics on E and V.

1.3. A Riemannian metric on E. It is convenient to choose a Riemannian metric on E to be compatible with a given Morse-Bott function h in the following sense.

We denote by $T^{\text{ver}}E$ the space of all vertical vectors in TE. For each $y \in E$, we have the canonical identification of E and $T^{\text{ver}}E$ (identification of fibers and vectors tangent to fibers). Via this identification the metric on E induces the metric g_{ver} on $T^{\text{ver}}E$.

We choose Euclidean connection ∇^{E^+} on E^+ and ∇^{E^-} on E^-. Then we define a connection on E as the direct sum of connections on E^+ and on E^-:

$$\nabla^E = \nabla^{E^+} \oplus \nabla^{E^-}.$$

We note that the splitting $E = E^+ \oplus E^-$ is parallel for our choice of ∇^E.

The choice of a connection defines the *horizontal distribution* $T^{\text{hor}}E \subset TE$ (see Section 1 of [Roe]), and thus the splitting of TE into complimentary vertical and horizontal subspaces $TE = T^{\text{ver}}E \oplus T^{\text{hor}}E$. Note that for each $y \in E$ we have an identification

$$p_* : T_y^{\text{hor}} \to T_{p(y)}M.$$

Thus we can lift g_M to the metric g_{hor} on $T^{\text{hor}}E$.

We define the metric g on TE to be

$$g = g_{\text{ver}} \oplus g_{\text{hor}} = g_{\text{ver}} \oplus p^* g_M.$$

1.4. The Bismut connection and a choice of basis on TE. In order to do computations we need to choose a basis and a connection on TE. As it will be explained in Appendix 1, the most computationally convenient choice of a connection is the Bismut connection. The *Bismut connection* $\bar{\nabla}$ on TE [Bis, Section 2] can be defined as a direct sum of two connections ∇^E and ∇^{TM}:

$$\bar{\nabla} = \nabla^E \oplus \nabla^{TM},$$

where ∇^E is the connection on $T^{\text{ver}}E$ (identified with E), and ∇^{TM} is the connection on $T^{\text{hor}}E$ (identified with TM). The properties of this connection are discussed in more detail in Appendix 1.

In order to choose a basis on TE, we first choose a basis on $TM \oplus E$ and then, we lift this basis to TE. Take $x \in M$. Let $\{a_i\}_{i=1,\dots n}$, $\{b_j\}_{j=1,\dots m}$ be orthogonal bases of E_x, T_xM. Let $\{a^i\}_{i=1,\dots n}$, $\{b^j\}_{j=1,\dots m}$ be the corresponding dual bases. Take $y \in E_x$. We can lift $\{a_i\}_{i=1,\dots n}$, $\{b_j\}_{j=1,\dots m}$ to TE. Since there is no risk of confusion we can assume as well that $\{a_i\}_{i=1,\dots n}$ is the basis of A_y and $\{b_j\}_{j=1,\dots m}$ is the basis of B_y.

1.5. A Euclidean metric on V. In order to choose a metric on V, we identify the manifold M with the zero section of E. Let $V_{|M}$ denote the restriction of V on M. Fix an arbitrary Euclidean metric q on $V_{|M}$, compatible with the flat connection on V. The flat connection on V defines a trivialization of V along the fibers of E and, therefore, gives a natural extension of q to an Euclidean metric q_V on V which is flat along the fibers of E.

1.6. The Witten differential and the Witten Laplacian. We define after Witten [Wi1] a one-parameter family of differentials $d(\alpha)$, $\alpha \geq 0$, by the formula

$$d(\alpha) = e^{-\alpha h} d e^{\alpha h} = d + \alpha dh \wedge . \tag{1.3}$$

It is easy to see that $d(\alpha) \circ d(\alpha) = e^{-\alpha h}(d \circ d)e^{\alpha h} = 0$.

The metric g on TE induces the metric on $\Lambda^* T^* E$ and, together with q_V, leads to an L^2-metric on $\Omega^\bullet(E, V)$.

Let $\Omega^\bullet_{(2)}(E, V)$ be the space of square integrable forms on E with values in V. If the bundle $E \to M$ is not orientable then we let $\Omega^\bullet_{(2)}(E, V)$ denote the space of square integrable forms, twisted by the orientation bundle of E.

The Witten Laplacian $\square(\alpha)$ on the bundle E, associated to the metrics g and q_V, is defined by the formula:

$$\square(\alpha) = (d(\alpha) + d^*(\alpha))^2 = d(\alpha)d^*(\alpha) + d^*(\alpha)d(\alpha), \tag{1.4}$$

where $d^*(\alpha)$ denotes the formal adjoint of $d(\alpha)$ with respect to the L^2-metric on $\Omega^\bullet_{(2)}(E, V)$. We denote by $\square^p(\alpha)$ the restriction of $\square(\alpha)$ to the space of p-forms.

1.7 The spectrum of the Witten Laplacian. In this section we formulate several results about the spectrum of $\square(\alpha)$.

First we need to define a self-adjoint extension of the Witten Laplacian We restrict $\square(\alpha)$ to the space $\Omega^\bullet_c(E, V)$ of smooth differential forms on E

with compact support. For any fixed α, $\square(\alpha)$ is a square of a first order symmetric elliptic operator $(d(\alpha) + d^*(\alpha))$.

Since (E, g) is complete, it follows that $\square(\alpha) : \Omega^\bullet_c(E, V) \to \Omega^\bullet_c(E, V)$ is essentially self-adjoint ([Bra, Theorem 1] or [Cher, Section 3]). Thus it has a unique self-adjoint extension [R-S, Chapter viii], which we will denote again as $\square(\alpha)$. Moreover, it follows from [Cher, Section 3] that the powers of $\square(\alpha)$, restricted to $\Omega^\bullet_c(E, V)$, are also essentially self-adjoint.

Theorem 1.1. *For any $\alpha > 0$ the spectrum of $\square^p(\alpha)$ is discrete. In particular,* $\ker \square^p(\alpha)$ *is finite-dimensional. Moreover, eigenforms of $\square(\alpha)$ form a complete basis for $\Omega^\bullet_{(2)}(E, V)$ in the L^2-topology, associated to the metric g on the tangent space TE.*

In order to prove Theorem 1.1 we observe that for any ω from the domain of $\square^p(\alpha)$ there exists a sequence $\{\omega_k\} \subset \Omega^p_s(E, V)$ such that $\|\omega_k - \omega\| \to 0$, and $\|\square^p(\alpha)\omega_k - \square^p(\alpha)\omega\| \to 0$ as $k \to \infty$. Then

$$\begin{aligned}\langle \square^p(\alpha)\omega, \omega\rangle &= \lim_{k\to\infty} \langle \square^p(\alpha)\omega_k, \omega_k\rangle \\ &= \lim_{k\to\infty} \left(\|d(\alpha)\omega_k\|^2 + \|d^*(\alpha)\omega_k\|\right) \geq 0.\end{aligned}$$

In Appendix 1 (Theorem A.1.5) we show that

$$\square(\alpha) = \square + \alpha^2|dh|^2 + \alpha A, \tag{1.5}$$

where $\square$ is the usual Laplacian associated to the metrics g and q_V, and A is an endomorphism of $\Lambda^\bullet(E, V)$. For any fixed $\alpha \geq 0$ zeroth order operator $\alpha^2|dh|^2 + \alpha A$ is symmetric and bounded below by a constant.

Theorem 1.2 (Bueler). *Let $x \to W(x)$ be a continuous map on a complete Riemannian manifold N with the image $W(x)$ a symmetric (zeroth order) operator on $\Lambda^* T^*_x \otimes V$. Assume that*

$$H = \square + W$$

is non-negative and essentially self-adjoint with core $\Omega^\bullet_c(E, V)$. Let $\nu(x)$ be the smallest eigenvalue of $W(x)$ on $\Lambda^* T^*_x \otimes V$. Assume that $\nu(x) \to \infty$, i. e. there exists $x_0 \in N$ such that for each $K > 0$ there exists $R \geq 0$ such that $\nu(x) \geq K$ if $x \in N/B_{x_0}(R)$. Then H has compact resolvent.

1.8. The Hodge theory for the Witten Laplacian. In this section we introduce the complex whose cohomology will be our main interest in the coming sections. We then show that even though the underlying manifold is non-compact, by using the deformation introduced in Section 1.6 this cohomology can be studied via Hodge theory.

We denote by $\Omega^\bullet(E,V,\alpha)$ the space of smooth (C^∞)-square integrable forms ω which have the property that $d(\alpha)\omega \in \Omega^\bullet_{(2)}(E,V)$ and $d(\alpha)^*\omega \in \Omega^\bullet_{(2)}(E,V)$. In sections 1 through 6, we study $H^\bullet(E,V,\alpha)$, the cohomology of the complex

$$0 \to \Omega^0(E,V,\alpha) \to \Omega^1(E,V,\alpha) \to \dots \to \Omega^{m+n}(E,V,\alpha) \to 0,$$

associated to the differential $d(\alpha)$. We have the following Hodge decomposition theorem for $d(\alpha)$.

Theorem 1.3. *For any $\alpha > 0$ we have an orthogonal decomposition*

$$\Omega^\bullet(E,V,\alpha) = \text{image } d(\alpha) \oplus \text{image } d^*(\alpha) \oplus \ker \Box(\alpha). \tag{1.6}$$

Proof. Since by Theorem 1.1 $\dim\ker\Box(\alpha) < \infty$, for any $\omega \in \Omega^\bullet(E,V)$ we have the decomposition $\omega = \beta + h$, where $\beta \in (\ker\Box(\alpha))^\perp$ and $h \in \ker\Box(\alpha)$.

Writing

$$\begin{aligned}\beta &= (d(\alpha)d^*(\alpha) + d^*(\alpha)d(\alpha))(\Box(\alpha))^{-1}\beta\\ &= d(\alpha)(d^*(\alpha)(\Box(\alpha))^{-1}\beta) + d^*(\alpha)(d(\alpha)(\Box(\alpha))^{-1}\beta)\end{aligned}$$

we have $\beta = \beta_1 + \beta_2$, where $\beta_1 \in$ image $d(\alpha)$ and $\beta_2 \in$ image $d^*(\alpha)$. Moreover, since $(d(\alpha))^2 = 0$, β_1 and β_2 are orthogonal. Thus for any $\omega \in \Omega^\bullet(E,V,\alpha)$ we have an orthogonal decomposition $\omega = h + \beta_1 + \beta_2$. $\Box$

We also have the following immediate corollary:

Corollary 1.4.

$$H^\bullet(E,V,\alpha) \cong \ker\Box(\alpha) = \ker d(\alpha) \cap \ker d^*(\alpha).$$

1.9. The space of rapidly decreasing forms. Until now, we have worked with the space of square integrable forms, but it will be convenient to work instead with the space of rapidly or decreasing or Schwartz forms. The purpose of this section is to show that restricting to this smaller space of forms does not change the cohomology of the complex defined in Section 1.7.

It is convenient to define the space of rapidly decreasing forms on E in terms of the Bismut connection on TE. In order to simplify the notation, let

$$\{a_1, \dots, a_n\} \cup \{b_1, \dots, b_m\} = \{e_1 \dots, e_{n+m}\}$$

be the basis, chosen in Section 1.4..

We now introduce the spaces $\Omega^p_s(E, V)$ of *smooth rapidly decreasing or Schwartz p-forms.* We say that $\omega \in \Omega^p_s(E, V)$ if $|y|^l \bar{\nabla}^\kappa \omega \in \Omega^p_{(2)}(E, V)$ for any $l \geq 0$ and any multi-index κ, $|\kappa| = 0, 1, \dots$. Here

$$\bar{\nabla}^\kappa = \bar{\nabla} e_{i_1} \circ \dots \circ \bar{\nabla} e_{i_\kappa}, \ \kappa = \{i_1, \dots, i_\kappa\}.$$

If $E = M \times \mathbb{R}^n$ with the product metric and the standard connection, then $\Omega^p_s(E, V)$ becomes the space of smooth Schwartz p-forms on E.

Since we can express d in terms of the Bismut connection, we conclude (see Theorem A.2.1) that $\Omega^\bullet_s(E, V)$ is invariant under $d(\alpha)$. Thus we can define $H^p_s(E, V, \alpha)$, the p-th cohomology of $(\Omega^p_s(E, V), \ d(\alpha))$.

Theorem 1.5. *For any large enough α and any $\omega \in \Omega^\bullet(E, V, \alpha)$, such that $\Box(\alpha)\omega = \lambda(\alpha)\omega$, we have $\omega \in \Omega^\bullet_s(E, V)$; i. e. the eigenforms of $\Box(\alpha)$ are Shwartz.*

For any $\alpha \geq 0$ we have an induced decomposition of $\Omega^\bullet_s(E, V)$:

$$\Omega^\bullet_s(E, V) = \operatorname{image} d(\alpha) \oplus \operatorname{image} d^*(\alpha) \oplus \ker \Box(\alpha). \tag{1.7}$$

The proof of (1.7) is similar to the proof of Theorem 1.3 if in addition we know (see Remark A.2.7) that if $\omega \in (\ker \Box(\alpha))^\perp \cap \Omega^\bullet_s(E, V)$ then $(\Box(\alpha))^{-1}\omega \in \Omega^\bullet_s(E, V)$.

Corollary 1.6. *For every $\alpha > 0$ $H^\bullet_s(E, V, \alpha)$ is isomorphic to $H^\bullet(E, V, \alpha)$.*

Remark 1.7. M. Shubin in Appendix to [Sh2] proved a theorem about the decay of eigenfunctions of matrix-valued differential operators on $\mathbb{R}^n$ similar to Theorem 1.5.

Remark 1.8. It can be shown that the space $\Omega^\bullet_s$ does not depend on the choice of horizontal distribution in the definition of the Bismut connection.

2. A deformation of the Witten Laplacian.

2.0. Introduction. In this section we fix our parameter $\alpha > 0$ to be large enough to ensure the conclusions of Theorem 1.5. For the sake of simplicity the notation $\hat{d}$ and $\hat{\Box}$ will indicate that the differential and the Witten Laplacian depend on a fixed α. Thus $d(\alpha) = \hat{d}$ and $\Box(\alpha) = \hat{\Box}$. We consider E to be a vector bundle over a non-compact manifold E^-.

In Section 2.1 we introduce for all $0 < \delta \leq 1$ an adiabatic deformation g_δ of metric g by expanding the metric in the directions orthogonal to fibers of $E \to E^-$.

For a fixed $\alpha > 0$ this deformation leads to a corresponding deformation $\hat{\Box}_\delta$ of the Witten Laplacian. To simplify the situation we remove the dependence of the metric on the parameter by introducing in Section 2.3 a new family $\hat{\Box}(\delta)$ of operators. This will be done in such a way that for all $\delta > 0$ the operators $\hat{\Box}_\delta$ and $\hat{\Box}(\delta)$ are isospectral.

In Section 2.2 we conclude that there is a natural bigrading on the space of differential forms on E, which is associated to an orthogonal decomposition of TE into horizontal and vertical vectors. This bigrading leads to a corresponding bigrading of the differential $\hat{d}$.

2.1. A one-parameter deformation of the metric on E. Let $\pi : E \to E^-$ be the projection. The tangent space TE has an orthogonal decomposition

$$TE = A \oplus B$$

where A is the set of all vectors tangent to fibers of $E \to E^-$ and

$$B = T^{\mathrm{hor}}E \oplus \{\text{vectors, tangent to fibers of } E^- \to M\}.$$

Note that

$$g = g_A \oplus g_B,$$

where g_A and g_B are the restrictions of g to A and B.

We define a one-parameter family of metrics on TE by setting

$$g_\delta = g_A \oplus \delta^{-2} g_B. \tag{2.1}$$

2.2. A bigrading on the space of forms. The decomposition $TE = A \oplus B$ leads to the corresponding decomposition of the dual space $T^*E = A^* \oplus B^*$. This decomposition in turn induces a bigrading on $\Omega^p(E,V)$ by

$$\Omega^p(E,V) = \otimes_{i=0}^{p} \Omega^{i,p-i}(E,V), \tag{2.2}$$

where $\Omega^{i,p-i}(E,V)=\Gamma(\Lambda^i A^*\oplus\Lambda^j B^*\oplus V)$.

Similarly, all operators on forms inherit a corresponding decomposition. In particular, the d-operator inherits a bigrading

$$d=d^{2,-1}+d^{1,0}+d^{0,1}+d^{-1,2},$$

where $d^{a,b}:\Omega^{i,j}(E,V)\to\Omega^{i+a,j+b}(E,V)$. Note that $d^{1,0}$ and $d^{0,1}$ are first order differential operators and $d^{2,-1}$ and $d^{-1,2}$ are zeroth order.

It follows from the integrability of A that $d^{2,-1}=0$ (see Corollary A.1.4).

For the operator $\hat{d}=d+\alpha dh\wedge$ we have a similar decomposition

$$\hat{d}=\hat{d}^{1,0}+\hat{d}^{0,1}+\hat{d}^{-1,2}. \tag{2.3}$$

Since $dh_{|T^{\mathrm{hor}}E}=0$ (it can easily be checked in local coordinates) and the metric g is chosen so that $T^{\mathrm{ver}}E^+$ is perpendicular to $T^{\mathrm{ver}}E^-$, we have $dh=dh^+-dh^-$, where $dh^+\wedge=d^{1,0}h^+\wedge$ is a $(1,0)$-operator and $dh^-\wedge=d^{1,0}h^-\wedge$ is a $(0,1)$-operator. Therefore,

$$\hat{d}^{1,0}=d^{1,0}+\alpha dh^+\wedge,\quad \hat{d}^{0,1}=d^{0,1}-\alpha dh^-\wedge,\quad \text{and}\quad \hat{d}^{-1,2}=d^{-1,2}.$$

The identity $(\hat{d}^2)=0$ yields the identities

$$0=(\hat{d}^2)^{2,0}=(\hat{d}^{1,0})^2, \tag{2.4}$$

$$0=(\hat{d}^2)^{1,1}=\hat{d}^{1,0}\hat{d}^{0,1}+\hat{d}^{0,1}\hat{d}^{1,0}, \tag{2.5}$$

$$0=(\hat{d}^2)^{0,2}=(\hat{d}^{0,1})^2+d^{-1,2}\hat{d}^{1,0}+\hat{d}^{1,0}d^{-1,2}, \tag{2.6}$$

$$0=(\hat{d}^2)^{-2,4}=(d^{-1,2})^2. \tag{2.7}$$

2.3. A one-parameter deformation of the Witten Laplacian. For each p and δ we have an induced Laplacian

$$\hat{\Box}^p_\delta=\hat{d}^*_{(g_\delta)}\hat{d}+\hat{d}\hat{d}^*_{(g_\delta)}:\Omega^p_s(E,V)\to\Omega^p_s(E,V),$$

where $\hat{d}^*_{(g_\delta)}$ is the adjoint of $\hat{d}$ with respect to the L^2-metric on $\Lambda^*T^*E\otimes V$ induced by the metrics g_δ and q_V.

The operator $\hat{\Box}^p_\delta$ depends on δ through the metric g_δ, which varies with δ. To simplify the situation we introduce an isometry

$$\rho_\delta:(\Omega^p_s(E,V),g_\delta)\to(\Omega^p_s(E,V),g),$$

where for each $\omega\in\Omega^{i,j}$,

$$\rho_\delta\omega=\delta^j\omega.$$

We define $\hat{\Box}^p(\delta)$ by the formula

$$\hat{\Box}^p(\delta) = \rho_\delta^{-1}\hat{\Box}^p_\delta \rho_\delta. \tag{2.8}$$

Clearly, operators $\hat{\Box}^p_\delta$ and $\hat{\Box}^p(\delta)$ are isospectral. We also observe that $\hat{d}^*_{g_\delta} = \rho_\delta^{-1}\hat{d}^*\rho_\delta$ and hence, for all p and δ

$$\dim\ker\hat{\Box}^p = \dim\ker\hat{\Box}^p_\delta \quad \text{and} \quad \ker\hat{\Box}^p_\delta = \ker\rho_\delta\hat{\Box}^p(\delta)\rho_\delta^{-1}. \tag{2.9}$$

Moreover, we have the following lemma, which is a simple calculation.

Lemma 2.1 ([Fo, Section 1]). *For any p*

$$\hat{\Box}^p(\delta) = \hat{d}_\delta\hat{d}^*_\delta + \hat{d}^*_\delta\hat{d}_\delta,$$

where

$$\hat{d}_\delta = \hat{d}^{1,0} + \delta\hat{d}^{0,1} + \delta^2 d^{-1,2} \quad \text{and} \quad \hat{d}^*_\delta = (\hat{d}^{1,0})^* + \delta(\hat{d}^{0,1})^* + \delta^2(d^{-1,2})^*$$

is the adjoint, where all adjoints are taken with respect to metrics g and q_V.

3. A Taylor analysis of zero eigenspaces and an associated nested sequence of spaces.

3.0. Introduction. In this section we start a Taylor analysis of the kernel of the operator $\hat{\Box}(\delta)$. We observe that by Hodge theory $\omega \in \ker\hat{\Box}(\delta)$ if and only if $\hat{d}_\delta\omega = 0$ and $\hat{d}^*_\delta\omega = 0$. This description of the kernel motivates an introduction in Section 3.1 of the nested sequence of spaces $\{E^p_k\}$ of "approximate solutions" to the equations above. We also define the appropriate differentials $\pi_k d_k \pi_k : E^p_k \to E^p_k$. Those spaces and differentials were first introduced in this form by R. Forman in [Fo]. Similar spaces were studied by Mazzeo and Melrose in [Ma-Me].

In this section we compute the spaces E^p_1 and E^p_2. In particular, in Section 3.2 we find that the restriction of a form from E^p_1 is a $\hat{d}^{0,1}$-harmonic forms in each fiber of $E \to E^-$. Furthermore, there is an isomorphism between E^p_2 and the de Rham cohomology $H^p_s(E^-, V_{|E^-}, \alpha)$ associated to the differential complex $(\Omega^p_s(E^-, V),\ d(\alpha) = d - \alpha dh^-\wedge)$.

3.1. A nested sequence of spaces. For each p we define a nested sequence of spaces

$$E^p_0 \supseteq E^p_1 \supseteq E^p_2 \supseteq \cdots,$$

where $E_0^p = \Omega_s^p(E,V)$, by

$$
\begin{aligned}
E_k^p = \{\omega \in \Omega_s^p(E,V) | \ \exists\, \omega_1, \dots, \omega_{k-1} \text{ with} \\
\hat{d}_\delta(\omega + \delta\omega_1 + \dots + \delta^{k-1}\omega_{k-1}) \in 0(\delta^k), \\
\hat{d}_\delta^*(\omega + \delta\omega_1 + \dots + \delta^{k-1}\omega_{k-1}) \in 0(\delta^k)\}, \ k = 1, 2, \dots.
\end{aligned}
\tag{3.1}
$$

We denote by π_k the orthogonal projection onto E_k^p. We also define an operator $\hat{d}_k$ on E_k^p by setting, for $\omega \in E_k^p$

$$
\hat{d}_k\omega = \lim_{\delta\to 0} \delta^{-k}\hat{d}_\delta(\omega + \delta\omega_1 + \dots + \delta^{k-1}\omega_{k-1}), \ k = 0, 1, 2, \dots.
\tag{3.2}
$$

We observe that generally the map $\hat{d}_k$ depends on ω_i's, however in the next section we are going to see that $\pi_k\hat{d}_k\pi_k$ does not. For example, $\pi_0\hat{d}_0\pi_0\omega = \hat{d}_0\omega = \hat{d}^{1,0}\omega$. In particular, we will show that $(\pi_k\hat{d}_k\pi_k)^2 = 0$ and $\pi_k\hat{d}_k\pi_k = 0$ for $k \geq 2$.

3.2. A computation of E_1^p. By definition

$$
E_1^p = \{\omega \in \Omega_s^p(E,V) | \ \hat{d}_\delta(\omega) \in 0(\delta), \hat{d}_\delta^*(\omega) \in 0(\delta)\}.
$$

Since $\hat{d}_\delta(\omega) = \hat{d}^{1,0}\omega + 0(\delta)$ and $\hat{d}_\delta^*(\omega) = (\hat{d}^{1,0})^*\omega + 0(\delta)$, we have

$$
\begin{aligned}
E_1^p &= \{\omega \in \Omega_s^p(E,V) | \ \hat{d}^{1,0}\omega = 0, (\hat{d}^{1,0})^*\omega = 0\} \\
&= \{\omega \in \Omega_s^p(E,V) | \ \pi_0\hat{d}_0\pi_0\omega = 0, (\pi_0\hat{d}_0\pi_0)^*\omega = 0\}.
\end{aligned}
\tag{3.3}
$$

Let $\omega \in E_1^p$. Since the differential $\hat{d}^{1,0}$ preserves the bigrading, we may assume that $\omega \in \Omega_s^{i,j}(E,V)$ for some i, j, $i+j = p$. Then ω can be considered as a j-form on E^- with the values in an infinite-dimensional bundle

$$
\pi : \Omega_s^i(A,V) \to E^-
$$

(where $\Omega_s^i(A,V)$ denotes the bundle whose fiber at $(x,y^-) \in E^-$ is $\Omega_s^i(\pi^{-1}(x,y),V)$).

We write ω as

$$
\omega = \gamma \otimes \pi^*\beta,
$$

where $\gamma \in \Omega_s^i(A, V_{|A})$, $\beta \in \Omega_s^j(E^-, V_{|E^-})$. Then we have

$$
\hat{d}^{1,0}\omega = (\hat{d}(\gamma \otimes \pi^*\beta))^{i+1,j} = \hat{d}_A\gamma \otimes \pi^*\beta.
$$

Similarly,

$$
(\hat{d}^{1,0})^*\omega = \hat{d}_A^*\gamma \otimes \pi^*\beta,
$$

and finally,

$$\hat{\Box}^{1,0}\omega = (\hat{\Box}_A\gamma) \otimes \pi^*\beta, \tag{3.4}$$

where $\hat{d}_A$, $\hat{d}^*_A$ and $\hat{\Box}_A$ are respectively the differential, codifferential and Laplacian on $\Omega^i_s(\pi^{-1}(x,y^-), V_{|\pi^{-1}(x,y^-)})$.

Thus

$$\ker \hat{\Box}^{1,0} = \Gamma\left(E^-, \Lambda^* T^* E^- \otimes \mathcal{H}(A,V)\right),$$

where $\mathcal{H}(A,V)$ denotes the vector bundle over E^- whose fibers at (x,y^-) are the $\hat{\Box}_A$-harmonic forms (i.e. elements of $\ker \hat{\Box}_A$) on $\pi^{-1}(x,y^-)$ with values in V.

The following lemma computes the kernel of $\hat{\Box}_A$ in the fiber.

Lemma 3.1. *Let $\hat{\Box}^i_A$ denote the restriction of the operator $\hat{\Box}_A$ to the the space $\Omega^i_s(\pi^{-1}(x,y^-),V)$. Then for every point $(x,y^-) \in E^-$, $\dim_V \ker \hat{\Box}^{(i)}_A = 1$, if $i=0$; $\dim_V \ker \hat{\Box}^{(i)}_A = 0$, otherwise.*

Moreover, if $\omega \in \ker \hat{\Box}^{(i)}_A$, then $\omega = \gamma \otimes \pi^*\beta$, where after an orthogonal change of coordinates in the fiber over $(x,y^-) \in E^-$,

$$\gamma_{|\Omega^i_s(\pi^{-1}(x,y^-),V)} = e^{-\alpha|Dy^+|^2} \otimes v,$$

for some diagonal matrix D.

Finally, the bundle $\mathcal{H}(A,V)$ is trivial.

Proof. Recall that we chose g, so that g_E and, hence, g_{E^+} are smooth fiber-wise Euclidean metrics. Therefore, for each $(x,y^-) \in E^-$ we can perform a calculation similar to [CFKS, Proposition 11.13] in the fiber over (x,y^-) to get

$$\hat{\Box}^i_A\gamma = (\Delta + \alpha^2|dh^+|^2 + \alpha B)\gamma,$$

where Δ is the Euclidean Laplacian in the fiber and B is a zeroth order operator. We have $\gamma = \phi \otimes v$, where $\phi \in \Omega^i_s(\pi^{-1}(x,y^-))$ and $v \in V$. Moreover, after an orthogonal change of coordinates in the fiber, the variables separate and $\hat{\Box}^i_A$ acts on the form γ by

$$\hat{\Box}^i_A\gamma = (\oplus_{k=1}^{n^+} H_k(\alpha)\phi) \otimes v.$$

The harmonic oscillators $H_k(\alpha)$ in the formula above can be expressed as

$$H_k(\alpha)\phi = \left(\Delta_k + \lambda_k(4\alpha^2 y_k^2 + 2\alpha B_k)\right)\phi,$$

where for each k, Δ_k is the one-dimensional non-negative Laplacian, $\{\lambda_k\}$ are some positive numbers, $B_k\phi = \phi$ if ϕ contains dy_k, and $B_k\phi = -\phi$ otherwise. It is easy to compute precisely eigenvalues and eigenforms of $H_k(\alpha)$ ([CFKS]). In particular, if $\gamma \in \ker \hat{\Box}_A^i$, then $\phi \in \ker H_k(\alpha)$ for all $k = 1, ..., n^+$; that is ϕ does not contain dy_k, $k = 1, ..., n^+$. So $\dim_V \ker \hat{\Box}_A^i$ is not zero if and only if $i = 0$. In this case there is a unique (up to multiplication) $\hat{\Box}_A$-harmonic 0-form in the fiber. This form can be represented as $\phi \otimes v$, where $\phi = e^{-\alpha|Dy^+|^2}$. □

The following useful corollary follows from part of Lemma 3.1.

Corollary 3.2. *For any p we have*

$$E_1^p \subset \Omega_s^{0,p}(E, V).$$

In particular, $E_1^p = \{0\}$ for $p > \dim E^-$.

We have the orthogonal (Hodge) decomposition in each fiber F, associated with the operator $\hat{d}_A$:

$$\Omega_s^\bullet(F, V) = \text{image } \hat{d}_A \oplus \text{image}(\hat{d}_A)^* \oplus \ker \hat{\Box}_A. \tag{3.5}$$

Lemma 3.3. *There exists a constant $c > 0$, so that for all $\omega \in \Omega_s^p(E, V)$,*

$$\langle \hat{\Box}^{1,0}\omega, \omega\rangle \geq c|(1 - \pi_1)\omega|^2.$$

Proof. It follows from (3.4) that

$$\begin{aligned} &\inf\{\lambda \in \text{spec}\hat{\Box}^{1,0} |\ \lambda > 0\} \\ &\quad = \inf_{(x,y^-)\in E^-} \Big[\inf \Big\{\lambda \in \text{spec } \Big(\hat{\Box}_A : \Omega^\bullet(\pi^{-1}((x, y^-), V)) \\ &\qquad\qquad \to \Omega^\bullet(\pi^{-1}((x, y^-)), V)\Big) \ |\ \lambda > 0\Big\}\Big]. \end{aligned}$$

The spectrum of $\hat{\Box}_A$ varies continuously over M, and the multiplicity of 0 is constant. Thus the smallest positive eigenvalue is a continuous function on M and is constant along the fibers of $E^- \to M$ and therefore achieves a positive minimum. This implies that the infimum above is positive, which is precisely the statement of Lemma 3.3 □

It follows from Lemma 3.3 that (3.5) induces a decomposition of $\Omega_s^\bullet(E,V)$.

$$\begin{aligned}\Omega_s^\bullet(E,V) &= \operatorname{image}\hat{d}^{1,0} \oplus \operatorname{image}(\hat{d}^{1,0})^* \oplus \ker\hat{\Box}^{1,0}\\ &= \operatorname{image}\hat{d}^{1,0} \oplus \operatorname{image}(\hat{d}^{1,0})^* \oplus \left(\ker\hat{d}^{1,0} \cap \ker(\hat{d}^{1,0})^*\right).\end{aligned} \tag{3.6}$$

In particular,

$$\Omega_s^{0,\bullet}(E,V) = \operatorname{image}(\hat{d}^{1,0})^* \oplus E_1^p.$$

3.3. A computation of E_2^p. Our next goal is to relate E_2^p to the cohomology $H_s^p(E^-,V,\alpha)$ of the differential complex $(\Omega_s^p(E^-,V),d(\alpha))$. To this extent we prove the following theorem:

Theorem 3.4. *For all p we have*

$$\dim E_2^p = \dim H_s^p(E^-,V,\alpha).$$

Moreover, if $n^-=0$ (i.e. $E=E^+$), then

$$\dim E_2^p = \dim H^p(M,V).$$

This theorem has an important corollary.

Corollary 3.5. *The spaces E_2^p are finite-dimensional.*

We prove Theorem 3.4 by a sequence of lemmas. The first lemma deals with the computation of E_2^p.

Lemma 3.6.

$$E_2^p = \{\omega \in E_1^p \mid \pi_1\hat{d}^{0,1}\pi_1\omega = 0,\ (\pi_1\hat{d}^{0,1}\pi_1)^*\omega = 0\}.$$

Proof. By (3.1)

$$\begin{aligned}E_2^p = \{\omega \in \Omega_s^p(E,V) \mid\ &\exists\ \omega_1 \text{ such that}\\ &\hat{d}_\delta(\omega+\delta\omega_1) = 0(\delta^2) \text{ and } (\hat{d}_\delta)^*(\omega+\delta\omega_1) = 0(\delta^2)\}.\end{aligned}$$

Since $\hat{d}_\delta(\omega+\delta\omega_1) = \hat{d}^{1,0}\omega + \delta(\hat{d}^{1,0}\omega_1 + \hat{d}^{0,1}\omega) + 0(\delta^2)$ and $(\hat{d}_\delta)^*(\omega+\delta\omega_1) = (\hat{d}^{1,0})^*\omega + \delta((\hat{d}^{1,0})^*\omega_1 + (\hat{d}^{0,1})^*\omega) + 0(\delta^2)$ we have

$$\begin{aligned}E_2^p = \{\omega \in E_1^p \mid\ &\exists\omega_1 \text{ such that}\\ &\hat{d}^{1,0}\omega_1 + \hat{d}^{0,1}\omega = 0 \text{ and } (\hat{d}^{1,0})^*\omega_1 + (\hat{d}^{0,1})^*\omega = 0\}.\end{aligned}$$

Let $\omega \in E_1^p$, then the equation $\hat{d}^{1,0}\omega_1 + \hat{d}^{0,1}\omega = 0$ can be solved if and only if

$$\hat{d}^{0,1}\omega \in \operatorname{image} \hat{d}^{1,0}.$$

We apply (2.5) to commute $\hat{d}^{0,1}$ and $\hat{d}^{1,0}$ and we find that

$$\hat{d}^{1,0}\hat{d}^{0,1}\omega = -\hat{d}^{0,1}\hat{d}^{1,0}\omega = 0.$$

Therefore, it follows from the decomposition (3.6) that $\hat{d}^{0,1}\omega \in \operatorname{image} \hat{d}^{1,0}$ if and only if the harmonic component of $\hat{d}^{0,1}\omega$ is 0, that is if and only if $\pi_1\hat{d}^{0,1}\omega = \pi_1\hat{d}^{0,1}\pi_1\omega = 0$.

Similarly, the equation $(\hat{d}^{1,0})^*\omega_1 + (\hat{d}^{0,1})^*\omega = 0$ can be solved if and only if $\pi_1(\hat{d}^{0,1})^*\omega = \pi_1(\hat{d}^{0,1})^*\pi_1\omega = 0$. □

Lemma 3.7. *Suppose γ is any locally constant section of the bundle $\mathcal{H}(A, V)$, then the following diagram is commutative:*

$$\begin{array}{ccc} E_1^p & \xrightarrow{\pi_1\hat{d}^{0,1}\pi_1} & E_1^{p+1} \\ \Big\uparrow{\scriptstyle \gamma\otimes\pi^*} & & \Big\uparrow{\scriptstyle \gamma\otimes\pi^*} \\ \Omega_s^p(E^-, V) & \xrightarrow{\hat{d}} & \Omega_s^{p+1}(E^-, V), \end{array} \tag{3.7}$$

where $\hat{d} = \hat{d}(\alpha) = d - \alpha dh^- \wedge$ is a differential on the bundle $E^- \to M$, and an isomorphism

$$\gamma \otimes \pi^* : \Omega_s^p(E^-, V) \to E_1^p$$

is defined by

$$\beta \mapsto \gamma \otimes \pi^*\beta.$$

Proof. Let $\beta \in \Omega_s^p(E^-, V)$. We want to check that

$$(\pi_1\hat{d}^{0,1}\pi_1) \circ (\gamma \otimes \pi^*)\beta = (\gamma \otimes \pi^*) \circ (\hat{d})\beta.$$

Indeed, we have equalities:

$$\begin{aligned} \pi_1\hat{d}^{0,1}\pi_1(\gamma \otimes \pi^*\beta) &= \pi_1(d^{0,1} - \alpha dh^-\wedge)(\gamma \otimes \pi^*\beta) \\ &= (d(\gamma \otimes \pi^*\beta))^{0,p+1} - \gamma \otimes (\alpha dh^- \wedge \pi^*\beta) \\ &= \gamma \otimes \pi^* d\beta - \gamma \otimes \pi^*(\alpha dh^- \wedge \beta) \\ &= \gamma \otimes \pi^*\hat{d}\beta, \end{aligned}$$

where the third equality holds since γ is locally constant and since a horizontal form dh^- commutes with π^*. $\square$

The following lemma completes the proof of Theorem 3.4.

Lemma 3.8. *We have:*

$$\dim\left(\ker(\pi_1 \hat{d}^{0,1}\pi_1) \cap \ker(\pi_1 \hat{d}^{0,1}\pi_1)^*\right) = \dim(\ker \hat{d} \cap \ker \hat{d}^*).$$

Proof. After taking the adjoints, we have a commutative diagram similar to (3.7):

$$\begin{tikzcd} E_1^p & E_1^{p+1} \arrow[l, "(\pi_1\hat{d}^{0,1}\pi_1)^*"'] \\ \Omega_s^p(E^-, V) \arrow[u, "\gamma\otimes\pi^*"'] & \Omega_s^{p+1}(E^-, V). \arrow[l, "(\hat{d})^*"] \arrow[u, "\gamma\otimes\pi^*"'] \end{tikzcd} \tag{3.8}$$

It follows from (3.7) and (3.8) that $\ker \pi_1\hat{d}^{0,1}\pi_1 \cong \ker \hat{d}$ and $\ker(\pi_1\hat{d}^{0,1}\pi_1)^* \cong \ker \hat{d}^*$. Thus we have equality in the lemma. $\square$

Now Theorem 3.4 follows from Lemma 3.8 and from the Hodge theoretic description of $H_s^*(E^-, V, \alpha)$ as

$$H_s^*(E^-, V, \alpha) = \ker \hat{d} \cap \ker \hat{d}^*.$$

4. An isomorphism between the cohomology of E and E^-.

4.0. Introduction and the main result of this section. The goal of this section is to relate for a fixed large α the cohomology $H_s^\bullet(E, V, \alpha)$ of the complex $(\Omega_s^\bullet(E, V), d(\alpha))$ to the cohomology $H_s^\bullet(E^-, V, \alpha)$ of the complex $(\Omega_s^\bullet(E^-, V), d(\alpha))$. The main result of this section is the following theorem:

Theorem 4.1. *For any p and for large enough α*

$$\dim H_s^p(E, V, \alpha) = \dim H_s^p(E^-, V, \alpha).$$

This theorem has an important corollary.

Corollary 4.2. *If $n^- = 0$ (i.e. $E = E^+$ and $E^- = M$), then*

$$\dim H^p_s(E, V, \alpha) = \dim H^p(M, V).$$

To prove Theorem 4.1 we show first that the nested sequence of spaces $E^p_0 \supseteq E^p_1 \supseteq E^p_2 \supseteq \ldots$ stabilizes at E^p_2, i.e. $E^p_2 = E^p_3 = \ldots = E^p_\infty$. Then the arguments from [Fo, p. 60], recalled in Section 4.3, lead to an isomorphism between E^p_∞ and $H^p_s(E, V, \alpha)$. This isomorphism together with Theorem 3.4 provides us with the chain of equalities

$$\dim H^p_s(E, V, \alpha) = \dim E^p_\infty = \dim E^p_2 = \dim H^p_s(E^-, V, \alpha).$$

Thus we have the conclusion of Theorem 4.1.

4.1. A preliminary result. In this section we will prove a preliminary result. Namely we will show that in the definition of spaces E^p_k we can choose $\omega_1, \ldots \omega_{k-1}$ to be from the space of rapidly decreasing forms. This result will allow us to do all computations completely inside $\Omega^\bullet_s(E, V)$.

Lemma 4.3. *Let $\omega_0 \in E^p_k$, and $\omega_1, \ldots, \omega_{k-1}$ be such that $\hat{d}_\delta(\omega_0 + \delta\omega_1 + \cdots + \delta^{k-1}\omega_{k-1}) = 0(\delta^k)$ and $\hat{d}^*_\delta(\omega_0 + \delta\omega_1 + \cdots + \delta^{k-1}\omega_{k-1}) = 0(\delta^k)$. Then $\omega_j \in \Omega^\bullet_s(E, V)$ for $j = 0, 1, \ldots, k-1$.*

Proof. We use induction on k.

$k = 1$. If $\omega_0 \in E^p_1$ then $\omega_0 \in \Omega^\bullet_s(E, V)$ by definition of E^p_1.

$k = i - 1$. Suppose the statement of Lemma 4.3 is true for $k = i - 1$.

$k = i$. Let $\omega_0 \in E^p_i$ and let $\omega_1, \ldots, \omega_{i-1}$ be such that

$$\hat{d}_\delta(\omega_0 + \delta\omega_1 + \cdots + \delta^{i-1}\omega_{i-1}) = 0(\delta^i) \tag{4.1}$$

$$\hat{d}^*_\delta(\omega_0 + \delta\omega_1 + \cdots + \delta^{i-1}\omega_{i-1}) = 0(\delta^i). \tag{4.2}$$

Then $\omega_1, \ldots, \omega_{i-2}$ are such that $\hat{d}_\delta(\omega_0 + \delta\omega_1 + \cdots + \delta^{i-2}\omega_{i-2}) = 0(\delta^{i-1})$ and $\hat{d}^*_\delta(\omega_0 + \delta\omega_1 + \cdots + \delta^{i-2}\omega_{i-2}) = 0(\delta^{i-1})$. By the previous step of the induction, $\omega_j \in \Omega^\bullet_s(E, V)$, $j = 0, \ldots, i-2$. Computing coefficients in front of δ^{k-1} in (4.1) and (4.2), we see that ω_{i-1} satisfies two equations:

$$\hat{d}^{1,0}\omega_{i-1} + \hat{d}^{0,1}\omega_{i-2} + d^{-1,2}\omega_{i-3} = 0,$$
$$(\hat{d}^{1,0})^*\omega_{i-1} + (\hat{d}^{0,1})^*\omega_{i-2} + (d^{-1,2})^*\omega_{i-3} = 0.$$

Hence, ω_{i-1} is a solution of

$$\begin{aligned}&(\hat{d}^{1,0})^*(\hat{d}^{1,0}\omega_{i-1}+\hat{d}^{0,1}\omega_{i-2}+d^{-1,2}\omega_{i-3})\\&+\hat{d}^{1,0}\left((\hat{d}^{1,0})^*\omega_{i-1}+(\hat{d}^{0,1})^*\omega_{i-2}+(d^{-1,2})^*\omega_{i-3}\right)=0.\end{aligned}$$

We rewrite this equation as

$$\hat{\Box}^{1,0}\omega_{i-1}=\chi, \tag{4.3}$$

where

$$\begin{aligned}\chi=&-(\hat{d}^{1,0})^*\left(\hat{d}^{0,1}\omega_{i-2}+d^{-1,2}\omega_{i-3}\right)\\&-\hat{d}^{1,0}\left((\hat{d}^{0,1})^*\omega_{i-2}+(d^{-1,2})^*\omega_{i-3}\right).\end{aligned}$$

Since both ω_{i-2} and ω_{i-3} are in $\Omega_s^\bullet(E,V)$, and $\Omega_s^\bullet(E,V)$ is invariant under $\hat{d}^{1,0}$ and $\hat{d}^{0,1}$ (see Theorem A.2.1), then $\chi\in\Omega_s^\bullet(E,V)$. We note that $\chi\in(\ker\hat{\Box}^{1,0})^\perp$. From (4.3) we can represent ω_{i-1} as

$$\omega_{i-1}=(\hat{\Box}^{1,0})^{-1}\chi+h,$$

where $h\in\ker\hat{\Box}^{1,0}$. Thus $\omega_{i-1}\in\Omega_s^\bullet(E,V)$, since $\chi\in\Omega_s^\bullet(E,V)$, $h\in\Omega_s^\bullet(E,V)$, and $(\hat{\Box}^{1,0})^{-1}$ leaves $\Omega_s^\bullet(E,V)$ invariant. $\square$

4.2. The spaces E_k^p for $k>1$. In Section 3 we defined the differentials $\pi_k\hat{d}_k\pi_k: E_k^p\to E_k^{p+1}$ and $(\pi_k\hat{d}_k\pi_k)^*: E_k^p\to E_k^{p-1}$.

Here we will show that these differentials equal zero for $k\geq 2$. This will lead (via the results of R. Forman [Fo, Theorem 2.5]) to the equality $E_2^p=E_\infty^p$. In other words, our nested sequence of spaces *stabilizes* at $k=2$.

We start with the lemma in the proof of which the dimension considerations of Corollary 3.2 are crucial.

Lemma 4.4. *For any $k\geq 2$, $\pi_k\hat{d}_k\pi_k=(\pi_k\hat{d}_k\pi_k)^*=0$.*

Proof. We prove that for any $k\geq 2$, $(\pi_k\hat{d}_k\pi_k)^*=0$. Then, after taking the adjoints, the identities $\pi_k\hat{d}_k\pi_k=0$ will follow.

Let $\omega\in E_k^p$, $k\geq 2$, then

$$(\pi_k\hat{d}_k\pi_k)^*\omega=\pi_k\hat{d}_k^*\omega.$$

Since $\omega \in E_k^p, \exists\, \omega_1, \ldots, \omega_{k-1}$, such that

$$\hat{d}_\delta^*(\omega + \delta\omega_1 + \cdots + \delta^{k-1}\omega_{k-1}) = \delta^k \left((\hat{d}^{0,1})^*\omega_{k-1} + (d^{-1,2})^*\omega_{k-2}\right) + 0(\delta^{k+1}).$$

Thus by definition of $\hat{d}_k^*$

$$\hat{d}_k^*\omega = \lim_{\delta\to 0} \delta^{-k}\hat{d}_\delta^*(\omega + \delta\omega_1 + \cdots + \delta^{k-1}\omega_{k-1}) = (\hat{d}^{0,1})^*\omega_{k-1} + (d^{-1,2})^*\omega_{k-2}.$$

Therefore,

$$\pi_k\hat{d}_k^*\pi_k\omega = \pi_k(\hat{d}^{0,1})^*\omega_{k-1} + \pi_k(d^{-1,2})^*\omega_{k-2}.$$

We consider each term in the right-hand side of the equality above separately.

Since $E_k^p \subseteq E_1^p$, $\pi_k(d^{-1,2})^*\omega_{k-1}$ is a $\hat{d}^{1,0}$-harmonic form. Then, according to Corollary 3.2, $\pi_k(d^{-1,2})^*\omega_{k-1} \in \Omega_s^{0,p}(E,V)$, i.e. the restriction of $\pi_k(d^{-1,2})^*\omega_{k-1}$ to the fiber must be a 0-form. However, the 0-degree component of $(d^{-1,2})^*\omega_{k-1}$ in a fiber must be 0, since $(d^{-1,2})^*\omega_{k-1}$ is a form in a fiber of degree at least 1. Thus $\pi_k(d^{-1,2})^*\omega_{k-1} = 0$.

Similarly, $\pi_k(\hat{d}^{0,1})^*\omega_{k-1} \in \Omega_s^{0,p}(E,V)$. Therefore,

$$\pi_k(\hat{d}^{0,1})^*\omega_{k-1} = \pi_k(\hat{d}^{0,1})^*\omega_{k-1}^{(0)},$$

where $\omega_{k-1}^{(0)}$ is the component of ω_{k-1} which belongs to $\Omega_s^{0,p-1}(E,V)$.

It follows from (3.6) that $\omega_{k-1}^{(0)}$ can be decomposed into $\hat{d}^{1,0}$-harmonic and $\hat{d}^{1,0}$-coexact components, $\omega_{k-1}^{(0)} = h_1 + (\hat{d}^{1,0})^*h_2$. Then

$$\pi_k(\hat{d}^{0,1})^*\omega_{k-1}^{(0)} = \pi_k(\hat{d}^{0,1})^*h_1 + \pi_k(\hat{d}^{0,1})^*(\hat{d}^{1,0})^*h_2.$$

Now, it follows from the description of E_2^p in Lemma 3.6 that

$$\pi_k(\hat{d}^{0,1})^*h_1 = \pi_k\pi_2\pi_1(\hat{d}^{0,1})^*\pi_1 h_1 = 0.$$

Also, from (2.5) and (3.6) we have

$$\pi_k(\hat{d}^{0,1})^*(\hat{d}^{1,0})^*h_2 = \pi_k\pi_1(\hat{d}^{1,0})^*(\hat{d}^{0,1})^*h_2 = 0.$$

□

In Section 3 we proved that $\dim E_2^p < \infty$. Thus there exists N, such that $E_0^p \supseteq E_1^p \supseteq \cdots \supseteq E_N^p = E_{N+1}^p = \ldots$. We denote such E_N^p as E_∞^p. Now we will show that the nested sequence of spaces stabilizes at $k = 2$.

We have the same situation as in [Fo].

Theorem 4.5 ([Fo, Theorem 2.5]). *For all $k \geq 0$*

(i) $(\pi_k \hat{d}_k \pi_k)^2 = 0$.

(ii) *The kernel of*

$$\hat{\Delta}_k = (\pi_k \hat{d}_k \pi_k)(\pi_k \hat{d}_k \pi_k)^* + (\pi_k \hat{d}_k \pi_k)^*(\pi_k \hat{d}_k \pi_k) : E_k^p \to E_k^p \tag{4.4}$$

is precisely E_{k+1}^p.

Note that we already proved part (i) of this theorem and we proved part (ii) for $k = 0$ and $k = 1$ in Section 3.

Corollary 4.6. $E_\infty^p = E_2^p$.

4.3. An isomorphism between E_∞^p and $H_s^p(E, V, \alpha)$. We outline the steps of the argument taken from [Fo, p. 60]. The proofs are given in [Fo, Sec. 3].

(i) For every $\omega \in E_\infty^p$, there is a formal power series

$$\omega_\delta = \omega + \delta\omega_1 + \delta^2\omega_2 + \ldots$$

such that, formally,

$$\hat{d}_\delta \omega_\delta = \hat{d}_\delta^* \omega_\delta = 0.$$

(ii) The ω_δ's arising in (i) form a basis, modulo the action of $\mathcal{T}$ (the ring of formal real Taylor series), for the cohomology of the complex $(\mathcal{T}[\Omega_s^p], \hat{d}_\delta)$. Here $\mathcal{T}[\Omega_s^p]$ denotes the space of formal Taylor series with coefficients in $\Omega_s^p(E, V)$.

(iii) The operator ρ_δ provides an isomorphism between $(\mathcal{T}[\Omega_s^p],\ \hat{d}_\delta)$ and $(\mathcal{T}[\Omega_s^p],\ d(\alpha))$.

(iv) The cohomology of $(\mathcal{T}[\Omega_s^p], d(\alpha))$ is canonically isomorphic to $\mathcal{T}[H_s^p(E, V, \alpha)]$ and hence, modulo $\mathcal{T}$, $H_s^p(E, V, \alpha)$ provides a basis.

Observations (i)-(iv) allow us to conclude, in particular, that for all p,

$$\dim E_\infty^p = \dim H_s^p(E, V, \alpha). \tag{4.5}$$

This fact completes the proof of Theorem 4.1.

Remark 4.6. In part (ii) of the argument above we used that all ω_i's arising in the expansion of ω_δ belong to $\Omega_s^\bullet(E, V)$.

5. The Hodge $*$-operator.

5.0. Introduction. In this section we use the Hodge $*$-operator as a convenient tool to study the cohomology $H_s^\bullet(E^-, V, \alpha) \equiv H_s^\bullet(E^-, V, d(\alpha))$ associated to the differential complex $(\Omega_s^\bullet(E^-, V), d(\alpha))$. This study allows us to compare $H_s^\bullet(E^-, V, d(\alpha))$ with the de Rham cohomology $H^\bullet(M, V)$ of M with values in V.

We note that Corollary 5.2 in this section can also be deduced without the help of the Hodge $*$-operator by methods of the previous sections.

5.1. The set up and the main result. First we want to compare $H_s^\bullet(E, V, d(\alpha))$ to the cohomology $H_s^\bullet(E, V, d(-\alpha))$ of the differential complex $(\Omega_s^p(E, V), d(-\alpha))$, where

$$d(-\alpha) = e^{\alpha h} d e^{-\alpha h} = d - \alpha dh \wedge .$$

We consider $d(-\alpha)$ as the differential, associated to a new Morse-Bott function $\tilde{h}$, where $\tilde{h} = -h$. Clearly, $\tilde{h}^+ = h^-$ and $\tilde{h}^- = h^+$.

This new Morse-Bott function $\tilde{h}$ on E leads to a new decomposition of E:

$$E = \tilde{E}^+ \oplus \tilde{E}^-,$$

where $\tilde{E}^+ \cong E^-$, $\tilde{E}^- \cong E^+$.

In this section we develop Poincare duality for the above situation. We denote as $o(E)$ and $o(E^-)$ orientation bundles of E and E^-. If E and E^- are orientable then $o(E)$ and $o(E^-)$ are trivial.

Theorem 5.1. *For any p,*

$$\dim H_s^p(E, V, d(\alpha)) = \dim H_s^{m+n-p}(E, V \otimes o(E), d(-\alpha)),$$

where $H_s^\bullet(E, V \otimes o(E), d(-\alpha))$ is the cohomology, twisted by the orientation bundle of E .

Before we prove this theorem we derive some important corollaries.

5.2. Some applications of Theorem 5.1. The main application of Theorem 5.1 is the proof of Theorem 5.3. Although, the first application of Theorem 5.1 is the following corollary:

Corollary 5.2. *Let $n^+ = 0$, that is $E = E^-$, $n = n^-$, and $H_s^p(E, V, \alpha) = H_s^p(E^-, V, \alpha)$. Then for all large enough α*

$$\dim H_s^p(E^-, V, \alpha) = \dim H^{m+n^- -p}(M, V \otimes o(E^-)).$$

Proof. By Theorem 5.1

$$\begin{aligned}\dim H^p_s(E^-, V, \alpha) &= \dim H^{m+n^- -p}_s(E^-, V \otimes o(E^-), d(-\alpha)) \\ &= \dim H^{m+n^- -p}_s(\tilde{E}^+, V \otimes o(E^-), \alpha).\end{aligned}$$

Then we can apply Corollary 4.2 to get an equality

$$\dim H^{m+n^- -p}_s(\tilde{E}^+, V \otimes o(E^-), \alpha) = \dim H^{m+n^- -p}(M, V \otimes o(E^-)).$$

□

Now we can prove

Theorem 5.3. *Let $E = E^+ \oplus E^- \to M$. Then for all $p \geq 0$ and large enough α*

$$\dim H^p(E, V, \alpha) = \dim H^{m+n^- -p}(M, V \otimes o(E^-)),$$

where $H^\bullet(M, V \otimes o(E^-))$ is the de Rham cohomology of M twisted by the orientation bundle of $E^- \to M$.

Proof. We have the following sequence of equalities

$$\begin{aligned}\dim H^p(E, V, \alpha) &= \dim H^p_s(E, V, \alpha) = \dim H^p_s(E^-, V, \alpha) \\ &= \dim H^{m+n^- -p}(M, V \otimes o(E^-)),\end{aligned}$$

where the first equality follows from Theorem 1.5, the second is Theorem 4.1 and the third is Corollary 5.2. □

Our next corollary easily follows from Theorem 5.3 and Poincare duality on M.

Corollary 5.4. *Let $E = E^+ \oplus E^- \to M$. Then for all large enough α*

$$\dim H^p(E, V, \alpha) = \dim H^{p-n^-}(M, V \otimes o(E^-)).$$

We will use Corollary 5.4 in the next section to give an analytic proof of the Thom isomorphism.

5.3. A proof of Theorem 5.1. We want to define the Hodge $*$-operator on E. If E is orientable, we choose an orientation on E by choosing a volume form on TE. Then we define the $*$–operator as in [CFKS, Proposition 11.9]. If E is not orientable, then instead of the volume form on TE we use the volume density.

Then we have

Lemma 5.5. *If* $\omega \in \Omega_s^p(E,V)$*, then*

$$d^*\omega = (-1)^{(m+n)(p+1)+1} * [d(*\omega)].$$

Proof of Lemma 5.5 is the same as in [CFKS, Theorem 11.10].

A proof of Theorem 5.1. From Corollary 1.4 we see that for all $p \geq 0$,

$$\dim H_s^p(E,V,d(\alpha)) = \dim\{\omega \in \Omega_s^p(E,V) \mid d(\alpha)\omega = 0, d(\alpha)^*\omega = 0\}$$

Similarly,

$$\begin{aligned}
&\dim H_s^{m+n-p}(E, V \otimes o(E), d(-\alpha)) \\
&\qquad = \dim\{\phi \in \Omega_s^{m+n-p}(E, V \otimes o(E)) \mid d(-\alpha)\phi = 0, d(-\alpha)^*\phi = 0\}
\end{aligned}$$

To finish the proof we only need to show that $\omega \in \Omega_s^p(E,V)$ is $d(-\alpha)$-harmonic if and only if $*\omega \in \Omega_s^{m+n-p}(E, V \otimes o(E))$ is $d(\alpha)$-harmonic. Indeed if

$$d(\alpha)\omega = e^{-\alpha h} d e^{\alpha h}\omega = 0,$$

then

$$\begin{aligned}
0 &= *e^{-\alpha h} d e^{\alpha h} * *\omega = e^{-\alpha h} * d * e^{\alpha h}(*\omega) \\
&= e^{-\alpha h} d^* e^{\alpha h}(*\omega) = d^*(-\alpha)(*\omega).
\end{aligned}$$

Thus $d(\alpha)\omega = 0$ if and only if $d^*(-\alpha)(*\omega) = 0$. Similarly, $d^*(\alpha)\omega = 0$ if and only if $d(-\alpha)(*\omega) = 0$. □

6. The Thom isomorphism.

6.0. Introduction. In this section we complete an analytic proof of the Thom isomorphism. We recall that the Thom isomorphism (see [Bott-Tu, Chapter 1.6]) relates compactly supported de Rham cohomology of the total space to the de Rham cohomology of the base.

The final step in our proof of the Thom isomorphism (Theorem 6.2) relates the compactly supported de Rham cohomology of E to the cohomology $H^\bullet(E, V, \alpha)$, associated to the differential complex $(\Omega^\bullet_s(E, V,), d(\alpha))$. Then our version of Thom isomorphism (for cohomology with values in a flat bundle V) will follow from Theorem 5.3.

6.1. The statement of the Thom isomorphism. Let $E \to M$ be a vector bundle of rank n over M. Let $V \to E$ be a flat vector bundle over E. We denote the V-valued cohomology of E with compact support in the vertical direction as $H^\bullet_c(E, V)$. Thus by definition $H^\bullet_c(E, V)$ is the cohomology associated to the differential complex $(\Omega^\bullet_c(E, V), d)$.

Theorem 6.1 (Thom isomorphism). *For all p*

$$\dim H^p_c(E, V) = \dim H^{p-n}\left(M, V \otimes o(E)\right).$$

6.2. A proof of the Thom isomorphism. We put a Morse-Bott function $h(y) = -h^-(y) = -|y|^2$, a metric g on E and a metric q on V. Thus $E = E^-$. In Section 6.3 we prove the following theorem:

Theorem 6.2. *For all large enough α and any p, $0 \le p \le \dim E$,*

$$\dim H^p_c(E, V) = \dim H^p(E, V, \alpha).$$

Theorem 6.2 together with Theorem 5.3 proves the Thom isomorphism via the following sequence of equalities:

$$\dim H^p_c(E, V) = \dim H^p(E, V, \alpha) = \dim H^{m+n-p}(M, V \times o(E)),$$

and the observation that by Poincare duality on M

$$\dim H^{m+n-p}(M, V \times o(E)) = \dim H^{p-n}(M, V \times o(E)).$$

6.3. A Proof of the Theorem 6.2. We prove Theorem 6.2 by using a sequence of lemmas.

We denote the space of $\{e^{\alpha h}\omega |\ \omega \in \Omega^\bullet_s(E, V)\}$ as $\tilde{\Omega}^\bullet(E, V, \alpha)$. Then the following lemma is an easy corollary of definition (1.3) of $d(\alpha)$.

Lemma 6.3. *For any large enough α and any p the following diagram is commutative:*

$$\begin{array}{ccc} \tilde{\Omega}^p(E, V, \alpha) & \xrightarrow{d} & \tilde{\Omega}^{p+1}(E, V, \alpha) \\ \Big\uparrow{\scriptstyle e^{\alpha h}} & & \Big\uparrow{\scriptstyle e^{\alpha h}} \\ \Omega^p_s(E, V) & \xrightarrow{d(\alpha)} & \Omega^{p+1}_s(E, V), \end{array}$$

where the vertical arrows are isomorphisms.

Let $f : \mathbb{R}^+ \to [0, 2)$ be a smooth increasing function, satisfying

$$f(t) = t, \text{ for } t \in [0, 1/2];$$
$$f(t) = \frac{4}{\pi}\arctan(t), \text{ for } t \in [1, \infty).$$

For all $(x, y) \in (M, \pi^{-1}(x))$ we also define a diffeomorphism ψ between manifolds E and D_2 by the formula

$$\psi(x, y) = \left(x, \frac{y}{|y|} f(|y|)\right).$$

Note that if $|y|^2$, then $\psi(x, y) = (x, y)$. The manifold D_2 is defined as the image of E under ψ. Observe that D_2 is a disc bundle over M with the fibers being open discs $\{y \mid |y| < 2\}$.

The diffeomorphism ψ^{-1} induces the map $(\psi^{-1})^*$ on the spaces of differential forms on E with values in V:

$$(\psi^{-1})^* : \Omega_c^p(E, V) \to \Omega_c^p(D_2, W),$$
$$(\psi^{-1})^* : \tilde{\Omega}^p(E, V, \alpha) \to (\psi^{-1})^*(\tilde{\Omega}^p(E, V, \alpha)),$$

where $W = (\psi^{-1})^* V$ is an induced flat bundle over D_2. We denote $(\psi^{-1})^*(\tilde{\Omega}^p(E, V, \alpha))$ as $\tilde{\Omega}^p(D_2, W, \alpha)$.

Since differential d and an isomorphism $(\psi^{-1})^*$ commute, we have the following lemma:

Lemma 6.4. *For all large enough α and any p*

$$\tilde{H}^p(E, V, \alpha) \cong \tilde{H}^p(D_2, W, \alpha),$$
$$H_c^p(E, V) \cong H_c^p(D_2, W).$$

Let $D_1 \subset D_2$ be a sub-bundle of D_2, where each fiber of $D_1 \to M$ is an open disc $\{y \mid |y| < 1\}$. Let $k : D_1 \to D_2$ be a diffeomorphism which restricts to be the multiplication by 2 map on each fiber.

For every $p \geq 0$ we have the following maps

$$i : \Omega_c^p(D_l, W) \to \tilde{\Omega}^p(D_l, W, \alpha), \ l = 1, 2,$$
$$j : \tilde{\Omega}^p(D_1, W, \alpha) \to \Omega_c^p(D_2, W),$$
$$k^* : \Omega_c^p(D_2, W) \to \Omega_c^p(D_1, W), \text{ and}$$
$$k^* : \tilde{\Omega}^p(D_2, W) \to \tilde{\Omega}^p(D_1, W),$$

where i is the inclusion, j is the extension maps and k^* is the map induced by the diffeomorphism k.

We now describe the map j. Let $\theta \in \tilde{\Omega}^p(D_1, W, \alpha)$, then $\theta = k^* \circ (\psi^{-1})^* \left(e^{-\alpha|y|^2}\omega\right)$, for some $\omega \in \Omega^p_s(E, V)$. Clearly θ and all its derivatives are zero on ∂D_1, since the form ω and all its derivatives decay rapidly at infinity. Thus, θ can be extended by zero to a form in $\Omega^p_c(D_2, W)$.

Since differential d commutes with the maps i, j and k^* commute, they induce the maps on the corresponding cohomology.

Lemma 6.5. *For $\alpha > 0$ and any p*

$$\begin{aligned} k^* \circ j \circ i &: H^p_c(D_1, W) \to H^p_c(D_1, W), \\ k^* \circ i \circ j &: \tilde{H}^p(D_1, W, \alpha) \to \tilde{H}^p(D_1, W, \alpha) \end{aligned}$$

are isomorphisms.

Moreover, since k^* is an isomorphism on the cohomology, so are i and j, and we have a corollary:

Corollary 6.6. *For any $\alpha > 0$ and any p*

$$H^p_c(D_2, W) \cong \tilde{H}^p(D_2, W).$$

This corollary together with Lemma 6.4. proves Theorem 6.2.

Remark 6.7. Corollary 6.6 can also be proved by methods of [Bott-Tu, Chapter 1]. Also see [Bue-P] for another proof of Corollary 6.6.

7. The asymptotic of the spectrum of the Witten Laplacian.

7.0. Introduction. For each $\alpha > 0$ and $p = 1, \ldots, m+n$, let $0 \le \lambda^p_1(\alpha) \le \lambda^p_2(\alpha) \le \ldots$ denote the eigenvalues of $\Box^p(\alpha)$. The goal of this section is to investigate the asymptotics of the eigenvalues of $\Box^p(\alpha)$ as $\alpha \to \infty$.

The main result of this section is Theorem 7.23. This theorem states that the bounded eigenvalues of $\Box(\alpha)$ approach the eigenvalues of the Laplacian

$$\triangle : \Omega^\bullet(M, V \otimes o(E^-)) \to \Omega^\bullet(M, V \otimes o(E^-))$$

on the space of V-valued differential forms on M, twisted by the orientation bundle of E^-. The theorem also estimates the rate of convergence of the spectrum.

In Section 7.1 we relate the spectrum of $\square(\alpha)$ to the spectrum of the adiabatic deformation $\hat{\square}(\delta)$ of the Witten Laplacian $\hat{\square} = \square(1)$. We observe that if $\delta = \alpha^{-1/2}$ then the operators $\square(\alpha)$ and $\delta^{-2}\hat{\square}(\delta)$ are isospectral. The isospectrality means that if $\{\lambda_j^p(\delta)\}$ denote the eigenvalues of $\hat{\square}^p(\delta)$, then for any p and $\alpha > 0$ we have $\lambda_j^p(\alpha) = \delta^{-2}\lambda_j^p(\delta)$, $j = 1, 2, \ldots$.

In Section 7.2 we study the kernel of $\hat{\square}(\delta)$. We conclude that for large enough δ the dimension of $\ker \hat{\square}(\delta)$ does not depend on δ and is equal to the dimension of $H^{p-n^-}(M, V_{|M} \otimes o(E^-))$. Moreover, by the Hodge theory on M we have

$$\dim H^{p-n^-}(M, V \otimes o(E^-)) = \dim \ker \triangle^{p-n^-}.$$

In Section 7.3 we start the Taylor analysis of the *small* spectrum of $\hat{\square}(\delta)$ (which corresponds via Theorem 7.1 to the bounded spectrum of $\hat{\square}(\alpha)$) by formulating several preliminary results.

In sections 7.4 and 7.6 we introduce a model operator $\triangle$ and prove the main result of this section. We use the classical variational approach as in [Du-Sc] to compare the small spectrum of the Witten Laplacian to the spectrum of the model operator.

The results of Section 7.4 are applied in Section 7.5 to study the eigenforms of the Witten Laplacian, which correspond to the small spectrum. In Section 7.5 we obtain the description of the limiting behavior of the eigenspaces similar to one in Section 5 of [Fo].

7.1. A rescaling of the eigenvalues. For a bigrading $E = E^+ \oplus E^-$ of a vector bundle $E \to M$ let $A = T^{\mathrm{ver}}E$, $B = T^{\mathrm{hor}}E$. Then $TE = A \oplus B$. As in Section 1.3 we choose the metric g on TE to be the sum of the metrics on A and B: $g = g_A \oplus g_B$.

We define a deformation g_δ $(0 < \delta \leq 1)$ of the metric g on TE by the formula

$$g_\delta = g_A \oplus \delta^{-2} g_B. \tag{7.1}$$

We also define the operators $\hat{d}$ and $\hat{d}_\delta$ by

$$\hat{d} = e^{-h} d e^h = d + dh\wedge \quad \text{and} \quad \hat{d}_\delta = d^{1,0} + \delta d^{0,1} + \delta^2 d^{-1,2} + dh \wedge .$$

We recall that the horizontal space B was chosen so that $dh_{|B} = 0$. Thus $dh\wedge$ is a $(1,0)$-operator. Therefore,

$$\hat{d}_\delta = \hat{d}^{1,0} + \delta d^{0,1} + \delta^2 d^{-1,2}, \text{ where } \hat{d}^{1,0} = d^{1,0} + dh \wedge . \tag{7.2}$$

As in Section 2.3

$$\hat{\square}^p(\delta) = \hat{d}_\delta \hat{d}_\delta^* + \hat{d}_\delta^* \hat{d}_\delta \tag{7.3}$$

denotes the adiabatic deformation of the Witten Laplacian $\hat{\square}$.

We have the following theorem which is proved by a simple rescaling argument:

Theorem 7.1. *Let $\delta = \alpha^{-1/2}$, then for all p*

$$\lambda_j^p(\alpha) = \delta^{-2}\lambda_j^p(\delta),\ j = 1, 2, \ldots. \tag{7.4}$$

In other words the operators $\square(\alpha)$ and $\delta^{-2}\hat{\square}(\delta)$ are isospectral.

7.2. Zero eigenvalues of $\hat{\square}(\delta)$. In this section we reproduce the results of Sections 3 and 4 in the setting of this section. We do not give proofs, but only indications of the necessary changes. The main result of this section states that for all p, $\dim\ker\hat{\square}^p(\delta)$ does not depend on δ and is equal to $\dim E_2^p = \dim H^{p-n^-}(M, V \otimes o(E^-))$.

As in Section 3 for each p we can define a nested sequence of spaces

$$E_0^p \supseteq E_1^p \supseteq E_2^p \supseteq \ldots$$

by

$$\begin{aligned} E_k^p = \{\omega \in \Omega_s^p(E, V) |\ & \exists \omega_1, \ldots, \omega_{k-1} \text{ with} \\ & \hat{d}_\delta(\omega + \delta\omega_1 + \cdots + \delta^{k-1}\omega_{k-1}) \in 0(\delta^k), \\ & \hat{d}_\delta^*(\omega + \delta\omega_1 + \cdots + \delta^{k-1}\omega_{k-1}) \in 0(\delta^k)\}, k = 1, 2, \ldots. \end{aligned} \tag{7.5}$$

Our computations of E_1^p and E_2^p work as before and we have

$$E_1^p = \{\omega \in \Omega_s^p(E, V) |\ \hat{d}^{1,0}\omega = 0, (\hat{d}^{1,0})^*\omega = 0\}, \tag{7.6}$$

$$E_2^p = \{\omega \in E_1^p |\ \pi_1 d^{0,1}\pi_1\omega = 0,\ (\pi_1 d^{0,1}\pi_1)^*\omega = 0\}. \tag{7.7}$$

Let $\omega \in E_1^p$. Since the differential $\hat{d}^{1,0}$ preserves the bigrading, we may assume that $\omega \in \Omega_s^{i,j}(E, V)$ for some $i, j,\ i + j = p$.

We write ω as

$$\omega = \gamma \otimes \pi^*\beta,$$

where $\gamma \in \Omega_s^i(A, V)$, $\beta \in \Omega^j(M, V)$. Then we have

$$\hat{\square}^{1,0}\omega = (\hat{\square}_A\gamma) \otimes \pi^*\beta, \tag{7.8}$$

where $\hat{\Box}_A$ is the Laplacian on $\Omega^i_s(\pi^{-1}(m), V)$.

Thus

$$\ker \hat{\Box}^{1,0} = \Gamma\left(M, \Lambda^\bullet T^* M \otimes \mathcal{H}(A, V)\right), \tag{7.9}$$

where $\mathcal{H}(A, V)$ denotes the vector bundle over M whose fibers at m are the $\hat{\Box}_A$-harmonic forms (i.e. elements of $\ker \hat{\Box}_A$) on $\pi^{-1}(m)$. Now we have the following lemma:

Lemma 7.2. *Let $\hat{\Box}^i_A$ denote the restriction of the operator $\hat{\Box}_A$ to the space $\Omega^i_s(\pi^{-1}(m), V)$. Then, for every $m \in M$, $\dim_V \ker \hat{\Box}^i_A = 1$, if $i = n^-$; $\dim_V \ker \hat{\Box}^i_A = 0$, otherwise. Furthermore, if $\omega \in \ker \hat{\Box}^i_A$, then $\omega = \gamma \otimes \pi^* \beta$. After an orthogonal change of coordinates in the fiber $\pi^{-1}(m)$ we have*

$$\gamma_{|(\Omega^i_s(\pi^{-1}(m), V_{|\pi^{-1}(m)})} = e^{-\alpha|Dy|^2} dy_{n^+ + 1} \wedge \cdots \wedge dy_n \otimes v$$

for some diagonal matrix D.

We note that $\mathcal{H}(A, V)$ is a rank-one bundle of the n^--forms. If $E^- \to M$ is not orientable, then the line bundle $\mathcal{H}(A, V) \to \Omega^{p-n^-}(M, V)$ is not trivial. On the other hand, the line bundle $\mathcal{H}(A, V) \to \Omega^{p-n^-}(M, V \otimes o(E^-))$ is trivial.

Now we have the following theorem:

Theorem 7.3. *For any p*

$$\dim E^p_2 = \dim H^{p-n^-}\left(M, V \otimes o(E^-)\right). \tag{7.10}$$

In a similar way as Theorem 3.4, this theorem follows from the following lemma:

Lemma 7.4. *Suppose γ is any locally constant section of the bundle $\mathcal{H}(A, V_{|A})$, then the following diagram is commutative:*

$$\begin{tikzcd}
E^p_1 \arrow[r, "\pi_1 d^{0,1} \pi_1"] & E^{p+1}_1 \\
\Omega^{p-n^-}\left(M, V \otimes o(E^-)\right) \arrow[u, "\gamma\otimes\pi^*"] \arrow[r, "d"] & \Omega^{p+1-n^-}\left(M, V \otimes o(E^-)\right), \arrow[u, "\gamma\otimes\pi^*"']
\end{tikzcd}$$

where an isomorphism

$$\gamma \otimes \pi^* : \Omega^{p-n^-}\left(M, V \otimes o(E^-)\right) \to E^p_1$$

is defined by

$$\gamma \otimes \pi^*(\beta) = \gamma \otimes \pi^* \beta.$$

Now we can prove our next theorem:

Theorem 7.5. *For all small enough δ and any p*

$$\dim\ker\hat{\Box}^p(\delta) = \dim E_2^p = \dim H^{p-n^-}\left(M, V\otimes o(E^-)\right).$$

Proof. By Theorem 7.3,

$$\dim E_2^p = \dim H^{p-n^-}\left(M, V\otimes o(E^-)\right). \tag{7.11}$$

Then by Corollary 5.4,

$$\dim H^{p-n^-}\left(M, V\otimes o(E^-)\right) = \dim H_s^p(E,V,\alpha) = \dim\ker\Box^p(\alpha), \tag{7.12}$$

where the last equality is Corollary 1.4. Finally, by Theorem 7.1

$$\dim\ker\Box^p(\alpha) = \dim\ker\hat{\Box}^p(\delta).$$

The statement of the theorem then follows from (7.11) and (7.12). □

Corollary 7.6. *For any p*

$$E_2^p = E_3^p = \ldots = E_\infty^p.$$

Proof. From (7.12) we have that $\dim E_2^p = \dim H_s^p(E,V,\alpha)$. Moreover, by applying the arguments from Section 4.3 to our setting we have $\dim E_\infty^p = \dim H_s^p(E,V,\alpha)$. Combining these two equalities we conclude that $\dim E_2^p = \dim E_\infty^p$. □

7.3. The asymptotics of the small eigenvalues of $\hat{\Box}^p(\delta)$. Preliminary results. We recall that $\pi_1 : \Omega_s^p(E,V) \to E_1^p$ denotes the orthogonal projection. We also denote as $\pi_1^\perp$ the orthogonal projection onto $(E_1^p)^\perp$. Then $\pi_1^\perp + \pi_1 = 1$.

Lemma 7.7. *There exists a constant $c > 0$, so that for all $\omega \in \Omega_s^p(E,V)$,*

$$\langle\hat{\Box}^{1,0}\omega,\omega\rangle \geq c\|\pi_1^\perp\omega\|^2. \tag{7.13}$$

The proof of Lemma 7.7 repeats arguments in the proof of Lemma 3.3.

Theorem 7.8. *There is $C > 0$ such that for all small enough δ*

$$\hat{\square}(\delta) \geq \frac{1}{2}\hat{\square}^{1,0} + \delta^2(\square^{0,1} - C) \geq \delta^2(\hat{\square}^{1,0} + \square^{0,1} - C). \tag{7.14}$$

The proof of this theorem is given in the Appendix 1 (see Theorem A.1.8).

The proof of the following corollary is a simple application of Lemma 7.7 and Theorem 7.8.

Corollary 7.9. *There exist $c_1 > 0$ and $c_2 > 0$, so that for any $\omega \in \Omega_s^\bullet(E, V)$*

$$\|\pi_1^\perp \omega\| \leq c_1 \|\hat{\square}(\delta)\omega\| + c_2\delta\|\omega\|.$$

Lemma 7.10.

$$\pi_1^\perp d^{0,1}\pi_1 = \pi_1^\perp (d^{0,1})^*\pi_1 = \pi_1 d^{0,1}\pi_1^\perp = \pi_1(d^{0,1})^*\pi_1^\perp = 0. \tag{7.15}$$

Proof. We will prove that $\pi_1^\perp d^{0,1}\pi_1 = 0$. It is enough to show that for any $\omega \in E_1^p$, $d^{0,1}\omega$ is $\hat{d}^{1,0}$-harmonic. By the commutativity relation (2.5) we have

$$\hat{d}^{1,0} d^{0,1}\omega = -d^{0,1}\hat{d}^{1,0}\omega = 0.$$

By another commutativity relation in the statement of Lemma A.1.6 part (1) we have

$$(\hat{d}^{1,0})^* d^{0,1}\omega = -d^{0,1}(\hat{d}^{1,0})^*\omega = 0.$$

Thus $d^{0,1}\omega \in E_1^{p+1} \Rightarrow \pi_1^\perp d^{0,1}\omega = 0$. □

7.4. The model operator. The asymptotics of δ-small eigenvalues of $\hat{\square}^p(\delta)$. We denote as

$$\Delta_M^p : \Omega^p(M, V \otimes o(E^-)) \to \Omega^p(M, V \otimes o(E^-)) \tag{7.16}$$

the standard Laplacian on M acting on $(V \otimes o(E^-))$-valued p-forms.

Theorem 7.11. *For any $p \geq n^-$ the operators $\pi_1 \square_p^{0,1} \pi_1 : E_1^p \to E_1^p$ and $\Delta_M^{p-n^-}$ are isospectral. For all p, $0 \leq p \leq n^-$, $\pi_1\square_p^{0,1}\pi_1 = 0$.*

Proof. It follows from the diagram in the Lemma 7.4 that operators $\Delta_m^{p-n^-}$ and

$$\pi_1 d^{0,1}\pi_1(\pi_1 d^{0,1}\pi_1)^* + (\pi_1 d^{0,1}\pi_1)^*\pi_1 d^{0,1}\pi_1$$

are isospectral. The statement of the theorem then follows from (7.15). $\square$

Let

$$0 \leq \mu_1^{p-n^-} \leq \mu_2^{p-n^-} \leq \dots \quad \text{and} \quad u_1, u_2, \dots \in E_1^p$$

denote the eigenvalues (counting multiplicities) of $\pi_1 \square_p^{0,1} \pi_1$ and the corresponding orthonormal eigenforms. By Theorem 7.11 these eigenvalues equal to the eigenvalues of $\triangle_M^{p-n^-}$.

It follows from Theorem 7.5 that

$$\dim E_2^p = \dim \ker(\pi_1 \square_p^{0,1} \pi_1) = \dim \ker(\hat{\square}^p(\delta)).$$

Therefore for small enough $\delta \geq 0$ the number of zero eigenvalues of $\hat{\square}^p(\delta)$ equals to the number of zero eigenvalues of $\triangle_M^{p-n^-}$.

Fix $0 < \epsilon < 1/2$. For every p and for each δ, $0 < \delta < 1$, we define the space $\hat{W}^p(\delta)$ to be the span of the eigenforms $\{\omega_j^p(\delta)\}_{j=1}^{\hat{k}_p(\delta)}$, satisfying

$$\hat{\square}^p(\delta)\omega_j^p(\delta) = \lambda_j^p(\delta)\omega_j^p(\delta) \text{ with } \lambda_j^p(\delta) < \delta^{2-\epsilon}, \quad j = 1, \dots, \hat{k}_p(\delta). \tag{7.17}$$

Since $\hat{\square}^p(\delta)$ has discrete spectrum, $\hat{k}_p(\delta) = \dim \hat{W}^p(\delta) < \infty$.

Similarly, we define the space $W^p(\delta) \subset E_1^p$ to be the span of the eigenforms $\{u_j^p\}_{j=1}^{k_p(\delta)}$ satisfying

$$\pi_1 \square_p^{0,1} \pi_1 u_j^p = \mu_j^{p-n^-} u_j^p \text{ with } \mu_j^{p-n^-} < \delta^{-\epsilon}, \quad j = 1, \dots, k_p(\delta). \tag{7.18}$$

Definition 7.12. We call an eigenvalue $\lambda_j^p(\delta)$ ($\mu_j^{p-n^-}$) δ-small if it satisfies the inequality

$$\lambda_j^p(\delta_1) < \delta^{2-\epsilon} \; (\mu_j^{p-n^-} < \delta^{-\epsilon}) \tag{7.19}$$

for all δ_1, $0 < \delta_1 < \delta$.

Our goal is to compare the δ-small eigenvalues of $\hat{\square}^p(\delta)$ to the δ-small eigenvalues of $\triangle^{p-n^-}$. First we want to estimate the eigenvalues of $\hat{\square}^p(\delta)$ in terms of the eigenvalues of $\triangle^{p-n^-}$. Our strategy is to show that the norm of the restriction of $\hat{\square}^p(\delta) - \pi_1 \square_p^{0,1} \pi_1$ to $W^p(\delta)$ is small.

Lemma 7.13. *Let $P(\delta)$ denote the orthogonal projection on $W^p(\delta)$. Then there exists a constant $C > 0$ such that for any p, and for all small enough $\delta > 0$ we have*

$$\|P(\delta)\hat{\square}^p(\delta)P(\delta) - \delta^2 P(\delta)\pi_1 \square_p^{0,1} \pi_1 P(\delta)\| \leq C\delta^{3-\epsilon}.$$

Corollary 7.14. *There exists a constant $C > 0$ such that for any p, and for all small enough δ, we have*

$$\lambda_j^p(\delta) \leq \delta^2 \mu_j^{p-n^-} + C\delta^{3-\epsilon}, \quad 1 \leq j \leq k_p(\delta). \tag{7.20}$$

Proof. Both operators $P(\delta)\hat{\Box}^p(\delta)P(\delta)$ and $\delta^2 P(\delta)\pi_1\Box_p^{0,1}\pi_1 P(\delta)$ are represented by symmetric matrices of the same dimension $k_p(\delta)$. Therefore, from Lemma 7.13 we can conclude that $P(\delta)\hat{\Box}^p(\delta)P(\delta)$ also has exactly $k_p(\delta)$ eigenvalues which we denote as $0 \leq \tilde{\lambda}_1^p(\delta) \leq \cdots \leq \tilde{\lambda}_{k_p(\delta)}^p(\delta)$. This eigenvalues satisfy

$$\tilde{\lambda}_j^p(\delta) \leq \delta^2 \mu_j^{p-n^-} + C\delta^{3-\epsilon}.$$

Then as a simple application of the classical min-max principle [Du-Sc] we conclude that

$$\lambda_j^p(\delta) \leq \tilde{\lambda}_j^p(\delta) \leq \delta^2 \mu_j^{p-n^-} + C\delta^{3-\epsilon},$$

which is the statement of the lemma . $\square$

Proof of Lemma 7.13. Since

$$\pi_1 \hat{d}^{1,0} = \hat{d}^{1,0}\pi_1 = \pi_1(\hat{d}^{1,0})^* = (\hat{d}^{1,0})^*\pi_1 = 0, \tag{7.21}$$

it follows from an explicit calculation of $\hat{\Box}^p(\delta)$ in (A.1.17) that

$$\begin{aligned} P(\delta)\hat{\Box}^p(\delta)P(\delta) &= P(\delta)\pi_1\hat{\Box}^p(\delta)\pi_1 P(\delta) \\ &= \delta^2 P(\delta)\pi_1\Box_p^{0,1}\pi_1 P(\delta) + \delta^3 P(\delta)K_3 P(\delta) \\ &\quad + \delta^4 P(\delta)\Box^{-1,2}P(\delta). \end{aligned} \tag{7.22}$$

Now we will estimate the second and the third terms in the right-hand side of (7.22). From an estimate

$$K_3 \leq \Box^{0,1} + \Box^{-1,2} \tag{7.23}$$

in (A.1.24) we conclude that

$$\delta^3\|P(\delta)K_3P(\delta)\| \leq \delta^3\|P(\delta)\Box_p^{0,1}P(\delta)\| + \delta^3\|P(\delta)\Box^{-1,2}P(\delta)\|. \tag{7.24}$$

Moreover, by the definition of $W^p(\delta)$

$$\|P(\delta)\Box_p^{0,1}P(\delta)\| \leq \delta^{-\epsilon}. \tag{7.25}$$

To estimate $\|P(\delta)\Box^{-1,2}P(\delta)\|$ we recall that (see A.1.21)

$$\Box^{-1,2} \leq c_1\hat{\Box}^{1,0} + c_2. \tag{7.26}$$

Therefore, since $P(\delta)\hat{\Box}^{1,0}P(\delta) = P(\delta)\pi_1\hat{\Box}^{1,0}\pi_1 P(\delta) = 0$ we conclude that

$$\|P(\delta)\Box^{-1,2}P(\delta)\| \leq \|P(\delta)(c_1\hat{\Box}^{1,0} + c_2)P(\delta)\| \leq c_2. \tag{7.27}$$

Now we use (7.22), (7.24), and (7.27) to get the following inequality

$$\begin{aligned} \|P(\delta)\hat{\Box}^p(\delta)P(\delta) &- \delta^2 P(\delta)\pi_1\Box_p^{0,1}\pi_1 P(\delta)\| \\ &\leq \delta^3\|P(\delta)\Box_p^{0,1}P(\delta)\| + 2\delta^3\|P(\delta)\Box^{-1,2}P(\delta)\| \\ &\leq \delta^{3-\epsilon} + 2c\delta^3 \leq C\delta^{3-\epsilon}. \end{aligned} \tag{7.28}$$

□

Our next goal is to estimate the δ-small eigenvalues of Δ^{p-n^-} in terms of the δ- small eigenvalues of $\hat{\Box}^p(\delta)$. As before the required result will be a corollary of the following lemma:

Lemma 7.15. *Let $\hat{P}(\delta)$ denote the orthogonal projection on $\hat{W}^p(\delta)$. Then there exists a constant $C > 0$ such that for any p and for all small enough $\delta > 0$ we have*

$$\|\hat{P}(\delta)\hat{\Box}^p(\delta)\hat{P}(\delta) - \delta^2\hat{P}(\delta)\pi_1\Box_p^{0,1}\pi_1\hat{P}(\delta)\| \leq C\delta^{3-3\epsilon/2}. \tag{7.29}$$

Corollary 7.16. *There exists a constant $C > 0$ such that for any p and for all small enough δ, we have*

$$\delta^2\mu_j^{p-n^-} \leq \lambda_j^p(\delta) + C\delta^{3-3\epsilon/2}, \quad 1 \leq j \leq \hat{k}_p(\delta). \tag{7.30}$$

In order to prove Lemma 7.15 we need several preliminary estimates which are the content of the next lemma.

Lemma 7.17. *There exists $C > 0$, such that for all δ small enough we have the following estimates:*

$$\|\hat{P}(\delta)\pi_1^{\perp}\hat{P}(\delta)\| \leq C\delta^{1-\epsilon/2}; \tag{7.31}$$

$$\|\hat{d}^{1,0}\hat{P}(\delta)\| \leq C\delta^{1-\epsilon/2}, \quad \|(\hat{d}^{1,0})^*\hat{P}(\delta)\| \leq C\delta^{1-\epsilon/2}; \tag{7.32}$$

$$\|d^{0,1}\hat{P}(\delta)\| \leq C\delta^{-\epsilon/2}, \quad \|(d^{0,1})^*\hat{P}(\delta)\| \leq C\delta^{-\epsilon/2}; \tag{7.33}$$

$$\|d^{0,1}\pi_1\hat{P}(\delta)\| \leq C\delta^{-\epsilon/2}, \quad \|(d^{0,1})^*\pi_1\hat{P}(\delta)\| \leq C\delta^{-\epsilon/2}; \tag{7.34}$$

$$\|d^{-1,2}\hat{P}(\delta)\| \leq C, \quad \|(d^{-1,2})^*\hat{P}(\delta)\| \leq C; \tag{7.35}$$

$$\|d^{-1,2}\pi_1\hat{P}(\delta)\| \leq C, \quad \|(d^{-1,2})^*\pi_1\hat{P}(\delta)\| \leq C. \tag{7.36}$$

Proof. Let ω be any form of norm one in $\hat{W}^p(\delta)$.

(7.31). It follows from Theorem 7.8 and Lemma 7.7 that

$$2\delta^{2-\epsilon} \geq 2\langle\hat{\Box}^p(\delta)\omega,\omega\rangle \geq \langle\hat{\Box}^{1,0}\omega,\omega\rangle - c_1\delta^2 \\ \geq c_2\|\pi_1^{\perp}\omega\|^2 - c_1\delta^2. \tag{7.37}$$

Inequality (7.31) easily follows from (7.37).

(7.32). We have an equality

$$\|\hat{d}^{1,0}\omega\|^2 + \|(\hat{d}^{1,0})^*\omega\|^2 = \langle\hat{\Box}^{1,0}\omega,\omega\rangle, \tag{7.38}$$

From Theorem 7.8 we can estimate $\hat{\Box}^{1,0}$. This estimate implies the following inequality:

$$\|\hat{d}^{1,0}\omega\|^2 + \|(\hat{d}^{1,0})^*\omega\|^2 \leq c_1\langle\hat{\Box}^p(\delta)\omega,\omega\rangle + c_2\delta^2 \leq c\delta^{2-\epsilon}. \tag{7.39}$$

(7.33). Similarly,

$$\delta^2\|d^{0,1}\omega\|^2 + \delta^2\|(d^{0,1})^*\omega\|^2 \leq \delta^2\langle\Box^{0,1}\omega,\omega\rangle \\ \leq \langle\hat{\Box}^p(\delta)\omega,\omega\rangle + c_2\delta^2 \leq c\delta^{2-\epsilon}. \tag{7.40}$$

Inequalities (7.33) follow from (7.40) after dividing both sides of (7.40) by δ^2.

(7.34). Since by Lemma 7.10 $\pi_1\Box^{0,1}\pi_1 + \pi_1^{\perp}\Box^{0,1}\pi_1^{\perp} = \Box^{0,1}$, we have

$$\pi_1\Box^{0,1}\pi_1 \leq \Box^{0,1}. \tag{7.41}$$

Now the inequality (7.34) follows from (7.40).

(7.35). We have an equality

$$\|d^{-1,2}\omega\|^2 + \|(d^{-1,2})^*\omega\|^2 = \langle\Box^{-1,2}\omega,\omega\rangle. \tag{7.42}$$

It follows from (7.26) that

$$\|d^{-1,2}\omega\|^2 + \|(d^{-1,2})^*\omega\|^2 \leq \langle\hat{\Box}^{1,0}\omega,\omega\rangle + c_1 \leq C. \tag{7.43}$$

Therefore both $\|d^{-1,2}\omega\|$ and $\|(d^{-1,2})^*\omega\|$ are bounded.

(7.36). The proof is similar to to the proof of the inequality (7.35). □

Proof of Lemma 7.15. Since

$$1 = \pi_1 + \pi_1^{\perp}, \tag{7.44}$$

we have

$$\hat{P}(\delta)\hat{\Box}^p(\delta)\hat{P}(\delta) = \hat{P}(\delta)\pi_1\hat{\Box}^p(\delta)\hat{P}(\delta) + \hat{P}(\delta)\pi_1^{\perp}\hat{\Box}^p(\delta)\hat{P}(\delta). \tag{7.45}$$

From (7.31) we have $\|\hat{P}(\delta)\pi_1^{\perp}\hat{P}(\delta)\| \le C\delta^{1-\epsilon/2}$. Moreover, since $\hat{P}(\delta)$ is an orthogonal projector on the space $\hat{W}(\delta)$ of δ-small forms, we have $\|\hat{\Box}^p(\delta)\hat{P}(\delta)\| \le \delta^{2-\epsilon}$. Therefore,

(7.46)
$$\begin{aligned}\|\hat{P}(\delta)\hat{\Box}^p(\delta)\hat{P}(\delta) - \hat{P}(\delta)\pi_1\hat{\Box}^p(\delta)\hat{P}(\delta)\| &= \|\hat{P}(\delta)\pi_1^{\perp}\hat{\Box}^p(\delta)\hat{P}(\delta)\| \\ &\le \|\hat{P}(\delta)\pi_1^{\perp}\hat{P}(\delta)\|\|\hat{\Box}^p(\delta)\hat{P}(\delta)\| \\ &\le C\delta^{1-\epsilon/2}\delta^{2-\epsilon} \le C\delta^{3-3\epsilon/2}.\end{aligned}$$

From an explicit calculation of $\hat{\Box}(\delta)$ in (A.1.17) and from (7.21) we conclude that

$$\begin{aligned}\hat{P}(\delta)\pi_1\hat{\Box}^p(\delta)\hat{P}(\delta) &= \delta^2 P(\delta)\pi_1\Box^{0,1}\hat{P}(\delta) + \delta^2 P(\delta)\pi_1 K_2\hat{P}(\delta) \\ &\quad + \delta^3 P(\delta)\pi_1 K_3\hat{P}(\delta) + \delta^4 P(\delta)\pi_1\Box^{-1,2}\hat{P}(\delta).\end{aligned} \tag{7.47}$$

We now estimate each term in the right-hand side of (7.47). From Lemma 7.10 we have an equality for the first term:

$$\pi_1\Box^{0,1} = \pi_1\Box^{0,1}\pi_1. \tag{7.48}$$

To estimate the next term we write

$$\delta^2\pi_1 K_2 = \delta^2\pi_1 d^{-1,2}(d^{0,1})^* + \delta^2\pi_1(d^{-1,2})^* d^{0,1}. \tag{7.49}$$

For any two norm one forms $\omega_1,\ \omega_2 \in \hat{W}^p(\delta)$ we have

$$\begin{aligned}|\langle\delta^2\pi_1 K_2\omega_1, \omega_2\rangle| &\le \delta^2|\langle(d^{1,0})^*\omega_1, (d^{-1,2})^*\pi_1\omega_2\rangle| + \delta^2|\langle d^{1,0}\omega_1, d^{-1,2}\pi_1\omega_2\rangle| \\ &\le \delta^2(\|(d^{1,0})^*\omega_1\|\|(d^{-1,2})^*\pi_1\omega_2\| + \|d^{1,0}\omega_1\|\|d^{-1,2}\pi_1\omega_2\|) \\ &\le C\delta^2\delta^{1-\epsilon/2} \le C\delta^{3-\epsilon/2}.\end{aligned} \tag{7.50}$$

We used (7.32) and (7.36) to get to the last line in the formula above. Therefore,

$$\delta^2 \|\hat{P}(\delta)\pi_1 K_2 \hat{P}(\delta)\| \le C\delta^{3-\epsilon/2}. \tag{7.51}$$

To estimate $\hat{P}(\delta)\pi_1 K_3 \hat{P}(\delta)$ we write

$$\begin{aligned}\langle \pi_1 K_3 \omega_1, \omega_2\rangle = {} & \langle d^{0,1}\omega_1, d^{-1,2}\pi_1\omega_2\rangle + \langle (d^{0,1})^*\omega_1, (d^{-1,2})^*\pi_1\omega_2\rangle \\ & + \langle d^{-1,2}\omega_1, d^{0,1}\pi_1\omega_2\rangle + \langle (d^{-1,2})^*\omega_1, (d^{0,1})^*\pi_1\omega_2\rangle.\end{aligned} \tag{7.52}$$

It follows from (7.33), (7.34), (7.35), and (7.36) that

$$|\langle \pi_1 K_3 \omega_1, \omega_2\rangle| \le C\delta^{-\epsilon/2}.$$

Therefore,

$$\delta^3 \|\hat{P}(\delta)\pi_1 K_3 \hat{P}(\delta)\| \le C\delta^{3-\epsilon/2}. \tag{7.53}$$

Similarly, we use (7.35) and (7.36) to deduce an estimate:

$$\delta^4 \|\hat{P}(\delta)\pi_1 \square^{-1,2} \hat{P}(\delta)\| \le C\delta^4. \tag{7.54}$$

Finally, by combining (7.46), (7.47), (7.48), (7.51), (7.53), and (7.54), we get the inequality in

$$\|\hat{P}(\delta)\hat{\square}^p(\delta)\hat{P}(\delta) - \hat{P}(\delta)\pi_1\hat{\square}^{0,1}\pi_1\hat{P}(\delta)\| \le C\delta^{3-2\epsilon/2},$$

which is the statement of Lemma 7.15. □

Now we combine Corollary 7.14 and Corollary 7.16 into a single theorem:

Theorem 7.18. (1) *For any p, $n^- \le p \le \dim E$, there exists C such that for all small enough δ we have*

$$|\lambda_j^p(\delta) - \delta^2 \mu_j^{p-n^-}| \le C\delta^{3-2\epsilon}, j = 1, \ldots, \min\{k^p(\delta), \hat{k}^p(\delta)\}. \tag{7.55}$$

As $\delta \to 0$, $\min\{k^p(\delta), \tilde{k}^p(\delta)\} \to \infty$.

(2) *If $p < n^-$, $\hat{\square}^p(\delta)$ does not have δ-small eigenvalues.*

Remark 7.19. Constants C in the statements of Lemmas 7.13, 7.15, 7.17, Corollaries 7.14, 7.16, and Theorem 7.18 do not depend on the choice of ϵ.

7.5. Asymptotics of the eigenforms of $\hat{\Box}^p(\delta)$. In this section we compare (rescaled) eigenforms of $\hat{\Box}^p(\delta)$ and the eigenforms of $\pi_1\Box_p^{0,1}\pi_1$.

Suppose μ is an eigenvalue of $\pi_1\Box_p^{0,1}\pi_1$. We denote as F_μ^p the corresponding eigenspace in E_1^p. For each δ we denote $F_\mu^p(\delta) = \mathrm{span}\{\omega_j(\delta)\}$, where $\hat{\Box}^p(\delta)\omega_j(\delta) = \lambda_j^p(\delta)\omega_j^p(\delta)$, with

$$|\lambda_j^p(\delta) - \delta^2\mu| \leq C\delta^3. \tag{7.56}$$

For all small enough δ, $\dim F_\mu^p(\delta) = \dim F_\mu^p$.

Theorem 7.20. *For any μ and p, and for all small enough δ we have*

$$F_\mu^p(\delta) = F_\mu^p + 0(\delta). \tag{7.57}$$

Proof. Let $\omega_j(\delta) \in F_\mu^p(\delta)$, $\|\omega_j(\delta)\| = 1$. From Corollary 7.9 we conclude that $\|\pi_1^\perp\omega_j(\delta)\| \leq C\delta$. Therefore,

$$\omega_j(\delta) = \pi_1\omega_j(\delta) + 0(\delta). \tag{7.58}$$

It follows from Remark 7.19 that we can take a limit $\epsilon \longrightarrow 0$ in the inequality (7.29). Therefore, we conclude that

$$\|\lambda_j^p(\delta)\omega_j(\delta) - \delta^2\pi_1\Box_p^{0,1}\pi_1\omega_j(\delta)\| \leq C\delta^3, \tag{7.59}$$

where the constant C depends on μ. We write

$$\lambda_j^p(\delta)\omega_j(\delta) = \lambda_j^p(\delta)\pi_1^\perp\omega_j(\delta) + \lambda_j^p(\delta)\pi_1\omega_j(\delta) - \delta^2\mu\pi_1\omega_j(\delta) + \delta^2\mu\omega_j(\delta). \tag{7.60}$$

Since $\|\lambda_j^p(\delta)\pi_1^\perp\omega_j(\delta)\| \leq C\delta^3$ and $\|(\lambda_j^p(\delta) - \delta^2\mu_j)\pi_1\omega_j(\delta)\| \leq C\delta^3$ after substitution of (7.60) into (7.59) we have

$$\begin{aligned} &\delta^2\|(\mu\pi_1 - \pi_1\Box_p^{0,1}\pi_1)\omega_j(\delta)\| \\ &\qquad \leq \|(\lambda_j^p(\delta) - \delta^2\pi_1\Box_p^{0,1}\pi_1)\omega_j(\delta)\| + C_1\delta^3 \leq C\delta^3. \end{aligned} \tag{7.61}$$

Let $P_\mu^\perp$ denote the orthogonal projection on $(F_\mu^p)^\perp$ in E_1^p, then

$$(\mu - \pi_1\Box_p^{0,1}\pi_1)\pi_1\omega_j(\delta) = P_\mu^\perp(\mu - \pi_1\Box_p^{0,1}\pi_1)P_\mu^\perp\omega_j(\delta).$$

From Theorem 7.11 it follows that there is a gap of a fixed non-zero length l between μ and the rest of the spectrum of $\pi_1\Box_p^{0,1}\pi_1$. Therefore, we have

$$\|P_\mu^\perp(\mu - \pi_1\Box_p^{0,1}\pi_1)P_\mu^\perp\omega_j(\delta)\| \geq l\|P_\mu^\perp\omega_j(\delta)\|. \tag{7.62}$$

Finally, we combine inequalities (7.61) and (7.62) to conclude:

$$P_{\mu}^{\perp}\omega_j(\delta) = 0(\delta).$$

Therefore,

$$\omega_j(\delta) = P_{\mu}\pi_1\omega_j(\delta) + 0(\delta),$$

which is the statement of the theorem. □

If $\mu = 0$, then $F_{\mu}^{p} = E_2^{p}$. Therefore, we have the following corollary of Theorem 7.20:

Corollary 7.21. *For any p and all small enough δ,*

$$\ker \hat{\Box}_{\delta}^{p} = \rho_{\delta}^{-1} \ker \Box^{p}(\delta) = \rho_{\delta}^{-1} E_2^{p} + 0(\delta),$$

where ρ_δ was defined in Section 2.3.

7.6. The main result about the spectrum of $\Box(\alpha)$. Now we reformulate Theorem 7.18 in terms of the spectrum of $\Box^{p}(\alpha)$ We recall from Section 7.1 that $\delta = \alpha^{-1/2}$ and $\lambda^{p}(\delta) = \alpha^{-1}\lambda^{p}(\alpha)$. Then we have the following definition, equivalent to Definition 7.12:

Definition 7.22. We call an eigenvalue $\lambda_j^{p}(\alpha)$ ($\mu_j^{p-n^-}$) α-bounded if it satisfies the inequality

$$\lambda_j^{p}(\alpha_1) < \alpha_1^{-\epsilon/2} \; (\mu_j^{p-n^-} < \alpha^{-\epsilon/2})$$

for all $\alpha_1 > \alpha$.

Theorem 7.16 can be reformulated as:

Theorem 7.23. (1) *For any p, $n^- \leq p \leq \dim E$ there exist constants $C > 0$ and $\alpha_0 = \alpha_0(k)$ such that for all $\alpha > \alpha_0$ we have*

$$|\lambda_j^{p}(\alpha) - \mu_j^{p-n^-}| \leq C\alpha^{-1/2+\epsilon}, j = 1, \dots k. \tag{7.63}$$

(2) *If $p \leq n^-$, $\Box^{p}(\alpha)$ does not have α-bounded eigenvalues.*

Remark 7.24. Theorem 7.20 and Corollary 7.21 can be easily reformulated to provide information about the eigenforms of $\Box(\alpha)$, which correspond to the small spectrum.

8. The asymptotic of the spectrum of Witten Laplacian on compact manifolds.

8.0. Introduction. In this section we observe that the asymptotics of the bounded eigenvalues of the Witten Laplacian

$$L(\alpha) : \Omega^\bullet(N, V) \to \Omega^\bullet(N, V)$$

can be calculated by restricting the operator $L(\alpha)$ onto tubular neighborhoods $\{E_i\}$ of connected components $\{M_i\}$ of the critical submanifold M and then applying Theorem 7.18.

In Section 8.1 we recall the definition of a Morse-Bott function and the statement of the Generalized Morse Lemma about the structure of a tubular neighborhood of a connected component of the critical submanifold.

In Section 8.2 we define a metric on N, which in the neighborhood of the critical submanifold comes from the metric on the tubular neighborhood defined in Section 1.3. In that section we also define the Witten Laplacian on N.

In Section 8.3 we prove Theorem 8.6. This theorem states that as $\alpha \to \infty$ the bounded eigenvalues of $L(\alpha)$ on N approach the eigenvalues of the standard Laplacian $\triangle$ on M, twisted by the orientation bundles $o(E^-)$. The theorem also contains the estimate of the rate of convergence of the eigenvalues of the Witten Laplacian $L(\alpha)$ on N.

8.1. The Generalized Morse Lemma. We start by giving a definition of a Morse-Bott function.

Let N be a compact smooth manifold without boundary. Let $h : N \to \mathbb{R}$ be a C^∞-function. We call a point $m \in M$ *a nondegenerate critical point of index* k if for any (or, equivalently, for some) submanifold $W \subset N$, which is transverse to M at m, the point m is a nondegenerate critical point of $h_{|W}$ of index k. A smooth submanifold M of N is called *critical* if every point of M is critical. A critical submanifold M is called *nondegenerate of index* k, if each point of M is nondegenerate of index k.

Definition 8.1. A C^∞-function $h : N \to \mathbb{R}$ is called a Morse-Bott function if all critical submanifolds of h are nondegenerate.

Definition 8.2. A *tubular neighborhood* of a submanifold $M \subset N$ is a pair (f, E), where $E \to M$ is a vector bundle over M and $f : E \to N$ is an embedding such that

(1) $f_{|M} = \mathrm{id}_{|M}$, where M is identified with the zero section of E;

(2) $f(E)$ is an open neighborhood of M in N.

Lemma 8.3 (Generalized Morse Lemma) [Hir]. *Let $h : N \to \mathbb{R}$ be a Morse–Bott function, M be a critical submanifold of h of index n^-. If M is connected, then there is a C^∞ tubular neighborhood $(f, E = E^+ \oplus E^-)$, $\dim E^- = n^-$, and a Euclidean structure on $E^+ \oplus E^-$ such that the composition $h \circ f : E^+ \oplus E^- \to \mathbb{R}$ is given by*

$$(y^+, y^-) \to |y^+|^2 - |y^-|^2 + C$$

for all $(y^+, y^-) \in E_m^+ \oplus E_m^-$, $m \in M$, and where $C = h(M)$.

8.2. The Witten Laplacian on N. Let N be a compact Riemannian manifold with the metric g_0. Let $h : N \to \mathbb{R}$ be a Morse-Bott function. We denote as $M_1, \ldots, M_\Lambda$ disjoint connected components of the critical submanifold M of h. For all j, $\mathrm{ind}(M_j) = n_j^-$. We assume in this section that all submanifolds M_j have positive dimension.

From Lemma 8.3 each M_j has a tubular neighborhood $(f_j, (p_j, E_j))$. Since submanifolds $M_1, \ldots, M_\Lambda$ are disjoint we can always assume that neighborhoods $U_1 = f_1(E_1), \ldots, U_\Lambda = f_\Lambda(E_\Lambda)$ are also disjoint.

On each E_j we put a metric g_j, chosen as in Section 1.3 . Let $\tilde{E}_j \subset E_j$ be the subset of all vectors in E_j with the norm less then 1 and $f_j(\tilde{E}_j) = \tilde{U}_j$. We choose a smooth non-negative partition of unity $\{\chi_j\}_{j=0,\ldots,\Lambda}$, such that $\mathrm{supp}(\chi_j) \subset U_j$ and $\chi_j = 1$ on $\tilde{U}_j$, $j = 1, \ldots \Lambda$. We define

$$\chi_0 = 1 - \sum_{j=1}^{\Lambda} \chi_j = 1 - \chi.$$

Finally, we put a new metric g on N, defined by

$$g = \chi_0 g_0 + \sum_{j=1}^{\Lambda} \chi_j (f_j^{-1})^* g_j. \tag{8.1}$$

Let

$$L(\alpha) : \Omega^\bullet(N, V) \to \Omega^\bullet(N, V)$$

be the Witten deformation of the Laplacian on N associated to h and g. That is

$$L(\alpha) = d(\alpha)d^*(\alpha) + d^*(\alpha)d(\alpha),$$

where

$$d(\alpha) = e^{-\alpha h} d e^{\alpha h}, \tag{8.2}$$

and $d^*(\alpha)$ is an adjoint of $d(\alpha)$ with respect to the metric g on N and the metric q_V on V.

For any p, $0 \le p \le \dim N$, and for any α the self-adjoint extension of the operator $L^p(\alpha)$ to L^2-integrable forms on N is an elliptic self-adjoint differential operator on compact manifold. Again we call this operator $L^p(\alpha)$. The spectrum of $L^p(\alpha)$ is discrete. We denote as $0 \le \nu_1^p(\alpha) \le \nu_2^p(\alpha) \ldots$ and $\phi_1^p(\alpha), \phi_2^p(\alpha), \ldots$ the eigenvalues (counting multiplicity) and the corresponding orthonormal eigenforms of $L^p(\alpha)$.

Our goal is to find the asymptotics of the bounded eigenvalues of $L(\alpha)$ as $\alpha \to \infty$.

8.3. The asymptotics of the bounded eigenvalues of $L(\alpha)$. We start by introducing some new notation. Let

$$\Box_j^p(\alpha) : \Omega_s^p(E_j, V) \to \Omega_s^p(E_j, V),$$

be the Witten Laplacian on E_j defined by

$$\Box_j^p(\alpha) = d(\alpha) d^*(\alpha) + d^*(\alpha) d(\alpha), \tag{8.3}$$

where

$$d(\alpha) = e^{-\alpha h_j} d e^{\alpha h_j}, \ h_j = f_j \circ h, \tag{8.4}$$

and $d^*(\alpha)$ is an adjoint of $d(\alpha)$ with respect to the metric g_j on E_j and the metric q_V on the flat vector bundle V over E_j (see Section 1). Here we do not distinguish between the vector bundle V on N and its push forward under f to a bundle over E.

We denote the disjoint union of E_j's as E:

$$E = E_1 \cup \cdots \cup E_\Lambda.$$

Then we define

$$\Box^p(\alpha) = \oplus_{j=1}^{\Lambda} \Box_j^p(\alpha) : \Omega_s^p(E, V) \to \Omega_s^p(E, V), \tag{8.5}$$

where

$$\Omega_s^p(E, V) = \oplus_{j=1}^{\Lambda} \Omega_s^p(E_j, V).$$

We denote as $0 \leq \lambda_1^p(\alpha) \leq \lambda_2^p(\alpha) \dots$ the eigenvalues of $\Box^p(\alpha)$ (counting multiplicities). Let $\omega_1^p(\alpha), \omega_2^p(\alpha), \dots$ be the corresponding orthonormal eigenforms. Then

$$\lambda_i^p(\alpha) = \sum_{j=1}^{\Lambda} \lambda_{l(i),j}^p(\alpha), \tag{8.6}$$

where $\lambda_{l(i),j}^p(\alpha)$ denotes the $l(i)$'s eigenvalue in the spectrum of $\Box_j^p(\alpha)$.

Fix $0 \leq \epsilon < 1$. As in Section 7.6 we give

Definition 8.4. We say that an eigenvalue $\nu_j^p(\alpha)$ ($\lambda_j^p(\alpha)$) is α-bounded if for all $\alpha_1 \geq \alpha$, $\nu_j^p(\alpha_1) < \alpha^\epsilon$ ($\lambda_j^p(\alpha_1) < \alpha^\epsilon$).

Let $k^p(\alpha)$ ($\hat{k}^p(\alpha)$) denote the number of α-bounded eigenvalues of $L(\alpha)$ ($\Box^p(\alpha)$).

The following theorem is the main result of this section. It shows that as $\alpha \to \infty$ the bounded eigenvalues of the Witten Laplacian $L(\alpha)$ on N converge to the bounded eigenvalues of the Witten Laplacian $\Box(\alpha)$ on the disjoint union of tubular neighborhoods of the critical submanifolds.

Theorem 8.5. *There exists a constant $C > 0$, such that for all large enough α and for any p we have*

$$|\lambda_j^p(\alpha) - \nu_j^p(\alpha)| \leq C\alpha^{-1/2}, j = 1, \dots, \min\{k^p(\alpha), \hat{k}^p(\alpha)\}. \tag{8.7}$$

Moreover, as $\alpha \to \infty$, $\min\{k^p(\alpha), \hat{k}^p(\alpha)\} \to \infty$.

The proof of this theorem is given in Appendix 3.

Fix ϵ with $0 \leq \epsilon < 1/2$. We now apply Theorem 7.23 to the setting of this section. To do so we need to introduce some new notation. For each critical submanifold M_j of $\operatorname{ind}(M_j) = n_j^-$ and $\dim M_j = m_j$ we denote by $\sigma_p(M_j)$ the spectrum of the standard Laplacian $\Delta^{p-n_j^-}$ on $\Omega^{p-n_j^-}(M_j, V_{|M_j} \otimes o(E_j^-))$. This spectrum is only defined for $n_j^- \leq p \leq m_j + n_j^-$. We define $\sigma_p(M_j)$ to be empty otherwise. We denote as $\sigma_p(M)$ the union of $\sigma_p(M_j)$ over all critical submanifolds M_j for which inequality $n_j^- \leq p \leq m_j + n_j^-$ is satisfied. We arrange all numbers from the set $\sigma_p(M)$ in non-decreasing order $0 \leq \mu_1^p \leq \mu_2^p \leq \dots$. In other words each μ_i^p is an eigenvalue of the standard Laplacian on $\Omega^{p-n_j^-}(M_j, V_{|M_j} \otimes o(E_j^-))$ for some $1 \leq j \leq \Lambda$.

By applying Theorem 7.23 to the Witten Laplacian on $E = E_1 \cup \dots \cup E_\Lambda$ we see that there is a constant C, such that for all large enough α

$$|\lambda_i^p(\alpha) - \mu_i^p| \leq C\alpha^{-1/2+\epsilon}, i = 1, \dots, \min\{k^p(\alpha), \hat{k}^p(\alpha)\}. \tag{8.8}$$

Inequalities (8.7), (8.8), and the fact that $\min\{k^p(\alpha), \hat{k}^p(\alpha)\} \to \infty$ prove the following theorem:

Theorem 8.6. *There exist constants $C > 0$ and $\alpha_0 = \alpha_0(k)$ such that for all $\alpha \geq \alpha_0$ and for all p*

$$|\nu_j^p(\alpha) - \mu_j^p| \leq C\alpha^{-1/2+\epsilon}, j = 1, \ldots k. \tag{8.9}$$

Remark 8.7. Theorem 8.6 stays true if for some j $\dim M_j = 0$ (i.e M_j is a point). In this case if $p = \mathrm{ind}(M_j)$ we set $\sigma_p(M_j) = \{0\}$ and $\sigma_p(M_j) = \emptyset$ otherwise.

9. The Morse-Bott Inequalities.

9.0. Introduction. In this section we prove the Morse-Bott inequalities as an application of the results of Section 8. The Morse-Bott inequalities (or the degenerate Morse inequalities of R. Bott) [Bott1] relate the Betti numbers of a compact manifold N to the Betti numbers of the connected components of the critical submanifold M of a Morse-Bott function h on N.

The non-degenerate Morse inequalities are a particular case of the Morse-Bott inequalities when a critical submanifold M is a union of a finite number of critical points.

In order to prove the Morse-Bott inequalities we only need the estimates on the number of zero eigenvalues of $L(\alpha)$ and the existence of the *spectral gap* separating zero eigenvalues of $L(\alpha)$ from the rest of the spectrum ([Bott2], [Bra-Far]).

9.1. A proof of the Morse-Bott inequalities. Let ν be an eigenvalue of $L^p(\alpha)$. Let $F_\nu^p(\alpha)$ be the corresponding eigenspace. For any non-negative number a we define $\Omega_a^p(\alpha)$ to be

$$\Omega_a^p(\alpha) = \oplus_{\nu \leq a} F_\nu^p(\alpha). \tag{9.1}$$

Lemma 9.1. *For any $a \geq 0$, $\alpha \geq 0$ the following sequence*

$$0 \to \Omega_a^0(\alpha) \to \Omega_a^1(\alpha) \to \cdots \to \Omega_a^n(\alpha) \to 0. \tag{9.2}$$

computes the V-valued de-Rham cohomology of N. In this sequence all the arrows are $d(\alpha)$'s .

Proof. Fix $\nu > 0$. We observe that $d(\alpha)$ commutes with $L(\alpha)$ and thus

$$d(\alpha)F^p_\nu(\alpha) \subseteq F^{p+1}_\nu(\alpha)$$

Therefore, we have the exact sequence:

$$0 \to F^0_\nu(\alpha) \to F^1_\nu(\alpha) \to \cdots \to F^n_\nu(\alpha) \to 0. \tag{9.3}$$

To see that (9.3) is exact, we recall that we have the following orthogonal Hodge decomposition:

$$\Omega^\bullet(N, V) = \operatorname{image} d(\alpha) \oplus \operatorname{image} d^*(\alpha) \oplus \ker L(\alpha). \tag{9.4}$$

Since both $d(\alpha)$ and $d^*(\alpha)$ commute with $L(\alpha)$, it follows from (9.4) that for any p,

$$F^p_\nu(\alpha) = d(\alpha)F^{p-1}_\nu(\alpha) \oplus d^*(\alpha)F^{p+1}_\nu(\alpha).$$

Therefore,

$$\ker(d(\alpha) : F^p_\nu(\alpha) \to F^{p+1}_\nu(\alpha)) = \operatorname{image}(d(\alpha) : F^{p-1}_\nu(\alpha) \to F^p_\nu(\alpha)). \tag{9.5}$$

Equality (9.5) proves the exactness of (9.4).

The following observation completes the proof of the lemma. From definition (8.2) of $d(\alpha)$ we conclude that the map

$$e^{-\alpha h} : \Omega^\bullet(N, V) \to \Omega^\bullet(N, V)$$

induces an isomorphism between $H^\bullet(N, V, \alpha)$, the cohomology of

$$(\Omega^\bullet(N, V),\ d(\alpha)),$$

and $H^\bullet(N, V)$. Therefore, for any p, $H^p(N, V, \alpha) = \ker \Box^p(\alpha)$ is isomorphic to $H^p(N, V)$. $\Box$

For every j, $0 \le j \le \Lambda$, we define the *twisted Betti numbers* $b^-_{p,j}$ of the critical submanifold M_j by

$$b^-_{p,j} = \dim \ker \left(\triangle^p_j :\ \Omega^p\left(M_j, V \otimes o(E^-_j)\right) \to \Omega^p\left(M_j, V \otimes o(E^-_j)\right)\right). \tag{9.6}$$

In particular, $b^-_{p,j} = 0$ for $p > \dim M_j$.

Choose a to be less than the smallest non-zero eigenvalue from

$$\bigcup_{p=1}^{k} \sigma_p(M),$$

then from Theorem 8.6 we conclude that for all large enough α

$$\dim \Omega^p_a(\alpha) = \sum_{i=1}^{\Lambda} b^-_{p-n_i,i}. \tag{9.7}$$

The following theorem is a standard consequence of Lemma 9.1 and (9.7).

Theorem 9.2. *For any p with $1 \leq p \leq \dim N$, the following inequality holds:*

$$\begin{aligned}\sum_{i=1}^{\Lambda}[b^-_{p-n_i^-,i} - b^-_{p-n_i^- -1,i} + \cdots + (-1)^{p-n_i^-} b^-_{0,i}] \\ \geq B_p - B_{p-1} + \cdots + (-1)^p B_0.\end{aligned} \tag{9.8}$$

If $p = n$ then (9.8) is an equality.

Let $P(t)$ be the Poincare polynomial for $H^\bullet(N.V)$. For each i, $1 \leq i \leq \Lambda$, let $P_i^-(t)$ be the Poincare polynomial for $H^\bullet\big(M_i, V \otimes o(E_i^-)\big)$, the cohomology of M_i twisted by the orientation bundle of E_i^-. The Morse-Bott inequalities [Bott1] say that there exists a polynomial $Q(t)$ given by $Q(t) = Q_0 + Q_1 t + \ldots$ with all non-negative coefficients, such that

$$\sum_{i=1}^{\Lambda}\left(t^{n_i^-} P_i^-(t) - P(t)\right) = Q(t)(1+t). \tag{9.9}$$

It is an easy observation that (9.8) and (9.9) are equivalent.

In order to recover the non-degenerate Morse inequalities, we assume that all M_i's are critical points and V is a bundle of rank one. Let m_k denote the number of critical points of index k. Then $b^{i,-}_{p-n_i^-} = 1$ if $p = n_i^-$ and $b^{i,-}_{p-n_i^-} = 0$ if $p \neq n_i^-$. The inequality (9.8) becomes

$$m_p - m_{p-1} + \cdots + (-1)^p m_0 \geq B_p - B_{p-1} + \cdots + (-1)^p B_0. \tag{9.10}$$

The inequalities (9.10) are called *the non-degenerate Morse inequalities* ([BFKS]).

Appendix 1. The Bismut connection and bounds on the Witten Laplacian.

A.1.0. Introduction. In this appendix we make computations with the Bismut connection on the tangent space TE of a vector bundle E. This connection is defined as the direct sum of a chosen Euclidean connection on E and the Levi-Civita connection on the base M. The Bismut connection is a more natural choice of a connection for our purposes than the Levi-Civita connection on TE because (as it will be seen in Section A.1.1) the Bismut connection preserves the decomposition of TE into horizontal and vertical subspaces. The drawback of the Bismut connection is that it has a non-trivial torsion.

The Bismut connection was introduced in [Bis]. It is also studied in [B-G-V].

In Section A.1.1 we define the Bismut connection and describe its torsion and curvature tensors.

In Section A.1.2 we choose a basis on TE and express the differential d in terms of the Bismut connection. In order to simplify proofs we assume in sections A.1.2 and A.1.3 that the bundle V is a one-dimensional trivial bundle.

In Section A.1.3 we give estimates on the Witten Laplacian $\hat{\Box}(\delta)$ by writing $\hat{\Box}(\delta)$ in terms of the Bismut connection. These estimates are used in Section 7.

A.1.1. The Bismut connection. The curvature and the torsion of the Bismut connection. Let ∇^E be a Euclidean connection on E chosen in Section 1.3, and let ∇^{TM} be the Levi–Civita connection on TM. Then we define the Bismut connection $\bar{\nabla}$ on TE by

$$\bar{\nabla} = \nabla^E \oplus \nabla^{TM}. \tag{A.1.1}$$

For any $e_1 \in E$, in order to parallel translate vector $X \in T_{e_1}E$ along the path $\gamma : [0,1] \to E$ with $\gamma(0) = e_1$, $\gamma(1) = e_2$, we identify X with the pair $(X^{\text{ver}}, X^{\text{hor}}) \in A \oplus B$, where $A \oplus B$ is identified by means of ∇^E with $E \oplus T_{p(e_1)}M$. Then the result of the parallel translation will be the pair $(\tilde{X}^{\text{ver}}, \tilde{X}^{\text{hor}}) \in E \oplus T_{p(e_2)}M$, where $\tilde{X}^{\text{ver}}$ is the result of the parallel translation of X^{ver} along $p\gamma$ with respect to ∇^E and $\tilde{X}^{\text{hor}}$ is the result of the parallel translation of X^{hor} along $p\gamma$ with respect to ∇^{TM}.

After Bismut [Bis] we denote as $\bar{T}(X,Y)$ the value of the torsion tensor $\bar{T}$ of $\bar{\nabla}$ on the vectors X, Y. Similarly, we denote as $\bar{L}(X,Y)$ the value of the curvature tensor $\bar{L}$ of $\bar{\nabla}$ on vectors X, Y. By definition

$$\bar{T}(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y], \tag{A.1.2}$$

$$\bar{L}(X,Y) = \bar{\nabla}_X \bar{\nabla}_Y - \bar{\nabla}_Y \bar{\nabla}_X - \bar{\nabla}_{[X,Y]}. \tag{A.1.3}$$

We denote as L and R respectively the curvature tensors associated to ∇^E and ∇^{TM}. That is R is just the Levi-Civita curvature of TM and $L \in \Omega^2(M, \mathrm{Hom}(E,E))$ is defined for $X \in TM$, $Y \in TM$ by

$$L(X,Y) = \nabla^E_X \nabla^E_Y - \nabla^E_Y \nabla^E_X - \nabla^E_{[X,Y]}. \tag{A.1.4}$$

Then it is a matter of simple calculations to get the following lemma.

Lemma A.1.1 ([Bis, Theorem 2.1]). *The metric g on TE is parallel for $\bar{\nabla}$. Moreover if $X, Y, Z \in T_yE$, then*

$$\bar{T}(X,Y) = [L(X^{hor}, Y^{hor})y] = -[X,Y]^{ver}, \tag{A.1.5}$$

$$\bar{L}(X,Y)Z = [L(X^{hor}, Y^{hor})Z^{ver}] + [R(X^{hor}, Y^{hor})Z^{hor}]. \tag{A.1.6}$$

A.1.2. A choice of basis. An expression of the Witten Laplacian in terms of the Bismut connection. Take $x \in M$. Let $\{a_i\}_{i=1,\dots,n}$, $\{b_j\}_{j=1,\dots,m}$ be orthogonal bases of E_x, T_xM. Let $\{a^i\}_{i=1,\dots,n}$, $\{b^j\}_{j=1,\dots,m}$ be the corresponding dual bases. Take $y \in E_x$. We can lift $\{a_i\}$ and $\{b_j\}$ to TE. Since there is no risk of confusion we can assume as well that $\{a_i\}_{i=1,\dots,n}$ is the basis of A_y and $\{b_j\}_{j=1,\dots,m}$ is the basis of B_y.

From the definition of the Bismut connection it follows that for all i and j

$$\bar{\nabla}_{a_i} a_j = 0 \text{ and } \bar{\nabla}_{a_i} b_j = 0. \tag{A.1.7}$$

In addition, we can choose the a vertical basis to be parallel in the horizontal direction. Then for all i and j

$$\bar{\nabla}_{b_i} a_j = 0. \tag{A.1.8}$$

Now we describe the formulas for the operators d and d^* in terms of the Bismut connection. In order to simplify the computations we assume that the bundle V is a trivial one-dimensional bundle over N.

We denote by $i(v)$ the operator of interior multiplication by the vector v. Then we have the following lemma:

Lemma A.1.2 ([Bis, Proposition 2.2]). *For any $(x,y) \in E$, we have*

$$\begin{aligned} d &= a^j \bar{\nabla}_{a_j} + b^j \bar{\nabla}_{b_j} + \frac{1}{2} b^k \wedge b^l i([L(b_k, b_l)y]), \\ d^* &= -i(a_j)\bar{\nabla}_{a_j} - i(b_j)\bar{\nabla}_{b_j} - \frac{1}{2}[L(b_k, b_l)y] i(b_k) i(b_l). \end{aligned} \tag{A.1.9}$$

The following corollary expresses operators $d^{1,0}$, $d^{0,1}$, $d^{-1,2}$ and their adjoints (defined in Section 7.1) in terms of the Bismut connection.

Corollary A.1.3.

$$d^{1,0} = a^j \bar{\nabla}_{a_j},\ (d^{1,0})^* = -i(a_j)\bar{\nabla}_{a_j}, \tag{A.1.10}$$

$$d^{0,1} = b^j \bar{\nabla}_{b_j},\ (d^{0,1})^* = -i(b_j)\bar{\nabla}_{b_j}, \tag{A.1.11}$$

$$\begin{aligned} d^{-1,2} &= \frac{1}{2} b^k \wedge b^l i([L(b_k, b_l)y]^{ver}), (d^{-1,2})^* \\ &= -\frac{1}{2}[L(b_k, b_l)y] i(b_k) i(b_l). \end{aligned} \tag{A.1.12}$$

Corollary A.1.4.

$$d^{2,-1} = (d^{2,-1})^* = 0.$$

Now we are ready to compute $\square(\alpha)$:

Theorem A.1.5. *For any α,*

$$\square(\alpha) = \square + \alpha^2 |dh|^2 + \alpha A, \tag{A.1.13}$$

where for all p the operator A is a bounded endomorphism of $\Lambda^p(E)$.

Proof. We have $d(\alpha) = d + \alpha dh\wedge$ and $d^*(\alpha) = d^* + \alpha i(\nabla h)$. Therefore,

$$\begin{aligned} \square(\alpha) &= \square + \alpha^2 \left((dh\wedge) i(\nabla h) + i(\nabla h)(dh\wedge)\right) \\ &\quad + \alpha \left(di(\nabla h) + i(\nabla h)d + d^*(dh\wedge) + (dh\wedge)d^*\right) \\ &= \square + \alpha^2 |dh|^2 + \alpha A. \end{aligned}$$

We recall that $dh = d^{1,0}h$ is a $(1,0)$-operator. By making computations at a point $(x, y) \in E$ and writing $dh = \frac{\partial h}{\partial y_j} a^j$, $i(\nabla h) = \frac{\partial h}{\partial y_j} i(a_j)$, it is easy to conclude that

$$d^{0,1} i(\nabla h) + i(\nabla h) d^{0,1} = 0,\ (d^{0,1})^*(dh\wedge) + (dh\wedge)(d^{0,1})^* = 0, \tag{A.1.14}$$

and

$$d^{-1,2} i(\nabla h) + i(\nabla h) d^{-1,2} = 0,\ (d^{-1,2})^*(dh\wedge) + (dh\wedge)(d^{-1,2})^* = 0. \tag{A.1.15}$$

Thus,

$$
\begin{aligned}
A &= d^{1,0}i(\nabla h) + i(\nabla h)d^{1,0} + (d^{1,0})^*(dh\wedge) + (dh\wedge)(d^{1,0})^* \\
&= \frac{\partial^2 h}{\partial y_j^2}(a^j i(a_j) - i(a_j)a^j) = 0.
\end{aligned}
$$

Since $a^j i(a_j) + i(a_j)a^j = 1$, $\frac{\partial^2 h}{\partial y_j^2} = 2$, for $j = 1,\dots,n^+$ and $\frac{\partial^2 h}{\partial y_j^2} = -2$, for $j = n^+,\dots,n$, we have

$$
(A.1.16) \qquad A = 2(n^- - n^+) + 2\sum_{j=1}^{n^+} a^j i(a_j) - 2\sum_{j=n^++1}^{n} a^j i(a_j).
$$

For any ω of the form $\phi a^{i_1}\wedge\cdots\wedge a^{i_k}\wedge b^{i_{k+1}}\wedge\cdots\wedge b^{i_p}$, $a^j i(a_j)\omega = 1$ if $j\in\{i_1,\dots,i_k\}$ and $a^j i(i_k) = 0$ otherwise. Explicit formula (A.1.16) shows that A is a bounded endomorphism of $\Lambda^p(E)$. $\square$

A.1.3. Some estimates for the Witten Laplacian. In order to estimate the adiabatic deformation of the Witten Laplacian we explicitly compute $\hat{\square}(\delta)$. Let $\square^{a,b}$ denote $d^{a,b}(d^{a,b})^* + (d^{a,b})^* d^{a,b}$, then

$$
\begin{aligned}
\hat{\square}(\delta) &= \hat{\square}^{1,0} + \delta^2\square^{0,1} + \delta^4\square^{-1,2} \\
&\quad + \delta((d^{1,0}(d^{0,1})^* + (d^{0,1})^* d^{1,0} + (d^{1,0})^* d^{0,1} + d^{0,1}(d^{1,0})^*)) \\
(A.1.17) &\quad + \delta^2(d^{1,0}(d^{-1,2})^* + (d^{-1,2})^* d^{1,0} + (d^{1,0})^* d^{-1,2} + d^{-1,2}(d^{1,0})^*) \\
&\quad + \delta^3(d^{0,1}(d^{-1,2})^* + (d^{-1,2})^* d^{0,1} + (d^{0,1})^* d^{-1,2} + d^{-1,2}(d^{0,1})^*),
\end{aligned}
$$

where $\hat{\square}^{1,0} = \square^{1,0} + |dh|^2 + A$. In our computation first we used the fact that the multiplication by e^{-h} commutes with $d^{0,1}$ and $d^{-1,2}$, and then equalities (A.1.14) and (A.1.15). To simplify notation we will write

$$
(A.1.18) \qquad \hat{\square}(\delta) = \hat{\square}^{1,0} + \delta^2\square^{0,1} + \delta^4\square^{-1,2} + \delta K_1 + \delta^2 K_2 + \delta^3 K_3,
$$

Lemma A.1.6.

(1) $d^{1,0}(d^{0,1})^* + (d^{0,1})^* d^{1,0} = 0$, $(d^{1,0})^* d^{0,1} + d^{0,1}(d^{1,0})^* = 0$.

(2) *The operator K_2 is bounded zeroth order.*

(3) *The operator $\square^{-1,2}$ is a zeroth order operator. Moreover for any $\omega\in\Omega^\bullet_s(E,V)$, $|\square^{-1,2}\omega|\le C|y|^2$.*

Proof. To prove part (1) it is enough to show that $(d^{1,0})^*d^{0,1}+d^{0,1}(d^{1,0})^*=0$. We use Corollary A.1.3 to write

$$\begin{aligned}(d^{1,0})^*d^{0,1}+d^{0,1}(d^{1,0})^* &= -i(a_j)\bar{\nabla}_{a_j}b^k\bar{\nabla}_{b_k}-b^k\bar{\nabla}_{b_k}i(a_j)\bar{\nabla}_{a_j}\\ &= -i(a_j)b^k\bar{\nabla}_{a_j}\bar{\nabla}_{b_k}-b^ki(a_j)\bar{\nabla}_{b_k}\bar{\nabla}_{a_j}\\ &= -i(a_j)b^k(\bar{\nabla}_{a_j}\bar{\nabla}_{b_k}-\bar{\nabla}_{b_k}\bar{\nabla}_{a_j})\\ &= -i(a_j)b^k(\bar{L}(a_j,b_k)+\bar{\nabla}_{[a_j,b_k]})=0.\end{aligned}$$

To get from the first to the second line in the formula above we use (A.1.7) and (A.1.8). Then we use the anti-commutativity relation $i(a_j)b^k+b^ki(a_j)=0$. To get the equality to 0 in the last line in the formula we use (A.1.5) and (A.1.6).

To prove part (2) and part (3) of the lemma we explicitly computeK_2 and $\Box^{-1,2}$ in terms of the Bismut connection. From [Bis, Proposition 2.6] we have

$$K_2=\frac{1}{2}\left(a^j\wedge[L(b_k,b_l)a_j]i(b_k)i(b_l)+i(a_j)i([L(b_k,b_l)a_j])b_k\wedge b_l\right). \tag{A.1.19}$$

and

$$\begin{aligned}\Box^{-1,2}=&-\frac{1}{4}(b^k\wedge b^li([L(b_k,b_l)y])[L(b_{k'},b_{l'})y]i(b_{k'})i(b_{l'})\\ &+[L(b_{k'},b_{l'})y]i(b_{k'})i(b_{l'})b^k\wedge b^li([L(b_k,b_l)y])).\end{aligned} \tag{A.1.20}$$

The statement of part (2) of the lemma then follows from the fact that the norm of $[L(b_k,b_l)a_j]$ is bounded.

To see (3) we observe that $[L(b_k,b_l)y]$ is linear in y. Therefore, we have an estimate $|[L(b_k,b_l)y]|\le c|y|$ which becomes an estimate on $\Box^{-1,2}$. $\Box$

Corollary A.1.7. *There exist constants c_1 and c_2 such that*

$$\Box^{-1,2}\le c_1\hat{\Box}^{1,0}+c_2. \tag{A.1.21}$$

Proof. The proof of the corollary is the following string of inequalities:

$$\Box^{-1,2}\le c|y|^2\le c_1|dh|^2\le c_1\hat{\Box}^{1,0}+c_2.$$

$\Box$

Next we estimate the operator $\hat{\Box}(\delta)$ from below in the following theorem:

Theorem A.1.8 (Same as Theorem 7.8). *There exists a constant $C > 0$ such that for all δ small enough*

$$\hat{\Box}(\delta) \geq \frac{1}{2}\hat{\Box}^{1,0} + \delta^2(\Box^{0,1} - C) \geq \delta^2(\hat{\Box}^{1,0} + \Box^{0,1} - C). \tag{A.1.22}$$

Proof. We observe that $\Box^{-1,2}$ is non-negative. Moreover, by Lemma A.1.6 the operator K_1 equals to zero, and K_2 is a bounded zeroth order operator. Therefore, there exists c, so that

$$\hat{\Box}(\delta) \geq \hat{\Box}^{1,0} + \delta^2\Box^{0,1} + \delta^3 K_3 - c\delta^2. \tag{A.1.23}$$

To estimate K_3 we observe that for any $\omega \in \Omega^\bullet_s(E, V)$ we have the inequality:

$$\langle \delta^3 K_3\omega, \omega\rangle = 2\langle \delta^{3/2}d^{0,1}\omega, \delta^{3/2}d^{-1,2}\omega\rangle + 2\langle \delta^{3/2}(d^{0,1})^*\omega, \delta^{3/2}(d^{-1,2})^*\omega\rangle,$$

so that

$$|\langle \delta^3 K_3\omega, \omega\rangle| \leq \delta^3\langle \Box^{0,1}\omega, \omega\rangle + \delta^3\langle \Box^{-1,2}\omega, \omega\rangle \tag{A.1.24}$$

Finally, by Corollary A.1.7

$$\langle \Box^{-1,2}\omega, \omega\rangle \leq c_1\langle \Box^{1,0}\omega, \omega\rangle + c_2\langle \omega, \omega\rangle. \tag{A.1.25}$$

Thus, for $\delta \leq 1/2$

$$\begin{aligned} \hat{\Box}(\delta) &\geq \hat{\Box}^{1,0} + \delta^2\Box^{0,1} - c\delta^3\hat{\Box}^{1,0} - C\delta^2 \\ &\geq 1/2\hat{\Box}^{1,0} + \delta^2(\Box^{0,1} - c). \end{aligned} \tag{A.1.26}$$

This inequality is the statement of the theorem. $\Box$

Appendix 2
The space of rapidly decreasing forms.

A.2.0. Introduction. It is useful to recall in a slightly different form the definition of the space of rapidly decreasing forms from Section 1.9. For any choice of numbers $a \geq 0$ and l, $l = 0, 1, 2, \ldots$, we define the space

$$S_{a,l} = \{\omega \in \Omega^\bullet(E, V) |\ \|(1 + |y|)^a(\bar{\nabla})^\kappa\omega\| \leq \infty,\ |\kappa| = 0, 1, 2, \ldots, l\}, \tag{A.2.1}$$

where $\bar{\nabla}$ is the Bismut connection, defined in the Section 1.4, and

$$(\bar{\nabla})^\kappa = \bar{\nabla}_{e_{i_1}} \circ \cdots \circ \bar{\nabla}_{e_{i_{|\kappa|}}}$$

for a multi-index $\kappa = \{i_1, \ldots, i_{|\kappa|}\}$. Here

$$\{e_1, \ldots, e_{m+n}\} = \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_m\}$$

is the basis of TE.

We say that ω belongs to the *space of rapidly decreasing forms* $\Omega^\bullet_s(E,V)$ if $\omega \in S_{a,l}$ for all $a \geq 0$ and $l \geq 0$. We denote as $S_{\infty,l}$ an intersection of $S_{a,l}$ for all $a \geq 0$. This is the same definition as in the Section 1.9. In this notation $\Omega_s(E,V) = S_{\infty,\infty}$.

The space of rapidly decreasing forms is very convenient to work with since, on one hand, for large $\alpha > 0$ the eigenforms of $\square(\alpha)$ are rapidly decreasing forms. On the other hand, $\Omega^\bullet_s(E,V)$ is invariant under operators $d^{1,0}$, $d^{0,1}$, $d^{-1,2}$ and under exterior multiplication by dh. In particular, $\Omega^\bullet_s(E,V)$ is also invariant under $d(\alpha)$ and $d^*(\alpha)$.

The goal of this appendix is to prove the following two theorems:

Theorem A.2.1. *The space of rapidly decreasing forms is invariant under operators $d^{1,0}$, $d^{0,1}$, $d^{-1,2}$, $dh\wedge$, and their adjoints. In particular, $\Omega^\bullet_s(E,V)$ is invariant under $d(\alpha)$ and $d^*(\alpha)$.*

Theorem 1.5. *For any large enough α and any $\omega \in \Omega^\bullet(E,V,\alpha)$, such that $\square(\alpha)\omega = \lambda(\alpha)\omega$, we have $\omega \in \Omega^\bullet_s(E,V)$; i. e. the eigenforms of $\square(\alpha)$ are rapidly decreasing forms.*

We will prove Theorem A.2.1 in Section A.2.1. The proof of Theorem 1.5 is contained in Section A.2.2.

A.2.1. A proof of Theorem A.2.1. To simplify the computations we assume that the vector bundle V is a trivial one-dimensional vector bundle.

We recall from Appendix 1 (Corollary A.1.3) the expression of the components of the differential d in terms of the Bismut connection.

$$d^{1,0} = a^j\bar{\nabla}_{a_j},\ (d^{1,0})^* = -i(a_j)\bar{\nabla}_{a_j}, \tag{A.2.1}$$

$$d^{0,1} = b^j\bar{\nabla}_{b_j},\ (d^{0,1})^* = -i(b_j)\bar{\nabla}_{b_j}, \tag{A.2.2}$$

$$d^{-1,2} = \frac{1}{2}b^k \wedge b^l i([L(b_k,b_l)y]^{ver}), \tag{A.2.3}$$

$$(d^{-1,2})^* = -\frac{1}{2}[L(b_k,b_l)y]i(b_k)i(b_l). \tag{A.2.4}$$

By definition, the space of rapidly decreasing forms is invariant under $\bar{\nabla}_{e_i}$. Therefore, we only need to show that $\Omega^\bullet_s(E,V)$ is invariant under exterior multiplications by the basis elements $\{e^j\}$ of T^*E, by dh, by $L(b_k, b_l)y$, and by all the adjoints of those operators.

Let $\omega \in \Omega_s^\bullet(E,V)$. As an example we show that $e^j \wedge \omega \in \Omega_s^\bullet(E,V)$. Since $\|e^j \wedge \omega\| = \|\omega\|$, and the operator $e^j\wedge$ commutes with the multiplication by $(1+|y|)^a$, we conclude that $e^j \wedge \omega \in S_{\infty,0}$ for all $j = 1,\dots,m+n$.

To show that $e^j \wedge \omega \in S_{\infty,\infty}$ we will use the simultaneous (for all j) induction in l. The case $l = 0$ is settled above.

Assume that $e^j \wedge \omega \in S_{\infty,l}$ for all $j = 1,\dots,m+n$.

We want to show that $e^j \wedge \omega \in S_{\infty,l+1}$ for all $j = 1,\dots,m+n$. It is equivalent to show that for any multi-index κ, $|\kappa| = l+1$, we have $\bar\nabla^\kappa(e^j \wedge \omega) \in S_{\infty,0}$. We write

$$\bar\nabla^\kappa(e^j \wedge \omega) = e^j \wedge \bar\nabla^\kappa\omega + [\bar\nabla^\kappa, e^j] \wedge \omega, \tag{A.2.5}$$

where $[\bar\nabla^\kappa, e^j] = \bar\nabla^\kappa e^j - e^j\bar\nabla^\kappa$ is the commutator. Since $\bar\nabla^\kappa\omega \in \Omega_s^\bullet(E,V)$, it follows that $e^j \wedge \bar\nabla^\kappa\omega \in S_{\infty,0}$.

The crucial step is to show that $[\bar\nabla^\kappa, e^j] \wedge \omega \in S_{\infty,0}$. This step easily follows from out next lemma.

Lemma A.2.2. *The commutator $[\bar\nabla^\kappa, e^j]$ can be represented as*

$$[\bar\nabla^\kappa, e^j] = e^r \wedge P_{\kappa,j,r}(\bar\nabla, y). \tag{A.2.6}$$

where each $P_{\kappa,j,r}(\bar\nabla, y)$ is a polynomial in y and $\bar\nabla$ of the form

$$\sum_{\beta,\gamma,|\gamma|\le l} C_{\beta,\gamma}(x) y^\beta \bar\nabla^\gamma$$

Here β and γ are multi-indeces, and coefficients $C_{\beta,\gamma}(x)$ depend only on the coordinate on the base.

The statement of the lemma above can be deduced from the calculation of the commutator $[\bar\nabla^\kappa, e^j]$ in local coordinates. Local expressions for the Bismut connection and the basis elements are all linear in y. Moreover, the transition functions between the coordinate neighborhoods $U \times \mathbb{R}^n$ which cover the bundle E are also linear in y.

A.2.2. A proof of Theorem 1.5. We start by observing that the local elliptic estimates imply that for any α all eigenforms of the Witten Laplacian $\Box(\alpha)$ are smooth.

The idea of the proof of Theorem 1.5 is to show by means of elliptic estimates that for large α the powers of the Witten Laplacian control the powers of $|y|$ and the powers of covariant derivatives with respect to the Bismut connection.

The following lemma is the first step in the proof of Theorem 1.5.

Lemma A.2.3. *For any $\alpha > 0$ and any $\omega \in \Omega^\bullet(E, V, \alpha)$, such that $\Box(\alpha)\omega = \lambda(\alpha)\omega$, we have $\omega \in S_{\infty,0}$, $d(\alpha)\omega \in S_{\infty,0}$, and $d^*(\alpha)\omega \in S_{\infty,0}$.*

Proof. We will start the proof by defining the family $\{J_t |\ t \geq 0\}$ of cut-off functions on E.

Let $\phi(s) : \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function defined by $\phi(s) = 1$ for $0 \leq s \leq 1$, $\phi(s) = 0$ for $s > 2$. We define the family $J_t(y) : E \to \mathbb{R}$ by $J_t(y) = \phi(|y|/t)$, where $|y|$ is the Euclidean norm of $y \in E$. Then for any $t > 0$ and any form $\omega \in \Omega^\bullet(E, V, \alpha)$, such that $\Box(\alpha)\omega = \lambda(\alpha)\omega$, we have

$$\lambda(\alpha)\|J_t|y|^a\omega\|^2 = \langle J_t^2|y|^{2a}\Box(\alpha)\omega, \omega\rangle. \tag{A.2.7}$$

After substituting $\Box(\alpha) = \Box + \alpha^2|dh|^2 + \alpha A$ into (A.2.7), we have

$$\begin{aligned}\lambda(\alpha)\|J_t|y|^a\omega\|^2 = \langle J_t^2|y|^{2a}\Box\omega, \omega\rangle + \alpha^2\langle J_t^2|y|^{2a}|dh|^2\omega, \omega\rangle \\ + \alpha\langle J_t^2|y|^{2a}A\omega, \omega\rangle.\end{aligned} \tag{A.2.8}$$

Moreover, we can integrate $\langle J_t^2|y|^{2a}\Box\omega, \omega\rangle$ by parts to get

$$\begin{aligned}\langle J_t^2|y|^{2a}\Box\omega, \omega\rangle &= \|J_t|y|^a d\omega\|^2 + \|J_t|y|^a d^*\omega\|^2 \\ &+ \langle d(J_t^2|y|^{2a}) \wedge \omega, d\omega\rangle + \langle d(J_t^2|y|^{2a}) \wedge d^*\omega, \omega\rangle.\end{aligned} \tag{A.2.9}$$

First we put $a = 0$ in (A.2.9). Since E is a vector bundle with compact base there exists $c > 0$ such that $|dh|^2 > c|y|^2$. Therefore, after substituting (A.2.9) into (A.2.8) we have:

$$\begin{aligned}&\|J_t d\omega\|^2 + \|J_t d^*\omega\|^2 + c\alpha^2\|J_t|y|^2\omega\|^2 \\ &\qquad \leq \|J_t d\omega\|^2 + \|J_t d^*\omega\|^2 + \alpha^2\langle J_t^2|dh|^2\omega, \omega\rangle \\ &\qquad = \lambda(\alpha)\|J_t\omega\|^2 - \alpha\langle J_t^2 A\omega, \omega\rangle - 2\langle J_t dJ_t \wedge \omega, d\omega\rangle \\ &\qquad\quad - 2\langle J_t dJ_t \wedge d^*\omega, \omega\rangle.\end{aligned}$$

We estimate further by means of Cauchy-Schwartz inequality:

$$\begin{aligned}&\|J_t d\omega\|^2 + \|J_t d^*\omega\|^2 + c\alpha^2\|J_t|y|^2\omega\|^2 \\ &\qquad \leq \lambda(\alpha)\|J_t\omega\|^2 + \alpha\|A\|\|J_t\omega\|^2 + 1/2\|J_t d\omega\|^2 \\ &\qquad\quad + 2\|dJ_t \wedge \omega\|^2 + 1/2\|J_t d^*\omega\|^2 + 2\|dJ_t \wedge \omega\|^2.\end{aligned}$$

Finally, since A is a bounded operator, and $|dJ_t| \le \frac{c}{t}$ for some positive constant c, there exists some positive constant $C(\alpha)$such that

$$\|J_t d\omega\|^2 + \|J_t d^*\omega\|^2 + \|J_t|y|^2\omega\|^2 \le C(\alpha)\|J_t\omega\|^2 \le C_1(\alpha)\|\omega\|^2.$$

Taking a limit as $t \to \infty$ in the inequality above we get that $d\omega$, $d^*\omega$ and $|y|\omega$ all belong to $S_{0,0}$. So, in particular, $\omega \in S_{1,0}$. Now we go back to (A.2.9). After differentiating $J_t|y|^{2a}$ and substituting of (A.2.9) into (A.2.8) we get

$$\begin{aligned}\lambda(\alpha)\|J_t|y|^a\omega\|^2 = {} & \|J_t|y|^a d\omega\|^2 + \|J_t|y|^a d^*\omega\|^2 + 2\langle J_t dJ_t \wedge |y|^{2a}\omega, d\omega\rangle \\ & + 2\langle J_t dJ_t \wedge |y|^{2a}\omega, d^*\omega\rangle + 2a\langle J_t^2|y|^{2a-1}d(|y|)\omega, d\omega\rangle \\ & + 2a\langle J_t^2|y|^{2a-1}d(|y|)\omega, d^*\omega\rangle \\ & + \alpha^2\langle J_t^2|y|^{2a}|dh|^2\omega,\omega\rangle + \alpha\langle J_t^2|y|^{2a}A\omega,\omega\rangle.\end{aligned} \tag{A.2.10}$$

Since there exists $c > 0$ such that $|dh|^2 \ge c|y|^2$, we have

$$\alpha^2\langle J_t^2|y|^{2a}|dh|^2\omega,\omega\rangle \ge c\alpha^2\langle J_t^2|y|^{2a+2}\omega,\omega\rangle. \tag{$A.2.11$}$$

After substituting (A.2.11) into (A.2.10) and taking into consideration the boundedness of J_t, dJ_t and $d|y|$, we conclude that there exists $C = C(\alpha, a)$, such that for all $t > 0$, $a \ge 1/2$, we will have

$$\begin{aligned}&\|J_t|y|^a d\omega\|^2 + \|J_t|y|^a d^*\omega\|^2 + c\alpha^2\|J_t|y|^{a+1}\omega\|^2 \\ &\quad \le C(\|J_t|y|^a\omega\|^2 + t^{-1}\||y|^{a-1/2}d\omega\|_{|\mathrm{supp}(dJ_t)}\||y|^{a+1/2}\omega\|_{|\mathrm{supp}(dJ_t)} \\ &\quad + \||y|^{a-1/2}d^*\omega\|_{|\mathrm{supp}(dJ_t)}\||y|^{a+1/2}\omega\|_{|\mathrm{supp}(dJ_t)} \\ &\quad + \|J_t|y|^{a-1/2}\omega\|\|J_t|y|^{a-1/2}d\omega\| \\ &\quad + \|J_t|y|^{a-1/2}\omega\|\|J_t|y|^{a-1/2}d^*\omega\|),\end{aligned} \tag{$A.2.12$}$$

where $\| \bullet \|_{|\mathrm{supp}(dJ_t)}$ means that in the definition of $\| \bullet \|$ we integrate only over the supp (dJ_t). Now we use (A.2.12) to prove the lemma. First, we assume that $a = 1/2$. Since we already know that $\omega \in S_{1,0}$, $d\omega \in S_{0,0}$, and $d^*\omega \in S_{0,0}$, we see that the right-hand side of (A.2.7) is bounded by a constant, which is independent on t. After taking a limit as $t \to \infty$, we conclude that $\omega \in S_{3/2,0}$, $d\omega \in S_{1/2,0}$, and $d^*\omega \in S_{1/2,0}$. At each step we can conclude that if $\omega \in S_{a+1/2,0}$, $d\omega \in S_{a-1/2,0}$, and $d^*\omega \in S_{a-1/2,0}$, then $\omega \in S_{a+1,0}$, $d\omega \in S_{a,0}$, and $d^*\omega \in S_{a,0}$. By iterating this process we deduce the statement of the lemma. $\square$

Our next goal is to get an estimate of norms of covariant derivatives with respect to the Bismut connection in terms of the Witten Laplacian.

Proposition A.2.4. *For large enough $\alpha > 0$ and for any form ω in the domain of $\Box(\alpha)$ there exists $C = C(\alpha) > 0$ such that the following elliptic estimate holds:*

$$(A.2.13) \qquad \sum_{i=1}^{n} \|\bar{\nabla}_{a_i}\omega\|^2 + \sum_{j=1}^{m} \|\bar{\nabla}_{b_j}\omega\|^2 \leq C(\langle \Box(\alpha)\omega, \omega\rangle + \|\omega\|^2).$$

In the proof of Proposition A.2.4 we will assume for simplicity that ω has compact support. If ω does not have compact support, then all calculations should be done for $J_t\omega$. In this case as in the proof of Lemma A.2.3. we can take a limit as $t \to \infty$. We have the following lemma:

Lemma A.2.5. *Assume that in Proposition A.2.4 formula (A.2.13) holds for all compactly supported forms ω. Then (A.2.13) holds for all ω, which satisfy conditions in Proposition A.2.4.*

Proof. Let ω be in the domain of $\Box(\alpha)$ then, by assumption,

$$(A.2.13) \quad \sum_{i=1}^{n} \|\bar{\nabla}_{a_i}J_t\omega\|^2 + \sum_{j=1}^{m} \|\bar{\nabla}_{b_j}J_t\omega\|^2 \leq C(\langle \Box(\alpha)J_t\omega, J_t\omega\rangle + \|J_t\omega\|^2).$$

We have $\bar{\nabla}_{a_i}(J_t\omega) = J_t\bar{\nabla}_{a_i}\omega + \frac{\partial J_t}{\partial y_i}\omega$ and $\bar{\nabla}_{b_j}(J_t\omega) = J_t\bar{\nabla}_{b_j}\omega$. Therefore, there exist constants $C_1 \geq 0$ and $C_2 \geq 0$ such that

$$\begin{aligned}
&\sum_{i=1}^{n} \|J_t\bar{\nabla}_{a_i}\omega\|^2 + \sum_{j=1}^{m} \|J_t\bar{\nabla}_{b_j}\omega\|^2 \\
&\qquad\leq C_1\left(\sum_{i=1}^{n} \|\bar{\nabla}_{a_i}J_t\omega\|^2 + \sum_{j=1}^{m} \|\bar{\nabla}_{b_j}J_t\omega\|^2 + \|\omega\|^2\right) \\
&\qquad\leq C_2(\langle \Box(\alpha)J_t\omega, J_t\omega\rangle + \|J_t\omega\|^2).
\end{aligned}$$

On the other hand,

$$\langle \Box(\alpha)J_t\omega, J_t\omega\rangle = \langle J_t^2\Box(\alpha)\omega, \omega\rangle + \|dJ_t \wedge \omega\|^2 + \|dJ_t \wedge *\omega\|^2 + \langle dJ_t \wedge \omega, J_tI\rangle,$$

where the term I is linear in $d\omega$ and $d^*\omega$. Moreover, we choose small $\epsilon \geq 0$ and estimate by Cauchy-Schwartz:

$$|\langle dJ_t \wedge \omega, J_tI\rangle| \leq 1/\epsilon\|dJ_t \wedge \omega\|^2 + \epsilon\|J_tI\|^2.$$

The norm of $J_t I$ can in turn be estimated by

$$C_3\left(\sum_{i=1}^{n}\|J_t\bar\nabla_{a_i}\omega\|^2+\sum_{j=1}^{m}\|J_t\bar\nabla_{b_j}\omega\|^2\right).$$

Finally, combining all the estimates together, we conclude that there exists $C>0$ such that

$$\begin{aligned}\sum_{i=1}^{n}\|J_t\bar\nabla_{a_i}\omega\|^2&+\sum_{j=1}^{m}\|J_t\bar\nabla_{b_j}\omega\|^2\\ &\leq C\left(\langle J_t^2\Box(\alpha)\omega,\omega\rangle+\|\omega\|^2\right)\\ &=C\left(\langle\Box(\alpha)\omega,\omega\rangle+\langle(1-J_t^2)\Box(\alpha)\omega,\omega\rangle+\|\omega\|^2\right).\end{aligned}$$

Inequality (A.2.13) now follows if you take $t\to\infty$ in the inequality above. $\Box$

Proof of Proposition A.2.4. After substituting $\Box(\alpha)=\Box+\alpha^2|dh|^2+\alpha A$ into $\langle\Box(\alpha)\omega,\omega\rangle$ and integrating by parts we get

$$(A.2.14)\quad \langle d\omega,d\omega\rangle+\langle d^*\omega,d^*\omega\rangle+\alpha^2\langle|dh|^2\omega,\omega\rangle+\alpha\langle A\omega,\omega\rangle=\langle\Box(\alpha)\omega,\omega\rangle.$$

We substitute $d^{1,0}+d^{0,1}+d^{-1,2}$ for d and $(d^{1,0})^*+(d^{0,1})^*+(d^{-1,2})^*$ for d^* in (A.2.14). After integrating by parts, we have

$$(A.2.15)\quad \begin{aligned}&\langle\Box^{1,0}\omega,\omega\rangle+\langle\Box^{0,1}\omega,\omega\rangle+\langle\Box^{-1,2}\omega,\omega\rangle+\text{ cross-terms}\\ &\qquad+\alpha^2\langle|dh|^2\omega,\omega\rangle+\alpha\langle A\omega,\omega\rangle=\langle\Box(\alpha)\omega,\omega\rangle\end{aligned}$$

We observe that according to Lemma A.1.5 the cross-terms which contain $d^{1,0}$ with $d^{0,1}$ and $(d^{1,0})^*$ with $(d^{0,1})^*$ will disappear. We will now estimate the rest of the cross-terms in terms of $\langle\Box^{1,0}\omega,\omega\rangle$, $\langle\Box^{0,1}\omega,\omega\rangle$ and $\langle\Box^{-1,2}\omega,\omega\rangle$. Namely,

$$\begin{aligned}\text{(A.2.16)}\quad 2|\langle d^{1,0}\omega,d^{-1,2}\omega\rangle|&+2|\langle(d^{1,0})^*\omega,(d^{-1,2})^*\omega\rangle|\\ &\leq 1/2\langle\Box^{1,0}\omega,\omega\rangle+2\langle\Box^{-1,2}\omega,\omega\rangle.\end{aligned}$$

Similarly,

$$\begin{aligned}\text{(A.2.17)}\quad 2|\langle d^{0,1}\omega,d^{-1,2}\omega\rangle|&+2|\langle(d^{0,1})^*\omega,(d^{-1,2})^*\omega\rangle|\\ &\leq 1/2\langle\Box^{0,1}\omega,\omega\rangle+2\langle\Box^{-1,2}\omega,\omega\rangle.\end{aligned}$$

We can use (A.2.16), (A.2.17) (estimating the cross-terms in (A.2.15) from below) to get the following estimate

$$(A.2.18)\qquad \begin{aligned}\langle\Box(\alpha)\omega,\omega\rangle \geq 1/2(\langle\Box^{1,0}\omega,\omega\rangle + \langle\Box^{0,1}\omega,\omega\rangle) - 3\langle\Box^{-1,2}\omega,\omega\rangle \\ + \alpha^2\langle|dh|^2\omega,\omega\rangle + \alpha\langle A\omega,\omega\rangle.\end{aligned}$$

In the basis defined in Section A.1.1 we calculate

$$(A.2.19)\qquad \langle\Box^{1,0}\omega,\omega\rangle = \sum_{i=1}^{n} \|\bar{\nabla}_{a_i}\omega\|^2.$$

On the other hand, for $\langle\Box^{0,1}\omega,\omega\rangle$ we have

$$(A.2.20)\qquad \begin{aligned}\langle\Box^{0,1}\omega,\omega\rangle = \sum_{j=1}^{m} \|\bar{\nabla}_{b_j}\omega\|^2 \\ + \sum_{k,l} \left\langle b^k \wedge i(b_l)\left(\bar{L}(b_l,b_k) - \bar{\nabla}_{[L(b_k,b_l)y]}\right)\omega,\omega\right\rangle.\end{aligned}$$

It follows from the description in Lemma A.1.1 that the operator

$$\sum_{k,l} b_k \wedge i(b_l)\left(\bar{L}(b_l,b_k) - \bar{\nabla}_{[L(b_k,b_l)y]}\right)$$

is of first order in $\bar{\nabla}_{a_i}$ and $\bar{\nabla}_{b_j}$ with the coefficients which are at most linear in y. Therefore, it can be estimated in terms of operators $\Box^{1,0}$, $\Box^{0,1}$ and $1+|y|^2$. That is, there is a constant $C_1 > 0$ such that

$$(A.2.21)\qquad \begin{aligned}|\sum_{k,l}\langle b^k \wedge i(b_l)\left(\bar{L}(b_l,b_k) - \bar{\nabla}_{[L(b_k,b_l)y]}\right)\omega,\omega\rangle| \\ \leq 1/4\langle\Box^{1,0}\omega,\omega\rangle + 1/4\langle\Box^{0,1}\omega,\omega\rangle + C_1(\|\omega\|^2 + \langle|y|^2\omega,\omega\rangle).\end{aligned}$$

After estimating (with the help of (A.2.21)) the first order part on the right-hand side of (A.2.20) from below, we get

$$\begin{aligned}\langle\Box^{0,1}\omega,\omega\rangle \geq \sum_{j=1}^{m} \|\bar{\nabla}_{b_j}\omega\|^2 - 1/4\langle\Box^{1,0}\omega,\omega\rangle - 1/4\langle\Box^{0,1}\omega,\omega\rangle \\ - C_1(\|\omega\|^2 + \langle|y|^2\omega,\omega\rangle).\end{aligned}$$

The inequality above is equivalent to

$$(A.2.22)\qquad \begin{aligned}\langle\Box^{0,1}\omega,\omega\rangle \geq 4/5\sum_{j=1}^{m} \|\bar{\nabla}_{b_j}\omega\|^2 - 1/5\langle\Box^{1,0}\omega,\omega\rangle - 4/5C_1(\|\omega\|^2 \\ + \langle|y|^2\omega,\omega\rangle).\end{aligned}$$

Next we substitute (A.2.22) into (A.2.18) to get

$$\begin{aligned}\langle\Box(\alpha)\omega,\omega\rangle \geq 1/2\langle\Box^{1,0}\omega,\omega\rangle &+ 2/5\sum_{j=1}^{m}\|\bar{\nabla}_{b_j}\omega\|^2 - 1/10\langle\Box^{1,0}\omega,\omega\rangle\\ &- 2/5C_1(\|\omega\|^2+\langle|y|^2\omega,\omega\rangle) - 3\langle\Box^{-1,2}\omega,\omega\rangle\\ &+\alpha^2\langle|dh|^2\omega,\omega\rangle+\alpha\langle A\omega,\omega\rangle.\end{aligned}\tag{A.2.23}$$

We recall that $|dh|^2 \geq c|y|^2$ for some positive $c \geq 0$, and that A is a bounded zeroth order operator. Therefore, for large enough α there exists $C_2 = C_2(\alpha)$, such that

$$\begin{aligned}-3\langle\Box^{-1,2}\omega,\omega\rangle &- 2/5C_1(\|\omega\|^2+\langle|y|^2\omega,\omega\rangle)\\ &+\alpha^2\langle|dh|^2\omega,\omega\rangle+\alpha\langle A\omega,\omega\rangle \geq -C_2\|\omega\|^2.\end{aligned}\tag{$A.2.24$}$$

Finally, we combine estimates (A.2.23) and (A.2.24) (substituting $\sum_{i=1}^{n}\|\bar{\nabla}_{a_i}\omega\|^2$ for $\langle\Box^{1,0}\omega,\omega\rangle$) to get

$$\langle\Box(\alpha)\omega,\omega\rangle \geq 2/5\sum_{i=1}^{n}\|\bar{\nabla}_{a_i}\omega\|^2 + 2/5\sum_{j=1}^{m}\|\bar{\nabla}_{b_j}\omega\|^2 - C_2\|\omega\|^2.\tag{$A.2.25$}$$

The statement of the lemma now easily follows from A.2.25. □

Corollary A.2.6. *For large enough α and for any form ω, such that $\Box(\alpha)\omega = \lambda(\alpha)\omega$, we have $\omega \in S_{0,1} \cap S_{\infty,0}$.*

The rest of the proof of Theorem 1.5 proceeds by induction in a and l.

For $0 \leq a < \infty$ and $l = 0,1,2,\dots$, we denote by $\|\omega\|_{a,l}$ the norm on $S_{a,l}$ defined by

$$\|\omega\|_{a,l}^2 = \sum_{\kappa,|\kappa|\leq l}\|(1+|y|)^a(\bar{\nabla})^{\kappa}\omega\|^2.\tag{$A.2.26$}$$

Let w be such that $\Box(\alpha)\omega = \lambda(\alpha)\omega$. Then by Corollary A.2.6 $\omega \in S_{0,1}\cap S_{\infty,0}$. This starts our induction.

Next by substituting $J_t|y|^a\omega$ instead of ω into the elliptic estimate (A.2.13) and by commuting $|y|^a$ with $\bar{\nabla}$ and $\Box(\alpha)$ we can get the following estimate

$$\|\omega\|_{a+1,1}^2 \leq C(a,\alpha)(\|\omega\|_{a+2,0}^2 + \|\omega\|_{a,1}^2).\tag{$A.2.27$}$$

From (A.2.27) we consequently conclude that $\omega \in S_{a,1}\cap S_{\infty,0}$ for $a = 1,2,\dots$. Therefore, $\omega \in S_{\infty,1}$.

By substituting $J_t\square(\alpha)\omega = \lambda(\alpha)J_t\omega$ instead of ω into the elliptic estimate (A.2.13) and by commuting $\square(\alpha)$ with $\bar{\nabla}$, we deduce that $\omega \in S_{0,2} \cap S_{\infty,1}$. Next we use induction to conclude that $\omega \in S_{\infty,2}$.

In this fashion we see that $\omega \in S_{\infty,k}$ for $k = 0, 1, 2, \ldots$, which is the statement of Theorem 1.5.

Remark A.2.7. The arguments similar to the proof of Theorem 1.5 show that all solutions ω of the equation

$$\square(\alpha)\omega = \gamma$$

are from the Schwartz space $\Omega_s^\bullet(E, V)$ if the right-hand side γ is Schwartz.

Appendix 3
A proof of Theorem 8.5.

A.3.0. Introduction. The goal of this appendix is to prove Theorem 8.5:

Theorem 8.5. *There exists a constant $C > 0$, such that for all large enough α and for any p we have*

$$|\lambda_j^p(\alpha) - \nu_j^p(\alpha)| \le C\alpha^{-1/2}, j = 1, \ldots, \min\{k^p(\alpha), \hat{k}^p(\alpha)\}.$$

Moreover, as $\alpha \to \infty$, $\min\{k^p(\alpha), \hat{k}^p(\alpha)\} \to \infty$.

The idea of the proof is to use the classical variational approach to eigenvalues of [Du-Sc]. On a small neighborhood around each component of the critical submanifold operators $L(\alpha)$ and $\square(\alpha)$ are equal. Since for large α the eigenforms of $L(\alpha)$ and $\square(\alpha)$ decay quickly away from critical submanifold, we can use the cut-offs of eigenforms of one operator to build test forms for the other.

The statement of Theorem 8.5 easily follows from two inequalities:

$$\begin{aligned} &\lambda_j^p(\alpha) \le \nu_j^p(\alpha) + C\alpha^{-1/2}, \ 1 \le j \le k^p(\alpha) \text{ and} \\ &\nu_j^p(\alpha) \le \lambda_j^p(\alpha) + C\alpha^{-1/2}, \ 1 \le j \le \hat{k}^p(\alpha). \end{aligned} \tag{A.3.1}$$

We will prove only the first inequality. The proof of the second inequality is similar.

In Section A.3.1. we introduce a variational characterization of the eigenvalues of $\square(\alpha)$ and we define the test forms for the variational approach. In sections A.3.2 through A.3.4 we complete our estimates.

A.3.1. Test forms and the min-max principle. We start by defining our test forms. First, we need to introduce a smooth family of partitions of unity $\{\chi_{R,j}\}_{j=0,\dots,\Lambda}$ on N. This family will depend on a parameter R. We define

$$\tilde{E}_{R,j} := \{(x,y) \in E_j \mid |y| < R\}, \quad \tilde{E}_R = \tilde{E}_{R,1} \cup \cdots \cup \tilde{E}_{R,\Lambda},$$

$$\tilde{U}_{R,j} := f_j(\tilde{E}_{R,j}), \quad \tilde{U}_R = \tilde{U}_{R,1} \cup \cdots \cup \tilde{U}_{R,\Lambda}.$$

Finally, we define the non-negative smooth functions $\chi_{R,j}$, $j = 1, \dots, \Lambda$, to be 1 on $\tilde{U}_{R,j}$ and to be 0 on $M - \tilde{U}_{2R,j}$. We further define:

$$\chi_{R,0} = 1 - \sum_{j=1}^{\Lambda} \chi_{R,j} = 1 - \chi_R.$$

As our test forms we pick $\psi^p_{R,i}(\alpha)$, $i = 1, \dots, j$, defined by

$$\psi^p_{R,i}(\alpha) := \sum_{l=1}^{\Lambda} f_l^*(\phi^p_i(\alpha)\chi_R) = f^*(\phi^p_i(\alpha)\chi_R) \in \Omega^p_s(E,V).$$

Then, provided that the test forms $\{\psi^p_{R,i}\}$ are linearly independent for $i = 1, \dots, j$, we can use the *min-max principle* in the form of the *Rayleigh quotient* ([Cha, Chapter 1] or [Du-Sc]). From the min-max principle we have

$$(A.3.2) \qquad \lambda^p_j(\alpha) \leq \frac{\langle \Box^p(\alpha)\psi^p_{R,i}(\alpha), \psi^p_{R,i}(\alpha)\rangle_E}{\langle \psi^p_{R,i}(\alpha), \psi^p_{R,i}(\alpha)\rangle_E}, \quad i = 1, \dots, j.$$

Fix small enough R. Then the metric on $\tilde{U}_R$ is the pullback of the metric on E under the diffeomorphism $f = (f_1, \dots, f_\Lambda)$. Therefore, for such R

$$(A.3.3) \qquad \begin{aligned} \langle \Box^p(\alpha)\psi^p_{R,i}(\alpha), \psi^p_{R,i}(\alpha)\rangle_E &= \langle \Box^p(\alpha) f^*(\phi^p_i(\alpha)\chi_R), f^*(\phi^p_i(\alpha)\chi_R)\rangle_E \\ &= \langle L^p(\alpha)(\phi^p_i(\alpha)\chi_R), \phi^p_i(\alpha)\chi_R\rangle_N. \end{aligned}$$

Similarly,

$$(A.3.4) \qquad \langle \psi^p_{R,i}(\alpha), \psi^p_{R,i}(\alpha)\rangle_E = \langle \phi^p_i(\alpha)\chi_R, \phi^p_i(\alpha)\chi_R\rangle_N.$$

Therefore, inequalities (A.3.2) become

$$(A.3.5) \qquad \lambda^p_j(\alpha) \leq \max_{1 \leq i \leq j} \frac{\langle L^p(\alpha)(\phi^p_i(\alpha)\chi_R), \phi^p_i(\alpha)\chi_R\rangle}{\langle \phi^p_i(\alpha)\chi_R, \phi^p_i(\alpha)\chi_R\rangle}.$$

A.3.2. A computation of the variational quotient. It is easy to compute ([CFKS, Proposition 11.13]) that

$$L(\alpha) = d^*d + dd^* + \alpha^2|dh|^2 + \alpha A, \tag{A.3.6}$$

where A is a zeroth order operator.

Since $d^*d + dd^* = *d*d + d*d*$, where $*$ denotes the Hodge $*$-operator on N, we have

$$\begin{aligned}(d^*d + dd^*)(\phi_i^p(\alpha)\chi) &= *d*d(\phi_i\chi) + d*d*(\phi_i\chi)\\ &= *d*(d\chi \wedge \phi_i + \chi d\phi_i) + d*(d\chi \wedge *\phi_i + \chi d(*\phi_i))\\ &= d^*(d\chi \wedge \phi_i) + *(d\chi \wedge *d\phi_i) + \chi d^*d\phi_i\\ &\quad + d*(d\chi \wedge *\phi_i) + d\chi \wedge d^*\phi_i + \chi dd^*\phi_i.\end{aligned} \tag{A.3.7}$$

After multiplying (A.3.7) by $\chi\phi_i^p(\alpha)$ on both sides, integrating by parts, and combining terms we have

$$\begin{aligned}\langle L\chi\phi_i, \chi\phi_i\rangle &= \langle \chi L\phi_i, \chi\phi_i\rangle + \langle d\chi \wedge \phi_i, d(\chi\phi_i)\rangle\\ &\quad + \langle *(d\chi \wedge *d\phi_i), \chi\phi_i\rangle + \langle *(d\chi \wedge *\phi_i), d^*(\chi\phi_i)\rangle\\ &\quad + \langle d\chi \wedge d^*\phi_i, \chi\phi_i\rangle.\end{aligned}$$

Recall that $L\phi_i = \nu_i\phi_i$. We rewrite the formula above as

$$\begin{aligned}\langle L\chi\phi_i, \chi\phi_i\rangle &= \nu_i\|\chi\phi_i\|^2 + \|d\chi \wedge \phi_i\|^2\\ &\quad + \langle d\chi \wedge \phi_i, \chi d\phi_i\rangle + \langle *(d\chi \wedge *d\phi_i), \chi\phi_i\rangle\\ &\quad + \langle *(d\chi \wedge *\phi_i), *(d\chi \wedge *\phi_i)\rangle + \langle *(d\chi \wedge *\phi_i), \chi d*\phi_i\rangle\\ &\quad + \langle d\chi \wedge d^*\phi_i, \chi\phi_i\rangle,\end{aligned} \tag{A.3.8}$$

where $\chi = \chi_R$.

We want to estimate the right hand side of this formula. We represent it as $\langle L\chi\phi_i, \chi\phi_i\rangle = \nu_i\|\chi_R\phi_i\|^2 + I_1 + I_2$, where

$$I_1 = \|d\chi_R \wedge \phi_i\|^2 + \|*(d\chi_R \wedge *\phi_i)\|^2, \tag{A.3.9}$$

$$\begin{aligned}I_2 &= \langle d\chi_R \wedge \phi_i, \chi_R d\phi_i\rangle + \langle *(d\chi_R \wedge *d\phi_i), \chi_R\phi_i\rangle\\ &\quad + \langle *(d\chi_R \wedge *\phi_i), \chi_R d*\phi_i\rangle + \langle d\chi_R \wedge d^*\phi_i, \chi_R\phi_i\rangle.\end{aligned} \tag{A.3.10}$$

A.3.3. An estimate of I_1. Since by definition of χ_R, supp $(d\chi_R) \subset \tilde{U}_{2R} \setminus \tilde{U}_R$, and $\|d\chi_R\| \leq$ const, we have

$$|I_1| \leq C_1\|(\phi_i)_{|\tilde{U}_{2R}\setminus\tilde{U}_R}\|^2. \tag{A.3.11}$$

Now we need the following

Lemma A.3.1. *There exists a constant $c = c(R)$ such that*

$$\|(\phi_i)_{|\tilde{U}_{2R}\setminus\tilde{U}_R}\|^2 \le c\alpha^{-1}. \tag{A.3.12}$$

Proof. After integrating by parts in (A.3.6) we have

$$\begin{aligned}\nu_i(\alpha) &= \langle L^p(\alpha)\phi_i, \phi_i\rangle \\ &= \|d\phi_i\|^2 + \|d^*\phi_i\|^2 + \alpha^2\langle |dh|^2\phi_i, \phi_i\rangle + \alpha\langle A\phi_i, \phi_i\rangle.\end{aligned} \tag{A.3.13}$$

Since the term $\|d\phi_i\|^2+\|d^*\phi_i\|^2$ is non-negative and A is a bounded operator, we have

$$\alpha^2\langle |dh|^2\phi_i, \phi_i\rangle \le \nu_i(\alpha) + c_1\alpha, \tag{A.3.14}$$

Therefore, we also have a similar bound for the restriction:

$$\alpha^2\langle (|dh|^2\phi_i)_{|\tilde{U}_{2R}\setminus\tilde{U}_R}, \phi_i\rangle \le \nu_i(\alpha) + c_1\alpha. \tag{A.3.15}$$

On the other hand $(|dh|^2)_{|\tilde{U}_{2R}\setminus\tilde{U}_R} \ge c_2$ for some positive $c_2 = c_2(R)$. Thus

$$\begin{aligned}\alpha^2 c_2\|(\phi_i)_{|\tilde{U}_{2R}\setminus\tilde{U}_R}\|^2 &\le \alpha^2\langle (|dh|^2\phi_i)_{|\tilde{U}_{2R}\setminus\tilde{U}_R}, \phi_i\rangle \\ &\le \nu_i(\alpha) + c_1\alpha.\end{aligned} \tag{A.3.16}$$

After dividing both parts of (A.3.16) by $\alpha^2 c_2$ we have

$$\|(\phi_i)_{|\tilde{U}_{2R}\setminus\tilde{U}_R}\|^2 \le \frac{\nu_i(\alpha) + c_1\alpha}{\alpha^2 c_2} \le c\alpha^{-1}. \tag{A.3.17}$$

In the last inequality we assumed that $\nu_i(\alpha)$ is α-bounded. See Chapter 8. □

We can now apply the lemma above to the inequality (A.3.11) to get

$$|I_1| \le C_2\alpha^{-1}. \tag{A.3.18}$$

A.3.4. An estimate of I_2. Since $\operatorname{supp}(d\chi_R) \subset \tilde{U}_{2R}\setminus\tilde{U}_R$, we have

$$|I_2| \le C_3\|(\phi_i)_{|\tilde{U}_{2R}\setminus\tilde{U}_R}\|\left(\|(d\phi_i)_{|\tilde{U}_{2R}\setminus\tilde{U}_R}\| + \|(d*\phi_i)_{|\tilde{U}_{2R}\setminus\tilde{U}_R}\|\right). \tag{A.3.19}$$

To estimate the right hand side of (A.3.19) we need the following

Lemma A.3.2. *There exists a constant $c = c(R)$ such that*

$$\|(d\phi_i)_{|\tilde{U}_{2R}\setminus\tilde{U}_R}\| \leq c, \quad \|(d * \phi_i)_{|\tilde{U}_{2R}\setminus\tilde{U}_R}\| \leq c. \tag{A.3.20}$$

Proof. We start with the equality (A.3.13):

$$\nu_i(\alpha) = \|d\phi_i\|^2 + \|d^*\phi_i\|^2 + \alpha^2\langle|dh|^2\phi_i, \phi_i\rangle + \alpha\langle A\phi_i, \phi_i\rangle. \tag{A.3.21}$$

Since the terms $\|d\phi_i\|^2$, $\|d^*\phi_i\|^2$, and $\alpha^2\langle|dh|^2\phi_i, \phi_i\rangle$ are all non-negative, A is a bounded operator, and $\nu_i(\alpha)$ is α-bounded, we conclude that for some $c_1 > 0$,

$$\|d\phi_i\|^2 \leq c_1\alpha, \quad \|d * \phi_i\|^2 \leq c_1\alpha. \tag{A.3.22}$$

Unfortunately, estimates (A.3.22) are not good enough, so we need to work a bit harder. We define a non-negative smooth characteristic function $\chi_{[R,2R]}$ by

$$\chi_{[R,2R]} = 1 \text{ on } \tilde{U}_{2R} \setminus \tilde{U}_R, \quad \operatorname{supp}(\chi_{[R,2R]}) \subset \tilde{U}_{4R} \setminus \tilde{U}_{R/2}. \tag{A.3.23}$$

Then, for $\chi = \chi_{[R,2R]}$,

$$\begin{aligned} B &= \|(d\phi_i)_{|\tilde{U}_{2R}\setminus\tilde{U}_R}\|^2 + \|(d * \phi_i)_{|\tilde{U}_{2R}\setminus\tilde{U}_R}\|^2 \\ &\leq \langle\chi d\phi_i, d\phi_i\rangle + \langle\chi d^*\phi_i, d^*\phi_i\rangle \\ &= \langle *d * (\chi d\phi_i), \phi_i\rangle + \langle d(\chi d^*\phi_i), \phi_i\rangle \\ &= \langle\chi(d^*d + dd^*)\phi_i, \phi_i\rangle + \langle *(d\chi \wedge *d\phi_i), \phi_i\rangle + \langle d\chi \wedge d^*\phi_i, \phi_i\rangle. \end{aligned} \tag{A.3.24}$$

Next, since ϕ_i is an eigenform for the eigenvalue ν_i, we observe that we have the equality:

$$\chi\nu_i\phi_i = \chi(d^*d + dd^*)\phi_i + \alpha^2\chi|dh|^2\phi_i + \alpha\chi A\phi_i. \tag{A.3.25}$$

By expressing $\chi(d^*d+dd^*)\phi_i$ from (A.3.25) and substituting it into (A.3.24) we get

$$\begin{aligned} B \leq &-\nu_i\langle\chi\phi_i, \phi_i\rangle - \alpha^2\langle\chi|dh|^2\phi_i, \phi_i\rangle - \alpha\langle\chi A\phi_i, \phi_i\rangle \\ &+ \langle *(d\chi \wedge *d\phi_i), \phi_i\rangle + \langle d\chi \wedge d^*\phi_i, \phi_i\rangle. \end{aligned} \tag{A.3.26}$$

Now we observe that, since $\chi|dh|^2$ is bounded from below by some positive constant, the term

$$-\alpha^2\langle\chi|dh|^2\phi_i, \phi_i\rangle - \alpha\langle\chi A\phi_i, \phi_i\rangle$$

is negative for large α. Therefore,

$$(A.3.27) \qquad \begin{aligned} B &\le -\nu_i\langle\chi\phi_i,\phi_i\rangle + \langle *(*d\chi\wedge *d\phi_i),\phi_i\rangle + \langle d\chi\wedge d^*\phi_i,\phi_i\rangle \\ &\le \nu_i(\alpha)\|(\phi_i)_{|\tilde U_{4R}\setminus U_{R/2}}\|^2 \\ &\quad + c\left(\|(\phi_i)_{|\tilde U_{4R}\setminus U_{R/2}}\|\|(d\phi_i)_{|\tilde U_{4R}\setminus U_{R/2}}\|\right). \end{aligned}$$

We use (A.3.22) and (A.3.12) to deduce from (A.3.26) the estimate in the statement of the lemma. ☐

It follows from Lemma A.3.1 and Lemma A.3.2 that for some $C_3 > 0$,

$$(A.3.28) \qquad \|I_2\| \le C_3\alpha^{-1/2}.$$

Now we estimate the denominator in (A.3.2). To do so we need the following lemma:

Lemma A.3.3. *There exists a constant $c = c(R)$ such that for all α large enough we have*

$$(A.3.29) \qquad |\langle\phi_i^p(\alpha)\chi_R,\phi_k^p(\alpha)\chi_R\rangle - \delta_{ik}| \le c\alpha^{-1}, \quad 1\le i,k\le k^p(\alpha)$$

where δ_{ik} is a Kronecker symbol.

Proof. We write $\chi_R = 1 + (\chi_R - 1)$. Then

$$\begin{aligned} \langle\phi_i^p(\alpha)\chi_R,\phi_k^p(\alpha)\chi_R\rangle &= \langle\phi_i^p(\alpha) + (\chi_R-1)\phi_i^p(\alpha),\phi_k^p(\alpha) + (\chi_R-1)\phi_k^p(\alpha)\rangle \\ &= \delta_{ik} + 2\langle(\chi_R-1)\phi_i^p(\alpha),\phi_k^p(\alpha)\chi_R\rangle \\ &\quad + \langle(\chi_R-1)\phi_i^p(\alpha),(\chi_R-1)\phi_k^p(\alpha)\rangle. \end{aligned}$$

Since $\mathrm{supp}(\chi_R - 1)\subset N\setminus\tilde U_R$,

$$|\langle\phi_i^p(\alpha)\chi_R,\phi_k^p(\alpha)\chi_R\rangle - \delta_{ik}| \le c\|\phi_i^p(\alpha)_{|N\setminus\tilde U_R}\|\|\phi_k^p(\alpha)_{|N\setminus\tilde U_R}\| \le c\alpha^{-1}.$$

In the formula above the last inequality on the right follows from the appropriately modified proof of Lemma A.3.1. ☐

Corollary A.3.4. *For large enough α the test functions in (A.3.2) are linearly independent.*

Now we are ready to finish the proof of the Theorem 8.5. From (A.3.5), (A.3.18), and (A.3.28) it follows that there exists $C_4 > 0$, such that

$$\begin{aligned}\lambda_j^p(\alpha) &\leq \frac{\nu_i^p(\alpha)\|\chi_R\phi_i^p(\alpha)\|^2 + I_1 + I_2}{\|\chi_R\phi_i^p(\alpha)\|^2} \\ &\leq \frac{\max_{1\leq i\leq j}(\nu_i^p(\alpha) + C_4\alpha^{-1/2})}{1 - c/\alpha}.\end{aligned}$$

Thus there exists $C > 0$, such that

$$\lambda_j^p(\alpha) - \nu_i^p(\alpha) \leq C\alpha^{-1/2}, i = 1, \ldots, j.$$

In particular, we have (A.3.1).

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