

# The Involutive Structure on the Blow-Up of $\mathbb{R}^n$ in $\mathbb{C}^n$

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## 1. Introduction.

Many interesting geometric structures can be defined by specifying a smooth subbundle  $\mathcal{V}$  of the complexified tangent bundle of the underlying smooth manifold, subject to an integrability condition. Examples include foliations, complex structures, and CR structures. In general such structures are called involutive, or formally integrable, and their study has been the starting point of far-reaching general investigations (see, for example, [T], [CT], and [HJ]). However, most naturally occurring examples, for instance all those mentioned above, have the property that  $\mathcal{V} \cap \overline{\mathcal{V}}$  has constant rank. In recent work on integral geometry ([BEGM], [E], and [BaE]), natural examples of involutive structures have arisen for which the rank of  $\mathcal{V} \cap \overline{\mathcal{V}}$  changes along a hypersurface. For these examples the underlying manifold is the real blow-up of  $\mathbb{R}\mathbb{P}^n$  in  $\mathbb{C}\mathbb{P}^n$  for various  $n$ . In this article we consider these new involutive structures from an analytic point of view.

Locally we may as well consider the blow-up of  $\mathbb{R}^n$  in  $\mathbb{C}^n$ . This blow-up  $B$  is a smooth real  $2n$ -manifold with a distinguished hypersurface  $\Sigma$ , the inverse image of  $\mathbb{R}^n$  under the blow-down map  $b : B \rightarrow \mathbb{C}^n$ . The blow-up is defined precisely so that the image under  $b$  of a neighborhood of a point of  $\Sigma$  is a localized wedge in  $\mathbb{C}^n$ , i.e. the product of an open set in  $\mathbb{R}^n$  with a localized cone in  $i\mathbb{R}^n$ . Away from  $\Sigma$ ,  $b$  is a diffeomorphism and the involutive structure is just the lift of the complex structure on  $\mathbb{C}^n \setminus \mathbb{R}^n$ ; the bundle  $\mathcal{V}$  of  $(0, 1)$  vectors extends smoothly across  $\Sigma$  but  $\dim(\mathcal{V} \cap \overline{\mathcal{V}}) = n - 1$  there.

A solution of an involutive structure is a function or distribution annihilated by all sections of  $\mathcal{V}$ . An important analytic problem is to understand

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the regularity of solutions of a given involutive structure. We show that the involutive structure on the blow-up of  $\mathbb{R}^n$  in  $\mathbb{C}^n$  is hypocomplex, which means that any solution is locally a holomorphic function of a basic set of independent solutions. From the point of view of the theory of involutive structures, hypocomplexity is the strongest possible regularity property for solutions; complex structures are hypocomplex. Our first proof of the hypocomplexity is an elementary power series argument. We also give another argument using the Edge of the Wedge Theorem of several complex variables. This is quite straightforward: a solution of the involutive structure near a point of  $\Sigma$  defines a holomorphic function on a localized wedge and the Edge of the Wedge Theorem provides the extension needed to deduce hypocomplexity. It turns out that this argument can be reversed; the Edge of the Wedge Theorem is an easy consequence of the hypocomplexity of the involutive structure. Thus, our power series argument provides a relatively simple new proof of the Edge of the Wedge Theorem. The idea of introducing the blow-up in this context seems to us particularly natural; the wedge blows up to an open set with boundary values along a hyperplane, where standard arguments can be applied.

In §2 we describe in detail the involutive structure on the blow-up of  $\mathbb{R}^n$  in  $\mathbb{C}^n$  and give the power series argument for hypocomplexity. We also use a similar power series argument to show that the associated inhomogeneous equations have infinite dimensional first local cohomology by showing that inhomogeneous terms of a particular form must be real-analytic if there is a solution. In §3 we explain the relationship with the Edge of the Wedge Theorem. Finally in §4 we review the integro-geometric considerations from [BEGM], [E], and [BaE] which gave rise to the compact version of the involutive structure on the blow-up in the first place.

## 2. The Involutive Structure.

We begin by reviewing some of the basic notions which will be relevant for us concerning involutive structures; see [T] for elaboration. An involutive structure on a smooth manifold  $M$  is a smooth complex subbundle  $\mathcal{V} \subset CTM$  satisfying the formal integrability condition  $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$ . Set  $d = \dim M - \dim \mathcal{V}$ . A *solution* of  $\mathcal{V}$  is a distribution  $f \in \mathcal{D}'(M)$  satisfying  $Lf = 0$  for all sections  $L$  of  $\mathcal{V}$ . Familiar special cases include the following:

1.  $\mathcal{V} = \bar{\mathcal{V}}$ . In this case  $\mathcal{V}$  is the complexification of a real Frobenius-integrable distribution on  $M$ , so  $\mathcal{V}$  defines a foliation. Locally one can find  $d$  independent smooth real-valued solutions  $x^1, \dots, x^d$  of  $\mathcal{V}$ ; any

solution of  $\mathcal{V}$  is of the form  $g(x^1, \dots, x^d)$  for some  $g \in \mathcal{D}'(\mathbb{R}^d)$ . Such involutive structures are called *real*.

2.  $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$  and  $\dim \mathcal{V} = \frac{1}{2} \dim M$ . In this case  $\mathcal{V}$  defines an integrable almost complex structure on  $M$ . By the Newlander-Nirenberg Theorem, locally one can find  $d$  independent smooth solutions  $z^1, \dots, z^d$  of  $\mathcal{V}$ ; then any solution of  $\mathcal{V}$  is of the form  $h(z^1, \dots, z^d)$  for some holomorphic function  $h$ . Such involutive structures are called *complex*.

The involutive structure  $\mathcal{V}$  is said to be *locally integrable* at  $m \in M$  if in some neighborhood of  $m$  one can find  $d$  smooth solutions  $f^1, \dots, f^d$  of  $\mathcal{V}$  with  $df^1, \dots, df^d$  linearly independent.  $\mathcal{V}$  is said to be locally integrable if it is locally integrable at every  $m \in M$ . The examples above are clearly locally integrable. It follows from the holomorphic version of the Frobenius Theorem by complexification that any real-analytic involutive structure on a real-analytic manifold is locally integrable.

An involutive structure is said to be a (generalized) CR structure if  $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$ . If additionally  $\dim \mathcal{V} = \frac{1}{2}(\dim M - 1)$ ,  $\mathcal{V}$  is said to be CR of hypersurface type. If such a  $\mathcal{V}$  is locally integrable at  $m \in M$ , then the image of the map  $f = (f^1, \dots, f^d) : M \rightarrow \mathbb{C}^d$  is a smooth hypersurface of  $\mathbb{C}^d$  and  $f_*(\mathcal{V}) = T^{0,1}\mathbb{C}^d \cap CTf(M)$ ; one says that the CR structure is embeddable. There are CR structures which are not embeddable; these provide examples of involutive structures which are not locally integrable.

One of our primary interests is the regularity of solutions of locally integrable involutive structures. Complex involutive structures exhibit the best possible regularity behavior: every distribution solution is a holomorphic function of a basic set of solutions. The hypocomplex involutive structures are defined by this regularity property:

**Definition 2.1.** Let  $\mathcal{V}$  be an involutive structure on  $M$ , locally integrable at  $m \in M$ , with  $f^1, \dots, f^d$  a set of independent smooth solutions of  $\mathcal{V}$  on a fixed neighborhood of  $m$  as above.  $\mathcal{V}$  is said to be hypocomplex at  $m$  if every solution of  $\mathcal{V}$  defined in some neighborhood of  $m$  is (on a possibly smaller neighborhood) of the form  $h(f^1, \dots, f^d)$  for a holomorphic function  $h$  defined in a neighborhood of  $(f^1(m), \dots, f^d(m)) \in \mathbb{C}^d$ .  $\mathcal{V}$  is hypocomplex on  $M$  if it is hypocomplex at each  $m \in M$ .

It is easily checked that this definition is independent of the choice of independent smooth solutions  $f^1, \dots, f^d$ .

Most involutive structures are not hypocomplex. A second family of hypocomplex examples consists of the CR structures induced on hypersur-

faces of  $\mathbb{C}^d$  with Levi form having at least one positive and at least one negative eigenvalue. The coordinates  $z^1, \dots, z^d$  restrict to form a set of independent smooth solutions of  $\mathcal{V}$ , and by the H. Lewy extension theorem, any solution of  $\mathcal{V}$  (i.e. any CR distribution) is the restriction of a holomorphic function defined in a neighborhood of the hypersurface.

We are interested in a particular involutive structure which arises naturally on the real blow-up of  $\mathbb{R}^n$  in  $\mathbb{C}^n$ . Recall that if  $N$  is a smooth submanifold of another smooth manifold  $M$ , then there is canonically associated another smooth manifold  $B$ , the real blow-up of  $N$  in  $M$ , together with a blow-down map  $b : B \rightarrow M$ . The space  $B$  is constructed by replacing each  $n \in N$  by the projectivized normal space  $\mathbb{P}(TM/TN)$  at  $n$ . The set  $\Sigma \equiv b^{-1}(N)$  is a hypersurface in  $B$ . The map  $b : B \setminus \Sigma \rightarrow M \setminus N$  is a diffeomorphism but, for  $n \in N$ , the fiber  $b^{-1}(n)$  is diffeomorphic to  $\mathbb{R}\mathbb{P}^{d-1}$ , where  $d$  is the codimension of  $N$  in  $M$ .

The prototypical example is the blow-up of  $\{0\} \subset \mathbb{R}^n$ . In this case local coordinates  $(y, t^1, \dots, t^{n-1})$  on  $B$  can be obtained in a neighborhood of the point of the fiber above  $\{0\}$  determined by the line through the first basis vector by writing a point of  $\mathbb{R}^n$  with nonzero first coordinate in the form  $(y, yt^1, \dots, yt^{n-1})$ . If we set  $t = (t^1, \dots, t^{n-1}) \in \mathbb{R}^{n-1}$ , then the blow-down map  $b$  takes the form

$$\begin{array}{ccc} (y, t) & \in & B \\ \downarrow & & \downarrow b \\ (y, yt) & \in & \mathbb{R}^n \end{array}$$

and in these coordinates  $\Sigma$  becomes  $\{y = 0\}$ . The image of this chart under  $b$  consists of the origin together with all points with nonzero first coordinate. The full fiber  $\Sigma = b^{-1}(0)$  can be covered by charts obtained similarly using each of the standard basis directions.

Coordinates for a general blow-up of  $N \subset M$  can be obtained by choosing appropriate local coordinates on  $M$ , applying the above construction for the coordinates transverse to  $N$ , and leaving the coordinates along  $N$  unchanged. For the case of interest here, that of  $\mathbb{R}^n \subset \mathbb{C}^n$ , this amounts to applying the above where  $(y, t)$  are the imaginary parts of the variables. So we write the real parts as  $(x, s)$  for  $x \in \mathbb{R}$  and  $s \in \mathbb{R}^{n-1}$  and set  $z = x + iy \in \mathbb{C}$ . It will turn out to be convenient to take our blow-down map to be a slight

modification of the one just described, namely

$$\begin{array}{ccc} (z, s, t) & \in & \mathbb{C} \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \subset B \\ \downarrow & & \\ (z, s + zt) & \in & \mathbb{C} \times \mathbb{C}^{n-1} \cong \mathbb{C}^n. \end{array}$$

This map is still exactly the standard blow-down in the imaginary parts and for fixed imaginary parts is simply an invertible linear transformation in the real parts, so certainly provides local coordinates realizing  $b : B \rightarrow \mathbb{C}^n$ . In these coordinates  $\Sigma$  again becomes  $\{y = 0\}$  and  $\Sigma$  is covered by charts obtained similarly using each of the standard basis directions in  $\mathbb{R}^n$ .

Since  $b : B \setminus \Sigma \rightarrow \mathbb{C}^n \setminus \mathbb{R}^n$  is a diffeomorphism, the complex structure on  $\mathbb{C}^n \setminus \mathbb{R}^n$  pulls back to a complex structure on  $B \setminus \Sigma$ . Although this complex structure does not extend across  $\Sigma$ , it does extend as an involutive structure.

**Proposition 2.2.** *The involutive structure on  $B \setminus \Sigma$  obtained by pulling back the complex structure on  $\mathbb{C}^n \setminus \mathbb{R}^n$  extends smoothly across  $\Sigma$  to determine an involutive structure on all of  $B$ .*

*Proof.* Denote points in  $\mathbb{C}^n$  by  $(z, w)$  with  $z \in \mathbb{C}$  and  $w \in \mathbb{C}^{n-1}$ . The complex structure on  $B \setminus \Sigma$  is given by  $\mathcal{V} = \ker\{b^*dz, b^*dw\}$ , where we interpret  $dw$  as vector-valued. Using the coordinate expression for  $b$  given above, this is  $\mathcal{V} = \ker\{dz, ds + tdz + zdt\}$ . As these forms clearly remain linearly independent across  $\Sigma$ ,  $\mathcal{V}$  extends smoothly across  $\Sigma$  as a vector bundle, so it follows that it determines an involutive structure on  $B$ .  $\square$

From the above it follows that in our  $(z, s, t)$  coordinates on  $B$ ,  $\mathcal{V}$  is spanned by  $\partial_{\bar{z}}$  and the  $\partial_t - z\partial_s$ . On  $\Sigma$  the latter are real and span  $\mathcal{V} \cap \bar{\mathcal{V}}$ , which agrees with the complexified tangent space to the fibers of  $b|_{\Sigma}$ . In particular  $\mathcal{V} \cap \bar{\mathcal{V}}$  has dimension  $n - 1$  on  $\Sigma$ . The coordinates  $z$  and  $w = s + zt$  are solutions of  $\mathcal{V}$ , so  $\mathcal{V}$  is locally integrable.

For  $n = 2$ , this involutive structure was introduced in [HJ] in different coordinates as an example as  $V_2$ , p. 501, and as (1.10), p. 502.

The main result of this section is the following.

**Theorem 2.3.** *The involutive structure on  $B$  is hypocomplex.*

*Proof.* Away from  $\Sigma$  the involutive structure is complex, so certainly it is hypocomplex there. Hence it suffices to establish hypocomplexity at

$(z, s, t) = (0, 0, 0)$  in the coordinate system introduced above. It must be shown that every distribution solution of  $\mathcal{V}$  near  $(z, s, t) = (0, 0, 0)$  is of the form  $h(z, s + zt)$  for some holomorphic function  $h$  defined in a neighborhood of  $(z, w) = (0, 0) \in \mathbb{C}^n$ . We present the details for  $C^1$  solutions of  $\mathcal{V}$  and then indicate the modifications required to extend the argument to distribution solutions.

Suppose that  $f(z, s, t)$  is a  $C^1$  solution of  $\partial_{\bar{z}}f = 0$  and  $(\partial_t - z\partial_s)f = 0$  in a neighborhood of the origin. Then  $f$  is, in particular, holomorphic in  $z$ , and so may be expanded as

$$(2.1) \quad f(z, s, t) = \sum_{k=0}^{\infty} a_k(s, t)z^k,$$

where the  $a_k$  are  $C^1$  functions near  $(s, t) = (0, 0)$ , and this series along with all first derivatives converges uniformly for  $(z, s, t)$  small. In particular, there are  $C, r$  and  $\epsilon > 0$  so that

$$(2.2) \quad |a_k(s, t)| \leq Cr^{-k} \quad \text{for } |s|, |t| \leq \epsilon.$$

Now  $\partial_t f = z\partial_s f$  gives

$$\partial_t a_0 = 0 \quad \text{and} \quad \partial_t a_k = \partial_s a_{k-1} \quad \text{for } k \geq 1.$$

From the first equation it follows that  $a_0(s, t) = b_0(s)$  for a  $C^1$  function  $b_0(s)$ . From the second equation with  $k = 1$  one then obtains  $a_1(s, t) = t \cdot \partial_s b_0(s) + b_1(s)$  for a  $C^1$  function  $b_1(s)$ . This gives also that  $\partial_s b_0$  is  $C^1$ , so  $b_0$  is  $C^2$ . From the equations with higher  $k$  one similarly deduces inductively that

$$(2.3) \quad a_k(s, t) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} t^\alpha \partial_s^\alpha b_{k-|\alpha|}(s)$$

for  $C^1$  functions  $b_k(s)$ , and the sum is over multiindices  $\alpha$ . Since  $a_k$  is  $C^1$  it follows that each  $\partial_s^\alpha b_{k-|\alpha|}$  is  $C^1$ , so by induction each  $b_k$  is actually  $C^\infty$ .

In order to deduce bounds on the  $b_k$  we use the following lemma, a real version of Cauchy estimates for polynomials.

**Lemma 2.4.** *For each  $n$  there is a constant  $R > 0$  so that if  $p(t) = \sum_{|\alpha| \leq k} c_\alpha t^\alpha$  is a polynomial on  $\mathbb{R}^n$  of degree  $k$ , then*

$$\max |c_\alpha| \leq R^k \sup_{|t_j| \leq 1} |p(t)|.$$

Here the sup is over all  $t \in \mathbb{R}^n$  with  $|t_1| \leq 1, \dots, |t_n| \leq 1$ .

Lemma 2.4 can be proved in a variety of ways. The case  $n > 1$  follows easily from the case  $n = 1$  by induction. For  $n = 1$ , the coefficients of  $p$  can be recovered from its values at  $k + 1$  points; taking these points to be equally spaced and spread over  $[-1, 1]$  and making straightforward estimates leads to a bound of the desired form (this argument was shown to us by Ken Bube). Alternatively, for fixed  $k$  and  $l$ ,  $\sup |c_l|$  is explicitly evaluated in terms of Chebyshev coefficients by a theorem of Markov, where the sup is over all polynomials  $p(t) = \sum_{l=0}^k c_l t^l$  satisfying  $\sup_{|t| \leq 1} |p(t)| \leq 1$  (see [BoE], p. 248). A bound as in Lemma 2.4 follows easily from this. We leave the details to the reader. By rescaling it is clear that in the statement of Lemma 2.4,  $\sup_{|t_j| \leq 1}$  can be replaced by  $\sup_{|t_j| \leq \epsilon}$ , but now  $R$  depends also on  $\epsilon > 0$ .

By (2.3),  $a_k$  is a polynomial in  $t$  of degree  $k$ ; applying Lemma 2.4 and using (2.2) one deduces the existence of constants  $C$  and  $M$  so that for all  $k$  and  $\alpha$ ,

$$\sup_{|s| \leq \epsilon} \frac{1}{\alpha!} |\partial^\alpha b_k(s)| \leq CM^{k+|\alpha|}.$$

In particular, each  $b_k$  is a real-analytic function of  $s$  and if we expand

$$(2.4) \quad b_k(s) = \sum_{\alpha \geq 0} \frac{1}{\alpha!} c_{k,\alpha} s^\alpha,$$

then  $\frac{1}{\alpha!} |c_{k,\alpha}| \leq CM^{k+|\alpha|}$ . Upon substituting (2.4) into (2.3) and then into (2.1) and simplifying, one obtains

$$f = \sum_{\substack{k \geq 0 \\ \alpha \geq 0}} \frac{1}{\alpha!} c_{k,\alpha} z^k (s + zt)^\alpha.$$

Thus  $f$  is of the desired form with  $h(z, w) = \sum \frac{1}{\alpha!} c_{k,\alpha} z^k w^\alpha$ .

The same sort of argument applies for distribution solutions except that the estimates must be made in different norms. We sketch the steps and leave the details to the reader. If  $f$  is a distribution solution then  $f$  is still holomorphic in  $z$ , so can be expanded in the form (2.1), where now the  $a_k$  are distributions in  $(s, t)$  and the series converges weakly in  $(z, s, t)$ . Also in some neighborhood of the origin  $f$  has finite order, so (2.2) can be replaced by the statement that for some  $\epsilon, N$ , and  $r$ , the set  $\{r^k a_k\}$  defines a bounded set in  $(C_{B_\epsilon \times B_\epsilon}^N)'$ . Here  $C_{B_\epsilon \times B_\epsilon}^N$  denotes the space of restrictions to  $B_\epsilon \times B_\epsilon = \{|s| \leq \epsilon\} \times \{|t| \leq \epsilon\}$  of  $C^N$  functions on  $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$  with compact support in  $B_\epsilon \times B_\epsilon$ , with the usual norm. The inductive calculations and

bootstrap leading to (2.3) proceed just as before, and it follows that the  $a_k$  are of the form (2.3) for  $C^\infty$  functions  $b_k(s)$ . It is not hard to prove a version of Lemma 2.4 for the  $(C^N)'$  norm; the statement is the same except that the right hand side is replaced by  $CR^k \|p\|_{(C_{B_1}^N)'}'$ , where  $C$  is another constant depending only on  $n$ . Applying this version of the Lemma now gives the bound

$$\frac{1}{\alpha!} \|\partial^\alpha b_k\|_{(C_{B_\epsilon}^N)'} \leq CM^{k+|\alpha|}.$$

From this it follows that each  $b_k$  is real-analytic and that  $f$  is of the desired form for a holomorphic function  $h(z, w)$  as before.  $\square$

**Remark 2.5.** F. Trèves has pointed out to us that the hypocomplexity also follows from microlocal regularity results for solutions of involutive structures given in [BT] or [BCT]. The result in [BT] characterizes the analytic wave-front set of solutions of involutive structures of tube type; the involutive structure on the blow-up is of this form. The result of [BCT] is that a point of the characteristic variety of an involutive structure is not in the analytic wave-front set of any solution if the Levi form at that point has a negative eigenvalue. In our case, at each point of the characteristic variety the Levi form has exactly one positive and one negative eigenvalue.

Next we show that the first local cohomology group of the involutive structure is infinite dimensional at a point of  $\Sigma$ . This amounts to studying solvability of the inhomogeneous equations

$$\partial_{\bar{z}} f = u \quad \text{and} \quad (\partial_{tj} - z\partial_{sj})f = v_j,$$

subject to the compatibility conditions

$$(\partial_{tj} - z\partial_{sj})u = \partial_{\bar{z}} v_j \quad \text{and} \quad (\partial_{tj} - z\partial_{sj})v_k = (\partial_{tk} - z\partial_{sk})v_j.$$

We will take  $u = 0$  and  $v_j = v_j(s)$  to depend only on  $s$ . In this case the first compatibility condition is automatic and the second reduces to  $\partial_{s_j} v_k = \partial_{s_k} v_j$ , which just says that the 1-form  $v_j ds^j$  is closed. We assume *a priori* only that the  $v_j$  are distributions in  $s$ .

**Theorem 2.6.** *If the equations  $\partial_{\bar{z}} f = 0$ ,  $(\partial_{tj} - z\partial_{sj})f = v_j(s)$  have a distribution solution  $f$  near  $(z, s, t) = (0, 0, 0)$ , then the  $v_j(s)$  are real analytic near  $s = 0$ .*



*Proof.* The argument is similar to the proof of Theorem 2.3. Again we present the details for the case in which there is a  $C^1$  solution  $f$ ; the modifications to deal with a distribution solution follow those outlined above.

The solution  $f$  must be holomorphic in  $z$ , so we have (2.1) and (2.2) just as before. The second equation gives

$$\partial_{t^j} a_0 = v_j(s) \quad \text{and} \quad \partial_{t^j} a_k = \partial_{s^j} a_{k-1} \quad \text{for } k \geq 1.$$

It follows that  $a_0(s, t) = t^j v_j(s) + b_0(s)$  for a function  $b_0(s)$ ; since  $f$  is  $C^1$  we deduce that  $v_j$  and  $b_0$  are both  $C^1$  as well. Inductively solving the remaining equations gives

$$(2.5) \quad a_k(s, t) = \sum_{|\alpha|=k+1} \frac{1}{\alpha!} t^\alpha (\partial_s^k v)_\alpha + \sum_{|\alpha| \leq k} \frac{1}{\alpha!} t^\alpha \partial_s^\alpha b_{k-|\alpha|}$$

for  $C^1$  functions  $b_k(s)$ , where  $(\partial_s^k v)_\alpha$  denotes the  $\alpha$ -multiindex component of the symmetric tensor  $\partial_s^k v$ . Again, one deduces inductively that  $v$  and all the  $b_k$  are  $C^\infty$ . Now apply Lemma 2.4 to the highest order coefficients of  $a_k$  as a polynomial in  $t$  and use (2.2) to deduce that for some constants  $C$  and  $M$ ,  $\sup_{|s| \leq \epsilon} \frac{1}{\alpha!} |\partial_s^\alpha v_j(s)| \leq CM^{|\alpha|}$ . It follows that each  $v_j$  is real-analytic.  $\square$

### 3. The Edge of the Wedge Theorem.

The hypocomplexity of the involutive structure on the blow-up of  $\mathbb{R}^n$  in  $\mathbb{C}^n$  is, for all intents and purposes, equivalent to the Edge of the Wedge Theorem. (See [VZS] for a discussion of the history and development of this result.)

**Edge of the Wedge Theorem.** *Let  $E \subset \mathbb{R}^n$  be open and let  $C \subset \mathbb{R}^n$  be an open cone such that  $C = -C$ . Set  $W = E + iC \subset \mathbb{C}^n$ . Let  $\mathcal{N}$  be an open set in  $\mathbb{C}^n$  with  $\mathcal{N} \cap \mathbb{R}^n = E$  and let  $f$  be a holomorphic function on  $\mathcal{N} \cap W$ . Suppose that  $\lim_{C \ni Y \rightarrow 0} f(\cdot + iY)$  exists in  $\mathcal{D}'(E)$ , and is independent of how  $Y \rightarrow 0$ . Then there is another open set  $\mathcal{N}' \subset \mathcal{N}$  with  $\mathcal{N}' \cap \mathbb{R}^n = E$  and a holomorphic function  $h$  on  $\mathcal{N}'$  such that  $h = f$  on  $\mathcal{N}' \cap W$ .*

The geometry is simplest in case  $n = 1$ . Setting  $\mathcal{N}_+ = \mathcal{N} \cap \{y > 0\}$  and  $\mathcal{N}_- = \mathcal{N} \cap \{y < 0\}$ , one has holomorphic functions  $f_+$  on  $\mathcal{N}_+$  and  $f_-$  on  $\mathcal{N}_-$  whose distribution boundary values agree on  $E$ . One proof of the Edge of the Wedge Theorem in this case is to observe that  $f_+$  and  $f_-$  together

define a distribution on  $\mathcal{N}$  which is a distribution solution of the Cauchy-Riemann equation. This observation follows from the following (standard) formulation of distribution boundary values of holomorphic functions:

(3.1)

Let  $E \subset \mathbb{R}$  and  $\mathcal{N} \subset \mathbb{C}$  be open with  $\mathcal{N} \cap \mathbb{R} = E$ , and set  $\mathcal{N}_+ = \mathcal{N} \cap \{y > 0\}$ . Suppose that  $f$  is a holomorphic function on  $\mathcal{N}_+$  and one of the following limits exists:

$$(i) \quad \tilde{f} = \lim_{\epsilon \rightarrow 0} f \chi_{\{y > \epsilon\}} \text{ in } \mathcal{D}'(\mathcal{N}).$$

$$(ii) \quad f_0 = \lim_{\epsilon \rightarrow 0} f(\cdot + i\epsilon) \text{ in } \mathcal{D}'(E).$$

Then the other exists, and  $\partial_{\bar{z}} \tilde{f} = \frac{i}{2} f_0(x) \delta_0(y)$ .

For boundary values from  $\mathcal{N}_-$  the corresponding equation for  $\partial_{\bar{z}} \tilde{f}$  has a minus sign. Hence  $\partial_{\bar{z}}(\tilde{f}_+ + \tilde{f}_-) = 0$ , and  $h = \tilde{f}_+ + \tilde{f}_-$  is the required holomorphic function.

For completeness, we briefly recall the proof of (3.1). If (i) exists, then certainly also  $\lim_{\epsilon \rightarrow 0} \partial_{\bar{z}}(f \chi_{\{y > \epsilon\}})$  exists in  $\mathcal{D}'(\mathcal{N})$ . However,  $\partial_{\bar{z}}(f \chi_{\{y > \epsilon\}}) = \frac{i}{2} f \delta_\epsilon(y)$ , so it follows that (ii) exists, and in the limit one obtains  $\partial_{\bar{z}} \tilde{f} = \frac{i}{2} f_0(x) \delta_0(y)$ . On the other hand, if (ii) exists, then  $y \rightarrow f(\cdot + iy)$  defines a weakly continuous function of  $y$  into  $\mathcal{D}'(E)$  up to  $y = 0$ . Using the uniform boundedness principle one deduces easily that (i) exists.

Although the geometry of the wedge is more complicated in higher dimensions, introduction of the blow-up reduces the geometry to exactly that in dimension one; namely solutions (of the involutive structure) on the two sides of a hypersurface with equal boundary values, and the above proof then goes right through. The main observation is simply that the image under the blow-down map  $b$  of a ball in  $B$  about a point of  $\Sigma$  is essentially a localized wedge as occurs in the Edge of the Wedge Theorem.

*Proof.* Choose a point of  $E$  and a line in  $C$ ; these determine a line in  $W$  and a point in the fiber in  $B$  above the point of  $E$ . Coordinates may be chosen so that the point is the origin in  $\mathbb{R}^n$  and the line is the first basis direction, so that in the coordinates introduced above the corresponding point of  $B$  is the origin  $(z, s, t) = (0, 0, 0)$ . Let  $\mathcal{M}$  be an open ball about the origin in  $B$  and set  $\mathcal{M}_+ = \mathcal{M} \cap \{y > 0\}$ ,  $\mathcal{M}_- = \mathcal{M} \cap \{y < 0\}$ ,  $\mathcal{M}_0 = \mathcal{M} \cap \{y = 0\}$ . Then for  $\mathcal{M}$  sufficiently small,  $b(\mathcal{M}_+ \cup \mathcal{M}_-) \subset \mathcal{N} \cap W$ . Consequently  $f \circ b$  defines solutions  $f_+$  and  $f_-$  of the involutive structure on  $\mathcal{M}_+$  and  $\mathcal{M}_-$ , resp. In the case that the boundary values of  $f$  are taken continuously, so

that  $f$  has a continuous extension to  $E \cup (\mathcal{N} \cap W)$ ,  $f \circ b$  clearly extends continuously to  $\mathcal{M}$  so the boundary values of  $f_+$  and  $f_-$  agree on  $\mathcal{M}_0$ . In the general case in which the boundary values of  $f$  are taken in the distribution sense, it is straightforward to check (again using the uniform boundedness principle) that the distribution boundary values of  $f_+$  and  $f_-$  exist in  $\mathcal{D}'(\mathcal{M}_0)$  and agree. Since  $f_+$  and  $f_-$  are annihilated by  $\partial_{\bar{z}}$ , one can now apply the exact argument as presented above in the  $n = 1$  case to conclude that  $\partial_{\bar{z}}(\tilde{f}_+ + \tilde{f}_-) = 0$ . However

$$(\partial_t - z\partial_s)\tilde{f}_+ = \lim(\partial_t - z\partial_s)(f_+ \chi_{\{y>\epsilon\}}) = 0,$$

and similarly for  $\tilde{f}_-$ . Thus  $\tilde{f}_+ + \tilde{f}_-$  is a distribution solution of the involutive structure in  $\mathcal{M}$ . From Theorem 2.3 it follows that there is a holomorphic function in a neighborhood of  $0 \in \mathbb{C}^n$  which agrees with  $f$  on  $b(\mathcal{M}') \setminus E$  for some ball  $\mathcal{M}' \subset \mathcal{M}$ . In particular,  $f$  extends continuously up to  $\mathbb{R}^n$ . Since a holomorphic function is determined by its restriction to  $\mathbb{R}^n$  and the boundary values are independent of the direction chosen, it follows that the holomorphic extensions obtained by varying the point of  $E$  and the direction in  $C$  all agree on overlaps, so they define one holomorphic function in a neighborhood of  $E$  as desired.  $\square$

It is also easy to deduce the hypocomplexity of the involutive structure from the Edge of the Wedge Theorem. Since on  $\Sigma$ , the  $\partial_t - z\partial_s$  are real and span the tangent space to the fibers of  $b|_\Sigma$ , any  $C^1$  solution of the involutive structure on a ball about a point of  $\Sigma$  is constant on these fibers so defines a holomorphic function on a localized wedge, continuous up to  $\mathbb{R}^n$ . The Edge of the Wedge Theorem provides the holomorphic extension to a neighborhood as required in the definition of hypocomplexity. This same argument works for distribution solutions as well, using the fact that any distribution solution of the involutive structure is transversally regular, i.e. it can be regarded as a continuous function of  $y$  with values in distributions on a piece of  $\Sigma$ .

### 4. Integral Geometry.

Consideration of the involutive structure on the blow-up of  $\mathbb{R}\mathbb{P}^n$  in  $\mathbb{C}\mathbb{P}^n$  arose in [BEGM], [E], and [BaE] in applying methods of complex integral geometry, specifically the Penrose transform, to problems in real integral geometry. We include here a description of the underlying geometric picture. This provides a natural realization of the blow-up  $B$  in this compactified

setting.

The real problem is to study the X-ray transform in  $\mathbb{R}^n$ , or really a compactified version thereof. Embed  $\mathbb{R}^n \subset \mathbb{R}\mathbb{P}^n$ ; then an affine line in  $\mathbb{R}^n$  is represented as a 2-plane through the origin in  $\mathbb{R}^{n+1}$ , so the space of such lines becomes a subset of  $\text{Gr}_2(\mathbb{R}^{n+1})$ . The correspondence is encoded in the double fibration

$$\begin{array}{ccc} & \mathbb{F}_{1,2}(\mathbb{R}^{n+1}) & \\ & \swarrow \quad \searrow & \\ \mathbb{R}\mathbb{P}^n & & \text{Gr}_2(\mathbb{R}^{n+1}), \end{array}$$

where  $\mathbb{F}_{1,2}(\mathbb{R}^{n+1})$  denotes the flag manifold of lines contained in planes in  $\mathbb{R}^{n+1}$ . The analogous complex correspondence is

$$\begin{array}{ccc} & \mathbb{F}_{1,2}(\mathbb{C}^{n+1}) & \\ & \swarrow \mu \quad \searrow \nu & \\ \mathbb{C}\mathbb{P}^n & & \text{Gr}_2(\mathbb{C}^{n+1}). \end{array}$$

The complex methods are applied to the real problem via the introduction of a hybrid correspondence. Define  $F = \nu^{-1}(\text{Gr}_2(\mathbb{R}^{n+1}))$ , so that

$$F = \{(L, P) : L \subset \mathbb{C}^{n+1} \text{ is a complex line, } P \subset \mathbb{R}^{n+1} \text{ is a real 2-plane, and } L \subset P + iP\}.$$

The hybrid correspondence is then

$$\begin{array}{ccc} & F & \\ & \swarrow \mu|_F \quad \searrow \nu|_F & \\ \mathbb{R}\mathbb{P}^n \subset \mathbb{C}\mathbb{P}^n & & \text{Gr}_2(\mathbb{R}^{n+1}). \end{array}$$

Of course  $\nu|_F$  is still a fibration, but this is not the case for  $\mu|_F$ . Write  $L \in \mathbb{C}\mathbb{P}^n$  as  $L = \text{span} \{ \xi + i\eta \}$  for  $\xi, \eta \in \mathbb{R}^{n+1}$ . If  $L \notin \mathbb{R}\mathbb{P}^n$  then  $\xi$  and  $\eta$  are linearly independent, so if  $L \subset P + iP$  for a real 2-plane  $P$ , necessarily  $P = \text{span} \{ \xi, \eta \}$ . Hence  $(\mu|_F)^{-1}(L)$  consists of a single point. On the other hand, if  $L \in \mathbb{R}\mathbb{P}^n$ , then  $L = \text{span} \{ \xi \}$  for some  $\xi \in \mathbb{R}^{n+1}$  so  $\mu^{-1}(\mathbb{R}\mathbb{P}^n) = \mathbb{F}_{1,2}(\mathbb{R}^{n+1})$  and  $(\mu|_F)^{-1}(L) \cong \mathbb{R}\mathbb{P}^{n-1}$ . This suggests the following:

**Proposition 4.1.** *The mapping  $\mu|_F : F \rightarrow \mathbb{C}\mathbb{P}^n$  realizes the blow-up of  $\mathbb{R}\mathbb{P}^n \subset \mathbb{C}\mathbb{P}^n$ .*

*Proof.* We show that, in suitable coordinates,  $\mu|_F$  is exactly the map  $b$  introduced previously. Coordinates in  $\mathbb{R}^{n+1}$  may be chosen to represent any fixed point of  $\mathbb{F}_{1,2}(\mathbb{R}^{n+1})$  as  $(\text{span}\{e_1\}, \text{span}\{e_1, e_2\})$ , where  $e_1, e_2$  are the first two basis vectors. For an arbitrary nearby point  $(L, P) \in F$ , we have

$$P = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ s \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ t \end{bmatrix} \right\} \text{ for some } s, t \in \mathbb{R}^{n-1},$$

and

$$L = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ s \end{bmatrix} + z \begin{bmatrix} 0 \\ 1 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ z \\ s + zt \end{bmatrix} \right\} \text{ for some } z \in \mathbb{C}.$$

Coordinates on  $F$  are  $(z, s, t)$ , and  $\mu|_F$  becomes  $(z, s, t) \rightarrow (z, s + zt)$  as desired. □

We refer to [BEGM], [E], and [BaE] for the application of this geometry to the inversion of real integral transforms.

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