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Seifert fibered manifolds and Dehn surgery III

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A Dehn surgery is called a Seifert fibering surgery if it yields a Seifert fibered manifold. It has been conjectured that nontrivial, Seifert fibering sugeries on knots in the 3-sphere are integral surgeries unless the knot is a trivial knot, a torus knot, or a cable of a torus knot. We first prove an analogous result for knots in a solid torus. As a corollary it is shown that the conjecture holds if a regular or exceptional fiber of the resulting Seifert fibered manifold is unknotted in the (original) 3-sphere; this assumption is verified for many Seifert fibering surgeries. As another application, we show that except for trivial examples, no periodic knots with period greater than 2 produce a Seifert fibered manifold with an infinite fundamental group by surgery.

1. Introduction.

Let K be a knot in a 3-manifold M. A slope of K is the isotopy class of a simple closed curve on $\partial N(K)$. The manifold obtained from M by Dehn surgery on a knot K with slope γ is denoted by $M(K;\gamma)$; if $M \cong S^3$, for simplicity we denote $M(K;\gamma)$ by $(K;\gamma)$. If $M \subset S^3$, then using the preferred meridian-longitude pair of $K \subset S^3$, we parametrize slopes γ of K by $r \in \mathbb{Q} \cup \{\infty\}$; then we also write $M(K;r)$ for $M(K;\gamma)$. A slope of K is *integral* if a representative of it intersects a meridian of K exactly once; for knots in S^3 integral slopes correspond to integers. A cable of a knot K is an essential, simple closed curve in $N(K)$ which is isotopic to a non-longitudinal curve in $\partial N(K)$.

If K is a trivial knot, a torus knot, or a cable of a torus knot, then $(K;r)$ is a Seifert fibered manifold for infinitely many $r \in \mathbb{Q} - \mathbb{Z}$. (Throughout this paper a trivial knot is not a torus knot.) It has been conjectured that:

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Conjecture 1.1. *If $(K;r)$ $(r \neq \infty)$ is a Seifert fibered manifold, then either r is an integer, or K is a trivial knot, a torus knot or a cable of a torus knot.*

The conjecture is proved for 2-bridge knots (Brittenham-Wu [6]) and satellite knots (Boyer-Zhang [5], Miyazaki-Motegi [25]). (See also [3].) The conjecture is also verified if $(K;r)$ admits a Seifert fibration over $\mathbb{R}P^2$ (Gordon-Luecke [16]), or a fibration over S^2 with less than three (Culler-Gordon-Luecke-Shalen [7]) or greater than three [5] exceptional fibers.

In this paper we first prove a result analogous to Conjecture 1.1 for knots in a solid torus. A *0-bridge braid* in a solid torus V is an essential simple closed curve isotopic to a curve in the boundary of V. A core of V is a 0-bridge braid, so that any other 0-bridge braid is a cable of a 0-bridge braid.

Theorem 1.2. *Let K be a knot in a solid torus V such that K is not contained in a 3-ball in V. Suppose that $V(K;\gamma)$ is a Seifert fibered manifold where the slope γ is not meridional. Then one of the following holds.*

(1) *K is a core of V or a cable of a 0-bridge braid in V.*

(2) *γ is an integral slope.*

Remark. Refer to Bleiler-Hodgson [2, Section 5] and Proposition 6.4 in Section 6.3 for examples of Seifert fibering surgeries on a knot in a solid torus.

This theorem implies Conjecture 1.1 under some hypothesis about "the position of a fiber" (Corollary 1.4 below). The hypothesis arises from the following question and examples: Suppose that $(K;r)$ is a Seifert fibered manifold. If a fiber in $(K;r)$ is contained in $S^3 - \operatorname{int} N(K)$, then it can be regarded as a knot in S^3. Which knot in S^3, then, becomes a fiber in $(K;r)$?

Example 1. Let K be a torus knot on the boundary of an unknotted solid torus in S^3, and c the core of the solid torus. If $(K;r)$ is a Seifert fibered manifold, it admits a Seifert fibration in which the trivial knot c is a fiber.

Example 2. Let K be the figure eight knot. Then $(K;r)$ is a Seifert fibered manifold if and only if $r = \pm 1, \pm 2, \pm 3$. The trivial knot c_1 in Figure 1.1 becomes an exceptional fiber in some Seifert fibrations of $(K;r)$ for $r = -1, -2, -3$; see Section 6.1 for proofs. The trivial knot c_2 also has this property.

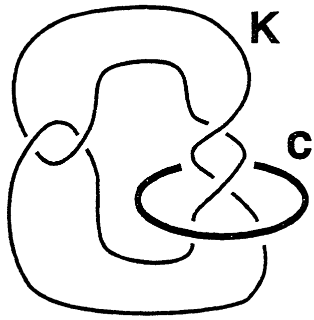

Figure 1.1

Example 3. Let K be a satellite knot such that $(K;r)$ is Seifert fibered. If $(K;r)$ is non-simple or $\pi_1((K;r))$ is finite, then [25] or [4], respectively shows that a torus knot is a companion of K; the Seifert fibration of the torus knot exterior extends over $(K;r)$. Thus, as in Example 1 a trivial knot in S^3 becomes an exceptional fiber.

Keeping these examples in mind, we conjecture that:

Conjecture 1.3. *Let K be a knot in S^3. If $(K;r)$ is a Seifert fibered manifold, then it admits a Seifert fibration such that a fiber of it is unknotted in (the original) S^3.*

Corollary 1.4. *Let K be a knot in S^3, and $(K;r)$ $(r \neq \infty)$ Seifert fibered. If $(K;r)$ satisfies Conjecture* 1.3, *then it satisfies Conjecture* 1.1.

In Section 6, Conjecture 1.3 will be verified for some Seifert fibering surgeries on the following knots: 2-bridge knots, Eudave-Muñoz' hyperbolic knots with non-hyperbolic surgeries [9], and some twisted torus knots. However, these surgeries are already known to be integral ones.

Concerning Seifert fibering surgery on hyperbolic knots, experiments via the computer program SnapPea written by Jeffrey Weeks suggest that shortest geodesics in knot complements become fibers (Section 7).

Remark. Hayashi [18] and Hayashi-Motegi [19] also obtained some estimates on Seifert fibering slopes under some assumption of the position of fibers.

Proof of Corollary 1.4. Let c be a trivial knot in S^3 given by Conjecture 1.3. Take a tubular neighborhood of c so that $N(c) \cap K = \emptyset$, and let

$V = S^3 - \operatorname{int} N(c)$. Since V is an unknotted solid torus in S^3, a 0-bridge braid in V is a trivial knot or a torus knot in S^3. We denote the natural image of c in $(K;r)$ by the same symbol c. Since $(K;r)$ admits a Seifert fibration in which c is a regular or exceptional fiber, $(K;r) - \operatorname{int} N(c)$ is Seifert fibered. Since $(K;r) = V(K;r) \cup N(c)$, we see that $V(K;r)$ is a Seifert fibered manifold.

If K is not contained in a 3-ball in V, Corollary 1.4 directly follows from Theorem 1.2. If K is contained in a 3-ball in V, then $V(K;r) \cong (K;r)\#V$. Since $V(K;r)$ is a bounded Seifert fibered manifold, it is irreducible, so that $(K;r) \cong S^3$ where $r \neq \infty$. By [15] K is then a trivial knot in S^3 as claimed in Conjecture 1.1. □

Another application of Theorem 1.2 is the study of Seifert fibering surgeries on periodic knots. A knot K in S^3 is called a *periodic knot* with period p if there is a homeomorphism $f : S^3 \to S^3$ such that $f(K) = K$, $f^p = \mathrm{id}$ $(p > 1)$, $\mathrm{Fix}(f) \cong S^1$, and $\mathrm{Fix}(f) \cap K = \emptyset$, where $\mathrm{Fix}(f)$ is the set of fixed points of f. We show that periodicity greater than 2 resists infinite, Seifert fibering surgery.

Theorem 1.5. *Let K be a periodic knot in S^3 with period $p > 2$. If $(K;r)$ is a Seifert fibered manifold with an infinite fundamental group, then K is a trivial knot, a torus knot, or a cable of a torus knot.*

Remarks. (1) The assumption $p > 2$ cannot be deleted. For example, the ± 1, ± 2, ± 3-surgeries on the figure eight knot, which is with period 2, is a Seifert fibered manifold with an infinite fundamental group.

(2) Using the orbifold geometrization conjecture, we can extend the theorem to manifolds with finite fundamental groups (Propositions 5.6, 5.9).

The paper is organized as follows. If a surgery of a solid torus is a Seifert fibered manifold, then homology calculation shows that the base space is the disk or the Möbius band. The former case is dealt with in Sections 2, 3, and the latter in Section 4. In Section 5, we discuss Seifert fibering surgeries on periodic knots. Using Theorem 1.2 and the geometric structures of Seifert fibered manifolds, we prove Theorem 1.5. In Section 6, we show that some Seifert fibering surgeries satisfy Conjecture 1.3. In Section 7, we consider Conjecture 1.3 from viewpoints of hyperbolic geometry.

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2. The case when base spaces are disks.

In Sections 2 and 3, we prove the following.

Proposition 2.1. *Let K be a knot in a solid torus V such that K is not contained in a 3-ball in V. Suppose that $V(K;\gamma)$ is a Seifert fibered manifold over the disk, where the slope γ is not meridional. Then one of the following holds.*

(1) *K is a core of V or a cable of a 0-bridge braid in V.*

(2) *γ is an integral slope.*

Proof. If $V(K;\gamma)$ contains at most one exceptional fiber, then $V(K;\gamma) \cong S^1 \times D^2$. Hence, by [11], [12] either K is a 0-bridge braid in V or γ is integral. In what follows we thus assume that $V(K;\gamma)$ is Seifert fibered over a disk with $n(\geq 2)$ cone points.

For convenience, we regard that V is a standardly embedded solid torus in S^3. Let (M, L) be a preferred meridian-longitude pair of $V \subset S^3$. A regular fiber on $\partial V(K;\gamma)(= \partial V)$ represents $pM + qL$ for some coprime integers p and q. We distinguish three cases whether $q = 0$, $|q| = 1$ or $|q| \geq 2$.

Attach a solid torus W to V in such a way that the meridional slope of W is identified with the slope $pM + qL$. Then we obtain a 3-manifold $V \cup W$ and denote the image of K in $V \cup W$ by K'. The 3-manifold $V \cup W$ is homeomorphic to $S^2 \times S^1$ if $q = 0$, S^3 if $|q| = 1$, or a lens space $L(q,p)$ if $|q| \geq 2$.

Let (μ, λ) be a preferred meridian-longitude pair of K, and μ' a meridian of K'. On $\partial N(K) = \partial N(K')$, we have $\mu = \mu'$, so that an integral slope on $\partial N(K)$ is also an integral slope on $\partial N(K')$ and vice versa.

In this paper, we exclude S^3 and $S^2 \times S^1$ from lens spaces.

Lemma 2.2. $(V \cup W)(K';\gamma) = V(K;\gamma) \cup W$ *is a connected sum of n lens spaces.*

Proof. Let $\pi : V(K;\gamma) \to D^2$ be a Seifert fibration. Take $n-1$ mutually non-parallel essential arcs a_i in $D^2 - \{$ cone points $\}$ $(i = 1, \dots, n-1)$. Let A_i be the vertical annulus $\pi^{-1}(a_i)$. Then the 2-spheres S_i obtained from A_i by capping off with two meridian disks of W define the required connected sum decomposition of $(V \cup W)(K';\gamma) = V(K;\gamma) \cup W$. □

For two slopes γ_1, γ_2 of a knot, the *distance* $\Delta(\gamma_1, \gamma_2)$ between γ_1 and γ_2 is defined to be their minimal geometric intersection number.

Lemma 2.3. *If $q = 0$, then γ is integral.*

Proof. Notice that $V(K;\mu) \cup W = V \cup W \cong S^2 \times S^1$, and $(V \cup W)(K';\gamma) = V(K;\gamma) \cup W$ is a connected sum of $n(\geq 2)$ lens spaces. If $V \cup W - \operatorname{int} N(K')$ is reducible, then the primeness of $S^2 \times S^1$ implies that K' is contained in a 3-ball in $V \cup W$. Therefore $(V \cup W)(K';\gamma)$ has $S^2 \times S^1$ as a connected summand, a contradiction. Thus $V \cup W - \operatorname{int} N(K')$ is irreducible. Apply [17] to conclude that $\Delta(\gamma, \mu') = \Delta(\gamma, \mu) = 1$; the slope γ is integral. □

Lemma 2.4. *If $|q| = 1$, then γ is integral.*

Proof. In this case $V \cup W \cong S^3$, and $(V \cup W)(K';\gamma)$ is a connected sum of $n(\geq 2)$ lens spaces. Hence by [14], $\Delta(\gamma, \mu') = \Delta(\gamma, \mu) = 1$. Thus γ is an integral slope. □

The last and the most difficult case is $|q| \geq 2$.

Lemma 2.5. *Suppose $|q| \geq 2$. Then either γ is integral or K is a cable of a 0-bridge braid in V.*

Proof. For simplicity set $X = V \cup W - \operatorname{int} N(K')$. Recall that $V \cup W = L(q,p)$. We divide the proof into three cases:

(1) $X = L(q,p) - \operatorname{int} N(K')$ is irreducible and not an atoroidal Seifert fibered manifold.

(2) X is an atoroidal Seifert fibered manifold.

(3) X is reducible.

In this section we settle cases (1) and (2). (In fact, we show that γ is an integral slope whatever K is.) Case (3) will be dealt with in the next section.

First we observe that

- $(V \cup W)(K'; \mu') = V \cup W = L(q,p)$.
- $(V \cup W)(K'; \gamma) = V(K; \gamma) \cup W$ is a connected sum of n lens spaces.

Case (1). A recent result of Boyer and Zhang states that:

Theorem 2.6 ([3, Theorem F(1)]). *Let M be a closed, orientable 3-manifold and K a knot in M such that the exterior $M - \operatorname{int} N(K)$ is irreducible. Assume that $M - \operatorname{int} N(K)$ is not an atoroidal Seifert fibered manifold. If $M(K;\gamma_1)$ is a reducible manifold and $M(K;\gamma_2)$ has a cyclic fundamental group, then $\Delta(\gamma_1, \gamma_2) \leq 1$.*

Hence, if $L(q,p) - \operatorname{int} N(K')$ is irreducible and not an atoroidal Seifert fibered manifold, the above observation and the theorem prove Lemma 2.5.
Case (2). Since X is an atoroidal Seifert fibered manifold, its base orbifold is either a disk with at most two cone points or the Möbius band with no cone points. In the second case, X is the twisted I-bundle over the Klein bottle, and hence X admits also a Seifert fibration over the disk with two cone points of indices 2, 2. Thus the second case reduces to the first.

If X has at most one exceptional fiber, then it is a solid torus. Hence $L(q,p)(K';\gamma)$ cannot be a connected sum of two lens spaces, a contradiction to Lemma 2.2. It follows that X has exactly two exceptional fibers. Let t be the slope of a regular fiber in $\partial N(K') \subset X$. Then, for any slope α on $\partial N(K)$, $L(q,p)(K';\alpha)$ is (i) a connected sum of two lens spaces if $\Delta(\alpha, t) = 0$, (ii) a lens space if $\Delta(\alpha, t) = 1$, or (iii) a Seifert fibered manifold over the 2-sphere with three cone points if $\Delta(\alpha, t) \geq 2$. The Seifert fibered manifolds in (iii) are neither lens spaces nor reducible (see [21]). Thus, we see that $\Delta(\gamma, t) = 0$ and $\Delta(\mu', t) = 1$, so that $\Delta(\gamma, \mu') = 1$ as claimed in Lemma 2.5.

3. Knots in lens spaces with reducible exteriors.

In this section we prove Lemma 2.5 in the case where $X = V \cup W - \operatorname{int} N(K')$ is reducible, and complete the proofs of Lemma 2.5 and Proposition 2.1.

Since a lens space $L(q,p)$ is irreducible, the reducibility of $L(q,p) - \operatorname{int} N(K')$ implies that K' is contained in a 3-ball $B \subset L(q,p)$. Let $\Sigma = \partial B$.

Since $V - \operatorname{int} N(K)$ is irreducible by the assumption of Proposition 2.1, Σ is not contained in V. Hence, we may assume that Σ intersects W with (non-empty) meridian disks of W. Now let us take such a 2-sphere Σ so that $|\Sigma \cap W|$ (= the number of components of $\Sigma \cap W$) is minimal. Since Σ separates $V \cup W$, $|\Sigma \cap W|$ is an even integer (≥ 2). Set $P = \Sigma \cap (V - \operatorname{int} N(K))$, a planar surface.

Lemma 3.1. $|\partial P| = 2$ *(i.e., P is an annulus) or γ is integral.*

Proof. Assume that $|\partial P| \geq 4$. Since Σ separates $L(q,p) = V \cup W$, P also separates V. Cutting V along P, we obtain two 3-manifolds M_1 and M_2. We assume that M_1 contains K. The minimality of $|\Sigma \cap W|$ guarantees that P is incompressible and boundary-incompressible in $V - \operatorname{int} N(K)$. In particular P is incompressible in both $M_1 - \operatorname{int} N(K)$ and M_2. There are two possibilities: (1) P is incompressible in $M_1(K;\gamma)$, (2) P is compressible in $M_1(K;\gamma)$.

(1) *P is incompressible in $M_1(K;\gamma)$.* Then P is also incompressible in the Seifert fibered manifold $V(K;\gamma) = M_1(K;\gamma) \cup_P M_2$. Since $|\partial P| \geq 4$, P is boundary-incompressible in $V(K;\gamma)$. Hence P is isotopic to a vertical (i.e., consisting of fibers) or a horizontal (i.e., transverse to fibers) surface [31]. Since each component of ∂P is a regular fiber in $V(K;\gamma)$, P cannot be isotopic to a horizontal surface. Thus P is isotopic to a vertical surface, and so P is an annulus, a contradiction.

(2) *P is compressible in $M_1(K;\gamma)$.*

Claim 3.2. *P is compressible also in $M_1 = M_1(K;\mu)$.*

Proof. If P is incompressible in M_1, then P is also incompressible in $V = M_1 \cup_P M_2$. This implies that a solid torus V contains an incompressible planar surface P with $|\partial P| \geq 4$, a contradiction. □

If there is no incompressible annulus in $M_1 - \operatorname{int} N(K)$ with one boundary component in P and the other in $\partial N(K)$, then Wu [33, Theorem 1] shows that $\Delta(\gamma,\mu) = 1$, and hence γ is integral as claimed in Lemma 3.1. In the following we thus assume that there is such an annulus, say A, in $M_1 - \operatorname{int} N(K)$. Write $\partial A = C_1 \cup C_2$, where $C_1 \subset \partial N(K)$ and $C_2 \subset P (\subset \Sigma)$. Since C_2 bounds a disk on the 2-sphere Σ, C_1 bounds a disk in the 3-ball B. This implies that K' is a trivial knot in B, and $\partial A \cap \partial N(K')$ has the preferred longitudinal slope, λ', of $K' \subset B$.

Then, [7, Theorem 2.4.3(b)] shows that $\Delta(\gamma, \lambda') \leq 1$ or $M_1 - \operatorname{int} N(K) \cong S^1 \times S^1 \times I$. The latter implies that the incompressible surface P in $M_1 - \operatorname{int} N(K)$ is a disk or an annulus, which contradicts our assumpton $|\partial P| \geq 4$. It follows that $\Delta(\gamma, \lambda') \leq 1$. This together with the triviality of $K' \subset B$ implies that either $B(K';\gamma) = B(K';1/n) \cong B^3$ or $B(K';\gamma) = B(K';0) \cong$ ($S^2 \times S^1$ with a puncture). Hence, $L(q,p)(K';\gamma) = (L(q,p) - B) \cup B(K';\gamma)$ is not a connected sum of $n(\geq 2)$ lens spaces. This contradicts Lemma 2.2, and proves Lemma 3.1. □

Lemma 3.3. *Suppose $|\partial P| = 2$ (i.e., P is an annulus). Then one of the following holds.*

(1) *K is a cable of a 0-bridge braid in V.*

(2) *γ is integral.*

Proof. Recall that each component of ∂P has the slope $pM + qL$. Since $|q| \geq 2$, P is incompressible in V, so it is a boundary-parallel annulus in V. The closure of the component of $V - P$ containing K is a solid torus. By shrinking it, we obtain a solid torus $V' \subset \operatorname{int} V$ such that $K \subset V'$; V' is a tubular neighborhood of a (p,q)-cable of the core of V. Since $V - \operatorname{int} N(K)$ is irreducible and boundary-irreducible, so is $V' - \operatorname{int} N(K)$.

First we assume that $T = \partial V'(K;\gamma)$ is incompressible in $V'(K;\gamma)$. Then T is incompressible in $V(K;\gamma)$. Since $V(K;\gamma)$ is a bounded Seifert fibered manifold, T cannot be isotopic to a horizontal torus, and hence T is isotopic to a vertical torus. (By isotoping Seifert fibration of $V(K;\gamma)$, we may assume that T is vertical.) It follows that T splits $V(K;\gamma)$ into two Seifert fibered manifolds C and $V'(K;\gamma)$, where C is a cable space. This implies that $V'(K;\gamma)$ is also a Seifert fibered manifold over the disk. Moreover, a fiber on $\partial V'(K;\gamma) \subset \partial C$ represents $pq\mu'_V + \lambda'_V$, where (μ'_V, λ'_V) is a preferred meridian-longitude pair of $V'(\subset V \subset S^3)$. Applying Lemma 2.4 to $V'(K;\gamma)$ shows that γ is integral.

Next assume that $\partial V'(K;\gamma)$ is compressible in $V'(K;\gamma)$. Then $V'(K;\gamma) \cong (S^1 \times D^2) \# M$ for some closed 3-manifold M. The irreducibility of $V(K;\gamma)$ implies that $M \cong S^3$. It follows that $V'(K;\gamma) \cong S^1 \times D^2$, and [11] shows that K is a 0- or 1-bridge braid in V'. If K is a 1-bridge braid in V', then by [12, Lemma 3.2] γ is integral. If K is a 0-bridge braid in V', then $K \subset V$ is a cable of a 0-bridge braid in V. □

Combining Lemmas 3.1 and 3.3 proves Lemma 2.5 in case (3), thus Lemma 2.5 is proved. Proposition 2.1 follows from Lemmas 2.3, 2.4 and 2.5. □□

4. The case when base spaces are Möbius bands.

Theorem 1.2 follows from Propositions 2.1 and 4.1 below. □

Proposition 4.1. *Let K be a knot in a solid torus V. Suppose that $V(K;\gamma)$ is a Seifert fibered manifold over the Möbius band. Then one of the following holds.*

(1) *K is a 0-bridge braid in V.*

(2) *γ is integral.*

Example 1. Let K be the $(1,2)$-cable of the core of V. Then $V(K;(2n \pm 2)/n)$ is a Seifert fibered manifold over the Möbius band with no cone points for any odd integer n.

Example 2. Let K be a $(2pq \pm 1, 2)$-cable of a (p,q)-cable of the core of V. Then $V(K;4pq)$ is a Seifert fibered manifold over a Möbius band with one cone point.

Proof. If K is contained in a 3-ball in V, then $V(K;\gamma) \cong V\#(K;\gamma)$ cannot be a Seifert fibered manifold over the Möbius band. Thus we assume that K is not contained in a 3-ball in V. We consider the torus decomposition of $V - \operatorname{int} N(K)$ ([22],[23]). Let P be the decomposing piece containing ∂V. Notice that P is Seifert fibered or hyperbolic (i.e., admits a complete hyperbolic structure in its interior) and that P is possibly $V - \operatorname{int} N(K)$ itself. Furthermore, if P is Seifert fibered, then P is a cable space or a composing space, each of which has a unique Seifert fibration [22].

Attach a solid torus W to V along their boundaries so that the slope $L + nM$ bounds a meridian disk D_W of W, where (M, L) is a meridian-longitude pair of V, and $n \in \mathbb{Z}$. We denote by K_n the image of K in the new 3-sphere $V \cup W \cong S^3$. Then $(K_n;\gamma) = V(K;\gamma) \cup_{L+nM=\partial D_W} W$ admits a Seifert fibration over $\mathbb{R}P^2$ unless $L + nM$ is the fiber slope of $V(K;\gamma)$. Assume that γ is not integral. [26, Corollary 1.4] states that if a surgery on a non-torus knot yields a Seifert fibered manifold over the projective plane, then the surgery slope is integral. Hence, K_n is a torus knot except possibly for one value of n. In the following we show that K is a 0-bridge braid in V, i.e., $V - \operatorname{int} N(K)$ is a cable space.

If P is hyperbolic, then for sufficiently large n, $P \cup_{L+nM=\partial D_W} W$ is also hyperbolic [30]. This implies that some torus knot exterior $S^3 - \operatorname{int} N(K_n)$ is hyperbolic (when $P = V - \operatorname{int} N(K)$) or $S^3 - \operatorname{int} N(K_n)$ contains an essential torus ($\subset \partial P - \partial V$). Since a torus knot exterior is an atoroidal Seifert fibered manifold, this is a contradiction. It follows that P is Seifert fibered. Let us assume that $P \neq V - \operatorname{int} N(K)$. Since torus knot exteriors contain no essential tori, $P \cup_{L+nM=\partial D_W} W$ is boundary-reducible and thus is a solid torus for infinitely many n. Then P is a cable space, and the distance between $L + nM$ and the slope of a fiber of P on ∂V is one; the latter shows that the fiber slope on ∂V must be M. But this implies that $S^2 \times S^1 \cong V \cup_{M=\partial D_W} W$ contains a lens space summand of $P \cup_{M=\partial D_W} W$, which is absurd. Hence, P is Seifert fibered and $P = V - \operatorname{int} N(K)$. It follows that $V - \operatorname{int} N(K)$ is a cable space as desired. □

5. Seifert fibering surgery on periodic knots.

Let K be a periodic knot in S^3 with an automorphism f with period p as described in the Introduction. By the positive answer to the Smith conjecture [27], f is a rotation of S^3 about the unknotted circle $\operatorname{Fix}(f)$. Let $N(K)$ be an f-invariant tubular neighborhood of K in S^3. Then we have a $\mathbb{Z}_p$-action on $E(K) = S^3 - \operatorname{int} N(K)$ generated by $f|E(K)$. We denote by $\bar{f}$ a periodic extension of $f|E(K)$ over $(K; r)$, which has also period p.

Proposition 5.1. *Let K be a periodic knot with an automorphism f with period p. Suppose that $(K; r)$ ($r \neq \infty$) admits a Seifert fibration with* $\operatorname{Fix}(f)$ *a fiber, and $\bar{f}$ preserves the Seifert fibration of $(K; r)$. Then K is a trivial knot, a torus knot, or a cable of a torus knot.*

If K is one of those knots stated in the conclusion, then $\operatorname{Fix}(f)$ is an exceptional fiber in a Seifert fibered manifold $(K; r)$.

Proof. By taking some power of f, if necessary, we may assume that f has a prime period p. (This f also satisfies the condition of Proposition 5.1.)

For simplicity, set $c = \operatorname{Fix}(f)$. Let K^* be the core of the glued solid torus in $(K; r)$. The circle c is fixed under $\bar{f}$, and we may assume $\operatorname{Fix}(\bar{f}^i) \subset c \cup K^*$ ($i = 1, \dots p-1$). Express $r = m/n$. In the following, (m, p) denotes the greatest common divisor of p and m. Since the period p is a prime number, there are two possibilities:

(1) $(m,p) = 1$; then K^* is not fixed under $\bar{f}^i$ for any integer $i = 1, \ldots, p-1$.

(2) $(m,p) = p$; then K^* is fixed under $\bar{f}$, so that $\mathrm{Fix}(\bar{f}) = c \cup K^*$.

Case (1). Choose an f-invariant tubular neighborhood $N(c)$ of c so that $N(c) \cap N(K) = \emptyset$. Let V be the solid torus $S^3 - \mathrm{int}\, N(c)$. Then the periodic map f induces p-fold coverings $V \to V_f$ and $K \to K_f$; $\bar{f}$ induces a p-fold covering $V(K;r) \to V_f(K_f;r_f)$, where $r_f = r/p$. Since $V(K;r)$ admits a Seifert fibration, $V_f(K_f;r_f)$ also admits a Seifert fibration [22, II.6.3.Theorem]. By Theorem 1.2, either (i) K_f is a core of V_f or a cable of a 0-bridge braid in V_f, or (ii) r_f is an integer. In case (i) the pull-back K is a core of V or a cable of a 0-bridge braid in V. Since V is unknotted in S^3, K is such a knot as claimed in the proposition. If r_f is an integer, then $m = npr_f$ is a multiple of p, contradicting $(m,p) = 1$. Case (ii) does not occur.

Case (2). Let $\pi : (K;r) \to B$ be a Seifert fibration. Take any fiber τ in $(K;r)$ meeting K^*. Since $\bar{f}$ fixes K^* and preserves the Seifert fibration of $(K;r)$, it follows $\bar{f}(\tau) = \tau$. Note that $\bar{f}|\tau : \tau \to \tau$ is the identity or a reflection.

Case (2)-(a). *The base space B is S^2.*

Claim 5.2. *$\bar{f}|\tau$ is the identity.*

Proof. It suffices to show $\bar{f}$ preserves the orientation of τ. The fiber preserving map $\bar{f}$ induces the automorphism φ of B. Then $\bar{f}|\tau$ preserves orientation if and only if φ preserves the orientation of $B = S^2$. Since $\bar{f}|c = \mathrm{id}$, φ restricts to a rotation (possibly an identity) on a small disk neighborhood of $\pi(c)$, and hence φ preserves orientation. □

By Claim 5.2 $K^* = \tau$. It follows that $S^3 - \mathrm{int}\, N(K) = (K;r) - \mathrm{int}\, N(K^*)$ is Seifert fibered. Hence K is a trivial knot or a torus knot in S^3.

Case (2)-(b). *B is $\mathbb{R}P^2$.*
We shall show that this case does not happen.

Claim 5.3. *$\bar{f}|\tau$ is a reflection of τ, and the period of $\bar{f}$ is* 2.

Proof. If $\bar{f}|\tau$ is the identity, then as in case (2)-(a) we see that $K^* = \tau$ and thus K is a trivial knot or a torus knot in S^3. However, the assumption of case (2)-(b) implies that $S^3 - \operatorname{int} N(K) = (K;r) - \operatorname{int} N(K^*)$ admits a Seifert fibration over the Möbius band. This is absurd. Since $\bar{f}^2$ fixes τ, $\bar{f}^2$ has a fixed point disjoint from $c \cup K^*$. Hence $\bar{f}$ and f are with period 2. □

The automorphism φ of B induced from $\bar{f}$ is the identity or an involution. If φ is the identity, the proof of Claim 5.2 shows that $\bar{f}$ fixes τ, contradicting Claim 5.3. Thus φ is an involution such that $\pi(K^*) \cup \pi(c) \subset \operatorname{Fix}(\varphi)$. Smith [29, p.414] shows that the fixed point set of an involution of $\mathbb{R}P^2$ consists of a point and a 1-sided simple loop. Hence $\pi(K^*)$ is a 1-sided simple loop on B, and $\operatorname{Fix}(\varphi) = \pi(K^*) \cup \pi(c)$.

Let D be a φ-invariant disk in $B - \pi(K^*)$ such that $\pi(c) \in D$, and no cone point lies in $D - \{\pi(c)\}$. $\overline{B - D}$ is a Möbius band with $\pi(K^*)$ a centerline. Let α be a non-separating arc properly embedded in $\overline{B - D} - \{$cone points$\}$ such that $\varphi(\partial\alpha) = \partial\alpha$, and α meets $\pi(K^*)$ in a single point. Then $\pi^{-1}(\alpha)$ is a vertical annulus properly embedded in $(K;r) - \operatorname{int} \pi^{-1}(D)$, and meets K^* in two points by Claim 5.3. Note that $\bar{f}$ exchanges the boundary components of $\pi^{-1}(\alpha)$. Let $A = \pi^{-1}(\alpha) - \operatorname{int} N(K^*)$, an annulus with two holes. Considering that $\pi^{-1}(D)$ is a tubular neighborhood of c in S^3, set $V = S^3 - \operatorname{int} \pi^{-1}(D)$, a solid torus. Then, A is properly embedded in $V - \operatorname{int} N(K)$ so that $A \cap \partial V = \partial\pi^{-1}(\alpha)$. The slope of a component of $A \cap \partial N(K)$ on $\partial N(K)$ is the surgery slope $r = m/n$. Since $\alpha \cup \beta$ is an orientation-reversing loop on B where β is a component of $\partial D - \partial\alpha$, $\pi^{-1}(\alpha \cup \beta)$ is a Klein bottle. Hence the oriented curves $A \cap \partial V$ are parallel on ∂V.

Now homological arguments lead us to a contradiction. Let (μ_V, λ_V) (resp. (μ_K, λ_K)) be a preferred meridian-longitude pair of V (resp. $N(K)$) in S^3. We denote the winding number of K in V by ω. Notice that $\lambda_K = \omega\lambda_V, \mu_V = \omega\mu_K$ in $H_1(V-K)$ where λ_V and μ_K generate $H_1(V-K) \cong \mathbb{Z}\oplus\mathbb{Z}$. Each component of $A\cap\partial V$ is essential in ∂V, and represents, say $x\mu_V + y\lambda_V$. If $[A\cap\partial N(K)] = 0 \in H_1(\partial N(K))$, then $A\cap\partial V$ represents zero in $H_1(V-K)$. It follows $2(x\omega\mu_K + y\lambda_V) = 0$ in $H_1(V-K)$, and hence $\omega = 0$. This violates the fact that the linking number of a knot with period 2 and its π-rotation axis is an odd number. (To observe this fact consider the 2-fold coverings $V \to V_f$ and $K \to K_f$ induced by f. Notice that ω, the linking number of K and c, is also the winding number of K_f in V_f. If ω is even, then the preimage of K_f has two components, a contradiction.) Hence, the oriented components of $A\cap\partial N(K)$ are parallel on $\partial N(K)$; each component represents $m\mu_K + n\lambda_K$. The surjectivity of $\pi_* : H_1((K;m/n)) \to H_1(B) \cong \mathbb{Z}_2$ implies that m is even. On the other hand, since f rotates V keeping a preferred longitude invariant and exchanges the components of $A \cap \partial V$, we see that x is an odd integer. Then the homology A between $A \cap \partial V$ and $A \cap \partial N(K)$ gives $m = -x\omega, n\omega = -y$. From the first equation and the parity of m and x, we see that ω is even. This again contradicts the fact that K is with period 2, and thus the proof of Proposition 5.1 is completed. □

Proof of Theorem 1.5. Assume that $(K; r)$ is a Seifert fibered manifold with an infinite fundamental group. If $(K; r)$ is reducible, then since $(K; r)$ is Seifert fibered and $H_1((K; r))$ is cyclic, $(K; r) \cong S^2 \times S^1$ [21, VI.7.Lemma]. Then K is a trivial knot and $r = 0$ [10, Corollary 8.3]. So we may assume $(K; r)$ is irreducible. From [28], such a manifold possesses a geometric structure modelled on $\mathbb{E}^3$, Nil, $H^2 \times \mathbb{R}$ or $\widetilde{SL_2\mathbb{R}}$. The geometry of $(K; r)$ is determined by $\chi(X)$, the Euler number of the base orbifold X and by $e(\eta)$, the Euler number of the Seifert bundle η whose total space is $(K; r)$. See the following table [28].

	$\chi(X) = 0$	$\chi(X) \neq 0$
$e(\eta) = 0$	$\mathbb{E}^3$	$H^2 \times \mathbb{R}$
$e(\eta) \neq 0$	Nil	$\widetilde{SL_2\mathbb{R}}$

Lemma 5.4. *Let K be a knot in S^3. If $(K; r)$ is a Seifert fibered manifold with the $\mathbb{E}^3$-geometry, then its base space is the 2-sphere, K is the trefoil knot, and $r = 0$.*

Proof. If the base orbifold X is $\mathbb{R}P^2$, the fact $\chi(X) = 0$ implies that X has two cone points of index 2. Then a homology calculation shows that $H_1(K; r)$ is non-cyclic, a contradiction. It follows that the base space is S^2. Regarding the Euler number of the bundle $\eta : (K; r) \to X$, a homology calculation shows that $|e(\eta)| = |H_1((K; r))|/\alpha_1 \cdots \alpha_n$, where α_i are the indices of cone points of X. Thus $e(\eta) = 0$ if and only if $r = 0$.

Let S be a non-separating incompressible surface in $(K; 0)$. Since S is non-separating, it is isotopic to a horizontal surface. Therefore $(K; 0)$ is a surface bundle over S^1 with a fiber S. On the other hand, since $\chi(X) = 0$, the branched cover S over X is a torus. From [10, Corollary 8.23], we see that K is the trefoil knot or the figure eight knot. If K is the figure eight knot, then $(K; 0)$ is a torus bundle over S^1 with hyperbolic monodromy. Such a manifold possesses the Sol-geometry by [28, Theorem 5.5], a contradiction. □

Next we consider the case where $(K;r)$ is modelled on Nil, $H^2 \times \mathbb{R}$ or $\widetilde{SL_2\mathbb{R}}$. By the hypothesis of the theorem, $(K;r)$ admits a $\mathbb{Z}_p$-action generated by $\bar{f}$, where $p > 2$. Under these geometries Meeks-Scott [24, Theorem 2.2] shows that $(K;r)$ has an $\bar{f}$-invariant Seifert fibration (see also [24, p.289]). Take a fiber t in such a fibration of $(K;r)$ so that t meets $c = \mathrm{Fix}(f)$. Since c is fixed by $\bar{f}$, we have $\bar{f}(t) = t$. If c were not a regular or exceptional fiber, then $\bar{f}|t$ would be a reflection. It follows $p = 2$, a contradiction. Hence c is a regular or exceptional fiber. Then, by Proposition 5.1 K is a trivial knot, a torus knot, or a cable of a torus knot as claimed in Theorem 1.5. □

In 1982, Thurston announced that:

Assertion 5.5. *If a closed, irreducible* 3*-manifold M admits an effective action of a finite group G with* $\dim \mathrm{Fix}(G) = 1$*, then M has a geometric decomposition. Furthermore, if M is also atoroidal, then G preserves the geometric structure of M.*

Using this, we can extend Theorem 1.5 to all Seifert fibered manifolds $(K;r)$. In fact, we prove:

Proposition 5.6. *Assume that Assertion* 5.5 *is valid. Let K be a periodic knot in S^3 with period greater than* 2*. If $\pi_1((K;r))$ is finite, then K is a trivial knot, a torus knot, or a cable of a torus knot.*

Remark. Using Assertion 5.5, Wang and Zhou [32] showed that a non-torus knot with a symmetry other than a strong inversion does not admit nontrivial, cyclic surgery.

Proof. Let $f : S^3 \to S^3$ be a periodic map of K given in the hypothesis of the proposition, and $\bar{f} : (K;r) \to (K;r)$ a periodic extension of $f|E(K)$.

If $(K;r)$ is reducible, then K is a trivial knot, a torus knot, or a cable knot by the positive solution to the cabling conjecture for symmetric knots (Gordon and Luecke, and Hayashi and Shimokawa [20]). But $\pi_1((K;r))$ are infinite for such K. (If K is a (p,q)-cable of a knot k ($q \geq 2$), then $r = pq$ and $(K;pq) = (k;p/q^2)\#(\text{lens space})$. Then by [7] $\pi_1((K;r))$ is infinite.) So assume that $(K;r)$ is irreducible. Then, by Assertion 5.5 $(K;r)$ with a finite fundamental group is geometric, so that $(K;r)$ is Seifert fibered. Assertion 5.5, together with the lemma below, implies that $\bar{f}$ preserves a

Seifert fibration of $(K; r)$. Then, the proof is completed by applying the arguments in the last paragraph in the proof of Theorem 1.5. □

Lemma 5.7. *Let M be a 3-manifold with the S^3-geometry which admits a periodic automorphism g with $\dim \mathrm{Fix}(g) = 1$. If g is an isometry of M, then it preserves a Seifert fibration of M.*

Proof. Let $G = \pi_1(M)$, the group of covering translations of $S^3 \to M$. Note that G is an isometry of the unit sphere $S^3 \subset \mathbb{R}^4$, i.e., $G \subset SO(4)$. Let $\mathcal{C} \subset S^3$ be the preimage of a component of $\mathrm{Fix}(g)$. Since g is an isometry of M, $\mathcal{C}$ consists of geodesics of S^3. Identify S^3 with the group of unit quaternions so that a component of $\mathcal{C}$ is the unit complex numbers S^1. Let $\widetilde{g} : S^3 \to S^3$ be the lifting of g such that $\mathrm{Fix}(\widetilde{g}) = S^1$; then $\widetilde{g}$ is a rotation about S^1. It suffices to prove:

Claim 5.8. *There is a Seifert fibration of S^3 which is preserved by $\widetilde{g}$ and the action of G.*

Proof of Claim 5.8. Define $\varphi : S^3 \times S^3 \to SO(4)$ to be $\varphi(p,q)(x) = px\bar{q}$ $(x \in S^3)$. Without loss of generality there is a one parameter subgroup H of S^3 such that $\varphi(H \times S^3) \supset G$ [28, Theorem 4.10]. We define $H = \{1\}$ if $\varphi(\{1\} \times S^3) \supset G$. Since $\mathrm{Ker}\varphi = \{\pm(1,1)\}$ [28, p.452], if $H \neq \{1\}$, then $H \times S^3 \supset \varphi^{-1}(G)$.

Case 1. $H = \{1\}$.

Since G acts on the components of $\mathcal{C}$ transitively, each component of $\mathcal{C}$ is written as $S^1 q$ for some $q \in S^3$. Hence the Hopf fibration $S^3 \to S^1 \backslash S^3$ contains $\mathcal{C}$ as fibers. Any rotation about S^1 leaves the fibration invariant. So the fibration is preserved by G and $\widetilde{g}$.

Case 2. $H \neq \{1\}$.

Let τ be the circle fibration $S^3 \to H\backslash S^3$. Since $\varphi(H \times S^3) \supset G$, the action of G leaves τ invariant. Let us show that $\widetilde{g}$ leaves τ invariant. Recall that $\widetilde{g}$ is a rotation about S^1, and thus $\widetilde{g}(x) = e^{i\theta} x e^{-i\theta}$, where θ is a half of the rotation angle of $\widetilde{g}$. By the covering space theory, if $\varphi(p,q)$ is a covering translation of $S^3 \to M$, i.e., $\varphi(p,q) \in G$, then so is $\widetilde{g} \circ \varphi(p,q) \circ \widetilde{g}^{-1}$. Note $\widetilde{g} \circ \varphi(p,q) \circ \widetilde{g}^{-1}(x) = e^{i\theta} p e^{-i\theta} x e^{i\theta} \bar{q} e^{-i\theta} = \varphi(e^{i\theta} p e^{-i\theta}, e^{i\theta} q e^{-i\theta})(x)$. Since $H \times S^3 \supset \varphi^{-1}(G)$, it follows $\widetilde{g}(p) = e^{i\theta} p e^{-i\theta} \in H$. In Case 2 there is $p \in H - \{\pm 1\}$ such that $\varphi(p,q) \in G$ for some $q \in S^3$, so the two geodesics $\widetilde{g}(H)$ and H have more than 2 points in common. Hence $\widetilde{g}(H) = H$. This implies that $\widetilde{g}(Hq) = H\widetilde{g}(q)$, i.e., $\widetilde{g}$ preserves τ invariant. □□

If $\pi_1((K;r))$ is isomorphic to the fundamental group of the Poincaré homology 3-sphere, Proposition 5.6 is improved as follows.

Proposition 5.9 (Property I for periodic knots). *Assume that Assertion 5.5 is valid. Let K be a periodic knot in S^3. If $\pi_1((K;r))$ is the binary icosahedral group I_{120}, then K is the trefoil knot.*

Remark. Using Assertion 5.5, Zhang [34] proved Proposition 5.9 for any periodic knot with period other than 2, 3, 5.

Proof. The proof of Proposition 5.6 shows that $(K;r)$ is Seifert fibered. We follow the proof of Claim 5.8. Recall that the group of unit quaternions S^3 contains a subgroup isomorphic to I_{120} as the pull-back of an icosahedral subgroup in $SO(3)$. Since two free, isometric actions of I_{120} on S^3 are conjugate in $O(4)$ ([28, Theorems 4.10, 4.11]), Case 1 in the proof occurs. Therefore, without assuming that K is with period greater than 2, $\mathrm{Fix}(f)$ is a fiber in $(K;r)$. By Proposition 5.1 K is a trivial knot, a torus knot, or a cable of a torus knot. It then follows from [34, Propositions 3.6 and 3.7] that K is the trefoil knot. □

6. Seifert fibering surgeries satisfying Conjecture 1.3.

6.1. Surgeries on 2-bridge knots.

Brittenham and Wu [6] showed that if a Dehn surgery on a non-torus 2-bridge knot produces a Seifert fibered manifold over S^2 with at most 3 exceptional fibers, then the knot is a twist knot $K_{\pm 2,2n}$ illustrated in Figure 6.1. We show that such surgeries satisfy Conjecture 1.3.

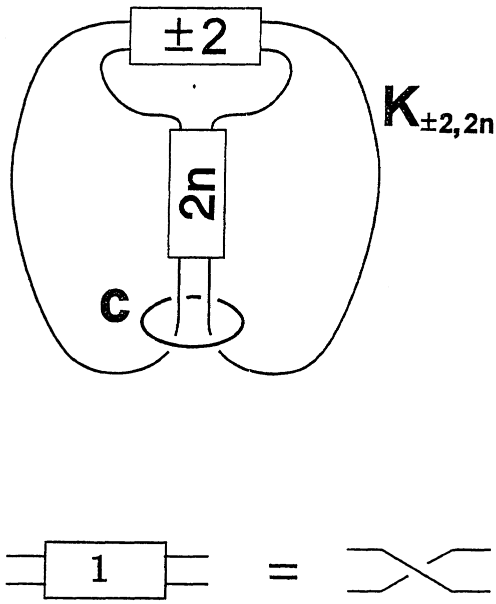

Figure 6.1

Proposition 6.1. *Let* $K_{\pm 2,2n}$ *be the twist knot in Figure* 6.1.

(1) [6] $M = (K_{\pm 2,2n}; r)$ *is a small Seifert fibered manifold for* $r = \mp 1, \mp 2, \mp 3$.

(2) *The trivial knot* c *in Figure* 6.1 *is an exceptional fiber in* M *if* $r = \mp 1, \mp 2, \mp 3$.

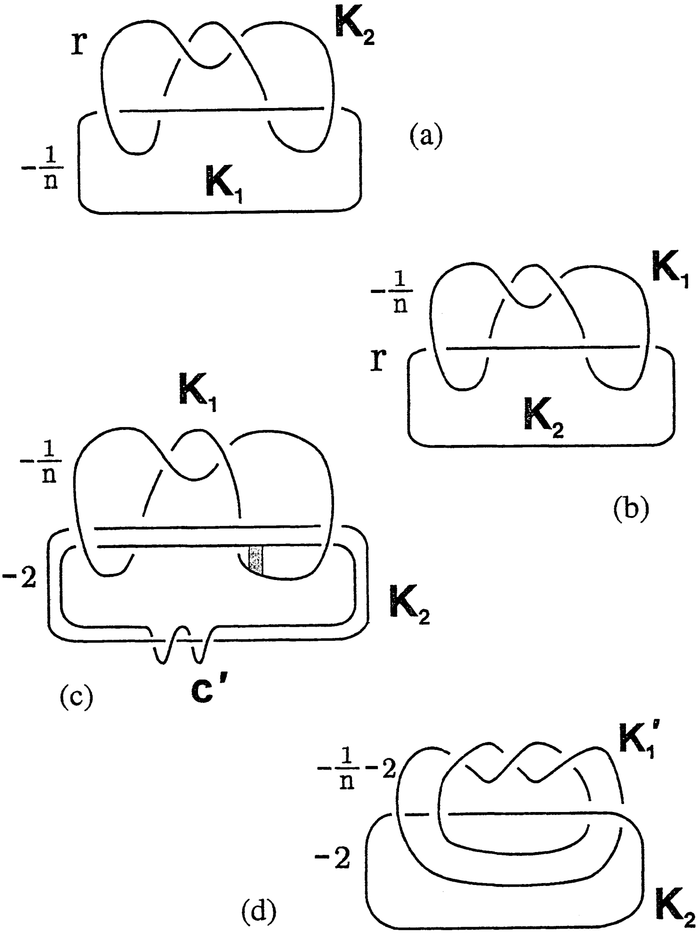

Figure 6.2 (a), (b), (c), (d)

Proof. Since $K_{-2,-2n}$ is the mirror image of $K_{2,2n}$, it suffices to consider r-surgeries of $K_{2,2n}$ for $r = -1, -2, -3$. Furthermore assume $r = -1, -2$. The proof for $r = -3$ is similar to the case $r = -2$. We follow the arguments in [6] which prove (1) of the above proposition. Let $K_1 \cup K_2$ be the link in Figure 6.2(a), and $L(r_1, r_2)$ the manifold obtained from S^3 by r_i-surgeries on K_i. Note that K_2 becomes $K_{2,2n}$ in $L(-\frac{1}{n}, \infty) = S^3$; a preferred longitude of K_1 becomes c in $L(-\frac{1}{n}, \infty)$. Since $lk(K_1, K_2) = 0$, we have $(K_{2,2n}; r) \cong L(-\frac{1}{n}, r)$. We show that a preferred longitude of K_1 is an exceptional fiber of the Seifert fibered manifold $L(-\frac{1}{n}, r)$ for $r = -1, -2$. First exchange the position of K_1 and K_2 by ambient isotopy of S^3 (Figure 6.2(b)).

Suppose $r = -1$. After the -1-surgery on K_2, K_1 becomes the left handed trefoil K' in $L(\infty, -1) = S^3$; then the framings of K_1 and K' are the same. It follows that $L(-\frac{1}{n}, -1) \cong (K'; -\frac{1}{n})$ is a Seifert fibered manifold. The glued solid torus in $(K'; -\frac{1}{n})$ is a tubular neighborhood of an exceptional fiber of $(K'; -\frac{1}{n})$. Since a preferred longitude of K' intersects a representative of the surgery slope $-\frac{1}{n}$ in a single point, it is a longitude of the glued solid torus in $(K'; -\frac{1}{n})$, and so isotopic to its core. Hence a preferred longitude of K_1 is an exceptional fiber in $L(-\frac{1}{n}, -1)$ after isotoping the fibration.

Suppose $r = -2$. The longitude c' of K_2 in Figure 6.2(c) bounds a meridian disk in the glued solid torus in $L(\infty, -2)$. Hence, in $L(\infty, -2)$ K_1 is isotopic to a band sum of K_1 and c'. Let K_1' be the band sum of K_1 and c' via the band described in Figure 6.2(c). Isotope K_1 to K_1' in $L(\infty, -2)$. Then a preferred longitude of K_1 becomes a longitude, λ, of K_1' with slope -2; the surgery coefficient for K_1' becomes $-\frac{1}{n} - 2$. An isotopy of $K_1' \cup K_2$ in S^3 gives Figure 6.2(d). The exterior of $L' = K_1' \cup K_2$ is a Seifert fibered manifold. It is checked in [6] that $L'(-\frac{1}{n} - 2, -2) \cong L(-\frac{1}{n}, -2)$ is also a Seifert fibered manifold. Each glued solid torus in $M = L'(-\frac{1}{n} - 2, -2)$ is a tubular neighborhood of an exceptional fiber. Since the distance between the slopes $-\frac{1}{n} - 2$ and -2 equals 1, λ is isotopic to an exceptional fiber in M. After isotoping the fibration of M, λ becomes an exceptional fiber. □

6.2. Surgeries on Eudave-Muñoz' knots.

Eudave-Muñoz [9] obtained a family of hyperbolic knots each of which has at least two integral, Seifert fibering surgeries and at least one non-integral surgery giving a toroidal manifold. His idea is to find a 2-string tangle which forms a trivial knot, a Montesinos link, or a sum of two prime tangles by adding adequate rational tangles. Then the double branched covering of such a tangle will be a knot exterior with several exceptional Dehn fillings.

The Eudave-Muñoz construction [9] starts with Figure 6.3. k is a trivial knot in S^3. Assume that p or n is 0, and $l \neq 0, \pm 1, m \neq 0$. If $p = 0$, assume that $(l, m) \neq (2, 1), (-2, -1)$, and $(m, n) \neq (1, 0), (-1, 1)$. If $n = 0$, assume that $m \neq 1, (l, m, p) \neq (-2, -1, 0), (2, 2, 1)$. B is a 3-ball such that $(B, B \cap k)$ is a 2-string trivial tangle, in particular, the 1/0-rational tangle according to the Bleiler's notation [1]. Let $\pi : S^3 \to S^3$ be the double covering of S^3 branched along k. Let $k(l, m, n, p) \subset S^3$ be a core of the solid torus $\pi^{-1}(B)$, and $c_i' \subset S^3$ be a component of $\pi^{-1}(c_i)$.

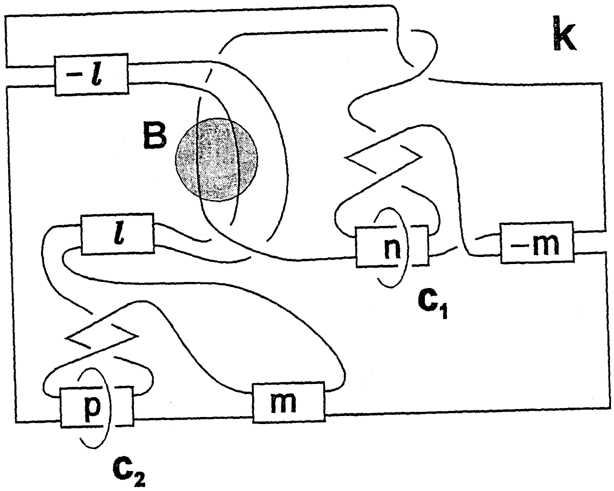

Figure 6.3

The letters in the boxes indicate the numbers of half twists as in Figure 6.1.

Proposition 6.2. *Let l, m, n and p be integers satisfying the above conditions. If $p = 0$ and $(l, m, n) \neq (2, 2, 0)$, set $c = c_1'$; otherwise, set $c = c_2'$. Then the following hold.*

(1) *c is a regular or exceptional fiber in any Seifert fibered manifold $(k(l, m, n, p); \gamma)$ obtained in* [9, Theorem 2.1].

(2) *c is a trivial knot in S^3.*

Proof. We only consider the case when $p = 0$ and $(l, m, n) \neq (2, 2, 0)$. Other cases can be settled in a similar manner.

Proof of (1). Replace the tangle $(B, B \cap k)$ in k with the $0/1$-rational tangle as in Figure 6.4(a).

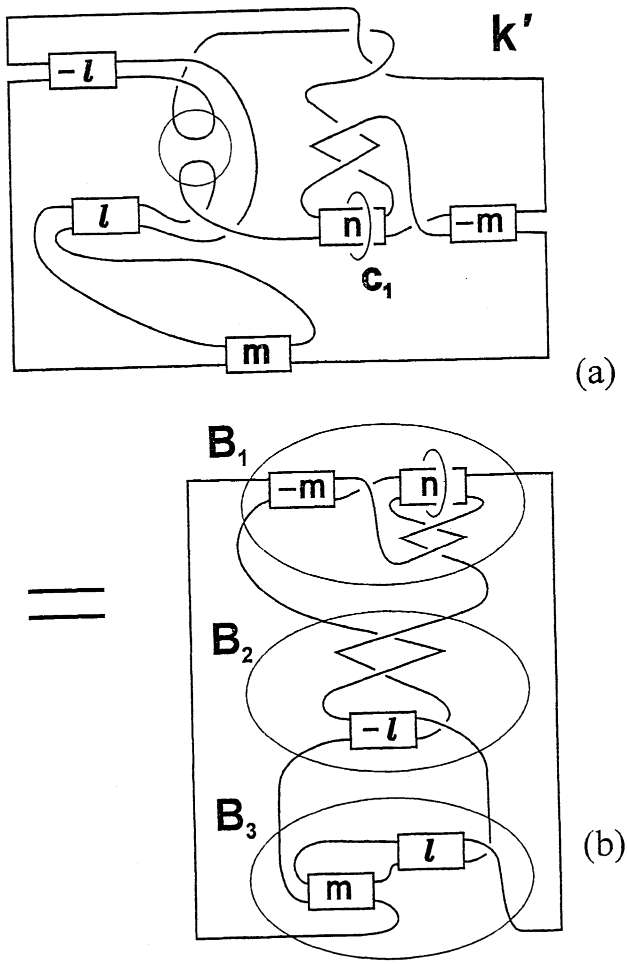

Figure 6.4(a), (b)

The resulting knot k' is a Montesinos link formed with three rational tangles $(B_i, B_i \cap k')$ (Figure 6.4(b)). Let $\pi' : W \to S^3$ be the double branched covering of S^3 along k'. Then W is obtained from S^3 by an integral surgery on $k(l, m, n, 0)$, and is a Seifert fibered manifold over a 2-sphere with at most three cone points. Let f be a core of ${\pi'}^{-1}(B_1)$; then f is a regular or exceptional fiber of W. We show that a component of ${\pi'}^{-1}(c_1)$ is isotopic to f. B_1' in Figure 6.5 is a 3-ball containing "n-half twists" with $c_1 \subset \partial B_1'$. It is easy to see that $(B_1 - \operatorname{int} B_1', (B_1 - \operatorname{int} B_1') \cap k')$ is homeomorphic to $(S^2 \times I, \{p_1, p_2, p_3, p_4\} \times I)$ where $p_i \in S^2$. Hence, the solid torus ${\pi'}^{-1}(B_1')(\subset {\pi'}^{-1}(B_1))$ is a tubular neighborhood of a core of ${\pi'}^{-1}(B_1)$. Since c_1 is a lattitude of the trivial tangle $(B_1', B_1' \cap k')$, a component of ${\pi'}^{-1}(c_1)$ is isotopic to the core f of $\pi'^{-1}(B_1')$. Hence, c_1' becomes an exceptional fiber after isotoping the fibration of W slightly. This proves (1) for the Seifert fibering surgery in Theorem 2.1(c) of [9]. In the same way, we can prove that c is a fiber for other Seifert fibering surgeries in [9].

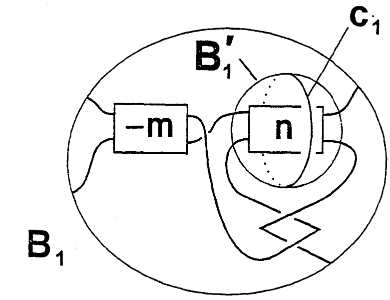

Figure 6.5

Proof of (2). Following [9], we will obtain an explicit description of $k(l,m,n,0)$. In Figure 6.6, the circles a_i $(i = \pm l, \pm m, n, p)$ indicate that we perform i-half twists along the disks bounded by the circles. The knot k in Figure 6.3 is the trivial knot k_0 in Figure 6.6 with these twistings. Note that the circles c_1, c_2 in Figure 6.3 are preferred longitudes of a_n, a_p in Figure 6.6, respectively.

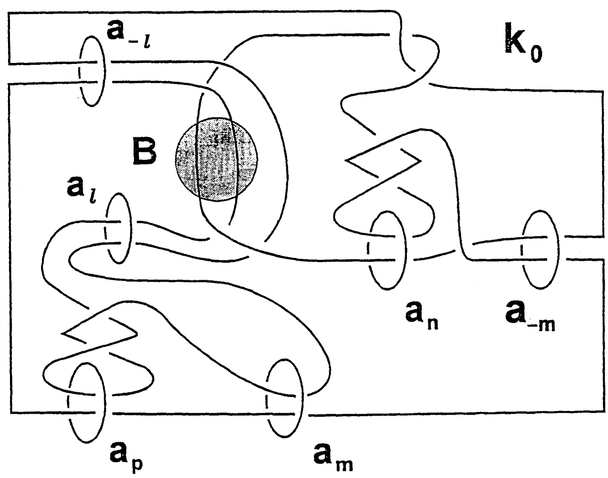

Figure 6.6

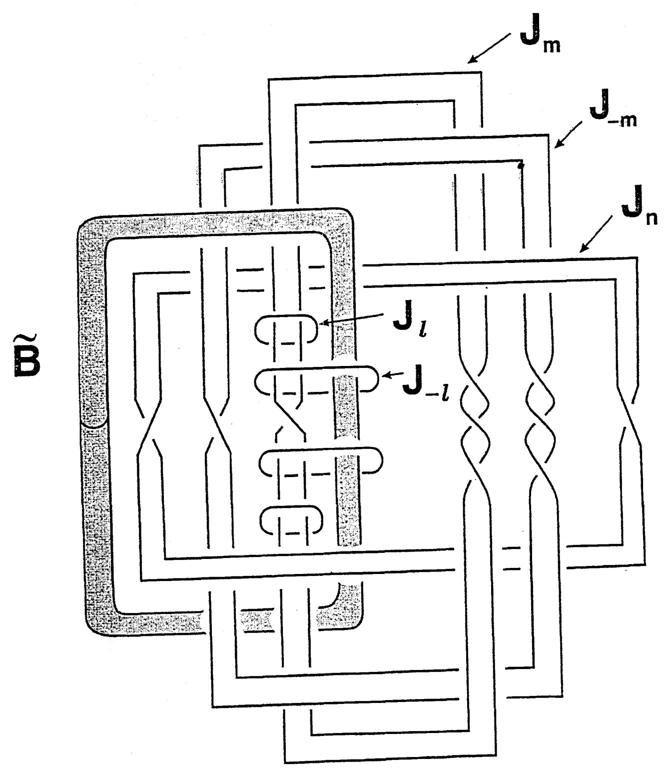

Figure 6.7

Now take the double covering of S^3 branched along k_0. The preimage of a_i $(i = \pm l, \pm m, n)$ consists of two simple closed curves, and let J_i denote one of them; a_p is deleted since p is assumed to be 0. Let $\widetilde{B}$ be the preimage of B. Eudave-Muñoz [9] obtained Figure 6.7. Let k_1 be a core of the unknotted solid torus $\widetilde{B}$ in S^3. The preimages of the evident disks bounded by a_i are annuli. Using these annuli, define framings of J_i. Note that J_i $(i = \pm l, n)$ are then given 0-framings. Hence c_1, a preferred longitude of a_n, lifts up to a preferred longitude of J_n. The knot $k(l, m, n, 0)$ is obtained from k_1 by doing $-1/i$-Dehn surgeries on J_i in terms of the meridian-longitude pairs determined by the framings of J_i. The curve $c = c_1'$ is thus obtained from a preferred longitude of J_n after performing the above surgeries. Eudave-Muñoz observes that these surgeries can be simplified as follows. The curves J_l and J_{-l} bound an annulus L intersecting $\widetilde{B}$ in a disk. Similarly, J_m and J_{-m} bound an annulus M such that $M \cap \widetilde{B} = \emptyset$, and $L \cap M$ is an arc. The annulus L (resp. M) also gives framings to $J_{\pm l}$ (resp. $J_{\pm m}$). However, these framings coincide with the ones we defined before. Therefore, doing

$\mp 1/l$-surgeries on $J_{\pm l}$ is equivalent to l-twisting along the annulus L (cf. Figure 6.8). The similar statement holds for the annulus M. We now have the following result.

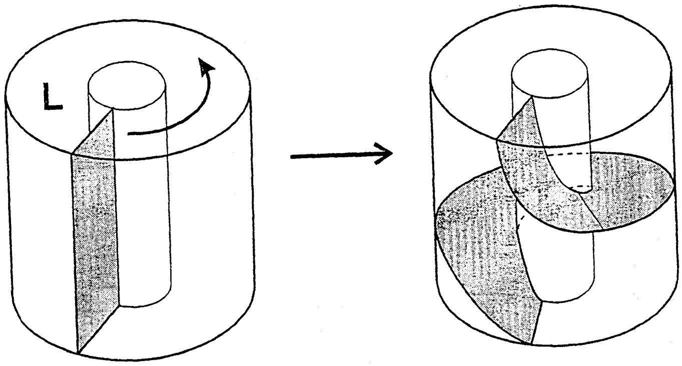

Figure 6.8
Twisting just once along the annulus L.

Lemma 6.3 (A special case of [9, Theorem 3.2]). *The knot $k(l,m,n,0)$ is obtained from the trivial knot k_1 in Figure 6.9 by doing l-twisting along the annulus L, m-twisting along the annulus M, and then $-1/n$-Dehn surgery on J_n.*

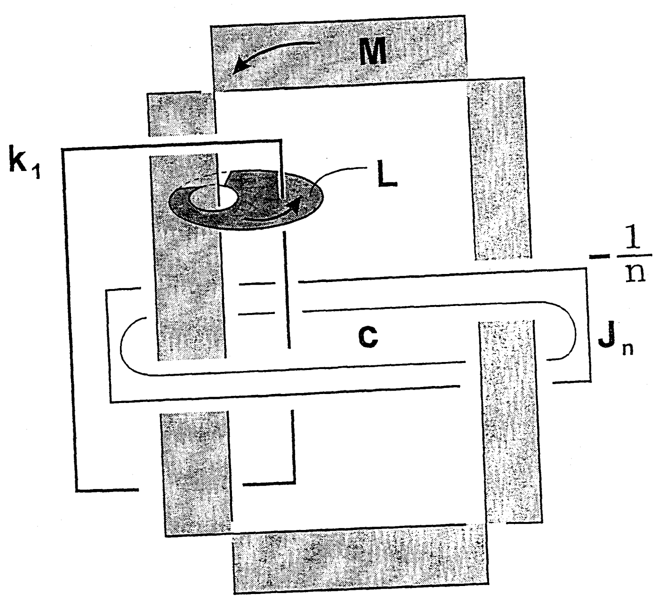

Figure 6.9
The arrows indicate the directions of twistings.

Since J_n does not intersect the annuli L, M, a preferred longitude of J_n is still a trivial knot after the operations in the lemma. This implies that c is a trivial knot in S^3. □

6.3. Surgeries on twisted torus knots.

Let W be a standardly embedded solid torus in S^3. Let $K_{p,q}$ be a simple loop on ∂W which winds around p times meridionally and q times longitudinally ($q > |p| \geq 2$). Given r, take an arc α in ∂W such that $\partial\alpha \cap K_{p,q} = \emptyset$, and α meets $K_{p,q}$ in r points of the same sign. We furthermore assume that α is disjoint from a meridian of W meeting $K_{p,q}$ in exactly q points. Note that two such α are isotoped to each other in ∂W so that $\partial\alpha$ keeps away from $K_{p,q}$ during the isotopy. Let $D = \alpha \times [-1, 1]$ be an embedded disk in S^3 such that D intersects ∂W transversely in $\alpha \times 0 = \alpha$. Let V be the unknotted solid torus $S^3 - \operatorname{int} N(\partial D)$ containing $K_{p,q}$ in its interior; a meridian disk of V intersects $K_{p,q}$ in r points.

Definition (twisted torus knot). Twist V n times along a meridian disk. Then the image of $K_{p,q}$ in the twisted solid torus V_n ($\subset S^3$) is called the *twisted torus knot* $K(p, q, r, n)$. Note that $K(p, q, r, 0)$ is just the torus knot $K_{p,q}$.

Recently in his thesis [8] Dean has studied Seifert fibering surgery of S^3 on $K(p, q, r, n)$ for $0 < r < \max\{|p|, q\}$ and $n = \pm 1$. In [25], we studied surgery of V on $K_{p,q}$ for $r = p + q$. Lemma 9.1 of [25] implies that:

Proposition 6.4. *Let $K = K(p, q, p+q, n) \subset V_n$. Let (M, L) be a preferred meridian-longitude pair of V_n in S^3. Then $X = V_n(K; pq + (p+q)^2 n)$ is a Seifert fibered manifold over the disk with two exceptional fibers of indices $|p|, q$. Furthermore, $L + nM \subset \partial X$ is a regular fiber of X.*

By extending the Seifert fibration of X to the complementary solid torus $S^3 - \operatorname{int} V_n$, we see that $(K; pq + (p+q)^2 n)$ is a Seifert fibered manifold over S^2 with three exceptional fibers of indices $|p|, q, |n|$. Note that a core of $S^3 - \operatorname{int} V_n$, which is unknotted in S^3, is an exceptional fiber of index $|n|$.

Claim 6.5. *A twisted torus knot $K = K(p, q, p + q, n)$ in S^3 is hyperbolic if $p + q \geq 2$ and $|n| > 5$.*

Proof. [25, Claim 9.2] states that $V - K_{p,q}$ admits a complete hyperbolic structure of finite volume in its interior. The manifold $S^3 - K$ is obtained from $V - K_{p,q}$ by $(-n\mu + \lambda)$-Dehn filling on ∂V, where (μ, λ) is a preferred meridian-longitude pair of $V \subset S^3$. If $n = 0$, then K is a torus knot and thus its complement is not hyperbolic. Since $V - \operatorname{int} N(K_{p,q})$ has two boundary components, by [13, Theorem 1.3] $S^3 - K$ admits a complete hyperbolic structure of finite volume for $|n| > 5$. □

7. Does a geodesic become a fiber after surgery?

Let K be a hyperbolic knot in S^3. If the hyperbolic structure of $S^3 - K$ degenerates to a Seifert fibering structure of $(K; r)$, then which curves in $S^3 - K$ become fibers of $(K; r)$? Conjecture 1.3 states that a trivial knot in S^3 becomes a fiber. Experiments via Weeks' computer program SnapPea suggest that a closed geodesic in the hyperbolic manifold $S^3 - K$ becomes a fiber. This section is a report on these experiments. In the following, verifying hyperbolicity, detecting geodesics, and calculating fundamental groups are done by using SnapPea unless otherwise indicated.

Question 7.1. *Suppose that K is a hyperbolic knot in S^3, and $(K; r)$ is Seifert fibered. Is there a closed geodesic c in $S^3 - K$ such that:*

(1) *c is unknotted in S^3, and*

(2) *c is a fiber in some Seifert fibration of $(K; r)$?*

Example 1. Let K be the figure eight knot. For $r = -1, -2, -3$, $(K; r)$ is a Seifert fibered manifold over S^2 with three exceptional fibers; the triple of indices are (2, 3, 7) for $r = -1$; (2, 4, 5) for $r = -2$; (3, 3, 4) for $r = -3$. Let c_1 and c_2 be the knots in $S^3 - K$ depicted in Figure 1.1. Then c_1 is the shortest geodesic and c_2 is the second shortest geodesic in $S^3 - K$.

Clearly both c_i are trivial knots in S^3. We can check by surgery calculus that each c_i becomes an exceptional fiber in $(K; r)$ for $r = -1, -2, -3$; the indices of c_i are as follows. (Cf. Section 6.1.)

	S^3	$(K; -1)$	$(K; -2)$	$(K; -3)$
c_1	unknot	fiber of index 7	fiber of index 5	fiber of index 4
c_2	unknot	fiber of index 7	fiber of index 4	fiber of index 3

As in this example, a shortest geodesic in $S^3 - K$ often serves as c in Question 7.1. We give more examples of this kind.

Example 2. Let $K_{2,2n}$ be the twist knot and c the trivial knot described in Figure 6.1; it is known that $K_{2,2n}$ is hyperbolic for $n \neq 0, 1$. As shown in Proposition 6.1, $(K_{2,2n}; r)$ is a Seifert fibered manifold with c an exceptional fiber in $(K_{2,2n}; r)$, where $r = -1, -2, -3$. Note that $K_{2,0} \cup c$ is the Whitehead link, and $K_{2,2n}$ is obtained as the image of $K_{2,0}$ by performing $-1/n$-surgery on c. From Thurston's hyperbolic Dehn surgery theory, c is the unique shortest geodesic in $S^3 - K_{2,2n}$ if $|n|$ is sufficiently large, and the length of c tends to 0 as $|n| \to \infty$. This result, together with tests by SnapPea for small n, suggests that c is the shortest geodesic in $S^3 - K_{2,2n}$ for any $n \neq 0, 1$.

Example 3. Let $l \neq 0, \pm 1$, $m \neq 0$, and $(l, m, n) \neq (2, 2, 0)$. Let $k(l, m, n, 0)$ and c be the knots in Proposition 6.2. Set $K_n = k(l, m, n, 0)$. Then c is a trivial knot and a regular or exceptional fiber of any Seifert fibered manifold $(K_n; \gamma)$ given in [9]. By Lemma 6.3 and Figure 6.9, K_n is obtained from K_0 after doing $-1/n$-surgery on c. We denote the core of the filled solid torus also by c. Using the fact that K_n $(n \neq 0, 1)$ is hyperbolic [9, Proposition 2.2], we prove that:

Claim 7.2. $K_0 \cup c$ is a hyperbolic link in S^3.

Hence, as in Example 2 above, after hyperbolic Dehn surgery the cusp c is the shortest geodesic in $S^3 - K_n$ if $|n|$ is sufficiently large.

Proof. As pointed out in [9], K_0 is a closed braid in the solid torus $S^3 - \operatorname{int} N(c)$. Since $lk(K_0, c) = 2lm - 1 \neq 0$, $S^3 - K_0 \cup c$ is irreducible.

Let $M = S^3 - \operatorname{int} N(K_0 \cup c)$; $S^3 - \operatorname{int} N(K_n)$ is obtained from M by $(\mu - n\lambda)$-Dehn filling on $\partial N(c)$, where (μ, λ) is a preferred meridian-longitude pair of c. Take the torus decomposition of M and let P be the piece containing $\partial N(c)$. First assume $P = M$. If P is hyperbolic, this is the desired case. If P is Seifert fibered, then $K_0 \subset S^3 - \operatorname{int} N(c)$ is a 0-bridge braid. So K_n is a torus knot for any n, a contradiction.

Next assume $P \neq M$ for a contradiction. Denote by $P(n)$ the manifold obtained from P by $(\mu - n\lambda)$-Dehn filling on $\partial N(c)$. If P is hyperbolic, $P(n)$ is hyperbolic for sufficiently large n [30]. Since $P(n)$ is boundary-irreducible, $S^3 - \operatorname{int} N(K_n)$ contains an essential torus, a contradiction. Now assume that P is Seifert fibered. Since K_0 is a closed braid in $S^3 - \operatorname{int} N(c)$, any incompressible torus in M separates $\partial N(c)$ and $\partial N(K_0)$. It follows

$\partial P \cap \partial M = \partial N(c)$. By choosing n so that a fiber slope on $\partial N(c) \subset P$ is not $\mu - n\lambda$, $P(n)$ is Seifert fibered for infinitely many n. If $P(n)$ is boundary-irreducible for some n, then the hyperbolic manifold $S^3 - \text{int}\, N(K_n)$ contains an essential torus, a contradiction. Hence, $P(n)$ is boundary-reducible and thus a solid torus for infinitely many n. By the assumption on P, it is a cable space. A fiber of P on $\partial N(c)$ represents, say $x\mu + y\lambda$. Since $P(n)$ is a solid torus for infinitely many n, the distance between the fiber slope and the surgery slope $|x + yn|$ is one for such n. It follows that $x = \pm 1$, $y = 0$; a fiber is a meridian of c. Then $P(\infty)$ contains a lens space summand, a contradiction. □

The following example shows that we cannot always take a shortest geodesic as the trivial knot c in Question 7.1.

Example 4. Let K be the twisted torus knot $K(3, 7, 10, -1)$ in Section 6.3, which is a hyperbolic knot. Then $(K; -79)$ is a lens space (see Proposition 6.4). Calculation of the fundamental group of complements implies that in S^3, the shortest geodesic c_1 in $S^3 - K$ is a trefoil knot and the second shortest geodesic c_2 is a trivial knot. In $(K; -79)$, c_1 is an exceptional fiber and c_2 is a regular fiber. Hence c_2 serves as c in Question 7.1.

Question 7.3. *Let K be a hyperbolic knot in S^3, and $(K; r)$ is Seifert fibered. Then does there exist a shortest geodesic in $S^3 - K$ which is a fiber in some Seifert fibration of $(K; r)$? In particular, is a shortest geodesic in $S^3 - K$ a trivial knot or a torus knot viewed in $S^3 = (K; \infty)$?*

Remarks. (1) Shortest geodesics in $S^3 - K$ are unknotted for any hyperbolic knot K with up to 11 crossings.

(2) Let K be the $(-2, 3, 7)$-pretzel knot. Then $(K; 17)$ is a Seifert fibered manifold. The shortest geodesic in $S^3 - K$ is an exceptional fiber in some Seifert fibration of $(K; 17)$, however the 5th shortest geodesic c_5 in $S^3 - K$ is not a fiber in any Seifert fibration of $(K; 17)$. In fact, $(K; 17) - c_5$ is hyperbolic.

Added in proof

A recent partial solution to Thurston's orbifold geometrization conjecture (Assertion 5.5) improves some results in this paper: Theorem 1.5 holds if $(K; r)$ has a finite fundamental group; and Propositions 5.6 and 5.9 hold without assuming Assertion 5.5. Assertion 5.5 is proved in the case when Fix (G) is a 1-manifold by Cooper, Hodgson and Kerckhoff [lecture series given at the Third MSJ Regional Workshop on Cone-Manifolds and Hyperbolic Geometry, July 1-10, 1998, Tokyo Institute of Technology, Tokyo] and Boileau and Porti [*Geometrization of* 3*-orbifolds of cyclic type*, preprint]. This case of Assertion 5.5 is what we need to prove Proposition 5.6.

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