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Bridge principle for constant and positive Gauss curvature surfaces.

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0. Introduction.

Parmi toutes les surfaces bordées par une même courbe de Jordan fermée, quelles sont celles qui minimisent l'aire? Que dire de leur régularité?

Ce problème porte le nom du physicien belge Joseph Antoine Ferdinand Plateau. En 1850, il observe expérimentalement des solutions en immergeant des structures métalliques à l'image de ces courbes dans de l'eau savonneuse.

Ces deux questions vont tenir en éveil les mathématiciens durant plus d'un siècle. Riemann, Schwarz, Weierstrass, Lamarle et beaucoup d'autres au 19^{eme} siècle obtiennent des résultats dans de nombreux cas particuliers.

1930 et 1931, l'approche moderne de Jesse Douglas et Tibor Radó par le calcul des variations permet une première réponse globale.

Ils montrent l'existence d'un minimum pour l'aire dans l'ensemble des surfaces homéomorphes au disque et bordées par une même courbe rectifiable γ de $\mathbb{R}^n$.

L'équation d'Euler associée à la variation de l'aire montre que toutes les solutions de ce problème sont des surfaces minimales, c'est à dire des surfaces de courbure moyenne nulle.

Les solutions de Douglas-Radó sont des surfaces minimales, mais dans de nombreux cas, pour certaines courbes, elles ne sont pas un minimum absolu pour l'aire.

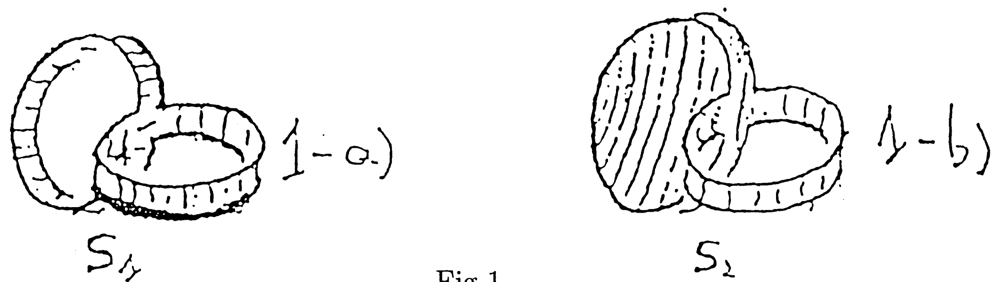

Fig 1

Il est facile par exemple de construire un contour comme dans la figure 1, où la surface de genre 1 (Voir Figure 1-a)) est d'aire plus petite que le disque (Voir Figure 1-b)) solution de Douglas-Radó.

Cet exemple suggère à J.Douglas une reformulation du problème de Plateau. On cherche dorénavant à construire une surface minimale bordée par une courbe de Jordan donnée en prescrivant la topologie de la solution.

Il faudra attendre les années 60 avec les travaux de Federer, Fleming et Almgren sur les courants pour bien comprendre l'existence et la régularité des surfaces d'aire minimisante.

Plus généralement, on s'intéresse à l'ensemble des surfaces minimales bordées par une même courbe. Quelle est leur nombre, leur topologie? Quelles sont celles qui minimisent l'aire?...

Dans le cadre de cette recherche, R.Courant dans son livre [5] en 1950, propose un principe du pont dont il suggère une preuve. Les deux exemples qui suivent illustrent bien l'intérêt de cette construction. Le principe du pont, c'est la possibilité de connecter par un ruban P d'épaisseur $\varepsilon > 0$, deux surfaces minimales S_1 et S_2, bordées par γ_1 et γ_2. Par petites deformations de $S_1 \bigcup S_2 \bigcup P$ on obtient une nouvelle surface minimale S^ε, bordée par une courbe γ_ε constituée de l'union de γ_1, γ_2 et de deux courbes parallèles éloignées d'au plus une distance $\varepsilon > 0$ (Voir Figure 2). De plus, on demande à S^ε de converger vers $S_1 \bigcup S_2$ quand $\varepsilon \to 0$.

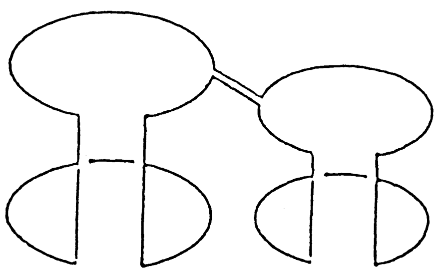

Fig 2

Certains contours de $\mathbb{R}^3$ possèdent au moins deux solutions de Douglas-Radó. Par exemple, pour deux cercles dans des plans parallèles, on peut ajuster la distance entre les plans pour que la caténoïde moins un petit ruban soit d'aire égale à la surface des deux disques plus le ruban (Voir Figure 3). Ces deux surfaces ont la topologie du disque et sont toutes deux minimales.

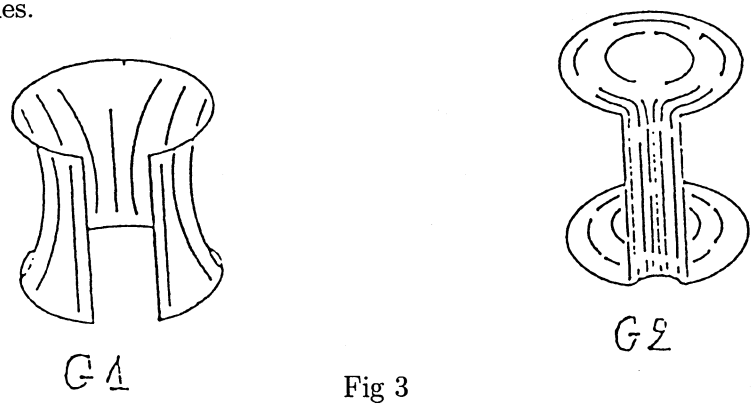

Fig 3

On va construire, grâce à ce principe, une courbe rectifiable qui va border un nombre infini de surfaces minimales. Elles auront toutes la topologie du disque.

On considère le contour γ_1 de la figure 3 et un contour γ_2 géométriquement similaire mais de taille différente. Chacune de ces deux courbes est le bord d'au moins deux surfaces minimales dont la topologie est celle du disque. Par le principe du pont, on obtient un contour (Voir Figure 2)) qui borde au moins quatre surfaces minimales, qui de part et d'autre du pont ont soit l'apparence de G_1, soit celle de G_2. On réitère cette opération N_0 fois. La courbe obtenue borde 2^{N_0} surfaces minimales. Par décroissance géométrique de la taille des contours et des ponts, on obtient à la limite une courbe rectifiable qui borde un nombre infini de surfaces minimales, toutes topologiquement des disques (Voir Figure 4).

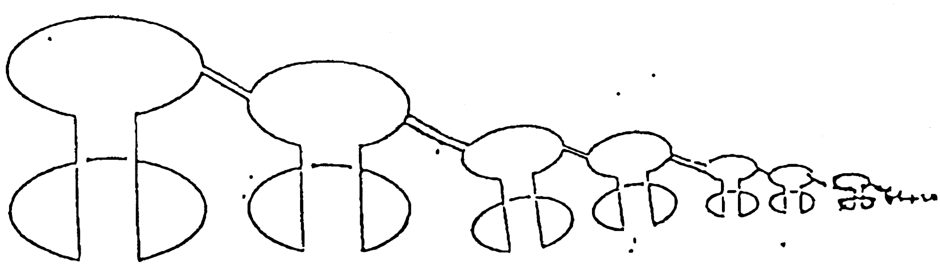

Fig 4

On doit à W.Fleming [7] l'exemple suivant sur la topologie d'une solution du problème de Plateau. A partir de la courbe de la figure 1, et d'une application itérée du principe du pont comme ci-dessus, on obtient la courbe γ_3 de la figure 5-a).

Cette courbe borde un disque minimisant en ne connectant que des surfaces de genre 0, de type S_2 comme dans la figure 5-b). En remplacant la première surface, par une surface de type S_1, on obtient une surface minimale dont la topologie est celle du disque à qui on a rajouté une anse (Voir Figure 5-c)). Cette dernière surface est d'aire plus petite que la première. Dans la figure 5-d), on obtient une autre surface d'aire inférieure et de genre 2. En réitérant cette opération, on a une suite minimisante de surfaces minimales pour l'aire et dont le genre augmente. Pour cette courbe rectifiable la solution du problème de Plateau est de genre infini.

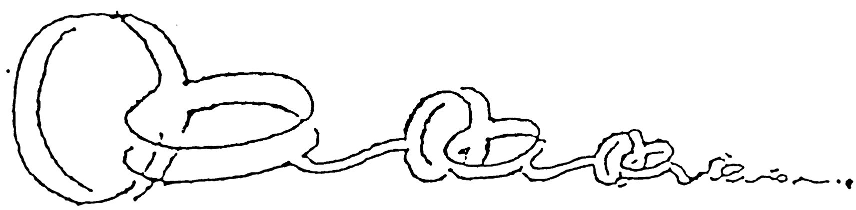

Fig 5-a

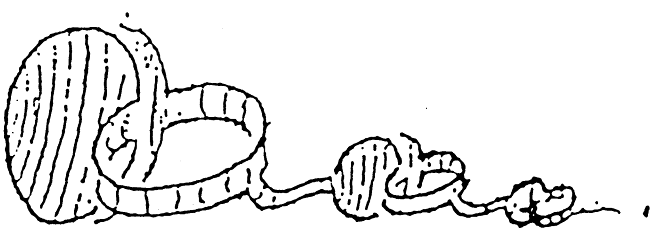

Fig 5-b

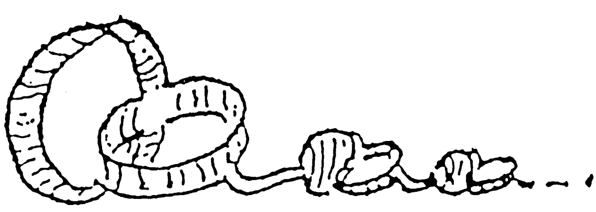

Fig 5-c

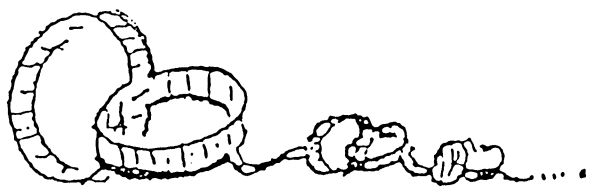

Fig 5-d [13]

Dans l'esprit de ces deux constructions, P.Lévy [14] en 1948, construit des exemples et offre des arguments pour l'existence du principe du pont. Il faudra attendre 1982, pour que W.Meeks et S-T Yau [16] publient une démonstration d'un principe du pont pour les surfaces stables et orientables de $\mathbb{R}^3$ qui justifie la construction de P.Lévy. En 1984, Lawson et Michelson [15] publient un principe du pont faible pour les surfaces orientables et stables de $\mathbb{R}^n$ qui ne permet pas de justifier les exemples de non unicité évoqués. En effet, leur construction nécessite que le ruban ou le pont arrive tangentiellement aux surfaces considérées. Si G_1 et G_2 sont bordées par la même courbe, le pont ne pourra pas être tangent aux deux à la fois. Jusqu'à présent, aucune de ces démonstrations ne convient pour un principe du pont dans le cas de surfaces instables. En 1987, N.Smale [20] publie un principe du pont faible pour toutes surfaces minimales régulières dans n'importe quelles dimensions et codimensions. Sa preuve reste valable dans le cas des surfaces instables de nullité 0 et pour les surfaces de courbure moyenne constante non nulle. Sa démonstration nécessite également un ruban tangent aux surfaces.

C'est B.White [23] et [?] qui établit le théorème du pont fort pour les surfaces minimales stables et instables qui justifie les constructions de P.Lévy, mais aussi de J.Hass [9] et P.Hall [10]. Il construit un pont en contraignant le bord du ruban à rester dans une sous- variété que l'on peut choisir transverse aux surfaces qu'il connecte.

Une question non résolue aujourd'hui concerne l'existence d'une surface minimale de genre 1 bordée par deux courbes convexes contenues dans deux plans parallèles (Voir Figure 6-a)). W.Meeks conjecture qu'une telle surface n'existe pas. Grâce au principe du pont fort, on peut montrer que de telles surfaces peuvent exister si les bords ne sont pas convexes. En prenant deux fois le même morceau de caténoïde stable, bordée par deux cercles dans des plans parallèles, on peut construire un pont en haut et un autre en bas (Voir Figure 6-b)). La surface minimale qui en résulte est bien de genre 1, mais elle est bordée en haut et en bas par des courbes non convexes en forme d'haltères (Voir Figure 6-c)).

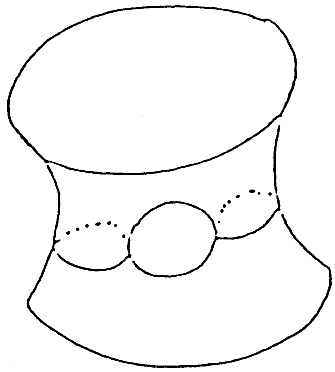

Fig 6-a

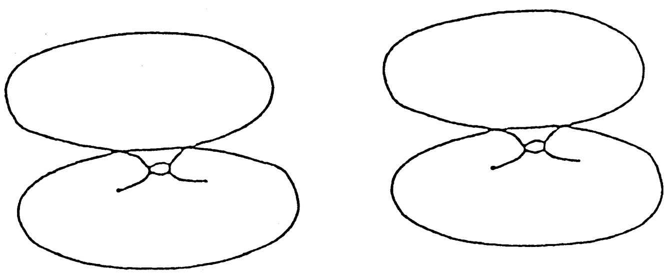

Fig 6-b

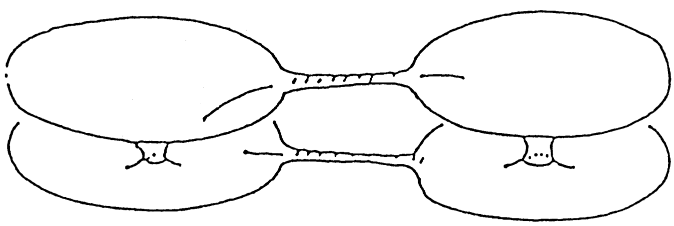

Fig 6-c

Tous ces theorèmes concernent les surfaces de courbure moyenne constante. Pour les surfaces de $\mathbb{R}^3$, il existe une autre fonction de courbure: la courbure Gaussienne ou courbure de Gauss. Alors que l'équation liée au problème de trouver une surface de courbure moyenne H prescrite est de type quasi-linéaire, la recherche d'une surface à courbure de Gauss $K > 0$ prescrite conduit à une équation fortement non linéaire, de type Monge-Ampère.

Le problème de Dirichlet pour les graphes de courbure de Gauss constante et positive est résolu sur des domaines convexes par Cafarelli, Nirenberg et Spruck [4] en 1984. Ils utilisent la méthode de continuité dont le point central est une estimée C^2 difficile à obtenir. En 1992, Hoffman, Rosenberg et Spruck [11] étendent ce résultat pour les graphes au-dessus des anneaux plans.

B.Guan et J.Spruck [2] montrent par la même méthode que deux courbes convexes dans des plans parallèles bordent une K-surface dont la topologie est celle de l'anneau. Ce résultat n'est toujours pas montré dans le cas des H-surfaces.

Cependant, sauf pour quelques cas particuliers, on ne connait pas le nombre de K-surfaces que borde une courbe. H.Brezis, J.M Coron [3] et M.Struwe [21] ont montré que pour une courbe donnée l'existence d'une "petite" H-surface, impliquait l'existence d'une "grande" H-surface.

H. Rosenberg conjecture que de nombreux théorèmes valables dans le cadre des H-surfaces, restent vrai dans le cadre des K-surfaces (et vice-versa). Dans l'esprit de ce principe, cette thèse montre qu'il existe un principe du pont faible pour les K-surfaces. Le ruban P devra être tangent aux surfaces. On ne peut pas contraindre le bord du pont à rester dans une sous-variété transverse aux surfaces. Par exemple, si une telle construction était possible, on connecterait deux calottes sphériques bordées par deux cercles situés dans un même plan (Voir Figure 7). On obtiendrait alors une K-surface bordée par une courbe plane possédant des points d'inflexions, obstruction à l'existence de surface de courbure de Gauss positive.

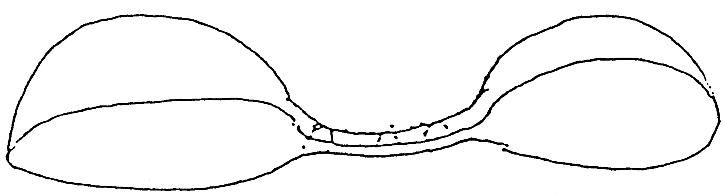

Fig 7

La preuve de ce théorème est similaire à celle de N.Smale [20], dans le cas des H-surfaces. On reformule le problème en un théorème de point fixe de Schauder en linéarisant l'équation. Ce point de vue a été introduit par R.Schoen et utilisé avec succès par Kapouleas[K] pour construire de nouvelles surfaces de courbure moyenne constante non nulle. Par les mêmes méthodes, M.Traizet [22] a construit des surfaces minimales complètes et plongées en recollant des surfaces de Scherk simplement périodiques. Il était

donc pertinent de s'intéresser à cette méthode dans le cadre des K-surfaces.
Remerciements. Je remercie Harold Rosenberg pour son soutien et ses nombreux conseils tout au long de ce travail.

1. Theorem.

In 1987 N. Smale proved [20] a bridge principle for constant mean curvature hypersurfaces of $\mathbb{R}^n$ without Jacobi fields. In this paper we prove, the same principle for surfaces immersed in $\mathbb{R}^3$ with constant and positive Gauss curvature without Jacobi fields. More precisely, we prove the following result:

Theorem 1.1. *Let S_1,S_2 be compact, immersed hypersurfaces of $\mathbb{R}^3$, with boundary, of constant Gauss curvature $K(S_i)_{i=1,2} = C > 0$, in particular they are strictly locally convex. Assume that $(S_i)_{i=1,2}$ are analytic up to their boundaries (with $0 < \beta < 1$) and these surfaces do not have normal Jacobi fields which vanish on the boundary.*

Let q_1 and q_2 be points of ∂S_1 and ∂S_2.

Then we can find an embedded path γ, connecting the boundaries of S_1, S_2, intersecting $S_1 \cup S_2$ only at its end points such that for any tubular neighborhood T of γ, there exists a bridge inside T, connecting ∂S_1 and ∂S_2 such that the resulting configuration spans M, a constant Gauss curvature surface with $K(M) = C > 0$, which is diffeomorphic to S_1 and S_2 joined by a thin strip.

Moreover M is a small perturbation of S_1 and S_2 joined by the strip.

Without loss of generality, up a dilation, we may assume that the Gauss curvature $K(S_i)_{i=1,2} = 1$. By a bridge, we mean two nearby curves and by a strip, a diffeomorphic copy of an interval$\times$interval.

To prove this theorem, we apply the same technique as N.Smale [20] for H-surfaces. We construct a family of approximate solutions M^ε of class $C^{4,\beta}$ with a certain path γ, and a tubular neighborhood T of γ. The parameter $\varepsilon > 0$ is the width of the strip S^ε. We attach this strip to $(S_i)_{i=1,2}$ in a neighborhood of $(q_i)_{i=1,2}$. We need the surfaces to be analytic up to their boundaries to make an analytic extension in a neighborhood of q_1, q_2 (see section 2). This will be the only place where we need analytic surfaces.

We look for a solution as a normal perturbation of M^ε by linearizing the Gauss curvature and reformulating the equations as a fixed point problem. Since the linearized operator is strictly elliptic when M^ε is strictly locally convex and the quadratic term is slighty different, we contruct M^ε very carefully.

We construct M^ε so that $|K-1|_{C^0(M^\varepsilon)} = |K-1|_{C^0(S^\varepsilon)} \le c\varepsilon^2$, in particular M^ε is convex (see section 3) and E(u) is small enough (see section 2).

We choose a path γ, which is alternatively plane and on spheres with radius 1 (see section 3).

For each γ_1, an embedded path as in theorem 1.1 and each tubular neighborhood T_1 of γ_1, we can construct a strip in T_1, along an embedded path $\gamma_0 \subset T_1$, which is composed of planar and spherical arcs by jumping from one sphere to an other by planar curves.

Notations.

For $\Psi : S \longrightarrow \mathbb{R}^3$ a compact $C^{4,\beta}$ hypersurface immersed in $\mathbb{R}^3$ with boundary $C^{4,\beta}$, TS and NS denote the tangent and normal bundles of S, T_qS and N_qS the corresponding fibers at $q \in S$.

N will be the unit normal vector field in $\mathbb{R}^3$ of S, which orients the surface, $N \in C^{3,\beta}(S,\mathbb{R}^3)$ and if $U \in C^{3,\beta}(NS)$ is a section, there exists $u \in C^{3,\beta}(S,\mathbb{R})$ such that $U = uN$.

A is the symmetric operator associated to the second fundamental form. Then there exists at each point q of S, an orthonormal basis $(e_i(q))_{i=1,2}$ of T_qS, eigenvectors associated to $(\kappa_i)_{i=1,2}$, the eigenvalues of the operator. A surface strictly locally convex satisfies $(\kappa_i)_{i=1,2} \neq 0$ and κ_1, κ_2 are of the same sign.

We orient the surface such that $(\kappa_i)_{i=1,2} > 0$.

K(q) is the Gauss curvature at q of S and $K \in C^{2,\beta}(S,\mathbb{R})$ with

$$K = detA = \kappa_1\kappa_2$$

Let $u \in C_0^{3,\alpha}(S)$ be a $C^{3,\alpha}$ real function on S, zero on the boundary ∂S with $0 < \alpha \le \beta$. If $|u|_{C^1}$ is sufficiently small then u induces a $C^{3,\alpha}$ immersion $\Psi_u = \Psi + uN : S \longrightarrow \mathbb{R}^3$ and $\Psi_u \equiv \Psi$ on ∂S. In the following we often identify S with $\Psi(S)$ and we note $S_u = \Psi_u(S)$ the small normal perturbation of S.

Let $\varepsilon_0 > 0$ be small enough so that for $u \in D = \{u \in C_0^{3,\alpha}(S) : |u|_{C^1} \le \varepsilon_0\}$, S_u is a surface immersed.

K(u) denote the Gauss curvature of S_u, $K(u) \in C^{1,\alpha}(S)$, moreover

$$K : D \subset C_0^{3,\alpha}(S) \longrightarrow C^{1,\alpha}(S)$$

is a C^∞ map of these Banach spaces. The Jacobi operator associated to the Gauss curvature of S is (see [19])

$$Lu = \frac{dK}{dt}(tu)_{|t=0} = \operatorname{div} T_1 \nabla u + 2KHu$$

where H is mean curvature on S and $T_1 = trace(A)I - A$. L is a uniformly elliptic operator, self adjoint, Fredholm of index 0 when S is strictly locally convex. A Jacobi field is a normal variation uN of $S, u = 0$ on ∂S, which is a solution of the equation $Lu = 0$.

2. Reformulation of the bridge theorem.

We reformulate theorem 1.1 as an elliptic boundary value problem on the approximate solution M^ε. We construct M^ε(see section 3) so that $\Psi(\varepsilon) : M^\varepsilon \longrightarrow \mathbb{R}^3$, the immersion, satisfies $C_0^{-1} \le |D\Psi(\varepsilon)| \le C_0$ and $|\Psi(\varepsilon)|_{C^{4,\beta}} \le C_0$ for a constant $C_0 > 0$ large enough, independent of $\varepsilon > 0$ the width of the strip.

Let $\varepsilon_0 > 0$ be chosen (depending only on C_0) so that $|u|_{C^1} \le \varepsilon_0$ implies that $\Psi(\varepsilon) + uN$ is an immersion and let $D = \{u \in C_0^{3,\alpha}(M^\varepsilon) : |u|_{C^1} \le \varepsilon_0\}$ with $0 < \alpha \le \beta$.

To prove theorem 1.1, we would like to find $u \in D \subset C_0^{3,\alpha}(M^\varepsilon)$ such that $K(u) = 1$ and show that u is small as $\varepsilon \to 0$.

Applying Taylor's theorem, we have for $u \in D$:

$$K(u) = K + Lu + E(u)$$

where

$$\begin{aligned} K = K(0) &= \text{ Gauss curvature of } M^\varepsilon, \\ Lu &= \frac{dK}{dt}(tu)|_{t=0} = \operatorname{div} T_1 \nabla u + 2KHu, \\ E(u) &= \int_0^1 (1-\tau)\frac{d^2K}{d\tau^2}(\tau u)d\tau. \end{aligned}$$

L is a uniformly elliptic operator since M^ε is strictly locally convex for each $\varepsilon > 0$ small with the coefficients of T_1 depending on the coefficients of the second fundamental form (see section 1). In particular if $support(u) \subset S_i$, i=1,2, then $Lu = L^{S_i}u$, where L^{S_i} is the Jacobi operator for S_i. As we are assuming L^{S_i} has no kernel we will see that L has no kernel for ε sufficiently small.

In section 6 we prove:

Lemma 6.1. *There exists positive constants* ε_1 *and* d_0 *such that for all* $\varepsilon < \varepsilon_1$

$$\mathrm{dist}(\mathrm{spec}(L), 0) \geq d_0 > 0$$

ε_1 *and* d_0 *depend only on* $Max_{M^\varepsilon}|2KH|$ *and* $\min_{i=1,2} \mathrm{dist}(\mathrm{spec}(L^{S_i}), 0), \mathrm{spec}(L) =$ *set of eigenvalues of* $L = \{\lambda \in \mathbb{R} | (L+\lambda)u = 0$ *has a non zero solution* $u \in C_0^{3,\alpha}(M^\varepsilon)\}$.

We also note that $L : C_0^{3,\alpha}(M^\varepsilon) \longrightarrow C^{1,\alpha}(M^\varepsilon)$ is a self adjoint uniformly elliptic (linear) operator. Then it is standard (see [18]), that spec(L) is a discrete set bounded from below and increasing to ∞.

Using lemma 6.1 we formulate the problem as a fixed point problem:

$$\begin{aligned} K(u) = 1 &\Leftrightarrow K + Lu + E(u) = 1 \\ &\Leftrightarrow u = -L^{-1}(K-1) - L^{-1}(E(u)) \Leftrightarrow u = J(u) \end{aligned}$$

where $J(u) = -L^{-1}(K-1) - L^{-1}(E(u))$.

By standard elliptic theory

$$L^{-1} : C^{1,\alpha}(M^\varepsilon) \longrightarrow C_0^{3,\alpha}(M^\varepsilon)$$

is a continuous map, for any $0 < \alpha \leq \beta, \varepsilon \leq \varepsilon_1$. $(K-1) \in C^{1,\alpha}(M^\varepsilon)$ and

$$E : D \subset C_0^{3,\alpha}(M^\varepsilon) \longrightarrow C^{1,\alpha}(M^\varepsilon)$$

is a continuous map for any $0 < \alpha \leq \beta$. Therefore:

$$J : D \subset C_0^{3,\alpha}(M^\varepsilon) \longrightarrow C_0^{3,\alpha}(M^\varepsilon)$$

is a continuous map for any $0 < \alpha \leq \beta, \varepsilon \leq \varepsilon_1$.

We recall the Schauder fixed point theorem (see [8]). If P is a convex, compact set of a banach space B and if $J : P \longrightarrow P$ is a continuous map, then J has a fixed point.

We solve $J(u) = u$ for some $u \in D$ by using the Schauder fixed point theorem.

The appropriate sets to look at are as follows. For $\varepsilon \leq \varepsilon_1, 0 < \alpha \leq \beta, 0 < \sigma \leq \frac{1}{2}$, we define $P(\varepsilon, \sigma, \alpha)$:

$$P(\varepsilon,\sigma,\alpha) = \Big\{ u \in C_0^{3,\alpha}(M^\varepsilon) : |u|_{C^0} \leq \varepsilon^{2,75-\sigma}, \\ |u|_{C^{3,\alpha}} \leq \varepsilon^{-0.25-\alpha-\sigma}, |u|_{H^2} \leq \varepsilon^{1,75-\sigma} \Big\}.$$

Clearly $P(\varepsilon, \sigma, \alpha)$ is a convex, closed and bounded set in $C_0^{3,\alpha}(M^\varepsilon)$. Now for $\alpha' < \alpha$, $P(\varepsilon, \sigma, \alpha) \subset C_0^{3,\alpha'}(M^\varepsilon)$ and is in fact compact as the inclusion

$C_0^{3,\alpha}(M^\varepsilon) \longrightarrow C_0^{3,\alpha'}(M^\varepsilon)$ is compact (see [8]). It will follow from lemma 5.3 that if $u \in P(\varepsilon, \sigma, \alpha)$ then $|u|_{C^1} \leq c\varepsilon^{1,75-\alpha-\sigma}$ for some constant depending only on C_0. So $P(\varepsilon, \sigma, \alpha) \subset D$ for $\varepsilon < \varepsilon_2$ for $\varepsilon_2 > 0$ small enough (so J is defined on $P(\varepsilon, \sigma, \alpha)$).

We reformulate theorem 1.1 as follows:

Theorem 2.1. *There exists $\bar{\alpha} > 0$, $\bar{\sigma} > 0$ and $\bar{\varepsilon} > 0$ depending on C_0, d_0, S_1, S_2 such that for all $\varepsilon < \bar{\varepsilon}$:*

$$J : P(\varepsilon, \bar{\sigma}, \bar{\alpha}) \longrightarrow P(\varepsilon, \bar{\sigma}, \bar{\alpha})$$

is a continuous map.

This implies theorem 1.1. Since $P(\varepsilon, \bar{\sigma}, \bar{\alpha})$ is a compact subset in $C_0^{3,\alpha'}(M^\varepsilon)$ for any $\alpha' < \bar{\alpha}$ and $J : D \subset C_0^{3,\alpha'}(M^\varepsilon) \longrightarrow C_0^{3,\alpha'}(M^\varepsilon)$ is continuous, it must have a fixed point $u \in P(\varepsilon, \bar{\sigma}, \bar{\alpha})$.

Now for $u \in P(\varepsilon, \sigma, \alpha)$, $Ju = v + w$ where v and w are solutions of the linear equations:

$$Lv = -(K - 1)$$
$$Lw = -E(u).$$

Thus proving theorem 2.1 involves estimating v and w in terms of ε in various norms (we must show that $v + w \in P(\varepsilon, \sigma, \alpha)$ if $u \in P(\varepsilon, \sigma, \alpha)$).

Since $(K - 1)$ does not depend on u, estimates of v gives bounds on $P(\varepsilon, \sigma, \alpha)$ and smallness of $|K-1|$ in norms $L^1, L^2, C^{0,\alpha}$ assure that v will be small in norms $C^{1,\alpha}, C^0$ and H^2. Recall that $|K-1|$ is a function depending on the construction of the approximate solution M^ε (see section 3).

We estimate the C^0 norm of v by a Sobolev inequality (see section 4) and by standard elliptic theory (see corollary 6.1). We use a Schauder estimate (see lemma 5.2) and the previous C^0 bound of v to obtain the $C^{2,\alpha}$ norm of v.

There can be some explosion of the $C^{2,\alpha}$ norm since the constant of the Schauder estimate behaves like ε^{-2} (see section 5); this is a consequence of the width of the strip and the fact that the Schauder estimates are local.

Interpolation inequalities (see lemma 5.1) give the $C^{1,\alpha}$ norm.

Then we can hope that E(u) will be smaller than $|K-1|$ in norm L^1, L^2 and $C^{0,\alpha}$ (see lemma 7.1) to obtain a better estimate for w (see the beginning of the section 7).

In the case of H-surfaces N.Smale constructs a family of approximate solutions and a set $P(\varepsilon, \sigma, \alpha)$ such that $v \in P(\varepsilon, \sigma, \alpha)$ and $u \in P(\varepsilon, \sigma, \alpha)$ implies that $|u|_{C^0} \leq \varepsilon^{1,5-\sigma}$, $|u|_{C^{2,\alpha}} \leq \varepsilon^{-0,5-\alpha-\sigma}$ and $|u|_{C^{1,\alpha}} \leq c\varepsilon^{0,5-\alpha-\sigma}$.

In his lemma 6, under the hypothesis $|u| < 1, |\nabla u| < 1$ he gets for a constant $C > 0$:

$$|E(u)| \leq C\left(\eta|u|^2 + |\nabla u|^2 + \eta|u||\nabla u| + |u||D^2u| + |\nabla u|^2|D^2u|\right) \text{ (pointwise)}$$

where η is a regular function on M^ε, such that $|\eta|_{C^0} \leq c\varepsilon^{-1}$. But η appears in the particular construction of the approximate solutions of N.Smale. Since terms $|D^2u|$ which behaves like $\varepsilon^{-0.5}$ are coupled with $|u|$ and $|\nabla u|^2$, $|E(u)|$ will be small enough ($|E(u)|_{C^0} \leq c\varepsilon^{0,5-3\sigma}$) to obtain a good estimate for w.

In the case of K-surfaces, under the hypothesis $|u| < 1, |\nabla u| < 1$ and $|D^2u| < 1$, we obtain in lemma 7.1:

$$|E(u)| \leq C\left(|u|^2 + |\nabla u|^2 + |D^2u|^2\right) \text{ (pointwise).}$$

We need $|u|_{C^{2,\alpha}}$ small enough to have a good estimate of $|E(u)|_{C^{0,\alpha}}$($|u|_{C^{2,\alpha}}$ can't behave like $\varepsilon^{-0,5}$).

$|K-1|_{C^0} \leq c\varepsilon^2$ implies that $|v|_{C^0} \leq c\varepsilon^{2,75}$(see (3.13) and section 7-(V1)) and thus $|v|_{C^{2,\alpha}} \leq c\varepsilon^{0.75-\alpha}$ (see lemma 5.2)and $|v|_{C^{3,\alpha}} \leq c\varepsilon^{-0.25-\alpha}$ (see lemma 5.4).

If $u \in P(\varepsilon, \sigma, \alpha)$, interpolation inequalities (lemma 5.3) gives that $|u|_{C^2} \leq c\varepsilon^{0.75-\sigma}$ and E(u) will be small enough (see section 7).

To obtain C^0 bounds of v, w we use L^1 and L^2 norms of $(K-1)$ and $E(u)$. Since $|E(u)|_{L^1} \leq c|u|^2_{H^2}$ and $|E(u)|_{L^2} \leq c|u|_{C^2}|u|_{H^2}$, we define $P(\varepsilon, \sigma, \alpha)$ with a H^2 bound. We use that $E(u) \in C^{1,\alpha}(M^\varepsilon)$ to get a bound of $|w|_{H^2}$. For these reasons we apply the Schauder fixed point theorem in $C_0^{3,\alpha}$ and we construct and attach the strip carefully in the section 3.

In section 4,5,6 we prove some technical lemmas which allow us to estimate solutions of $Lu = F$, $F \in C^{1,\alpha}(M^\varepsilon)$ as well as the eigenvalue bounds. Next, we use these lemmas to estimate v and w in section 7.

3. Construction of approximate solutions.

Let q_1 and q_2 be points of ∂S_1 and ∂S_2. We extend $(S_i)_{i=1,2}$ to $(S_i')_{i=1,2}$ an open neighborhood of $(q_i)_{i=1,2}$ with $K = 1 > 0$ by analyticity of the constant Gauss curvature surfaces.

Let e_1, e_2 be unit tangent vectors of S_1', S_2' at q_1 and q_2, orthogonal to ∂S_1, ∂S_2.

Let $P_1 = (N_{q_1}, e_1)$ and $P_2 = (N_{q_2}, e_2)$ be planes of $\mathbb{R}^3$, and let B_1,B_2 be spheres of radius 1 with their centers respectively at P_1 and P_2, such that $(B_i)_{i=1,2} \cap (S_1 \cup S_2) = \emptyset$. $(B_i)_{i=1,2}$ are K-surfaces with $K = 1$.

Let $\gamma(t)$ be a smooth embedded path parametrized by arclength on $[0, l_0]$ such that:

1. $\gamma : I = [0, l_0] \longrightarrow \mathbb{R}^3$ with $\gamma(0) = q_1, \gamma(l_0) = q_2$ and $\gamma \cap (S_1 \cup S_2) = \{q_1, q_2\}$ for $t \in [0, l_0]$.

2. $\gamma'(0) = e_1, \gamma'(l_0) = e_2$.

3. $0 < |\gamma''(t)|$.

4. There exists t_1, t_2, t_3, t_4 some values in $[0, l_0]$ and $r_0 > 0, r_1 > 0$ small enough, such that

   - $\forall t \in [0, 2r_0], \gamma(t) \subset P_1 \cap S_1'$
   - $\forall t \in [2r_0, t_1], \gamma(t) \subset P_1$
   - $\forall t \in [t_1 - r_1, t_2 + r_1], \gamma(t) \subset B_1$
   - $\forall t \in [t_2, t_3], \gamma(t)$ is a planar curve.
   - $\forall t \in [t_3 - r_1, t_4 + r_1], \gamma(t) \subset B_2$
   - $\forall t \in [t_4, l_0 - 2r_0], \gamma(t) \subset P_2$
   - $\forall t \in [l_0 - 2r_0, l_0], \gamma(t) \subset P_2 \cap S_2'$.

We note:

$$\begin{aligned} n(t) &= \frac{\gamma''(t)}{|\gamma''(t)|} \\ b(t) &= \gamma'(t) \wedge n(t) \\ \gamma''(t) &= k(t)n(t) \end{aligned}$$

with $k(t) \neq 0$ by 3). When the curve is plane, we have $n'(t) = -k(t)\gamma'(t)$ and b(t) a constant vector.

Let T be any tubular neighborhood of γ. We construct a strip as a $C^{4,\beta}$ foliation of curves along γ in T.

Let $P_t = N_{\gamma(t)}\gamma(t)$ the normal 2-dimensional space to $\gamma'(t)$ at $\gamma(t)$, with the orthonormal frame $(\gamma(t), n(t), b(t))$. Then for each t, we construct a piece of curve C_t passing through $\gamma(t)$ in P_t parametrized by arclength on $[-r_2, r_2]$ with $C_t(0) = 0$ where $r_2 > 0$ is small enough such that $C_t \subset T$.

Then, let x, y be functions $C^{4,\beta}$ on $[0, l_0] \times [-r_2, r_2]$ such that for t fixed and $s \in [-r_2, r_2]$, $C_t(s) = x(t, s)n(t) + y(t, s)b(t)$.

We consider the $C^{4,\beta}$ surface S_{r_2} parametrized by:

$$\begin{aligned} F_{r_2} : [0, l_0] \times [-r_2, r_2] &\longrightarrow \mathbb{R}^3 \\ (t, s) &\longrightarrow \gamma(t) + x(t, s)n(t) + y(t, s)b(t) \end{aligned}$$

with $F_{r_2}(t,0)=\gamma(t)$. Let N(t) be the normal vector to S_{r_2} on $\gamma(t)$. Then

$$N(t)=\frac{F_t(t,0)\wedge F_s(t,0)}{|F_t(t,0)\wedge F_s(t,0)|}.$$

Let K(t,s) be the Gauss curvature of S_{r_2} at $F_{r_2}(t,s)$ and $K\in C^{2,\beta}(S_{r_2},\mathbb{R})$. We will need an estimate of $|K-1|_{C^0(S_{r_2})}$ in terms of r_2.

Since

$$K(t,s)=K(t,0)+sK_s(t,0)+\int_0^s (s-h)\frac{d^2K}{dh^2}(t,hs)dh; \tag{3.1}$$

if $K(t,0)=1$ and $K_s(t,0)=0$ we have

$$|K-1|_{C^0}\le\left|\frac{d^2K}{ds^2}\right|_{C^0} r_2^2.$$

In the following lemma we study, conditions on x,y to have $K(t,0)=1$ and $K_s(t,0)=0$. For $r_2>0$ small enough, S_{r_2} will be strictly locally convex $(K>0)$. Then we construct a bridge in T which satisfy these conditions.

Lemma 3.1. *Let S be a compact surface in $\mathbb{R}^3$, $C^{4,\beta}$ up to the boundary, strictly locally convex and let γ be a planar embedded curve of S parametrized by arclength of class $C^{4,\beta}$*

$$\gamma(t):[0,r_0]\longrightarrow\mathbb{R}^3$$

Assume also that the unit normal vector to S at $\gamma(t)$ is

$$N(t)=\cos\alpha(t)n(t)+\sin\alpha(t)b(t)$$

where $\alpha(t)$ is a $C^{3,\beta}$ function

$$\alpha(t):[0,r_0]\longrightarrow\mathbb{R}$$

with $0\le\alpha(t)\le\frac{\pi}{4}$.

We can locally parametrize the surface S for $r_0>0$ small enough by

$$\begin{aligned}G:[0,r_0]\times[-r_0,r_0]&\longrightarrow\mathbb{R}^3\\ (t,s)&\longrightarrow\gamma(t)+X(t,s)n(t)+Y(t,s)b(t).\end{aligned}$$

Where X,Y are $C^{4,\beta}$ functions on $[0,r_0]\times[-r_0,r_0]$ such that the curve $C_t(s)=X(t,s)n(t)+Y(t,s)b(t)$ is parametrized by arclength with $C_t(0)=0$.

Let $K(t,s)$ be the Gauss curvature of S at G(t,s).

In the following we note f for f(t), a function which depends only on t, and its derivative f'.

1. If S is a K-surface with $K=1$, then with $R = \dfrac{k\cos\alpha}{1+(\alpha')^2}$ we have:

(a)
$$X(t,0) = Y(t,0) = 0$$

(b)
$$X_t(t,0) = Y_t(t,0) = 0$$

(c)
$$X_s(t,0) = \sin\alpha$$
$$Y_s(t,0) = -\cos\alpha$$

(d)
$$X_{tt}(t,0) = Y_{tt}(t,0) = 0$$

(e)
$$X_{ss}(t,0) = \frac{1}{R}\cos\alpha$$
$$Y_{ss}(t,0) = \frac{1}{R}\sin\alpha$$

(f)
$$X_{st}(t,0) = \alpha'\cos\alpha$$
$$Y_{st}(t,0) = \alpha'\sin\alpha$$

(g)
$$X_{ttt}(t,0) = Y_{ttt}(t,0) = 0$$

(h)
$$X_{tts}(t,0) = \alpha''\cos\alpha - (\alpha')^2\sin\alpha$$
$$Y_{tts}(t,0) = \alpha''\sin\alpha + (\alpha')^2\cos\alpha$$

(i)
$$X_{sst}(t,0) = -\frac{R'}{R^2}\cos\alpha - \frac{\alpha'}{R}\sin\alpha$$
$$Y_{sst}(t,0) = -\frac{R'}{R^2}\sin\alpha + \frac{\alpha'}{R}\cos\alpha$$

(j)
$$X_{sss}(t,0) = \frac{3k}{R}\sin\alpha\cos\alpha - \frac{\alpha''}{kR} - \frac{2\alpha' R'}{kR^2} - 4\sin\alpha$$
$$Y_{sss}(t,0) = \frac{1}{R^2\cos\alpha} + \frac{3k}{R}\sin^2\alpha - \frac{\alpha''\sin\alpha}{kR\cos\alpha} - \frac{2\alpha' R'\sin\alpha}{kR^2\cos\alpha} - \frac{4\sin^2\alpha}{\cos\alpha}$$

2. *Reciprocally if S is a surface with Gauss curvature K not necessarily constant and if X, Y verify a)-j), with N as above then $K(t,0) = 1$ and $K_s(t,0) = 0$ for $t \in [0, r_0]$.*

Proof. 1)

- $C_t(0) = 0 \Rightarrow a)$ and $a)$ implies b), d), g).
- $C_t(s)$ parametrized by arclength implies $X_s^2 + Y_s^2 = 1$ and
$$G_s(t,0) = X_s n + Y_s b$$
orthogonal to N(t) implies c).
- $c) \Rightarrow f) \Rightarrow h)$.
- $e) \Rightarrow i)$.

Then we have to prove e) and j).

Since
$$X_s^2 + Y_s^2 = 1$$
we have on S
$$X_s X_{ss} + Y_s Y_{ss} = 0 \text{ and } X_{ss}^2 + Y_{ss}^2 + X_s X_{sss} + Y_s Y_{sss} = 0,$$
which implies by c) for each point (t,0):
$$X_{ss} \sin\alpha - Y_{ss} \cos\alpha = 0 \tag{3.2}$$
and by a), b), c), d), f) and e) for each point (t,0):
$$\frac{1}{R^2} + X_{sss} \sin\alpha - Y_{sss} \cos\alpha = 0. \tag{3.3}$$
We write K(t,s) in terms of X,Y,DX,DY,D^2X, D^2Y and by a), b), c), d), f) we have:
$$K(t,0) = k\cos\alpha(X_{ss}\cos\alpha + Y_{ss}\sin\alpha) - (\alpha')^2 = 1 \tag{3.4}$$
which gives e) with (3.2). Now by e) we have (3.3).

Since S is a K-surface, we have $K_s(t,s) = 0$. As above we write $K_s(t,0)$ in terms of X,Y,DX,DY, D^2X, D^2Y, D^3X, D^3Y, and by a)-i) we have:
$$\begin{aligned} K_s(t,0) = {} & \frac{\alpha''}{R} - \frac{2k^2}{R}\cos\alpha\sin\alpha - \frac{k}{R^2}\sin\alpha + \\ & k\cos\alpha(X_{sss}\cos\alpha + Y_{sss}\sin\alpha - \frac{k}{R}\sin\alpha) + \frac{2\alpha' R'}{R^2} + 4k\sin\alpha = 0 \end{aligned} \tag{3.5}$$

which gives us j) with (3.3).

• proof of e).

By direct computation with $\gamma'' = kn$, $n' = -k\gamma'$ ($k > 0$ since S is strictly locally convex) and $b' = 0$ (since γ is a planar curve):

$$
\begin{aligned}
G_t(t,s) &= (1-kX)\gamma' + X_t n + Y_t b \\
G_s(t,s) &= X_s n + Y_s b \\
G_t \wedge G_s(t,s) &= (X_t Y_s - Y_t X_s)\gamma' - Y_s(1-kX)n + X_s(1-kX)b \\
G_{tt}(t,s) &= -(k_t X + 2kX_t)\gamma' + (k(1-kX) + X_{tt})n + Y_{tt} b \\
G_{ss}(t,s) &= X_{ss} n + Y_{ss} b \\
G_{st}(t,s) &= -kX_s\gamma' + X_{st} n + Y_{st} b \\
g_{11}(t,s) &= \langle G_t, G_t\rangle = (1-kX)^2 + X_t^2 + Y_t^2 \\
g_{22}(t,s) &= \langle G_s, G_s\rangle = X_s^2 + Y_s^2 \\
g_{12}(t,s) &= \langle G_t, G_s\rangle = X_t X_s + Y_t Y_s
\end{aligned}
\tag{3.6}
$$

and

$$
K(t,s) = \frac{eg - f^2}{(g_{11}g_{22} - g_{12}^2)^2} \tag{3.7}
$$

(see [DoC]) since $|G_t \wedge G_s| = \sqrt{g_{11}g_{22} - g_{12}^2}$ with

$$
\begin{aligned}
e(t,s) &= \langle G_t \wedge G_s, G_{tt}\rangle = -(k'X + 2kX_t)(X_t Y_s - Y_t X_s) \\
&\quad - Y_s(1-kX)(k(1-kX) + X_{tt}) + X_s Y_{tt}(1-kX) \\
g(t,s) &= \langle G_t \wedge G_s, G_{ss}\rangle \\
&= X_s Y_{ss}(1-kX) - Y_s X_{ss}(1-kX) \\
f(t,s) &= \langle G_t \wedge G_s, G_{st}\rangle = -kX_s(X_t Y_s - Y_t X_s) \\
&\quad - Y_s X_{st}(1-kX) + X_s Y_{st}(1-kX)
\end{aligned}
\tag{3.8}
$$

Then by a), b) c),d), f), we have:

$$
\begin{aligned}
g_{11}(t,0) &= 1 \\
g_{22}(t,0) &= 1 \\
g_{12}(t,0) &= 0 \\
e(t,0) &= k\cos\alpha \\
g(t,0) &= X_{ss}\cos\alpha + Y_{ss}\sin\alpha \\
f(t,0) &= \alpha'
\end{aligned}
$$

Thus (3.4):

$$K(t,0) = k\cos\alpha(X_{ss}\cos\alpha + Y_{ss}\sin\alpha) - (\alpha')^2.$$

Then with (3.2) we have the following system:

$$\begin{cases} X_{ss}\sin\alpha - Y_{ss}\cos\alpha = 0 \\ k\cos\alpha(X_{ss}\cos\alpha + Y_{ss}\sin\alpha) - (\alpha')^2 = 1 \end{cases}$$

Thus:

$$\begin{cases} Y_{ss} = X_{ss}\dfrac{\sin\alpha}{\cos\alpha} \\ X_{ss} = \dfrac{1+(\alpha')^2}{k\cos\alpha}\cos\alpha \end{cases}$$

which gives us e).

- proof of j).

By derivation in s of (3.6) we have:

$$\begin{aligned} g_{11s}(t,s) &= -2kX_s(1-kX) + 2X_{ts}X_t + 2Y_{ts}Y_t \\ g_{22s}(t,s) &= 0 \\ g_{12s}(t,s) &= X_{ts}X_s + X_tX_{ss} + Y_{ts}Y_s + Y_tY_{ss} \end{aligned}$$

and by a)-f):

$$\begin{aligned} g_{11s}(t,0) &= -2k\sin\alpha \\ g_{22s}(t,0) &= 0 \\ g_{12s}(t,0) &= 0 \end{aligned}$$

By derivation in s of (3.8) we have:

$$\begin{aligned} e_s(t,s) = {} & -(k'X + 2kX_t)_s(X_tY_s - Y_tX_s) - (k'X + 2kX_t)(X_tY_s - Y_tX_s)_s \\ & - Y_{ss}(1-kX)(k(1-kX) + X_{tt}) \\ & + kY_sX_s(k(1-kX) + X_{tt}) - Y_s(1-kX)(-k^2X_s + X_{tts}) \\ & + X_{ss}Y_{tt}(1-kX) + X_sY_{tts}(1-kX) - kX_s^2Y_{tt} \\ g_s(t,s) = {} & (1-kX)(X_sY_{sss} - Y_sX_{sss}) + kX_s(Y_sX_{ss} - X_sY_{ss}) \\ f_s(t,s) = {} & -kX_{ss}(X_tY_s - Y_tX_s) - kX_s(X_tY_{ss} - Y_tX_{ss}) \\ & + (1-kX)(X_{ss}Y_{st} + X_sY_{sts} - Y_{ss}X_{st} - Y_sX_{sts}) \end{aligned}$$

and by a)-f):

$$\begin{aligned} e_s(t,0) &= 2k^2X_sY_s - kY_{ss} - Y_sX_{tts} + X_sY_{tts} \\ &= \alpha'' - 2k^2\cos\alpha\sin\alpha - \frac{k}{R}\sin\alpha \\ g_s(t,0) &= X_{sss}\cos\alpha + Y_{sss}\sin\alpha - \frac{k}{R}\sin\alpha \\ f_s(t,0) &= (X_{ss}Y_{st} + X_sY_{sts} - Y_{ss}X_{st} - Y_sX_{sts}) = -\frac{R'}{R^2} \end{aligned}$$

By derivation in s of (3.7) we have:

$$K_s(t,0) = (e_sg + eg_s - 2ff_s) - 2(g_{11}g_{22} - g_{12}^2)_s(eg - f^2).$$

Then:

$$\begin{aligned} K_s(t,0) = &\frac{\alpha''}{R} - \frac{2k^2}{R}\cos\alpha\sin\alpha - \frac{k}{R^2}\sin\alpha \\ &+ k\cos\alpha\left(X_{sss}\cos\alpha + Y_{sss}\sin\alpha - \frac{k}{R}\sin\alpha\right) \\ &+ \frac{2\alpha'R'}{R^2} + 4k\sin\alpha. \end{aligned}$$

Thus with (3.3) we get the following system:

$$\begin{cases} X_{sss}\cos\alpha + Y_{sss}\sin\alpha &= -\dfrac{\alpha''}{kR\cos\alpha} + \dfrac{3k}{R}\sin\alpha + \dfrac{\sin\alpha}{R^2\cos\alpha} \\ &\quad -\dfrac{2\alpha'R'}{kR^2\cos\alpha} - \dfrac{4\sin\alpha}{\cos\alpha} \\ \dfrac{1}{R^2} + X_{sss}\sin\alpha - Y_{sss}\cos\alpha = 0 \end{cases}$$

and

$$\begin{cases} Y_{sss} = \dfrac{1}{R^2\cos\alpha} + X_{sss}\dfrac{\sin\alpha}{\cos\alpha} \\ X_{sss} = \dfrac{3k}{R}\sin\alpha\cos\alpha - \dfrac{\alpha''}{kR} - \dfrac{2\alpha'R'}{kR^2} - 4\sin\alpha \end{cases}$$

gives j).

2) Reciprocally, K(t,0) and $K_s(t,0)$ depending only of a)-j),k, k', the calculus of (3.4), (3.5) above is true. ■

Now we construct the bridge S_{r_2} by piecewise.

1) For $t \in I_1 = [0, t_1] \cup [t_2, t_3] \cup [t_4, l_0]$, γ is a planar curve. For $s \in [-r_0, r_0]$ we define $C_t(s) = x(t,s)n + y(t,s)b$ with $R = k$ by:

$$x(t,s) = R\left(1 - \cos\frac{s}{R}\right)$$
$$y(t,s) = -R\sin\left(\frac{s}{R}\right).$$

We consider the embedding:

$$F_1 : I_1 \times [-r_2, r_2] \longrightarrow \mathbb{R}^3$$
$$(t,s) \longrightarrow \gamma + x(t,s)n + y(t,s)b$$

Since γ is a smooth curve, F_1 is a $C^{4,\beta}$ embedding of $I_1 \times [-r_2, r_2]$ for $r_2 > 0$ small enough, and $|F_1|_{C^{4,\beta}} \leq C_0$, $|DF_1| \geq C_0^{-1}$ for $C_0 > 0$ large enough independent of r_2. Moreover F_1 verifies lemma 3.1-2).

We have:

$$(a) \qquad x(t,0) = y(t,0) = 0$$

and $a)$ implies $b), d), g)$ of lemma 3.1.

$$x_s(t,s) = \sin\frac{s}{R} \text{ and } y_s(t,s) = -\cos\frac{s}{R}$$

implies

$$x_s(t,0) = 0 \text{ and } y_s(t,0) = -1.$$

Then

$$F_t(t,0) = \gamma' \text{ and } F_s(t,0) = -b$$

which implies $N(t) = n$.

Then x, y verify a)-j) of lemma 3.1 with $\alpha = \alpha' = \alpha'' = 0$ and $R = k$ for $t \in I_1$: we have a), b), c), d), g) and $c) \Rightarrow f) \Rightarrow h)$.

$$x_{ss}(t,s) = \frac{1}{R}\cos\frac{s}{R} \text{ and } y_{ss}(t,s) = \frac{1}{R}\sin\frac{s}{R}$$

implies

$$(e) \qquad x_{ss}(t,0) = \frac{1}{R} \text{ and } y_{ss}(t,0) = 0$$

e) implies i).

$$x_{sss}(t,s) = -\frac{1}{R^2}\sin\frac{s}{R} \text{ and } y_{sss}(t,s) = \frac{1}{R^2}\cos\frac{s}{R}$$

implies

$$(j) \qquad x_{sss}(t,s) = 0 \text{ and } y_{sss}(t,s) = \frac{1}{R^2}.$$

Then for $t \in I_1$, we have $K(t,0) = 1$ and $K_s(t,0) = 0$ by lemma 3.1.

2) For $t \in [t_1 - r_0, t_2 + r_0] \cup [t_3 - r_0, t_4 + r_0]$, $\gamma(t)$ is a curve of the K-surface $B_1 \cup B_2$, with $K = 1$. Then we take C_t parametrized as above of length $2r_2$ such that $C_t \subset P_t \cap (B_1 \cup B_2)$ and $C_t(0) = 0$. Then

$$\begin{aligned} F_2 : I_2 \times [-r_2, r_2] &\longrightarrow \mathbb{R}^3 \\ (t,s) &\longrightarrow \gamma + x(t,s)n + y(t,s)b \end{aligned}$$

is a $C^{4,\beta}$ embedded for $r_2 > 0$ small enough and

$$F_2(I_2 \times [-r_2, r_2]) \subset B_1 \cup B_2.$$

Then

$$K(t,s) = 1 \text{ for } t \in I_2.$$

Let

$$F_{r_2} = \begin{cases} F_1 \text{ if } (t,s) \in I_1 \times [-r_2, r_2] \\ F_2 \text{ if } (t,s) \in I_2 \times [-r_2, r_2] \end{cases}$$

Since for $t \in I_1 \cap I_2$, γ is a piece of great circle of the spheres B_1 or B_2 (Their centers are in the plan P_1, P_2), then $F_1 = F_2$ for $t \in I_1 \cap I_2$ and F_{r_2} is a $C^{4,\beta}$ embedding of $[0, l_0] \times [-r_2, r_2]$.

Clearly $|F_{r_2}|_{C^{4,\beta}} \le C_0$ and $|DF_{r_2}| > C_0^{-1}$ for $C_0 > 0$ large enough.

Now we smooth out the embedding in small neighborhood of q_1 and q_2, so that the strip is smoothly attached to $S_1 \cup S_2$.

Let F_ε be the restriction of $F_{r_2} = F$:

$$F_\varepsilon = F_{|[2r_0, l_0 - 2r_0] \times [-\varepsilon, \varepsilon]} \text{ with } 0 < r_0 < 1 \text{ small enough.}$$

We claim that we can extend F_ε to $[-\varepsilon, \varepsilon] \times [0, l_0]$ so that it's image is a $C^{4,\beta}$ extension of $S_1 \cup S_2$ and

$$|F_\varepsilon|_{C^{4,\beta}} \le C_0, \qquad |DF_\varepsilon| > C_0^{-1}$$

for $C_0 > 0$ large enough independent of ε.

3) For $t \in I_3 = [0, r_0]$, γ is a planar curve of S_1'. Let $N(t) = \cos\alpha n + \sin\alpha b$ be the normal vector on S_1' at $\gamma(t)$ defined on $[0, 2r_0]$. Then $\alpha(t) : [0, 2r_0] \longrightarrow \mathbb{R}$

is an analytic map, since a K-surface is analytic in its' interior. Moreover $\alpha(0) = 0$ implies $0 \leq \alpha \leq \pi/4$ for $r_0 > 0$ small enough.

We can locally parametrize S_1' around γ by:

$$\begin{aligned} G : [0, 2r_0] \times [-\varepsilon, \varepsilon] &\longrightarrow \mathbb{R}^3 \\ (t.s) &\longrightarrow \gamma + X(t,s)n + Y(t,s)b \end{aligned}$$

where X,Y satisfy Lemma 3.1-1).

Let x, y be functions defined on $[0, 2r_0] \times [-\varepsilon, \varepsilon]$ by:

$$x(t,s) = R\left(\cos\alpha - \cos\left(\alpha + \frac{s}{R}\right)\right) + \frac{s^3}{6}\left(X_{sss}(t,0) + \frac{1}{R^2}\sin\alpha\right)$$
$$y(t,s) = R\left(\sin\alpha - \sin\left(\alpha + \frac{s}{R}\right)\right) + \frac{s^3}{6}\left(Y_{sss}(t,0) - \frac{1}{R^2}\cos\alpha\right)$$

with $R = \dfrac{k\cos\alpha}{1 + (\alpha')^2}$ and $X_{sss}(t,0), Y_{sss}(t,0)$ defined in j) of Lemma 3.1.

We claim that x, y verifies a)-j) of lemma 3.1:

Since

$$(a) \qquad x(t,0) = y(t,0) = 0$$

we have b), d), g).

$$x_s(t,s) = \sin\left(\alpha + \frac{s}{R}\right) + \frac{s^2}{2}\left(X_{sss}(t,0) + \frac{1}{R^2}\sin\alpha\right)$$
$$y_s(t,0) = -\cos\left(\alpha + \frac{s}{R}\right) + \frac{s^2}{2}\left(Y_{sss}(t,0) - \frac{1}{R^2}\cos\alpha\right)$$

implies

$$(c) \qquad x_s(t,0) = \sin\alpha \text{ and } y_s(t,0) = -\cos\alpha$$

and we have f),h).

$$x_{ss}(t,s) = \frac{1}{R}\cos\left(\alpha + \frac{s}{R}\right) + s\left(X_{sss}(t,0) + \frac{1}{R^2}\sin\alpha\right)$$
$$y_{ss}(t,s) = \frac{1}{R}\sin\left(\alpha + \frac{s}{R}\right) + s\left(Y_{sss}(t,0) - \frac{1}{R^2}\cos\alpha\right)$$

implies

$$(e) \qquad x_{ss}(t,0) = \frac{1}{R}\cos\alpha \text{ and } y_{ss}(t,0) = \frac{1}{R}\sin\alpha$$

and we have i).

$$x_{sss}(t,s) = -\frac{1}{R^2}\sin\left(\alpha + \frac{s}{R}\right) + X_{sss}(t,0) + \frac{1}{R^2}\sin\alpha$$
$$y_{sss}(t,s) = \frac{1}{R^2}\cos\left(\alpha + \frac{s}{R}\right) + Y_{sss}(t,0) - \frac{1}{R^2}\cos\alpha$$

implies

$$(e) \qquad x_{sss}(t,0) = X_{sss}(t,0) \text{ and } y_{sss}(t,0) = Y_{sss}(t,0)$$

Thus

$$\begin{array}{llll} x(t,0) & = X(t,0) & y(t,0) & = Y(t,0) \\ Dx(t,0) & = DX(t,0) & Dy(t,0) & = DY(t,0) \\ D^2x(t,0) & = D^2X(t,0) & D^2y(t,0) & = D^2Y(t,0) \\ D^3x(t,0) & = D^3X(t,0) & D^3y(t,0) & = D^3Y(t,0) \end{array}$$

Let ϕ_1 be a cutoff function on $\mathbb{R}$ such that:

$$\phi_1 : \mathbb{R} \longrightarrow \mathbb{R} \text{ and } 0 \le \phi_1(t) \le 1$$
$$\phi_1(t) = 1 \text{ for } t \le \frac{r_0}{2}$$
$$\phi_1(t) = 0 \text{ for } t \ge \frac{3r_0}{4}$$

and $\max|D^k\phi_1| \le c_1$, $|D^k\phi_1|_{(\alpha)} \le c_1$ for k=1,2,3,4 and $0 < \alpha < 1$, for some constant c_1 independent of ε.

Let F_3 be the embedding of $[0,r_0] \times [-\varepsilon,\varepsilon]$ defined by:

$$F_3 : [0,r_0] \times [-\varepsilon,\varepsilon] \longrightarrow \mathbb{R}^3$$
$$(t,s) \longrightarrow \gamma + (\phi_1 X + (1-\phi_1)x)n + (\phi_1 Y + (1-\phi_1)y)b.$$

Then using a Taylor expansion for x, y, X, Y in s, it's easy to see that F_3 verify the lemma 3.1 and:

$$|DF_3| \ge C_0^{-1}$$
$$|F_3|_{C^{4,\beta}} \le C_0.$$

4) For $t \in I_4 = [r_0, 2r_0]$, $\gamma \in P_1 = (N(0), e_1) \cap S_1'$.

Let ϕ_2 be a cutoff function on $\mathbb{R}$ such that:

$$\phi_2 : \mathbb{R} \longrightarrow \mathbb{R} \text{ and } 0 \le \phi_2(t) \le 1$$
$$\phi_2(t) = 1 \text{ for } t \le \frac{3r_0}{2}$$
$$\phi_2(t) = 0 \text{ for } t \ge \frac{7r_0}{4}$$

and $\max|D^k\phi_2| \le c_3$, $|D^k\phi_2|_{(\alpha)} \le c_3$ for k=1,2,3,4 and $0<\alpha<1$, for some constant c_3 independent of ε.

Let F_4 be the embedding of $[r_0, 2r_0]\times[-\varepsilon,\varepsilon]$ defined by:

$$\begin{aligned} F_4 : [r_0, 2r_0]\times[-\varepsilon,\varepsilon] &\longrightarrow \mathbb{R}^3 \\ (t,s) &\longrightarrow \gamma + xn + yb \end{aligned}$$

with

$$\begin{aligned} x(t,s) &= R\left(\cos\beta - \cos\left(\beta + \frac{s}{R}\right)\right) + \frac{s^3}{6}\left(f_1(t) + \frac{1}{R^2}\sin\beta\right) \\ y(t,s) &= R\left(\sin\beta - \sin\left(\beta + \frac{s}{R}\right)\right) + \frac{s^3}{6}\left(f_2(t) - \frac{1}{R^2}\cos\beta\right) \end{aligned}$$

where

$$\begin{aligned} \beta &= \phi_2\alpha \\ R &= \frac{k\cos\beta}{1+(\beta')^2} \\ f_1(t) &= \frac{3k}{R}\sin\beta\cos\beta - \frac{\beta''}{kR} - \frac{2\beta' R_t}{kR^2} - 4\sin\beta \\ f_2(t) &= \frac{1}{R^2\cos\beta} + \frac{3k}{R}\sin^2\beta - \frac{\beta''\sin\beta}{kR\cos\beta} - \frac{2\beta' R_t\sin\beta}{kR^2\cos\beta} - \frac{4\sin^2\beta}{\cos\beta} \end{aligned}$$

Since α and ϕ_2 are C^∞, then β is C^∞ and F_4 is a $C^{4,\beta}$ embedding of $[r_0, 2r_0]\times[-\varepsilon,\varepsilon]$ for $\varepsilon>0$ small enough. Moreover it's clear that x, y verify a)-j) of lemma 3.1 with $N(t) = \cos\beta n + \sin\beta b$.

We define the $C^{4,\beta}$ extension of F_ε near q_1 by:

$$\Psi_\varepsilon = \begin{cases} F_1 \text{ if } (t,s) \in ([2r_0, t_1]\cup[t_2,t_3]\cup[t_4, l_0 - 2r_0])\times[-\varepsilon,\varepsilon] \\ F_2 \text{ if } (t,s) \in ([t_1,t_2]\cup[t_3,t_4])\times[-\varepsilon,\varepsilon] \\ F_3 \text{ if } (t,s) \in [0,r_0]\times[-\varepsilon,\varepsilon] \\ F_4 \text{ if } (t,s) \in [r_0,2r_0]\times[-\varepsilon,\varepsilon] \end{cases}$$

Similarly, extending F_ε near q_2, we get a $C^{4,\beta}$ embedding

$$\Psi_\varepsilon : [0, l_0]\times[-\varepsilon,\varepsilon] \longrightarrow \mathbb{R}$$

with $K(t,0)=1$ and $K_s(t,0)=0$, with the $C^{4,\beta}$ bound

$$|D\Psi_\varepsilon| \ge C_0^{-1} \text{ and } |\Psi_\varepsilon|_{C^{4,\beta}} \le C_0. \tag{3.9}$$

Now to construct a $C^{4,\beta}$ approximate solution M^{ε} we need to move the boundary of the strip towards γ in a sufficiently nice way:

Let f be a function on $[0,\infty[$ whose graph is smooth such that:

$$\begin{cases} f(0) = 1 \\ f(t) = \dfrac{1}{2} \text{ for } t > \dfrac{1}{4} \\ f'(t) < 0 \text{ for } 0 < s < \dfrac{1}{4} \end{cases}$$

and the graph of f is smoothly tangent to the y-axis.

Let

$$f_{\varepsilon}(t) = \begin{cases} \varepsilon f\left(\dfrac{t}{\varepsilon}\right) & t \in \left[0, \dfrac{\varepsilon}{4}\right] \\ \dfrac{\varepsilon}{2} & t \in \left[\dfrac{\varepsilon}{4}, l_0 - \dfrac{\varepsilon}{4}\right] \\ \varepsilon f\left(\dfrac{l_0 - t}{\varepsilon}\right) & t \in \left[l_0 - \dfrac{\varepsilon}{4}, l_0\right] \end{cases} \tag{3.10}$$

Now we get a family of thin strips S^{ε} defined by

$$S^{\varepsilon} = \{(t,s) \in [0,l_0] \times [-\varepsilon,\varepsilon] / |s| \leq f_{\varepsilon}(t)\}$$

and $M^{\varepsilon} = S_1 \cup S_2 \cup S^{\varepsilon}$. Thus M^{ε} is a $C^{4,\beta}$ surface up to the boundary ∂M^{ε}, that is S_1 connected to S_2 by a strip of width ε that contains γ, and is itself contained in T. We will often identify S^{ε} with $\Psi_{\varepsilon}(S^{\varepsilon})$. M^{ε} is embedded if S_1 and S_2 are embedded.

Since Ψ_{ε} parametrize the strip, we can find a finite collection of parametrizations of M^{ε}, $((\Omega_j, \Psi_j))_{j=1\ldots J}$ with $\Psi_1 = \Psi_{\varepsilon}, \Omega_1 = S^{\varepsilon}$ and $\Psi_j : \Omega_j \longrightarrow M^{\varepsilon}$, $i = 1\ldots J$ covering $S_1 \cup S_2$.

Of course $C_0^{-1} \leq |D\Psi_1| \leq C_0$ so lengths and volumes are equivalent in Ω_1 and $\Psi_1(\Omega_1)$. If g_{ij} is the metric tensor induced by Ψ_1 and $(g^{ij}) = (g_{ij})^{-1}$ then we have

$$\begin{aligned} &\sum_{i,j} \left(|g_{ij}|_{C^{3,\beta}(S^{\varepsilon})} + |g^{ij}|_{C^{3,\beta}(S^{\varepsilon})}\right) \leq C_0 \\ &\lambda^{-1} I \leq (g_{ij}) \leq \lambda I \qquad\qquad I = \text{ identity matrix} \\ &\lambda^{-1} I \leq (g^{ij}) \leq \lambda I \end{aligned}$$

For $C_0 > 0$ and $\lambda > 0$ some constant independent of ε. M^{ε} is an approximate solution in the sense that the Gauss curvature of M^{ε} is near 1 in the norm

$C^{1,\alpha}$. $K=1$ on $S_1 \cup S_2$ and on S^ε we have by construction and (3.1):

$$K(t,s) = 1 + \int_0^s (s-h)\frac{d^2K}{ds^2}(t,hs)dh$$

Since $\left|\frac{d^2K}{ds^2}\right|_{C^0(S^\varepsilon)}$ depends on a bound of Ψ_ε in the norm C^4 we have:

$$|K-1|_{C^0(M_\varepsilon)} \le c_1\varepsilon^2 \tag{3.11}$$

Then

$$\int_{M^\varepsilon} |K-1|^p = \int_{S_\varepsilon} |K-1|^p \le c_1\varepsilon^{2p}Vol(S_\varepsilon) \le c_2\varepsilon^{2p}\varepsilon \tag{3.12}$$

Thus

$$\begin{aligned} |K-1|_{L^1(M^\varepsilon)} &\le c_3\varepsilon^3 \\ |K-1|_{L^2(M^\varepsilon)} &\le c_3\varepsilon^{2,5} \end{aligned} \tag{3.13}$$

Moreover since $K(t,0)=1$ we have in the same way:

$$|K-1|_{C^{1,\alpha}(M_\varepsilon)} \le c_4\varepsilon^{1-\alpha} \text{ and } |K-1|_{C^1(M^\varepsilon)} \le c_5\varepsilon \tag{3.14}$$

for some constants c_1, c_2, c_3, c_4, c_5 independent of ε.

4. Technical lemma.

As in [20], we prove a technical lemma to obtain a C^0 estimate. We use the same technique and obtain a $|u|_{L^p}$ terms to the right hand side since the mean curvature is only bounded.

Lemma 4.1. *Let Γ be any positive constant. Then for $u \in C^2(M^\varepsilon)$ with $u_{|\partial M^\varepsilon}$ identically zero and $|u|_{L^1} \le \Gamma$ we have*

$$|u|_{C^0} \le C_p \left\{ |\operatorname{div}(T_1\nabla u)|_{L^p} + |u|_{L^p} \right\}^{1-e_p}$$

for any $p>1$, where C_p depends only on Γ, Volume(M^ε) and p, and where $e_p > 0$ with $e_p \to 0$ as $p \to 1$.

Proof. We prove this using a Sobolev inequality and a standard iteration procedure. In the proof $c_1, c_2...$ will be constants independent of ε. By a result of Allard and Michael-Simon (see [1], [17]) we know that

$$\left(\int |v|^2\right)^{\frac{1}{2}} \le c_1 \int (|\nabla v| + |H||v|)$$

for $v \in C_0^1(M^\varepsilon)$, where H is the mean curvature, ∇v is the gradient of v and c_1 depends only on n. Applying this to $v = u^s$ with $u \in C_0^1(M^\varepsilon)$ and $s = q/(2-q)$ for $1 \leq q < 2$, one easily derives the L^q Sobolev inequality

$$(4.1)\qquad \left(\int |_n v|^{2q/(2-q)}\right)^{(2-q)/2q} \leq c_2 \left(\int |\nabla_n v|^q\right)^{1/q} + \left(\int |_n v|^q\right)^{1/q}$$

since $|H|$ is bounded on M^ε compact.

Moreover for $u \in C^2(M^\varepsilon)$ we have:

$$(4.2)\qquad -\operatorname{div}(T_1 \nabla |u|) \leq |\operatorname{div}(T_1 \nabla u)|$$

to prove (4.2), let x_1, x_2 be a system of local orthonormal coordinates. Then

$$\begin{aligned}\operatorname{div}(T_1 \nabla |u|) &= \operatorname{div}\left(T_{1ij} \frac{\partial u}{\partial x_i} \frac{u}{|u|}\right) \\ &= \frac{\partial}{\partial x_j}\left(T_{1ij} \frac{\partial u}{\partial x_i}\right) \frac{u}{|u|} + T_{1ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{1}{|u|} \\ &\quad - T_{1ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{u^2}{|u|^3} = \operatorname{div}(T_1 \nabla u) \frac{u}{|u|}\end{aligned}$$

So we have (4.2).

Now let $k \in \mathbb{R}$, $k > 2$ and integrate by parts to get:

$$\begin{aligned}\int |u|^{k-1} \operatorname{div}(T_1 \nabla |u|) &= -(k-1) \int |u|^{k-2} (\nabla |u|)(T_1 \nabla |u|) \\ &\leq -c_3(k-1) \int |u|^{k-2} |\nabla |u||^2\end{aligned}$$

since T_1 is positive definite. And so by (4.2):

$$(4.3)\qquad c_3(k-1) \int |u|^{k-2} |\nabla |u||^2 \leq \int |u|^{k-1} |\operatorname{div}(T_1 \nabla u)|$$

(Integration by parts is justified as the integral on the right is finite). Now

$$\int |\nabla |u|^{k/2}|^2 = \frac{k^2}{4} \int |u|^{k-2} |\nabla |u||^2$$

combining this with (4.3) we get

$$(4.4)\qquad \int |\nabla |u|^{k/2}|^2 \leq |c_4 k \int |u|^{k-1} |\operatorname{div}(T_1 \nabla u)||.$$

For $1 \leq q < 2$, we apply Hölder's inequality to get

$$\int |\nabla |u|^{k/2}|^q \leq c_5 \left(\int |\nabla |u|^{k/2}|^2 \right)^{q/2}.$$

Now apply (4.4) to the right hand sides and (4.1) to the left hand side of the above inequality and we get with $t = q/(2-q)$:

(4.5)

$$\begin{aligned}\left(\int |u|^{kt} \right)^{\frac{1}{t}} &\leq c_6 \left(\left(\int |\nabla |u|^{k/2}|^q \right)^{1/q} + \left(\int |u|^{kq/2} \right)^{1/q} \right)^2 \\ &\leq c_7 \left(\left(\int |\nabla |u|^{k/2}|^2 \right)^{1/2} + \left(\int |u|^k \right)^{1/2} \right)^2 \\ &\leq c_8 \left(\int |\nabla |u|^{k/2}|^2 + \int |u|^k \right) \\ &\leq c_9 \left(k \int |u|^{k-1} |\operatorname{div}(T_1 \nabla u)| + \int |u|^{k-1}|u| \right).\end{aligned}$$

Let $\delta > 0$ be small, and apply Hölder's inequality twice to the right hand side of (4.5):

$$\begin{aligned}\left(\int |u|^{kt} \right)^{1/t} &\leq c_9 \left(k \int |u|^{k-1} |\operatorname{div}(T_1 \nabla u)| + \int |u|^{k-1}|u| \right) \\ &\leq c_{10} \left(\int |u|^{(k-1)t^{1-\delta}} \right)^{1/t^{1-\delta}} (k |\operatorname{div}(T_1 \nabla u)|_{L^{t_\delta}} + |u|_{L^{t_\delta}}) \\ &\leq c_{11} \left(\int |u|^{kt^{1-\delta}} \right)^{(k-1)/kt^{1-\delta}} (k |\operatorname{div}(T_1 \nabla u)|_{L^{t_\delta}} + |u|_{L^{t_\delta}})\end{aligned}$$

where t_δ is the conjugate exponent $t^{1-\delta}/(t^{1-\delta} - 1)$ of $t^{1-\delta}$. Now, letting $k = t^{\delta i}$, i=1,2...(t with q close enough to 2 to have $k = t^\delta > 2$) and $I_i = |u|_{L^{t^{\delta i+1}}}$, the above inequality gives us the recursion formula with $\xi = \xi_i = t^{\delta i}$:

(4.6)
$$\begin{aligned}I_i &\leq (c_{11})^{1/\xi} I_{i-1}^{\frac{\xi-1}{\xi}} (\xi |\operatorname{div}(T_1 \nabla u|_{L^{t_\delta}} + |u|_{L^{t_\delta}})^{\frac{1}{\xi}} \\ &\leq (c_{11}\xi)^{\frac{1}{\xi}} I_{i-1}^{\frac{\xi-1}{\xi}} (|\operatorname{div}(T_1 \nabla u|_{L^{t_\delta}} + |u|_{L^{t_\delta}}))^{\frac{1}{\xi}}.\end{aligned}$$

Since $\xi > 1$, iterating (4.6) gives us:

(4.7)
$$I_i \leq c_{12} t_i (|\operatorname{div}(T_1 \nabla u|_{L^{t_\delta}} + |u|_{L^{t_\delta}}))^{\theta_i} I_0^{s_{i1}}$$

where,

$$\begin{aligned} s_{ij} &= \frac{(t^{\delta i}-1)(t^{\delta(i-1)}-1)...(t^{\delta j}-1)}{t^{\delta i}t^{\delta(i-1)}...t^{\delta(j-1)}} \quad (1 \le j \le i), \\ &= \frac{1}{t^{\delta i}} \quad (j = i+1), \\ t_i &= \prod_{j=2}^{i+1} (t^{\delta(j-1)})^{s_{ij}}, \\ \theta_i &= \sum_{j=2}^{i+1} s_{ij}. \end{aligned}$$

The idea is to let $i \longrightarrow \infty$ in (4.7) and use the fact that $|u|_{C^0} = limsupI_i$ as $i \longrightarrow \infty$. It is easy that $logt_i \le logt^{\delta} \sum_{j=1}^{\infty} jt^{-\delta j} < \infty$ and so $c_{10}t_i \le c_{11}$ (which depends on δ). Also, $0 < s_{i1} < 1$ and thus $I_0^{s_{i1}} \le c_{12} \le \max(1, \Gamma)$. Finally it is not hard to show that θ_i is increasing in i and satisfies $\frac{1}{t^{\delta}} \le \theta_i \le 1$. Denoting $\theta = \theta_{\delta} = lim_{i \to \infty}\theta_i$, and letting $i \longrightarrow \infty$ in (4.7) we get

$$|u|_{C^0} \le c_{13}(|\operatorname{div}(T_1 \nabla u)|_{L^{t_\delta}} + |u|_{L^{t_\delta}}))^{\theta_\delta}.$$

Letting $p = t_{\delta} = t^{1-\delta}/(t^{1-\delta} - 1)$ and recalling that $t = q/q - 2$ the above inequality implies lemma 2. ■

5. Schauder estimates.

Let M^{ε} be a $C^{3,\beta}$ surface of $\mathbb{R}^3$, with ∂M^{ε} of class $C^{3,\beta}$ for $0 < \varepsilon \le \varepsilon_1$, $0 < \beta \le 1$ and L an elliptic differential operator on $C^{2,\alpha}(M^{\varepsilon})$. We assume:

a) There exists a finite collection of $C^{3,\beta}$ parametrizations $U = \{(\Omega_j, \Psi_j)\}$ j=1,...J, $\Psi_j : \Omega_j \longrightarrow M^{\varepsilon}$.

b) For each $x \in \Omega_j$ there is a closed ball of radius ε, B_{ε} with $x \in B_{\varepsilon} \subset \Omega_j$.

c) $\sum_j |\Psi_j|_{C^k} \le C_0 \varepsilon^{1-k}$ and $|D\Psi_j|_{C^0} \ge C_0^{-1}$ for $k = 1, 2, 3$ and for $0 < \alpha \le \beta$, $\sum_j |\Psi_j|_{C^{k,\alpha}} \le C_0 \varepsilon^{1-k-\alpha}$ for some constant $C_0 > 0$ independent of $\varepsilon > 0$.

d) The metric on Ω_k satisfies

$$\lambda I \le (g_{ij}) \le \lambda^{-1} I$$
$$\lambda I \le (g^{ij}) \le \lambda^{-1} I$$

for some constant λ.

e) (Straightening out the boundary) If $x_0 \in \Omega \cap \Psi^{-1}(\partial M^\varepsilon)$, $(\Omega, \Psi) \in U$, there exists a $C^{3,\beta}$ parametrization Φ, straightening out the boundary with $\Phi(x_0) = 0$

$$\Phi : B_{\varepsilon/4}(x_0) \cap \Omega \longrightarrow \mathbb{R}^2_+ = \{x \in \mathbb{R}^2 / x_2 \geq 0\}$$
$$\Phi(B_{\varepsilon/4}(x_0) \cap \Psi^{-1}(\partial M^\varepsilon)) \subset \mathbb{R} \times \{0\}$$
$$|\Phi|_{C^k} \leq C_0 \varepsilon^{1-k}$$
$$|\Phi|_{C^{k,\alpha}} \leq C_0 \varepsilon^{1-k-\alpha} \qquad \text{for } k = 1,2,3 \text{ and for } 0 < \alpha \leq \beta$$
$$|D\Phi|_{C^0} \geq C_0^{-1}$$

f) If $u \in C^{2,\alpha}(M^\varepsilon)$ and $x = (x_1, x_2)$ are coordinates for $(\Omega, \Psi) \in U$ then

$$Lu = a_{ij} u_{x_i x_j} + b_i u_{x_i} + cu$$

in Ω, where the coefficients a_{ij}, b_i, c are $C^{0,\alpha}$ and satisfy

$$\lambda I \leq (a_{ij}) \leq \lambda^{-1} I$$
$$\sum_{ij} (|a_{ij}|_{C^0} + \varepsilon^\alpha |a_{ij}|_{C^{0,\alpha}}) \leq C_0$$
$$\sum_i (\varepsilon |b_i|_{C^0} + \varepsilon^{1+\alpha} |b_i|_{C^{0,\alpha}}) \leq C_0$$
$$(\varepsilon^2 |c|_{C^0} + \varepsilon^{2+\alpha} |c|_{C^{0,\alpha}}) \leq C_0,$$

the constant λ and C_0 don't depend on ε.

Explosions as in c) and f) require that in the case of H-surfaces we attach the strip S^ε, in a neighborhood of $(q_i)_{i=1,2}$ with a cut off function satisfying $|D^k \phi| \leq c\varepsilon^{-k}$ and then Ψ_j verify these assumptions.

Clearly M^ε constructed in section 3 satisfies a), b), c), d) since the $C^{3,\beta}$ bounds of Ψ_j(j=1...J) is independent of ε (see 3.9) and the width of the strip is ε.

Condition e) is obviously true when x is a boundary point of S_1 or S_2 as $(S_i)_{i=1,2}$ don't depend on ε. If $x_0 \in \partial\Omega_1 = \partial S^\varepsilon$, then e) follows from (3.10) and the definition of S^ε, since ∂S^ε is locally a graph over $[0, l_0]$ of a function of the type $x \longrightarrow \varepsilon F(x/\varepsilon)$ where F is $C^{3,\beta}$ and $C_0^{-1} \leq |DF| \leq C_0$.

$Lu = \operatorname{div} T_1 \nabla u + 2KHu$ is elliptic since M^ε is strictly locally convex. Since the coefficients of T_1 depend on the coefficients of the second fundamental form (see section 1), their bounds in $C^{0,\alpha}$ depend on bounds of $|\Psi|_{C^{3,\alpha}}$ and we have f).

Finally, under the above assumptions N.Smale proved.

Lemma 5.1. *Let $\sigma > 0$. Then for $u \in C^{2,\alpha}(M^\varepsilon)$ we have the following estimates, for $0 < \alpha \leq \beta$*

$$\begin{aligned}|u|_{C^2} &\leq \sigma\varepsilon^\alpha |u|_{C^{2,\alpha}} + C_\sigma \varepsilon^{-2}|u|_{C^0}\\ |u|_{C^{1,\alpha}} &\leq \sigma\varepsilon |u|_{C^{2,\alpha}} + C_\sigma \varepsilon^{-1-\alpha}|u|_{C^0}\\ |u|_{C^1} &\leq \sigma\varepsilon^{1+\alpha} |u|_{C^{2,\alpha}} + C_\sigma \varepsilon^{-1}|u|_{C^0}\end{aligned}$$

where C_σ is a constant depending on σ and C_0.

Lemma 5.2. *If $u \in C_0^{2,\alpha}(M^\varepsilon)$ and $Lu=F$, $0 < \alpha \leq \beta$ then*

$$|u|_{C^{2,\alpha}} \leq C_\alpha \{\varepsilon^{-\alpha}|F|_{C^0} + |F|_{(\alpha)} + \varepsilon^{-2-\alpha}|u|_{C^0}\}$$

for all $\varepsilon < \varepsilon(\alpha)$, where C_α depends on α, C_0, and λ.

Even if the construction of M^ε satisfies better bounds than assumptions a), c), d), e), f) we can't have an explosion smaller than in Lemma 5.2. This explosion comes from b) and a necessary cut off function on B_ε which verify $|D^k\phi| \leq c\varepsilon^{-k}$ to prove the local Schauder estimate (see [20]).

Since we want to apply, the Schauder fixed point theorem in the Banach space $C_0^{3,\alpha}(M^\varepsilon)$, we establish two Lemmas of the same character.

A direct application of local interpolation inequalities (see [8]) (As in lemma 5.1) gives us:

Lemma 5.3. *Let $\sigma > 0$. Then for $u \in C^{3,\alpha}(M^\varepsilon)$ we have the following estimates, for $0 < \alpha \leq \beta$*

$$\begin{aligned}|u|_{C^3} &\leq \sigma\varepsilon^\alpha |u|_{C^{3,\alpha}} + C_\sigma \varepsilon^{-3}|u|_{C^0}\\ |u|_{C^{2,\alpha}} &\leq \sigma\varepsilon |u|_{C^{3,\alpha}} + C_\sigma \varepsilon^{-2-\alpha}|u|_{C^0}\\ |u|_{C^2} &\leq \sigma\varepsilon^{1+\alpha} |u|_{C^{3,\alpha}} + C_\sigma \varepsilon^{-2}|u|_{C^0}\\ |u|_{C^{1,\alpha}} &\leq \sigma\varepsilon^2 |u|_{C^{3,\alpha}} + C_\sigma \varepsilon^{-1-\alpha}|u|_{C^0}\\ |u|_{C^1} &\leq \sigma\varepsilon^{2+\alpha} |u|_{C^{3,\alpha}} + C_\sigma \varepsilon^{-1}|u|_{C^0}\\ |u|_{C^{0,\alpha}} &\leq \sigma\varepsilon^3 |u|_{C^{3,\alpha}} + C_\sigma \varepsilon^{-\alpha}|u|_{C^0}\end{aligned}$$

where C_σ is a constant depending on σ and C_0.

In our case $Lu = \operatorname{div} T_1 \nabla u + 2KHu$, we have $a_{ij} = T_{1ij}, b_i = \partial_j T_{1ij}, c = 2KH$ and M^ε of class $C^{4,\beta}$ implies:

(f') a_{ij}, b_i, c are $C^{1,\alpha}$ and satisfy:

$$\sum_{ij}(|a_{ij}|_{C^1} + |a_{ij}|_{C^{1,\alpha}}) \leq C_0$$

$$\sum_{i}(|b_i|_{C^1} + |b_i|_{C^{1,\alpha}}) \leq C_0$$

$$(|c|_{C^1} + |c|_{C^{1,\alpha}}) \leq C_0$$

Which gives us:

Lemma 5.4. *If $u \in C_0^{3,\alpha}(M^\varepsilon)$ and Lu=F, $0 < \alpha \leq \beta$ then*

$$|u|_{C^{3,\alpha}} \leq C_\alpha\{\varepsilon^{-\alpha}|F|_{C^1} + |\nabla F|_{(\alpha)} + \varepsilon^{-3-\alpha}|u|_{C^0}\}$$

for all $\varepsilon < \varepsilon(\alpha)$, where C_α depends on α, C_0, and λ

Proof. We have by differentiating the equation $Lu = F$

$$L(\nabla u) = \nabla F - (\nabla a_{ij})u_{x_i x_j} - (\nabla b_i)u_{x_i} - (\nabla c)u$$

By applying lemma 5.2, we have:

$$\begin{aligned}|\nabla u|_{C^{2,\alpha}} \leq C_\alpha\{&\varepsilon^{-\alpha}(|F|_{C^1} + |a_{ij}|_{C^1}|u|_{C^2} + |b_i|_{C^1}|u|_{C^1} + |c|_{C^1}|u|_{C^0}) \\ &+ |\nabla F|_{(\alpha)} + |\nabla a_{ij}|_{(\alpha)}|u|_{C^2} + |\nabla b_i|_{(\alpha)}|u|_{C^1} + |\nabla c|_{(\alpha)}|u|_{C^0} \\ &+ |a_{ij}|_{C^1}|D^2u|_{(\alpha)} + |b_i|_{C^1}|\nabla u|_{(\alpha)} + |c|_{C^1}|u|_{(\alpha)} \\ &+ \varepsilon^{-2-\alpha}|\nabla u|_{C^0}\}\end{aligned}$$

and by (f')):

$$\begin{aligned}|\nabla u|_{C^{2,\alpha}} \leq\ & C_\alpha\{\varepsilon^{-\alpha}|F|_{C^1} + |\nabla F|_{(\alpha)}\} \\ &+ C_0C_\alpha(\varepsilon^{-\alpha}|u|_{C^2} + |D^2u|_{(\alpha)}) \\ &+ C_0C_\alpha(\varepsilon^{-\alpha}|u|_{C^1} + |\nabla u|_{(\alpha)}) \\ &+ C_0C_\alpha(\varepsilon^{-\alpha}|u|_{C^0} + |u|_{(\alpha)}) \\ &+ C_\alpha\varepsilon^{-2-\alpha}|u|_{C^1}\end{aligned}$$

and by lemma 5.3 we have:

$$\begin{aligned}\varepsilon^{-\alpha}|u|_{C^2} + |D^2u|_{(\alpha)} &\leq \sigma\varepsilon|u|_{C^{3,\alpha}} + C_\sigma\varepsilon^{-2-\alpha}|u|_{C^0} \\ \varepsilon^{-\alpha}|u|_{C^1} + |\nabla u|_{(\alpha)} &\leq \sigma\varepsilon^2|u|_{C^{3,\alpha}} + C_\sigma\varepsilon^{-1-\alpha}|u|_{C^0} \\ \varepsilon^{-\alpha}|u|_{C^0} + |u|_{(\alpha)} &\leq \sigma\varepsilon^3|u|_{C^{3,\alpha}} + C_\sigma\varepsilon^{-\alpha}|u|_{C^0} \\ \varepsilon^{-2-\alpha}|u|_{C^1} &\leq \sigma|u|_{C^{3,\alpha}} + C_\sigma\varepsilon^{-3-\alpha}|u|_{C^0}\end{aligned}$$

Then with $\sigma = 1/2C_0C_\alpha$, by substracting the $|u|_{C^{3,\alpha}}$ terms on the right, the lemma is proved. ■

6. Eigenvalue bounds and L^2 estimates.

Lemma 6.1 is proved as in [20] and we redo here for a complete proof of the theorem.

Lemma 6.1. *There exists positive constants* ε_1 *and* d_0 *such that for all* $\varepsilon < \varepsilon_1$

$$\mathrm{dist}(\mathrm{spec}(L), 0) \geq d_0$$

ε_1 *and* d_0 *depend only on* $\max_{M^\varepsilon} |2KH|$ *(i.e. on* C_0*) and* $\min_{i=1,2}(d_i), d_i = \mathrm{dist}(\mathrm{spec}(L^{S_i}), 0)$.

Proof. If lemma 6.1 were false, then there would exist a sequence of eigenvalues $\lambda_k = \lambda_k(e_k)$ $k = 1, 2...$ such that $\lambda_k \to 0$ as $k \to \infty$ (and $\varepsilon_k \to 0$), and a sequence of eigenfunctions u_k such that

$$Lu_k = -\lambda_k u_k \text{ and } \int |u_k|^2 = 1.$$

$c_1, c_2...$ will be constants depending only on C_0, d_1 and d_2. First we show that $|u_k|_{C^0}$ is bounded. This follows easily from lemma 4.1:

$$|u_k|_{C^0} \leq C_p \left\{ | - \lambda_k u_k - 2KHu_k|_{L^p} + |u_k|_{L^p} \right\}^{1-e_p} \text{ for } p > 1$$

(Note that lemma 4.1 applies here as $|u_k|_{L^1} \leq c_1$). Now

$$| - \lambda_k u_k - 2KHu_k|_{L^p} \leq (|\lambda_k| + |2KH|_{C^0})|u_k|_{L^p},$$

so $|u_k|_{C^0} \leq c_2 |u_k|_{L^p}^{1-e_p}$ and $|u_k|_{L^2} = 1$ imply $|u_k|_{C^0} \leq c_2$.

Also integrating the eigenvalues equation by parts, we have:

$$\int u_k \operatorname{div} T_1 \nabla u_k + \int 2KHu_k^2 = -\lambda_k \int |u_k|^2$$

$$c_0 \int |\nabla u_k|^2 \leq (|\lambda_k| + |2KH|_{C^0}) \int |u_k|^2$$

$$|\nabla u_k|_{L^2}^2 \leq c_3 |u_k|_{L^2}^2 \leq c_3.$$

Let $(B_{5C_0\varepsilon}(q_i))_{i=1,2}$ be small geodesic ball about q_i of radius $5C_0\varepsilon$. Since $C_0^{-1} \leq |D\Psi_j| \leq C_0$, $B_{2C_0\varepsilon}(q_i)$ contains deformations of the boundary where the strip is attached (see section 3). Let $\bar{S^\varepsilon} = \Psi_1(S^\varepsilon) \cup B_{5C_0\varepsilon}(q_1) \cup B_{5C_0\varepsilon}(q_2)$

Since $Vol(\bar{S}^{\varepsilon}) \leq c_4\varepsilon$ and $\int |u_k|^2 = 1$, we can assume that

$$1/4 \leq \int_{S_1/B_{5C_0\varepsilon}} |u_k|^2$$

(by chosing a subsequence if necessary). Let Φ be a cut off function,defined on S_1 by

$$\Phi(x) = \begin{cases} 0 & r < \xi \\ \dfrac{log(r/\xi)}{log(1/\xi^{1/2})} & \xi \leq r \leq \xi^{1/2} \\ 1 & \xi^{1/2} < r \end{cases}$$

where $\xi^{1/2} = 3C_0\varepsilon$ and $r(x) =$ geodesic distance from q_1 to x. So Φ is lipschitz and thus H^1. By integrating in geodesic polar coordinates

$$\int_{B_{\xi^{1/2}}} |\nabla\Phi|^2\sqrt{g}dx \leq \frac{c}{log(\xi^{-1/2})^2}\int_{r=\xi}^{\xi^{1/2}}\frac{dr}{r} \leq \frac{c}{log(\xi^{-1/2})}$$

$$|\nabla\Phi|_{L^2} \leq c(log\xi^{-1/2})^{-1/2} \to 0 \text{ as } \xi, \varepsilon \to 0.$$

(Here $\sqrt{g}dx$ is the volume measure in linear coordinates on the geodesic ball B_ρ; $\sqrt{g}dx = \sqrt{g}rdrd\theta$ and we used the fact that $\sqrt{g}$ is bounded). Now let $\Phi_j \in C^2(S_1)$ j=1,2... so that $\Phi_j \to \Phi$ in C^0 and H^1 norms as $j \to \infty$ and $\Phi_j \equiv 0$ on $B_{2C_0\varepsilon}(q_1)$. Also let λ_k and u_k be as above. Then $v_{k,j} = \Phi_j u_k \in H_0^1(S_1)$ and

$$|v_{k,j}|_{C^0} \leq c_4$$
$$|v_{k,j}|_{L^2} \geq 1/4.$$

We have the orthogonal decomposition under L

$$L^2(S_1) = H_+ \oplus H_-$$

where $H_+(H_-)$ is the direct sum of the eigenspaces of L^{S_1} with positive (negative) eigenvalues. For $w \in L^2(S_1)$, let w^+, w^-, be the orthogonal projections onto H_+, H_- respectively. Now

$$\int |v_{k,j}|^2 = \int |v_{k,j}^+|^2 + \int |v_{k,j}^-|^2 \leq c_1$$

and so $|v_{k,j}^-|_{C^0} \leq c_5$ since H_- is finite dimensional. Thus $|v_{k,j}^+|_{C^0} \leq c_6$ since $|v_{k,j}|_{C^0} \leq c_4$ and $v_{k,j} = v_{k,j}^+ + v_{k,j}^-$. Letting $c_7^{-1} = \min_{i=1,2}(d_i)$ we have by standard Hilbert theory:

$$1/4 \leq \int |v_{k,j}^+|^2 + \int |v_{k,j}^-|^2 \leq c_7\left(\left|\int v_{k,j}^+ Lv_{k,j}\right| + \left|\int v_{k,j}^- Lv_{k,j}\right|\right)$$

We show that the terms on the right goes to zero, getting a contradiction when $j \to \infty$ and $k \to \infty$. Now

$$\left|\int v_{k,j}^{-} L v_{k,j}\right| \leq \left|\int v_{k,j}^{-}(-\lambda_k v_{k,j} + 2T_1 \nabla u_k \nabla \Phi_j + u_k \operatorname{div} T_1 \nabla \Phi_j)\right|$$

$$\leq c_8 \left(\lambda_k + |\nabla u_k|_{L^2} |\nabla \Phi_j|_{L^2} |v_{k,j}^{-}|_{C^0} + \left|\int v_{k,j}^{-} u_k \operatorname{div} T_1 \nabla \Phi_j\right|\right)$$

$$\leq c_9 \left(\lambda_k + |\nabla u_k|_{L^2} |\nabla \Phi_j|_{L^2} + \left|\int v_{k,j}^{-} u_k \operatorname{div} T_1 \nabla \Phi_j\right|\right)$$

If we let $j \to \infty$ and $k \to \infty$, the first two terms tend to zero since $|\nabla u_k|_{L^2}$ is bounded and $|\nabla \Phi_j|_{L^2} \to |\nabla \Phi|_{L^2}$ as $j \to \infty$ and $|\nabla \Phi|_{L^2} \to 0$ as $k \to \infty$ $(\varepsilon \to 0)$.

So we must show that the third term on right tends to zero.

Integrating by parts

$$\left|\int v_{k,j}^{-} u_k \operatorname{div} T_1 \nabla \Phi_j\right| \leq \left|\int \nabla v_{k,j}^{-} u_k T_1 \nabla \Phi_j\right| + \left|\int v_{k,j}^{-} \nabla u_k T_1 \nabla \Phi_j\right|$$

$$\leq c_{10} \left(|\nabla v_{k,j}^{-}|_{L^2} |\nabla \Phi_j|_{L^2} + |\nabla u_k|_{L^2} |\nabla \Phi_j|_{L^2}\right)$$

as remarked above, the second term on the right tends to zero. Since $|\nabla \Phi_j|_{L^2} \to 0$, it suffices to show that $|\nabla v_{k,j}^{-}|_{L^2}$ is bounded. However, this follows from the fact that H_- is finite dimensional and that $|\nabla v_{k,j}^{-}|_{L^2}$ is bounded. For example $v_{k,j}^{-} = a_1 \Lambda_1 + ... + a_I \Lambda_I$, I = index of S_1, $\{\Lambda_i\}$ $i = 1, ... I$ a smooth orthonormal basis for H_- and $(\sum a_i^2)^{1/2} \leq c$. Then $\nabla v_{k,j}^{-} = a_1 \nabla \Lambda_1 ... a_I \nabla \Lambda_I$ is clearly bounded in L^2.

Thus we've shown that $\int v_{k,j}^{-} L v_{k,j}$ tends to zero. The proof that $\int v_{k,j}^{+} L v_{k,j}$ tends to zero is identical, except that $|\nabla v_{k,j}^{+}|_{L^2}$ is bounded since $|\nabla v_{k,j}^{-}|_{L^2}$ and $|\nabla v_{k,j}|_{L^2}$ are bounded (independent of ε). This completes the proof of lemma 6.1. ■

For the rest of the paper H_+, H_- will denote the direct sum of the eigenspaces of L in $L^2(M^\varepsilon)$ corresponding to the positive(negative) eigenvalues. So $L^2(M^\varepsilon) = H_+ \oplus H_-$ is an orthogonal sum, invariant under L. As above, if $u \in L^2(M^\varepsilon), u = u^+ + u^-$ for unique $u^+ \in H_+, u^- \in H_-$. We have the following:

Corollary 6.1. *For* $u \in C_0^{3,\alpha}(M^\varepsilon)$ *we have*

(a) $$|u|_{L^2}^2 \leq C_1(|u^+|_{C^0} + |u^-|_{C^0})|Lu|_{L^1}$$

(b) $$|u|_{H^1} \leq C_2 |Lu|_{L^2}$$

where C_1, C_2 *depends only on* d_0 *and* c_0 *the constant of ellipticity of* Lu.

Proof. By standard Hilbert space theory

$$\int |u|^2 = \int |u^+|^2 + \int |u^-|^2 \leq \frac{1}{d_0}\left(\left|\int u^+Lu\right| + \left|\int u^-Lu\right|\right)$$

from which a) follows. Thus

$$|u|_{L^2}^2 \leq \frac{1}{d_0}\left(|u^+Lu|_{L^1} + |u^-Lu|_{L^1}\right) \leq C_1|u|_{L^2}|Lu|_{L^2}$$

and so $|u|_{L^2} \leq C_1|Lu|_{L^2}$.

Using a) and integrating

$$u \operatorname{div} T_1\nabla u + u2KHu = uLu$$

by parts, we get:

$$|\nabla u|_{L^2}^2 \leq c_1(|u|_{L^2}^2 + |u|_{L^2}|Lu|_{L^2}) \leq c_2|Lu|_{L^2}^2$$

Which gives us b). ■

7. Proof of the theorem 2.1.

Let $\gamma_1, \gamma_2, \gamma_3$ be the following constants: $\gamma_1 = 2,75, \gamma_2 = -0,25, \gamma_3 = 1,75$. Let $u \in P(\varepsilon, \sigma, \alpha)$, $0 < \varepsilon \leq \varepsilon_1, 0 < \sigma \leq 1/2, 0 < \alpha \leq \beta$, and recall that $Tu = v + w$, where $Lv = -(K-1)$ and $Lw = -E(u)$. To prove theorem 2.1, we claim that it suffices to establish the following estimates:

(V1) $$|v|_{C^0} \leq K_1\varepsilon^{\gamma_1-\theta}$$

(V2) $$|v|_{C^{3,\alpha}} \leq K_2\varepsilon^{\gamma_2-\alpha-\theta}$$

(V3) $$|v|_{H^2} \leq K_3\varepsilon^{\gamma_3-\theta}$$

(W1) $$|w|_{C^0} \leq J_1\varepsilon^{\beta_1-\rho_1}$$

(W2) $$|w|_{C^{3,\alpha}} \leq J_2\varepsilon^{\beta_2-\rho_2}$$

(W3) $$|w|_{H^2} \leq J_3\varepsilon^{\beta_3-\rho_3}$$

for all $\varepsilon < \varepsilon(\alpha, C_0, d_0, S_1, S_2)$ where K_1, K_2, K_3 are constants that may depend on $\theta, \alpha, C_0, d_0, S_1, S_2$ and where θ can be taken arbitrarily small;

J_1, J_2, J_3 may depend on $\sigma, \alpha, C_0, d_0, S_1, S_2$; $\beta_1, \beta_2, \beta_3$ are constants such that

$$\beta_1 > 2,75, \quad \beta_2 > -0.25 \quad \beta_3 > 1,75$$

and finally ρ_1, ρ_2, ρ_3 are functions of σ and α such that $\rho_i \to 0$ as $\sigma \to 0$ and $\alpha \to 0$.

(V1)-(V3) and (W1)-(W3) implies theorem 2.1 for the following reasons: Fix $\sigma = \bar{\sigma}$ and $\alpha = \bar{\alpha}$ so small that $\beta_i - \rho_i > \gamma_i$ i=1,2,3 and then fix $\theta = \bar{\sigma}/2$. Now of course K_i and J_i are fixed constants. Letting $\bar{K} = Max\{K_1, K_2, K_3, J_1, J_2, J_3\}$, we can take $\bar{\varepsilon}$ so small that:

$$K_i \varepsilon^{\gamma_i - \bar{\sigma}/2} + J_i \varepsilon^{\beta_i - \rho_i} \leq 2\bar{K}\varepsilon^{\gamma_i - \bar{\sigma}/2} \leq \varepsilon^{\gamma_i - \bar{\sigma}} \qquad i \neq 2$$
$$K_2 \varepsilon^{\gamma_2 - \bar{\sigma}/2 - \bar{\alpha}} + J_2 \varepsilon^{\beta_2 - \rho_2} \leq 2\bar{K}\varepsilon^{\gamma_2 - \bar{\sigma}/2 - \bar{\alpha}} \leq \varepsilon^{\gamma_2 - \bar{\sigma} - \bar{\alpha}}$$

for all $\varepsilon < \bar{\varepsilon}$. That is, the right hand inequalities are true for $\varepsilon < \bar{\varepsilon} = (2\bar{K})^{-2/\bar{\sigma}}$. Clearly then, by the definition of $P(\varepsilon, \sigma, \alpha)$, $v + w \in P(\varepsilon, \bar{\sigma}, \bar{\alpha})$, for all $\varepsilon < \bar{\varepsilon}$,since the left hand sides of the above inequalities are just upper bounds for $v + w$ in the norms defining $P(\varepsilon, \bar{\sigma}, \bar{\alpha})$. So it suffices to establish (V1)-(V3) and (W1)-(W3).

Estimations of (V1)-(V3).

Again $c_1, c_2, \ldots$ will be constants independent of ε. We first establish a L^2 estimate of v:

From corollary 6.1-b and (3.13)

$$|v|_{H^1} \leq c_1 |K - 1|_{L^2} \leq c_2 \varepsilon^{2,5} \tag{7.1}$$

and so by lemma 4.1, applying to v^+ and v^-, gives us

$$|v^{\pm}|_{C^0} \leq C_p \{|\operatorname{div}(T_1 \nabla v^{\pm})|_{L^p} + |v^{\pm}|_{L^p}\}^{1-e_p} \text{ for } p > 1$$

but for $1 < p \leq 2$

$$|\operatorname{div}(T_1 \nabla v^{\pm})|_{L^p} + |v^{\pm}|_{L^p} \leq c_3(|(K-1)^{\pm}|_{L^p} + |v^{\pm}|_{L^p})$$
$$\leq c_4(|(K-1)^{\pm}|_{L^2} + |v^{\pm}|_{L^2}) \leq c_5 \varepsilon^{2,5}$$

since M^{ε} is compact and bounds on 2KH depend only on C^2 bounds of parametrizations ($|v|_{L^2} \leq c_2 \varepsilon^{2,5}$ due to (7.1)). Thus $|v^{\pm}|_{C^0} \leq c_6 \varepsilon^{2,5(1-e_p)}$ and $c_6 = c_6(p)$ (recall $e_p \to 0$ as $p \to 1$).

Combining this, corollary 6.1-a gives us

$$|v|_{L^2}^2 \leq c_7 \varepsilon^{2,5(1-e_p)} |K - 1|_{L^1} \leq c_8 \varepsilon^{5,5-2,5e_p}$$

and

$$(7.2) \qquad |v|_{L^2} \leq c_9 \varepsilon^{2,75-\theta} \text{ with } \theta \text{ close to } 0 \text{ as } p \to 1.$$

Now we prove (V.1) using lemma 4.1 and (7.2).

$$|v|_{C^0} \leq C_p \{ |\operatorname{div}(T_1 \nabla v)|_{L^p} + |v|_{L^p} \}^{1-e_p} \leq c_{10} \{ |K-1|_{L^p} + |v|_{L^2} \}^{1-e_p}$$

For p close to 1, $|K-1|_{L^p} \leq c_{11} \varepsilon^x$ with x close to 3 (see 3.12).

Then when $p \to 1$, $|K-1|_{L^p} \leq c_{11} \varepsilon^{2,75-\theta}$ and by (7.2):

$$|v|_{C^0} \leq c_{12} \varepsilon^{(2,75-\theta)(1-e_p)}$$

which gives (V1) for p close to 1:

$$(V1) \qquad |v|_{C^0} \leq K_1 \varepsilon^{2,75-\theta}$$

(V2) follows from lemma 5.4, (V1) and (3.14):

$$|v|_{C^{3,\alpha}} \leq C_\alpha \{ \varepsilon^{-\alpha} |K-1|_{C^1} + |\nabla(K-1)|_{(\alpha)} + \varepsilon^{-3-\alpha} |v|_{C^0} \}$$

then

$$(V2) \qquad |v|_{C^{3,\alpha}} \leq K_2 \varepsilon^{-0,25-\alpha-\theta}$$

Now lemma 5.3, combined with (V1) and (V2) imply that

$$|\nabla v|_{C^0} \leq c_{13} \varepsilon^{1,75-\mu_1}$$
$$|D^2 v|_{C^0} \leq c_{14} \varepsilon^{0,75-\mu_1}$$

where, $c_{13} = c_{13}(\mu_1), c_{14} = c_{14}(\mu_1)$ and $\mu_1 > 0$ can be chosen arbitrarily small.

Now we prove (V3). First of all, by standard elliptic theory (see [8], [18]), there exists a constant c_{15}, depending only on S_1 and S_2 such that

$$(7.3) \qquad |Z|_{H^2} \leq c_{15} \{ |LZ|_{L^2} + |Z|_{L^2} \}$$

for all $Z \in H^2(S_i) \cap H_0^1(S_i), i = 1, 2$. Let ϕ be a cut off function on S_1 satisfying

$$\phi(x) = 1 \text{ if } x \text{ outside } B_{5C_0\varepsilon}(q_1)$$
$$\phi(x) = 0 \text{ if } x \in B_{3C_0\varepsilon}(q_1)$$

and $|\nabla \phi|_{C^0} \leq c_{16} \varepsilon^{-1}, |D^2 \phi|_{C^0} \leq c_{17} \varepsilon^{-2}$.

Let $B_i = B_{5C_0\varepsilon(q_i)}, i = 1,2$. We will estimate $\int |D^2v|^2$ separately, on the pieces S_i/B_i, B_i i=1,2 and S^ε. Applying (7.3) to $Z = \phi v$ and using the fact that support of $(\nabla^k\phi), k = 1,2$ is contained in B_1 with $Vol(B_1) \leq c\varepsilon^2$. We have:

$$\begin{aligned}|v|_{H^2(S_1/B_1)} &\leq c_{15}\{|\phi(K-1) + 2T_1\nabla v\nabla\phi + v\operatorname{div}T_1\nabla\phi|_{L^2} + |\phi v|_{L^2}\}\\ &\leq c_{18}\{\varepsilon^{2,5} + \varepsilon^{-1}|\nabla v|_{C^0}Vol(B_1)^{1/2}\\ &\quad + |v|_{C^0}(\varepsilon^{-1}+\varepsilon^{-2})Vol(B_1)^{1/2} + \varepsilon^{2,75}\}\\ &\leq c_{19}\varepsilon^{1,75-\mu_2}, \quad c_{19} = c_{19}(\mu_2)\end{aligned}$$

One can derive the same estimate for S_2/B_2 and so

$$|v|_{H^2(M^\varepsilon/\bar{S^\varepsilon})} \leq c_{20}\varepsilon^{1,75-\mu_3}, \quad c_{20} = c_{20}(\mu_3) \tag{7.4}$$

With $\bar{S^\varepsilon} = B_1 \cup B_2 \cup \Psi_1(S^\varepsilon)$.

Now for $i = 1,2$,

$$|v|_{H^2(B_i)} \leq c_{21}|v|_{C^2}\left(Vol(B_i)\right)^{1/2} \leq c_{21}\varepsilon^{1,75-\mu_4}, \quad c_{21} = c_{21}(\mu_4) \tag{7.5}$$

So we must estimate

$$\int_{\Psi_1(S^\varepsilon)} |D^2v|^2 = \int_{S^\varepsilon} |D^2v|^2\sqrt{g}dtds$$

For (Ω_1, Ψ_1) we express L as section 5-f') and as in the proof of lemma 5.4, by differentiating $Lv = -(K-1)$ with respect to t we get on S^ε:

$$-L(v_t) = K_t + \partial_t a_{ij}v_{x_ix_j} + \partial_t b_i v_{x_i} + \partial_t(2KH)v \tag{7.6}$$

with the coordinates $x = (x_1, x_2) = (t, s)$.

Integrating

$$v_tLv_t = v_t\operatorname{div}T_1\nabla v_t + v_t 2KHv_t$$

by parts on S^ε we get:

$$\int_{S^\varepsilon}|\nabla v_t|^2 \leq c_{22}\left\{\left|\int_{S^\varepsilon} v_tLv_t\right| + \left|\int_{S^\varepsilon} v_t2KHv_t\right| + \left|\int_{\partial S^\varepsilon} v_t(T_1\nabla v_t).e\right|\right\} \tag{7.7}$$

where e is the outward unit normal to ∂S^ε. But now v is zero on the boundary which is parallel to the t-axis.

Then $v_t(t,\varepsilon/2) = v_t(t,-\varepsilon/2) = 0$ for $t \in [\varepsilon, l_0 - \varepsilon]$ (see (3.10)). Thus

$$\int_{\partial S^\varepsilon} v_t(T_1\nabla v_t).e = \int_{\partial S_1^\varepsilon} v_t(T_1\nabla v_t).e + \int_{\partial S_2^\varepsilon} v_t(T_1\nabla v_t).e$$

where

$$\partial S_1^\varepsilon = \{(t,s) \in \partial S^\varepsilon / t = 0 \text{ and } t = l_0\}$$
$$\partial S_2^\varepsilon = \{(t,s) \in \partial S^\varepsilon / 0 \leq t \leq \varepsilon \text{ and } l_0 - \varepsilon \leq t \leq l_0\}$$

Then $Vol(\partial S_1^\varepsilon) \leq c_{23}\varepsilon$ and $Vol(\partial S_2^\varepsilon) \leq c_{23}\varepsilon$ therefore

$$\left| \int_{\partial S^\varepsilon} v_t(T_1 \nabla v_t).e \right| \leq c_{24}\varepsilon |\nabla v|_{C^0} |D^2 v|_{C^0} \leq c_{25}\varepsilon^{3,5-\mu_5}, \quad c_{25} = c_{25}(\mu_5)$$

and (7.7) becomes

$$\text{(7.8)} \qquad \begin{aligned} \int_{S^\varepsilon} |\nabla v_t|^2 &\leq c_{22} \left| \int_{S^\varepsilon} v_t L v_t \right| + c_{26} |\nabla v|^2_{C^0}\varepsilon + c_{27}\varepsilon^{3,5-\mu_5} \\ &\leq c_{22} \left| \int_{S^\varepsilon} v_t L v_t \right| + c_{28}\varepsilon^{3,5-\mu_5}, \quad c_{28} = c_{28}(\mu_5). \end{aligned}$$

The first term on the right hand side of (7.8) is estimated by using (7.6) and (V1)-(V3).

$$\left| \int_{S^\varepsilon} v_t L v_t \right| \leq c_{29} \left\{ \int_{S^\varepsilon} |v_t||K_t| + |v_t||\partial_t a_{ij} v_{x_i x_j}| + |v_t||\partial_t b_i v_{x_i}| + |v_t||\partial_t c v| \right\}.$$

Then by (3.14) and by section $5 - f'$):

$$\begin{aligned} \left| \int_{S^\varepsilon} v_t L v_t \right| &\leq c_{30} \left\{ |\nabla v|_{C^0} |K-1|_{C^1}\varepsilon + |v|_{C^1}|v|_{C^2}\varepsilon + |v|^2_{C^1}\varepsilon + |v|_{C^1}|v|_{C^0}\varepsilon \right\} \\ &\leq c_{31}\varepsilon^{3,5-\mu_6}, \quad c_{31} = c_{31}(\mu_6) \end{aligned}$$

so together with (7.8)

$$\text{(7.9)} \qquad \int_{S^\varepsilon} |\nabla v_t|^2 \leq c_{32}\varepsilon^{3,5-\mu_7}, c_{32} = c_{32}(\mu_7).$$

Now, we claim that for $x \in S^\varepsilon$

$$\text{(7.10)} \qquad |D^2 v| \leq c_{33}\{|\nabla v_t| + |\nabla v| + |v| + |K-1|\}$$

This follows from the equation $Lv = -(K-1)$ and the fact that L is uniformly elliptic (independent of ε).

$Lv = -(K-1)$ is of the form $a_{ij}u_{x_i x_j}$ + lower order terms $= F$. Since $a_{22} \geq \lambda_0$, we get

$$|u_{ss}| \leq c_{34}\{|u_{tt}| + |u_{ts}| + |\text{lower order terms}| + |F|\}.$$

However

$$|u_{tt}| + |u_{ts}| \leq c_{35}|\nabla v_t|$$

and so (7.10) follows.

Combining (7.9) with (7.10) we get

$$\int_{S^\varepsilon} |D^2 v|^2 \leq c_{36}\varepsilon^{3,5-\mu_8}, c_{36} = c_{36}(\mu_8). \tag{7.11}$$

Finally putting together (7.4), (7.5) and (7.11) we get

$$|v|_{H^2} \leq K_3 \varepsilon^{1,75-\mu_9} \tag{$V3$}$$

where μ_9 can be taken arbitrarily small.

Estimation of (W1)-(W3).

We prove (W1)-(W3) in a manner similar to the proofs of (V1)-(V3) and since $Lw = -E(u)$, we will therefore need to estimate $E(u)$ in the $C^0, C^{0,\alpha}, C^1, C^{1,\alpha}$ and L^p norms.To do this we prove the following:

Lemma 7.1. *For $u \in P(\varepsilon, \sigma, \alpha)$ we have:*

a) $|E(u)| \leq K_4 \left(|u|^2 + |\nabla u|^2 + |D^2 u|^2\right)$ *(pointwise)*

b) $|\nabla E(u)| \leq K_5 \left(|u|^2 + |\nabla u|^2 + |D^2 u|^2 + |D^3 u|(|u| + |\nabla u| + |D^2 u|)\right)$ *(pointwise)*

c) $|E(u)|_{C^0} \leq K_6 \varepsilon^{1,5-2\sigma}$

d) $|E(u)|_{C^1} \leq K_7 \varepsilon^{0,5-2\sigma}$

e) $|\nabla E(u)|_\alpha \leq K_8 \varepsilon^{0,5-2\sigma-\alpha}$

for all $\varepsilon < \varepsilon_1, \varepsilon_1$ sufficiently small and $K_4, ...K_8$ are constants depending only on C_0, λ_0 and α.

Proof. First note that c), d) follows from a), b) and the definition of $P(\varepsilon, \sigma, \alpha)$. To prove a), b), e) recall that

$$E(u) = \int_0^1 (1 - \tau) \frac{d^2 K}{d\tau^2} (\tau u) d\tau$$

So it suffices to show that $|\frac{d^2K}{d\tau^2}(\tau u)|$, $|\nabla \frac{d^2K}{d\tau^2}(\tau u)|$ and $|\nabla \frac{d^2K}{d\tau^2}(\tau u)|_{(\alpha)}$ are bounded by the quantities on the right hand side of a),b) and e), respectively for $0 \leq \tau \leq 1$.

Let $(\Omega, \Psi) \in U$. Since $\psi(\tau) = \psi + \tau u N$ is a local parametrization of $M_{\tau u}$, we have by (3.7):

$$K(\tau) = \frac{b_{11}(\tau) b_{22}(\tau) - b_{12}(\tau)^2}{(g_{11}(\tau) g_{22}(\tau) - g_{12}(\tau)^2)^2}$$

where we've abbreviated $K(\tau) = K(\tau u)$, with

$$g_{ij}(\tau) = \langle \psi_{x_i}(\tau), \psi_{x_j}(\tau) \rangle$$
$$b_{ij}(\tau) = \langle \psi_{x_1}(\tau) \wedge \psi_{x_2}(\tau), \psi_{x_i x_j}(\tau) \rangle$$

where

$$\psi_{x_i}(\tau) = \psi_{x_i} + \tau u N_{x_i} + \tau u_{x_i} N.$$

Then

$$\frac{d^2 K}{d\tau^2}(\tau u) = K(\tau u)'' = (g^{-2})'' b + 2(g^{-2})' b' + (g^{-2}) b''$$
$$\nabla \frac{d^2 K}{d\tau^2}(\tau u) = \nabla K(\tau u)'' = b \nabla (g^{-2})'' + (g^{-2})'' \nabla b$$
$$+ 2b' \nabla (g^{-2})' + 2(g^{-2})' \nabla b' + b'' \nabla (g^{-2}) + (g^{-2}) \nabla b''$$

where $g = det(g_{ij}(\tau))$ and $b = det(b_{ij}(\tau))$.

And by appendix 1g)-1i) and 1s)-1u) we obtain a); by appendix 1g)-1l) and 1s)-1x) we prove b) and e). ■

Proof of (W1)-(W3). In order to prove (W1)-(W3) we will first have to estimate $E(u)$ in L^1, L^2 and $L^p, p \geq 1, p$ near of 1. ν_1, ν_2... will denote functions of σ and α that tend to zero as σ and α tend to zero. c_1, c_2... will be constants independent of ε.

By lemma 5.3, we know that

$$|\nabla u|_{C^0} \leq c_1 \varepsilon^{1,75-\sigma}$$

and

$$|D^2 u|_{C^0} \leq c_1 \varepsilon^{0,75-\sigma}.$$

First we estimate $|E(u)|_{L^1}$.

By lemma 7.1-a) we get by the definition of $P(\varepsilon, \sigma, \alpha)$

$$\begin{aligned} |E(u)|_{L^1} &\leq K_4 \int \left(|u|^2 + |\nabla u|^2 + |D^2 u|^2\right) \\ &\leq c_2 |u|^2_{H^2} \leq c_3 \varepsilon^{3,5-2\sigma} \end{aligned}$$

which implies

$$|E(u)|_{L^1} \leq c_3 \varepsilon^{3,5-\nu_1};$$

with $p = 1 + \delta$ it's easy to see that

$$|E(u)|_{L^p} \leq c_4 \varepsilon^{3,5-\nu_2}$$

with c_4 depending on δ.

$$\begin{aligned} |E(u)|_{L^2} &\leq c_5 \left(|u|^2_{L^4} + |\nabla u|^2_{L^4} + |D^2 u|^2_{L^4}\right) \\ &\leq c_5 \left(|u|_{C^0}|u|_{L^2} + |\nabla u|_{C^0}|\nabla u|_{L^2} + |D^2 u|_{C^0}|D^2 u|_{L^2}\right) \\ &\leq c_6 |u|_{C^2}|u|_{H^2} \end{aligned}$$

which implies that

$$|E(u)|_{L^2} \leq c_7 \varepsilon^{2,5-\nu_3}.$$

First we prove a L^2 estimate of w.

By corollary 6.1-b):

$$|w|_{H^1} \leq c_8 |E(u)|_{L^2} \leq c_9 \varepsilon^{2,5-\nu_3} \tag{7.12}$$

applying lemma 4.1 to $w^{\pm}$ we get, after setting $p = 1 + \delta(\delta$ close to 0):

$$\begin{aligned}|w^{\pm}|_{C^0} &\leq C_p\{|E(u)|_{L^2} + |w|_{L^2}\}^{1-e_p}\\ &\leq c_{10}(\varepsilon^{2,5-\nu_3})^{1-e_p} \leq c_{11}\varepsilon^{2,5-\nu_4}\end{aligned}$$

Now we use corollary 6.1-a):

$$\begin{aligned}|w|^2_{L^2} &\leq c_{12}\varepsilon^{2,5-\nu_4}|E(u)|_{L^1}\\ &\leq c_{13}\varepsilon^{6-\nu_5}.\end{aligned}$$

Then

$$|w|_{L^2} \leq c_{14}\varepsilon^{3-\nu_6}. \tag{7.13}$$

Now we prove (W1) using lemma 4.1 and (7.13)

$$|w|_{C^0} \leq c_{15}\{|E(u)|_{L^p} + |w|_{L^2}\}^{1-e_p}$$

which give (W1) after setting $p = 1 + \delta$, with $\beta_1 = 3$ and $\rho_1 = \nu_7$.

$$|w|_{C^0} \leq J_1\varepsilon^{3-\nu_7} \tag{$W1$}$$

(W2) follows from lemma 5.4 and lemma 7.1:

$$\begin{aligned}|w|_{C^{3,\alpha}} &\leq c_{16}\{\varepsilon^{-\alpha}|E(u)|_{C^1} + |\nabla E(u)|_{(\alpha)} + \varepsilon^{-3-\alpha}|w|_{C^0}\}\\ &\leq c_{17}\varepsilon^{-\alpha-\nu_8}\end{aligned}$$

which give (W2) with $\beta_2 = -\alpha$ and $\rho_2 = \nu_8$.

We prove (W3) like (V3).

Lemma 5.3 implies that

$$\begin{aligned}|\nabla w|_{C^0} &\leq c_{18}\varepsilon^{2-\nu_9}\\ |D^2 w|_{C^0} &\leq c_{19}\varepsilon^{1-\nu_{10}}\end{aligned}$$

As in (V3) it follows by (7.3):

$$\begin{aligned}|w|_{H^2(S_1/B_1)} &\leq c_{20}\{|\phi E(u) + 2T_1\nabla w\nabla\phi + w\,\mathrm{div}\,T_1\nabla\phi|_{L^2} + |\phi w|_{L^2}\}\\ &\leq c_{21}\varepsilon^{2-\nu_{11}}\end{aligned} \tag{7.14}$$

and

$$\begin{aligned}|w|_{H^2(B_i)} &\leq c_{22}|D^2 w|_{C^0}\varepsilon\\ &\leq c_{23}\varepsilon^{2-\nu_{12}}.\end{aligned} \tag{7.15}$$

Since

$$|w|_{H^2(S^\varepsilon)} \leq c_{24}|w|_{C^2}\varepsilon^{1/2} \leq c_{25}\varepsilon^{1,5-\nu_{13}},$$

we must estimate $\int_{\psi_1(S^\varepsilon)} |D^2w|^2$ and we proceed as in (V3).

By differentiating $Lw = -E(u)$ with respect to t we get on S^ε:

$$-Lw_t = \partial_t E(u) + \partial_t a_{ij} w_{x_i x_j} + \partial_t b_i w_{x_i} + \partial_t(2KH)w. \tag{7.16}$$

And integrating by parts on S^ε, $w_t L w_t$ we get:

$$\begin{aligned}\int_{S^\varepsilon} |\nabla w_t|^2 \leq c_{26} \Bigg\{ \left|\int_{S^\varepsilon} w_t L w_t\right| + \left|\int_{S^\varepsilon} w_t 2KH w_t\right| \\ + \left|\int_{\partial S^\varepsilon} w_t(T_1 \nabla w_t).e\right| \Bigg\}\end{aligned} \tag{7.17}$$

and as in (V3):

$$\left|\int_{\partial S^\varepsilon} w_t(T_1\nabla w_t).e\right| \leq c_{27}\varepsilon|\nabla w|_{C^0}|D^2w|_{C^0} \leq c_{28}\varepsilon^{4-\nu_{14}} \tag{7.18}$$

and

$$\left|\int_{S^\varepsilon} w_t 2KH w_t\right| \leq c_{29}|\nabla w|^2_{C^0}\varepsilon \leq c_{30}\varepsilon^{4-\nu_{15}}. \tag{7.19}$$

Moreover by (7.16)

$$\begin{aligned}\left|\int_{S^\varepsilon} w_t L w_t\right| \leq c_{31}\{&|\nabla w|_{L^2}|\nabla E(u)|_{L^2} + |\nabla w|_{C^0}|D^2 w|_{C^0}\varepsilon \\ &+ |\nabla w|^2_{C^0}\varepsilon + |\nabla w|_{C^0}|w|_{C^0}\varepsilon\}.\end{aligned}$$

By lemma 7.1-b) we get

$$\begin{aligned}|\nabla E(u)|_{L^2} &\leq c_{32}\left(|u|^2_{L^4} + |\nabla u|^2_{L^4} + |D^2u|^2_{L^4}\right. \\ &\qquad \left. + \left|\left(|u| + |\nabla u| + |D^2 u|\right)|D^3 u|\right|_{L^2}\right) \\ &\leq c_{33}\left(|u|_{C^2}|u|_{H^2} + |D^3u|_{C^0}|u|_{H^2}\right) \\ &\leq c_{34}(\varepsilon^{0,75-\sigma}\varepsilon^{1,75-\sigma} + \varepsilon^{-0,25-\sigma}\varepsilon^{1,75-\sigma}) \leq c_{35}\varepsilon^{1,5-2\sigma}.\end{aligned}$$

Then by (7.12) and the previous estimate:

$$\begin{aligned}\left|\int_{S^\varepsilon} w_t L w_t\right| &\leq c_{36}(\varepsilon^{2,5-\nu_3}\varepsilon^{1,5-\nu_{16}} + \varepsilon^{2-\nu_9}\varepsilon^{1-\nu_{10}}\varepsilon + \varepsilon\varepsilon^{4-\nu_{17}} \\ &\qquad + \varepsilon\varepsilon^{2-\nu_9}\varepsilon^{3-\nu_7}) \\ &\leq c_{37}\varepsilon^{4-\nu_{18}}.\end{aligned} \tag{7.20}$$

Then (7.17), (7.18), (7.19) and (7.20) give

$$\int_{S^\varepsilon} |\nabla w_t|^2 \leq c_{38}\varepsilon^{4-\nu_{19}}. \tag{7.21}$$

Since (by (7.10))

$$|D^2 w| \leq c\{|\nabla w_t| + |\nabla w| + |w| + |E(u)|\}$$

we get by (7.21), (W1) and lemma 7.1-c):

$$\begin{aligned}\int_{S^\varepsilon} |D^2 w|^2 &\leq c_{39}\left\{\int_{S^\varepsilon} |\nabla w_t|^2 + |\nabla w|^2_{C^0}\varepsilon + |w|^2_{C^0}\varepsilon + |E(u)|^2_{C^0}\varepsilon\right\}\\ &\leq c_{40}\varepsilon^{4-\nu_{20}}\end{aligned}$$

which gives (W3) by (7.14), (7.15) with $\beta_3 = 2$.

8. Appendix.

Proposition 1. *There exists a constant K_9 depending only on C_0 and λ_0 such that the following estimates hold, for $0 \leq \tau \leq 1$, and for ε sufficiently small:*

(a)
$$\sum_{i,j} |g_{ij}| \leq K_9$$
$$\sum_{i,j} |g_{ij}|_{(\alpha)} \leq K_9$$

(b)
$$\sum_{i,j} |g'_{ij}| \leq K_9(|u| + |\nabla u|^2)$$
$$\sum_{i,j} |g'_{ij}|_{(\alpha)} \leq K_9\varepsilon^{2,75-\sigma-\alpha}$$

(c)
$$\sum_{i,j} |g''_{ij}| \leq K_9(|u|^2 + |\nabla u|^2)$$
$$\sum_{i,j} |g''_{ij}|_{(\alpha)} \leq K_9\varepsilon^{3,5-2\sigma-\alpha}$$

(d)
$$\sum_{i,j} |g_{ijx_k}| \leq K_9$$
$$\sum_{i,j} |g_{ijx_k}|_{(\alpha)} \leq K_9$$

(e)
$$\sum_{i,j} |g'_{ijx_k}| \le K_9(|u| + |\nabla u|)$$
$$\sum_{i,j} |g'_{ijx_k}|_{(\alpha)} \le K_9 \varepsilon^{1,75-\sigma-\alpha}$$

(f)
$$\sum_{i,j} |g''_{ijx_k}| \le K_9(|u|^2 + |u||\nabla u| + |D^2 u||\nabla u|)$$
$$\sum_{i,j} |g''_{ijx_k}|_{(\alpha)} \le K_9 \varepsilon^{2,5-2\sigma-\alpha}$$

(g)
$$\sum_{i,j} |g| \le K_9$$
$$\sum_{i,j} |g|_{(\alpha)} \le K_9$$

(h)
$$\sum_{i,j} |g'| \le K_9(|u| + |\nabla u|^2)$$
$$\sum_{i,j} |g'|_{(\alpha)} \le K_9 \varepsilon^{2,75-\sigma-\alpha}$$

(i)
$$\sum_{i,j} |g''| \le K_9(|u|^2 + |\nabla u|^2)$$
$$\sum_{i,j} |g''|_{(\alpha)} \le K_9 \varepsilon^{3,5-2\sigma-\alpha}$$

(j)
$$\sum_{i,j} |g_{x_k}| \le K_9$$
$$\sum_{i,j} |g_{x_k}|_{(\alpha)} \le K_9$$

(k)
$$\sum_{i,j} |g'_{x_k}| \le K_9(|u| + |\nabla u|)$$
$$\sum_{i,j} |g'_{x_k}|_{(\alpha)} \le K_9 \varepsilon^{1,75-\sigma-\alpha}$$

(l)
$$\sum_{i,j} |g''_{x_k}| \leq K_9(|u|^2 + |\nabla u|^2 + |\nabla u||D^2u|)$$
$$\sum_{i,j} |g''_{x_k}|_{(\alpha)} \leq K_9\varepsilon^{2,5-2\sigma-\alpha}$$

(m)
$$\sum_{i,j} |b_{ij}| \leq K_9$$
$$\sum_{i,j} |b_{ij}|_{(\alpha)} \leq K_9$$

(n)
$$\sum_{i,j} |b'_{ij}| \leq K_9(|u| + |\nabla u| + |D^2u|)$$
$$\sum_{i,j} |b'_{ij}|_{(\alpha)} \leq K_9\varepsilon^{0,75-\sigma-\alpha}$$

(o)
$$\sum_{i,j} |b''_{ij}| \leq K_9(|u|^2 + |\nabla u|^2 + |u||D^2u| + |\nabla u||D^2u|)$$
$$\sum_{i,j} |b''_{ij}|_{(\alpha)} \leq K_9\varepsilon^{2,5-2\sigma-\alpha}$$

(p)
$$\sum_{i,j} |b_{ijx_k}| \leq K_9(1 + |D^3u|)$$
$$\sum_{i,j} |b_{ijx_k}|_{(\alpha)} \leq K_9\varepsilon^{-0,25-\sigma-\alpha}$$

(q)
$$\sum_{i,j} |b'_{ijx_k}| \leq K_9(|u| + |\nabla u| + |D^2u| + |D^3u|)$$
$$\sum_{i,j} |b'_{ijx_k}|_{(\alpha)} \leq K_9\varepsilon^{-0,25-\sigma-\alpha}$$

(r)
$$\sum_{i,j} |b''_{ijx_k}| \leq K_9(|u|^2 + |\nabla u|^2 + |D^2u|^2 + |u||D^3u| + |\nabla u||D^3u|)$$
$$\sum_{i,j} |b''_{ijx_k}|_{(\alpha)} \leq K_9\varepsilon^{1,5-2\sigma-\alpha}$$

(s)
$$\sum_{i,j} |b| \leq K_9$$
$$\sum_{i,j} |b|_{(\alpha)} \leq K_9$$

(t)
$$\sum_{i,j} |b'| \leq K_9(|u| + |\nabla u| + |D^2 u|)$$
$$\sum_{i,j} |b'|_{(\alpha)} \leq K_9 \varepsilon^{0,75-\sigma-\alpha}$$

(u)
$$\sum_{i,j} |b''| \leq K_9(|u|^2 + |\nabla u|^2 + |D^2 u|^2)$$
$$\sum_{i,j} |b''|_{(\alpha)} \leq K_9 \varepsilon^{1,5-2\sigma-\alpha}$$

(v)
$$\sum_{i,j} |b_{x_k}| \leq K_9(1 + |D^3 u|)$$
$$\sum_{i,j} |b_{x_k}|_{(\alpha)} \leq K_9 \varepsilon^{-0,25-\sigma-\alpha}$$

(w)
$$\sum_{i,j} |b'_{x_k}| \leq K_9(|u| + |\nabla u| + |D^2 u| + |D^3 u|)$$
$$\sum_{i,j} |b'_{x_k}|_{(\alpha)} \leq K_9 \varepsilon^{-0,25-\sigma-\alpha}$$

(x)
$$\sum_{i,j} |b''_{x_k}| \leq K_9(|u|^2 + |\nabla u|^2 + |D^2 u|^2 + |u||D^3 u| + |\nabla u||D^3 u| + |D^2 u||D^3 u|)$$
$$\sum_{i,j} |b''_{x_k}|_{(\alpha)} \leq K_9 \varepsilon^{0,5-2\sigma-\alpha}$$

Proof of the proposition 1.

As usual $c_1, c_2 \ldots$ will be constants depending only on C_0 and λ_0. Since $u \in P(\varepsilon, \sigma, \alpha)$, it follows from lemma 5.3 that:

$$\begin{aligned}
|u|_{C^3} &\le c_1 \varepsilon^{-0,25-\sigma} \\
|u|_{C^{2,\alpha}} &\le c_1 \varepsilon^{0,75-\sigma-\alpha} \\
|u|_{C^2} &\le c_1 \varepsilon^{0,75-\sigma} \\
|u|_{C^{1,\alpha}} &\le c_1 \varepsilon^{1,75-\sigma-\alpha} \\
|u|_{C^1} &\le c_1 \varepsilon^{1,75-\sigma} \\
|u|_{C^{0,\alpha}} &\le c_1 \varepsilon^{2,75-\sigma-\alpha}
\end{aligned}$$

Moreover we get from the expression of $\psi_{x_i}(\tau)$ (see lemma 7.1):

$$\begin{aligned}
g_{ij}(\tau) = \langle \psi_{x_i}, \psi_{x_j} \rangle &+ \tau u \left(\langle \psi_{x_i}, N_{x_j} \rangle + \langle N_{x_i}, \psi_{x_j} \rangle \right) \\
&+ \tau^2 u_{x_i} u_{x_j} + \tau^2 u^2 \langle N_{x_i}, N_{x_j} \rangle
\end{aligned}$$

Since $N_{x_i} \in TM$.

Using the fact that $|u| \le 1, |\nabla u| \le 1$ and $|D^2 u| \le 1$ so that higher powers of these are dominated by lower powers, using $|\Psi|_{C^{4,\beta}}$ is bounded (independent of ε) it is straightforward to establish 1a)-1f) by derivation.

By example:

$$\begin{aligned}
g'_{ijx_k} = u_{x_k} \left(\langle \psi_{x_i}, N_{x_j} \rangle + \langle N_{x_i}, \psi_{x_j} \rangle \right) &+ u \left(\langle \psi_{x_i}, N_{x_j} \rangle + \langle N_{x_i}, \psi_{x_j} \rangle \right)_{x_k} \\
+ 2\tau u_{x_i x_k} u_{x_j} &+ 2\tau u_{x_i} u_{x_j x_k} + 4\tau u_{x_k} u \langle N_{x_i}, N_{x_j} \rangle \\
+ 2\tau u^2 \left(\langle N_{x_i}, N_{x_j} \rangle \right)_{x_k}
\end{aligned}$$

Then for $0 \le \tau \le 1$

$$|g'_{ijx_k}| \le c_2 \left(|u| + |\nabla u| + |\nabla u||D^2 u| + |u||\nabla u| + |u|^2 \right) \le K_g(|u| + |\nabla u|)$$

which gives us e).

The proofs of 1g)-1l) are done exactly in the same way with $g = g_{11}g_{22} - g_{12}^2$ combining with 1a)-1f).

We have (see lemma 7.1):

$$b_{ij}(\tau) = \langle \psi_{x_1}(\tau) \wedge \psi_{x_2}(\tau), \psi_{x_i x_j}(\tau) \rangle$$

Then

$$\begin{aligned}\psi_{x_1}(\tau)\wedge\psi_{x_2}(\tau) &= (\psi_{x_1}\wedge\psi_{x_2}) + \tau u(\psi_{x_1}\wedge N_{x_2} + N_{x_1}\wedge\psi_{x_2}) \\ &\quad + \tau u_{x_1}(N\wedge\psi_{x_2}) + \tau u_{x_2}(\psi_{x_1}\wedge N) \\ &\quad + \tau^2u^2(N_{x_1}\wedge N_{x_2}) + \tau^2 u u_{x_2}(N_{x_1}\wedge N) \\ &\quad + \tau^2 u u_{x_1}(N\wedge N_{x_2}) \\ \psi_{x_ix_j}(\tau) &= \psi_{x_ix_j} + \tau u N_{x_ix_j} + \tau u_{x_j}N_{x_i} + \tau u_{x_i}N_{x_j} + \tau u_{x_ix_j}N\end{aligned}$$

Thus

$$\begin{aligned}|\psi_{x_ix_j}(\tau)| &\le c_3 \\ |\psi_{x_ix_j}(\tau)'| &\le c_4(|u| + |\nabla u| + |D^2u|) \\ |\psi_{x_ix_j}(\tau)''| &= 0 \\ |\psi_{x_1}(\tau)\wedge\psi_{x_2}(\tau)| &\le c_5 \\ |(\psi_{x_1}(\tau)\wedge\psi_{x_2}(\tau))'| &\le c_6(|u| + |\nabla u|) \\ |(\psi_{x_1}(\tau)\wedge\psi_{x_2}(\tau))''| &\le c_7(|u|^2 + |u||\nabla u|)\end{aligned}$$

Thus we get 1m)-1o) and 1s)-1u) by derivation of $b_{ij}(\tau)$ like for $g_{ij}(\tau)$.

Moreover

$$\begin{aligned}|\psi_{x_ix_jx_k}(\tau)| &\le c_8(1 + |D^3u|) \\ |\psi_{x_ix_jx_k}(\tau)'| &\le c_9(|u| + |\nabla u| + |D^2u| + |D^3u|) \\ |\psi_{x_ix_jx_k}(\tau)''| &= 0 \\ |(\psi_{x_1}(\tau)\wedge\psi_{x_2}(\tau))_{x_k}| &\le c_{10} \\ |(\psi_{x_1}(\tau)\wedge\psi_{x_2}(\tau))'_{x_k}| &\le c_{11}(|u| + |\nabla u| + |D^2u|) \\ |(\psi_{x_1}(\tau)\wedge\psi_{x_2}(\tau))''_{x_k}| &\le c_{12}(|u|^2 + |\nabla u|^2 + |u||D^2u|)\end{aligned}$$

Then the proof of 1p)-1r) and 1v)-1x) are done exactly in the same way with $b = b_{11}b_{22} - b_{12}^2$.

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