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Degeneration of Kähler-Einstein Metrics on Complete Kähler Manifolds

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0. Introduction.

According to algebraic geometers, a degeneration of projective varieties is a smooth holomorphic family $\pi : \mathcal{X} \to \Delta$ with the following property: the fiber $X_t = \pi^{-1}(t)$ are smooth except for $t = 0$. Assume that the central fiber X_0 is a reduced divisor with normal crossings. In [T], G. Tian proved the convergence of complete Kähler-Einstein metrics as $t \to 0$ for two cases: 1) On X_t when X_t has ample canonical line bundle for $t \neq 0$, 2) On $X_t \setminus \mathcal{D}$ when $K_{X_t} + \mathcal{D} \cap X_t$ is ample for $t \neq 0$, where $\mathcal{D}$ is a divisor of $\mathcal{X}$. In case 1) the result can be stated as

Theorem (Tian). *Let $g_{E,t}$ be Kähler-Einstein metric with*

$$\mathrm{Ric}(g_{E,t}) = -g_{E,t}$$

on X_t. Assume that the central fiber X_0 is the union of smooth hypersurfaces, say $X_{01}, \cdots, X_{0m}$, with normal crossings and each line bundle $K_{X_{0i}} + \sum_{j \neq i} X_{0j}$ is ample on X_{0i}, $1 \leq i \leq m$. Further assume that no three of divisors X_{0i} have non-empty intersection.

Then $g_{E,t}$ converge to a complete Kähler-Einstein metric on $X_0 \setminus \mathrm{Sing}(X_0)$ in the sense of Cheeger-Gromov.

In this paper, we prove the same result without assuming that no three divisors have nonempty intersection. The key observation is that Lemma 1.5 in [T] can be weakened. We will prove our result in a larger setting. Before stating the main theorem of this paper we make several definitions.

Definition 0.1. Let M be a complex manifold of dimension n and C_j are smooth hypersurfaces in M ($j = 1, \cdots, n_2$) such that $\sum_{j=1}^{n_2} C_j$ is a normal crossing divisor. Let m_j be natural numbers. Complex V-manifold $M\left(\sum \frac{1}{m_j} C_j\right)$ is defined in the following way.

(i) As a topological space, $M\left(\sum \frac{1}{m_j} C_j\right)$ is M,

(ii) For a point $p \in M \setminus \cup C_j$, we take a small neighborhood U disjoint from $\cup C_j$, and consider (U, id) as a local uniformization,

(iii) For a point p in some C_j, without loss of generality, assume $p \in (\cap_1^k C_j) \setminus (\cup_{k+1}^{n_2} C_j)$ for some k. We take a small neighborhood U of p with coordinate system $(z^1, \cdots, z^n)$ such that C_j is defined by $z^j = 0$ for $j = 1, \cdots, k$, and $C_j \cap U = \emptyset$ for $j = k+1, \cdots, n_2$. We define the uniformization of U to be $p_U : \tilde{U} \to U$ with

$$p_U(w^1, \cdots, w^n) = ((w^1)^{m_1}, \cdots, (w^k)^{m_k}, w^{k+1}, \cdots, w^n).$$

Given a Kähler metric g, we will denote its Kähler form by ω_g and its Ricci form by $\mathrm{ric}(g)$.

Definition 0.2. Let $M\left(\sum \frac{1}{m_j} C_j\right)$ be a complex V-manifold.

(i) A Kähler metric g on $M \setminus \cup C_j$ is called a Kähler V-metrics on $M\left(\sum \frac{1}{m_j} C_j\right)$ if for each $p \in \cup C_j$, $p_U^* g$ can be extended smoothly to $\tilde{U}$ as a metric.

(ii) A family of Kähler V-metrics g_t on $M\left(\sum \frac{1}{m_j} C_j\right)$ is said to converge in C^2-topology if g_t converge in C^2-topology on $M \setminus (\cup C_j)$ and $p_U^* g_t$ converge in C^2-topology on $\tilde{U}$ for each $p \in \cup C_j$.

(iii) A Kähler V-metric g on $M\left(\sum \frac{1}{m_j} C_j\right)$ is called complete if g is a complete metric on $M \setminus \cup U$ with boundary $\partial(\cup U)$ and the extension of $p_U^* g$ is a complete metric on $\tilde{U}$ with boundary $p_U^{-1}(\partial U)$.

(iv) A Kähler V-metric g on $M\left(\sum \frac{1}{m_j} C_j\right)$ is called Kähler-Einstein if $\mathrm{ric}(g) = -c \cdot \omega_g$ on $M \setminus \cup C_j$ for some constant c.

Definition 0.3. Let M be a smooth projective variety of dimension n and D be a $\mathbb{Q}$-divisor on M.

(i) D is called numerically effective(nef in short) if for any curve C in M the intersection number $D \cdot C$ is non-negative. Such a divisor is called big if $D^n > 0$.

(ii) D is called to be ample modulo another divisor E if for every effective reduced curve C on X which is not contained in E, $D \cdot C > 0$.

Let m_j be a family of natural numbers$(1 \leq j \leq n_2)$. Let $\pi : \mathcal{X} \to \Delta$ be a degeneration of projective varieties with two divisors $\mathcal{C}$ and $\mathcal{D}$ satisfying the following assumptions.

(1) X_t is smooth for $t \neq 0$ and X_0 is a union of smooth hypersurfaces $X_{0i}(1 \leq i \leq n_1)$ in $\mathcal{X}$ with normal crossings,

(2) $\mathcal{C} + \mathcal{D}$ is a reduced divisor with normal crossings. Divisor $\mathcal{C}$ consists of smooth components $\mathcal{C}_1, \cdots, \mathcal{C}_{n_2}$ and divisor $\mathcal{D}$ consists of smooth components $\mathcal{D}_1, \cdots, \mathcal{D}_{n_3}$. We further assume that the restriction maps $d\pi|_{\mathcal{C}_j}$ and $d\pi|_{\mathcal{D}_k}$ are surjective$(1 \leq j \leq n_2, 1 \leq k \leq n_3)$,

(3) The divisor $\mathcal{C}+\mathcal{D}$ intersects both central fiber X_0 and its singular part $\mathrm{Sing}(X_0)$ transversally.

(4) Let $C_t = X_t \cap \mathcal{C}, C_{jt} = X_t \cap \mathcal{C}_j$ and $D_t = X_t \cap \mathcal{D}, D_{kt} = X_t \cap \mathcal{D}_k$, then each C_{jt} and D_{kt} is smooth. For $t \neq 0$, line bundle

$$K_{X_t} + \sum_{k=1}^{n_3} D_{kt} + \sum_{j=1}^{n_2} \frac{m_j - 1}{m_j} C_{jt}$$

is nef, big and ample modulo D_t, $t \neq 0$. It follows that there is a unique complete Kähler-Einstein V-metric $g_{E,t}$ on

$$(X_t \setminus D_t) \left(\sum_{j=1}^{n_2} \frac{1}{m_j} C_{jt} \right)$$

with $\mathrm{ric}(g_{E,t}) = -\omega_{g_{E,t}}$ (see [TY], Theorem 2.1 for existence and [Y2], p.474 for completeness).

The following is the main theorem of this paper which provides a sufficient condition on the convergence of this family of metrics $g_{E,t}$ as t goes to 0.

Theorem 0.1. *Let $\pi : \mathcal{X} \to \Delta$ be the degeneration family with properties (1)-(4) given above. We assume that for each $i, 1 \leq i \leq n_1$, line bundle*

$$K_{X_{0i}} + \sum_{l=1, l \neq i}^{n_1} X_{0l} + \sum_{k=1}^{n_3} D_{k0} + \sum_{j=1}^{n_2} \frac{m_j - 1}{m_j} C_{j0}$$

is nef, big and ample modulo $\sum_{k=1}^{n_3} D_{k0}$ *on* X_{0i}. *Then the complete Kähler-Einstein V-metric* $g_{E,t}$ *on* $(X_t \setminus D_t)\left(\sum_{j=1}^{n_2} \frac{1}{m_j} C_{jt}\right)$ *converges to the unique complete Kähler-Einstein V-metric* $g_{E,0}$ *on*

$$(X_0 \setminus (\mathrm{Sing}(X_0) \cup D_0)) \left(\sum_{j=1}^{n_2} \frac{1}{m_j} C_{j0} \right)$$

in the sense of Cheeger-Gromov: there are an exhaustion of compact sets $F_\beta \Subset X_0 \setminus (\mathrm{Sing}(X_0) \cup D_0)$ *and diffeomorphisms* $\phi_{\beta,t}$ *from* F_β *to into* X_t *satisfying:*

i) $X_t \setminus (D_t \cup (\cup_\beta \phi_{\beta,t}(F_\beta)))$ *consists of finite union of submanifolds of real codimension 1,*

ii) $\phi_{\beta,t}$ *maps* C_0 *into* C_t *and* $\phi^*_{\beta,t} g_{E,t}$ *is a V-metric on* F_β,

iii) *for each fixed* β, *V-metrics* $\phi^*_{\beta,t} g_{E,t}$ *converge to* $g_{E,0}$ *on* F_β *in* C^2-*topology as t goes to 0.*

In section 1 we construct families of Kähler V-metrics on $(\mathcal{X} \setminus (\mathcal{D} \cup \mathrm{Sing}(X_0)))\left(\sum_{j=1}^{n_2} \frac{1}{m_j} C_{j0}\right)$ with prescribed asymptotic behavior near $X_0 \setminus (\mathrm{Sing}(X_0) \cup D_0)$. In section 2 we prove Theorem 0.1 using the estimates in [TY] and results from section 1.

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1. Construction of family of Kähler V-metrics with asymptotic behavior.

In this section we adopt the notations used in introduction. For each $i (1 \leq i \leq n_1)$, we choose a neighborhood U_i of X_{0i} in $\mathcal{X}$ such that $\bar{U}_{i_1} \cap \bar{U}_{i_2} = \emptyset$ when $X_{0i_1} \cap X_{0i_2} = \emptyset$, where $\bar{U}_{i_1}$ and $\bar{U}_{i_2}$ denote the closure of U_{i_1} and U_{i_2} respectively. We fix a relative volume $\tilde{V}$ on $\mathcal{X}$. Without loss of generality, we assume $\mathcal{X} = \cup_{i=1}^{n_1} U_i$. Let $\tilde{V}_i$ be the local representation of the relative volume form $\tilde{V}$ on U_i, in particular, for each $t \in \Delta$, $\tilde{V}_i|_{X_t}$ is the volume form of $X_t \cap U_i$. We denote by $U_{i_1 \cdots i_l}$ the intersection $U_{i_1} \cap \cdots \cap U_{i_l}$ for each tuple $(i_1, \cdots, i_l)$. It is a neighborhood of $X_{0i_1 \cdots i_l} = X_{0i_1} \cap \cdots X_{0i_l}$.

Now we begin to construct a family of Kähler V-metrics with the asymptotic behavior. Let s_i be the defining section of line bundle $[X_{0i}]$. By Lemma 1.1 and 1.2 in [T], there are Hermitian metrics $\| \ \|_i$ of line bundles $[X_{0i}]$ on $\mathcal{X}$ satisfying$(1 \leq i \leq n_1)$:

(1) $\|s_i\|_i \equiv 1$ outside U_i,

(2) $\|s_1\|_1 \cdots \|s_{n_1}\|_{n_1} \equiv |t|$ on $\mathcal{X}$,

(3) $\| \ \|_{i_1}^2 \cdot \tilde{V}_{i_1} = \| \ \|_{i_2}^2 \cdot \tilde{V}_{i_2}$ on $U_{i_1} \cap U_{i_2}$, $1 \le i_1, i_2 \le n_1$.

Without loss of generality, we may assume $\|s_i\|_i^2 \le 3$ in $\mathcal{X}$. Assume the defining sections of C_j and D_k in $\mathcal{X}$ are u_j and v_k respectively. We equip line bundles $[C_j]$ and $[D_k]$ with Hermitian metrics $\| \ \|_{j,2}$ and $\| \ \|_{k,3}$ respectively. Let $\mu_1, \cdots, \mu_{n_3}$ be rational numbers in $[0,1]$ and ε be a small positive number. We will specify them later. Now we define a relative volume V on $\mathcal{X} \setminus (\mathcal{C} \cup \mathcal{D} \cup \operatorname{Sing}(X_0))$ as follows. For $t \in \Delta \setminus \{0\}$,

$$V_{it} = \frac{\tilde{V}_i}{\prod_{k=1}^{n_3}\left[\varepsilon\|v_k\|_{k,3}^{2\mu_k} \cdot (-\log \varepsilon\|v_k\|_{k,3}^2)^2\right]}$$
$$\cdot \frac{1}{\prod_{j=1}^{n_2}\left[\varepsilon\|u_j\|_{j,2}^{\frac{2m_j-2}{m_j}} \cdot \left(1-\varepsilon\|u_j\|_{j,2}^{\frac{2}{m_j}}\right)^2\right] \cdot \prod_{l=1,l\ne i}^{n_1} \varepsilon\|s_l\|_l^2}$$
$$\cdot \prod_{l=1,l\ne i}^{n_1}\left[\frac{-\pi}{\log \varepsilon|t|} csc \frac{\pi \log \varepsilon\|s_l\|_l^2}{2\log \varepsilon|t|}\right]^2 \cdot \left[\frac{-\pi}{\log \varepsilon|t|} csc \frac{\pi \log \varepsilon \prod_{l=1,l\ne i}^{n_1}\|s_l\|_l^2}{2\log \varepsilon|t|}\right]^2$$
$$\text{on } U_i \cap (X_t \setminus (C_t \cup D_t))$$

and

$$V_{i0} = \frac{2^{2n_1}\tilde{V}_i}{\prod_{k=1}^{n_3}\left[\varepsilon\|v_k\|_{k,3}^{2\mu_k} \cdot (-\log \varepsilon\|v_k\|_{k,3}^2)^2\right]}$$
$$\cdot \frac{1}{\prod_{j=1}^{n_2}\left[\varepsilon\|u_j\|_{j,2}^{\frac{2m_j-2}{m_j}} \cdot \left(1-\varepsilon\|u_j\|_{j,2}^{\frac{2}{m_j}}\right)^2\right] \cdot \prod_{l=1,l\ne i}^{n_1} \varepsilon\|s_l\|_l^2}$$
$$\cdot \frac{1}{\prod_{l=1,l\ne i}^{n_1}(-\log \varepsilon\|s_l\|_l^2)^2 \cdot (-\log \varepsilon \prod_{l=1,l\ne i}^{n_1}\|s_l\|_l^2)^2}$$
$$\text{on } U_i \cap (X_0 \setminus (C_0 \cup D_0 \cup \operatorname{Sing}(X_0))).$$

In order to see that these volume forms V_{it} can be glued together to give a global volume form V_t, we simply observe that on

$$U_{i_1} \cap U_{i_2} \cap X_t \qquad (1 \le i_1, i_2 \le n_1),$$

$$
\begin{aligned}
V_{i_1 t} = & \frac{\tilde{V}_{i_1}}{\varepsilon \|s_{i_2}\|_{i_2}^2} \cdot \frac{1}{\prod_{k=1}^{n_3} \left[\varepsilon \|v_k\|_{k,3}^{2\mu_k} \cdot \left(-\log \varepsilon \|v_k\|_{k,3}^2 \right)^2 \right]} \\
& \cdot \frac{1}{\prod_{j=1}^{n_2} \left[\varepsilon \|u_j\|_{j,2}^{\frac{2m_j-2}{m_j}} \cdot \left(1 - \varepsilon \|u_j\|_{j,2}^{\frac{2}{m_j}} \right)^2 \right] \cdot \prod_{l=1, l\neq i_1, i_2}^{n_1} \varepsilon \|s_l\|_l^2} \\
& \cdot \left[\frac{-\pi}{\log \varepsilon |t|} csc \frac{\pi \log \varepsilon \|s_{i_2}\|_{i_2}^2}{2 \log \varepsilon |t|} \right]^2 \prod_{l=1, l\neq i_1, i_2}^{n_1} \left[\frac{-\pi}{\log \varepsilon |t|} csc \frac{\pi \log \varepsilon \|s_l\|_l^2}{2 \log \varepsilon |t|} \right]^2 \\
& \cdot \left[\frac{-\pi}{\log \varepsilon |t|} csc \frac{\pi \log \varepsilon \prod_{l=1, l\neq i_1}^{n_1} \|s_l\|_l^2}{2 \log \varepsilon |t|} \right]^2 \\
= & \frac{\tilde{V}_{i_2}}{\varepsilon \|s_{i_1}\|_{i_1}^2} \cdot \frac{1}{\prod_{k=1}^{n_3} \left[\varepsilon \|v_k\|_{k,3}^{2\mu_k} \cdot \left(-\log \varepsilon \|v_k\|_{k,3}^2 \right)^2 \right]} \\
& \cdot \frac{1}{\prod_{j=1}^{n_2} \left[\varepsilon \|u_j\|_{j,2}^{\frac{2m_j-2}{m_j}} \cdot \left(1 - \varepsilon \|u_j\|_{j,2}^{\frac{2}{m_j}} \right)^2 \right] \cdot \prod_{l=1, l\neq i_1, i_2}^{n_1} \varepsilon \|s_l\|_l^2} \\
& \cdot \left[\frac{-\pi}{\log \varepsilon |t|} csc \left(\pi - \frac{\pi \log \left(\varepsilon \prod_{l=1, l\neq i_2}^{n_1} \|s_l\|_l^2 \right)}{2 \log \varepsilon |t|} \right) \right]^2 \\
& \cdot \prod_{l=1, l\neq i_1, i_2}^{n_1} [\frac{-\pi}{\log \varepsilon |t|} csc \frac{\pi \log \varepsilon \|s_l\|_l^2}{2 \log \varepsilon |t|}]^2 \cdot [\frac{-\pi}{\log \varepsilon |t|} csc (\pi - \frac{\pi \log \varepsilon \|s_{i_1}\|_{i_1}^2}{2 \log \varepsilon |t|})]^2 \\
= & V_{i_2 t}.
\end{aligned}
$$

Define

$$
\begin{aligned}
\omega_t &= \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log V_t && \text{on } X_t \setminus (C_t \cup D_t) \qquad (t \neq 0), \\
\omega_0 &= \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log V_0 && \text{on } X_0 \setminus (C_0 \cup D_0 \cup \mathrm{Sing}(X_0)).
\end{aligned}
$$

Simple computations show: for $t \neq 0$, on $U_i \cap (X_t \setminus (C_t \cup D_t))$,

$$
\begin{aligned}
\omega_t = & -\mathrm{ric}(\tilde{V}_t) + \sum_{k=1}^{n_3} \mu_k \, \mathrm{ric}(\| \cdot \|_{k,3}) + \sum_{j=1}^{n_2} \frac{m_j - 1}{m_j} \mathrm{ric}(\| \cdot \|_{j,2}) + \sum_{l=1, l\neq i}^{n_1} \mathrm{ric}(\| \cdot \|_l) \\
& + \sum_{k=1}^{n_3} \left[\frac{2 \, \mathrm{ric}(\| \cdot \|_{k,3})}{\log \varepsilon \|v_k\|_{k,3}^2} + \frac{\sqrt{-1}}{\pi} \cdot \frac{\partial \log \|v_k\|_{k,3}^2 \wedge \bar{\partial} \log \|v_k\|_{k,3}^2}{(-\log \varepsilon \|v_k\|_{k,3}^2)^2} \right]
\end{aligned}
$$

$$+\sum_{j=1}^{n_2}\left[\frac{2\varepsilon\cdot\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\|u_j\|_{j,2}^{\frac{2}{m_j}}}{1-\varepsilon\|u_j\|_{j,2}^{\frac{2}{m_j}}}+\frac{2\varepsilon^2\cdot\frac{\sqrt{-1}}{2\pi}\partial\|u_j\|_{j,2}^{\frac{2}{m_j}}\wedge\bar{\partial}\|u_j\|_{j,2}^{\frac{2}{m_j}}}{(1-\varepsilon\|u_j\|_{j,2}^2)^2}\right]$$
$$+\sum_{l=1,l\neq i}^{n_1}\frac{\pi}{\log\varepsilon|t|}ctg\frac{\pi\log\varepsilon\|s_l\|_l^2}{2\log\varepsilon|t|}\cdot\mathrm{ric}(\|\cdot\|_l)$$
$$+\sum_{l=1,l\neq i}^{n_1}\left(\frac{\pi}{2\log\varepsilon|t|}csc\frac{\pi\log\varepsilon\|s_l\|_l^2}{2\log\varepsilon|t|}\right)^2\cdot\frac{\sqrt{-1}}{\pi}\partial\log\|s_l\|_l^2\wedge\bar{\partial}\|s_l\|_l^2$$
$$+\frac{\pi}{\log\varepsilon|t|}ctg\frac{\pi\log(\varepsilon\prod_{l=1,l\neq i}^{n_1}\|s_l\|_l^2)}{2\log\varepsilon|t|}\cdot\sum_{l=1,l\neq i}^{n_1}\mathrm{ric}(\|\cdot\|_l)$$
$$+\left(\frac{\pi}{2\log\varepsilon|t|}csc\frac{\pi\log(\varepsilon\prod_{l=1,l\neq i}^{n_1}\|s_l\|_l^2)}{2\log\varepsilon|t|}\right)^2$$
$$\cdot\frac{\sqrt{-1}}{\pi}\partial\log\prod_{l=1,l\neq i}^{n_1}\|s_l\|_l^2\wedge\bar{\partial}\prod_{l=1,l\neq i}^{n_1}\|s_l\|_l^2,$$

where $\mathrm{ric}(\tilde{V}_t)$, $\mathrm{ric}(\|\cdot\|_i)$, $\mathrm{ric}(\|\cdot\|_{j,2})$, and $\mathrm{ric}(\|\cdot\|_{k,3})$ denote the curvature tensor of the volume forms $\tilde{V}_t=\tilde{V}|_{X_t}$ and the Hermitian metrics $\|\cdot\|_i$, $\|\cdot\|_{j,2}$, and $\|\cdot\|_{k,3}$ respectively. Also, on $X_{0i}\setminus(C_0\cup D_0\cup\mathrm{Sing}(X_0))$,

$$\omega_0=-\mathrm{ric}(\tilde{V}_0)+\sum_{k=1}^{n_3}\mu_k\,\mathrm{ric}(\|\cdot\|_{k,3})+\sum_{j=1}^{n_2}\frac{m_j-1}{m_j}\mathrm{ric}(\|\cdot\|_{j,2})+\sum_{l=1,l\neq i}^{n_1}\mathrm{ric}(\|\cdot\|_l)$$
$$+\sum_{k=1}^{n_3}\left[\frac{2\,\mathrm{ric}(\|\cdot\|_{k,3})}{\log(\varepsilon\|v_k\|_{k,3}^2)}+\frac{\sqrt{-1}}{\pi}\cdot\frac{\partial\log\|v_k\|_{k,3}^2\wedge\bar{\partial}\log\|v_k\|_{k,3}^2}{(-\log\varepsilon\|v_k\|_{k,3}^2)^2}\right]$$
$$+\sum_{j=1}^{n_2}\left[\frac{2\varepsilon\cdot\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\|u_j\|_{j,2}^{\frac{2}{m_j}}}{1-\varepsilon\|u_j\|_{j,2}^{\frac{2}{m_j}}}+\frac{2\varepsilon^2\cdot\frac{\sqrt{-1}}{2\pi}\partial\|u_j\|_{j,2}^{\frac{2}{m_j}}\wedge\bar{\partial}\|u_j\|_{j,2}^{\frac{2}{m_j}}}{(1-\varepsilon\|u_j\|_{j,2}^2)^2}\right]$$
$$+\sum_{l=1,l\neq i}^{n_1}\left[\frac{2\cdot\mathrm{ric}(\|\cdot\|_l)}{\log\varepsilon\|s_l\|_l^2}+\frac{\frac{\sqrt{-1}}{\pi}\cdot\partial\log\|s_l\|_l^2\wedge\bar{\partial}\log\|s_l\|_l^2}{(\log\varepsilon\|s_l\|_l^2)^2}\right]$$
$$+\frac{2\cdot\sum_{l=1,l\neq i}^{n_1}\mathrm{ric}(\|\cdot\|_l)}{\log\varepsilon\prod_{l=1,l\neq i}^{n_1}\|s_l\|_l^2}$$
$$+\frac{\frac{\sqrt{-1}}{\pi}\partial\log\prod_{l=1,l\neq i}^{n_1}\|s_l\|_l^2\wedge\bar{\partial}\log\prod_{l=1,l\neq i}^{n_1}\|s_l\|_l^2}{(\log\varepsilon\prod_{l=1,l\neq i}^{n_1}\|s_l\|_l^2)^2}.$$

Assume that $K_{X_t}+\sum_{k=1}^{n_3} D_{kt}+\sum_{j=1}^{n_2} \frac{m_j-1}{m_j}C_{jt}$ is nef, big and ample modulo D_t, $t \neq 0$. By a result of Y. Kawamata([K]), there are constants $0 < \mu_k < 1 (1 \leq k \leq n_3)$ such that for properly chosen volume $\tilde{V}$,

$$-\operatorname{ric}(\tilde{V}_t) + \sum_{k=1}^{n_3} \mu_k \operatorname{ric}(\|\cdot\|_{k,3}) + \sum_{j=1}^{n_2} \frac{m_j-1}{m_j} \operatorname{ric}(\|\cdot\|_{j,2})$$

is a Kähler V-metric on $(X_t \setminus D_t)\left(\sum \frac{1}{m_j} C_{jt}\right)$. Assume that for each $i, 1 \leq i \leq n_1$, line bundle

$$K_{X_{0i}} + \sum_{k=1}^{n_3} D_{k0} + \sum_{j=1}^{n_2} \frac{m_j-1}{m_j} C_{j0} + \sum_{l=1, l\neq i}^{n_1} X_{0l}$$

is nef, big and ample modulo D_0 on X_{0i}, then

$$-\operatorname{ric}(\tilde{V}_0) + \sum_{k=1}^{n_3} \mu_k \operatorname{ric}(\|\cdot\|_{k,3}) + \sum_{j=1}^{n_2} \frac{m_j-1}{m_j} \operatorname{ric}(\|\cdot\|_{j,2}) + \sum_{l=1, l\neq i}^{n_1} \operatorname{ric}(\|\cdot\|_l)$$

is a Kähler V-metric on $(X_{0i} \setminus (D_0 \cup \operatorname{Sing}(X_0)))\left(\sum \frac{1}{m_j} C_{j0}\right)$ for the same reason. The following lemma follows directly from the formulas of ω_t and ω_0 above.

Lemma 1.1. *Assume that* $K_{X_t}+\sum_{k=1}^{n_3} D_{kt}+\sum_{j=1}^{n_2} \frac{m_j-1}{m_j}C_{jt}$ *is nef, big and ample modulo* D_t, $t \neq 0$, *and assume that for each* $i, 1 \leq i \leq n_1$, *line bundle*

$$K_{X_{0i}} + \sum_{k=1}^{n_3} D_{k0} + \sum_{j=1}^{n_2} \frac{m_j-1}{m_j} C_{j0} + \sum_{l=1, l\neq i}^{n_1} X_{0l}$$

is nef, big and ample modulo D_0 *on* X_{0i}.

Then by choosing the volume form $\tilde{V}$ *properly and a small* ε, ω_t *is the Kähler forms of complete Kähler V-metrics* g_t *for* t *sufficiently small. Moreover, the Kähler V-metrics* g_t *converges to* g_0 *outside* $D_0 \cup \operatorname{Sing}(X_0)$ *in the sense of Cheeger-Gromov: there are an exhaustion of compact subsets* $F_\beta \Subset X_0 \setminus (D_0 \cup \operatorname{Sing}(X_0))$ *and diffeomorphisms* $\phi_{\beta,t}$ *from* F_β *into* X_t *satisfying:*

(1) $X_t \setminus \cup_\beta \phi_{\beta,t}(F_\beta)$ *consists of finite union of submanifolds of real codimension 1,*

(2) $\phi_{\beta,t}$ *map* $F_\beta \cap C_0$ *into* C_t *and* $\phi_{\beta,t}^* g_t$ *are V-metrics,*

(3) *for each fixed β, $\phi^*_{\beta,t}g_t$ converge to g_0 on F_β in C^2-topology on the space of V-metrics as t goes to 0.*

Before we state the next lemma, we need a couple of definitions.

Definition 1.1. Let $M\left(\sum \frac{1}{m_j}C_j\right)$ be the V-manifold in Definition 0.1. Let B be a ball in $\mathbb{C}^n$ and ψ is a holomorphic map from B into M. ψ is called a quasi-coordinate map if either $\psi(B)\cap(\cup C_j) = \emptyset$ and ψ is of maximal rank everywhere, or $\psi(B)$ is contained in some uniformizing neighborhood U, the lifting $\psi_U : B \to \tilde{U}$ is of maximal rank everywhere, and ψ satisfies $\psi = p_U \circ \psi_U$. (B, ψ) is called a local quasi-coordinate of the V-manifold.

Definition 1.2. Let $M\left(\sum \frac{1}{m_j}C_j\right)$ be a V-manifold and g is a smooth Kähler V-metric on it. We call that $(M\left(\sum \frac{1}{m_j}C_j\right), g)$ has bounded geometry of order $k+\beta$, where $k \in \mathbb{N}, \beta \in [0,1)$, if the following conditions hold: There are a system of quasi-coordinates $\{(B_\alpha, \psi_\alpha)\}$ such that:

(i) Every $x \in M$ is the image of the center of some B_α,

(ii) There are positive numbers ε and δ independent of α such that the radius of B_α is between ε and δ.

(iii) There exists constant C such that

$$0 < C^{-1}(\delta_{ij}) \le (g_{\alpha i\bar{j}}) \le C(\delta_{ij})$$
$$\left|\frac{\partial^{|p|+|q|}g_{\alpha i\bar{j}}}{\partial z_\alpha^p \partial \bar{z}_\alpha^q}\right|_{C^\beta(B_\alpha)} \le C$$

for all multi-indexes p, q with $|p| + |q| \le k$, where $g_{\alpha i\bar{j}}$ is the pullback metric on B_α and $|\cdot|_{C^\beta(B_\alpha)}$ is the standard Hölder norm.

Lemma 1.2. *Same assumptions as Lemma 1.1. Then*

(i) *the Kähler V-metric g_0 in Lemma 1.1 has bounded geometry of order $4+\beta$.*

(ii) *the Kähler V-metrics g_t in Lemma 1.1 have uniformly bounded curvature tensors on the uniformization coordinate systems of the V-manifold for $t \neq 0$.*

Proof. (i) we prove that $(X_{0i}\setminus(D_0\cup \mathrm{Sing}(X_0)))\left(\sum_j \frac{1}{m_j}C_{0j}\right)$ has bounded geometry of order $4+\beta$ for each $i=1,\cdots,n_1$. In [TY] it has been proved that g_0 has bounded geometry on $(X_{0i}\setminus(D_0\cup U(\mathrm{Sing}(X_0))))\left(\sum_j \frac{1}{m_j}C_{0j}\right)$, where $U(\mathrm{Sing}(X_0))$ is a small neighborhood of $\mathrm{Sing}(X_0)$ in X_{0i}. We only need to prove that g_0 has bounded geometry on $U(\mathrm{Sing}(X_0))$.

Pick up a point $x_0\in \mathrm{Sing}(X_0)\cap X_{0i}$, for simplicity, we assume that

$$x_0\in\left(\bigcap_{l=1}^{i_1}X_{0l}\right)\cap X_{0i}\cap\left(\bigcap_{j=1}^{j_1}C_{j0}\right)\cap\left(\bigcap_{k=1}^{k_1}D_{k0}\right),$$

$x_0\notin X_{0l}$, for $l>i_1, l\neq i$, $x_0\notin C_{j0}$, for $j>j_1$, and $x_0\notin D_{k0}$, for $k>k_1$. Take a neighborhood U_{x_0} of x_0 with coordinate system $(z^1,\cdots,z^n)$ of X_{0i} such that X_{0l} is defined by $z^l=0, l=1,\cdots,i_1$, D_{k0} is defined by $z^{i_1+k}=0, k=1,\cdots,k_1$, and C_{j0} is defined by $z^{i_1+k_1+j}=0, j=1,\cdots,j_1$. If we choose U_{x_0} small enough, we may assume that $\|s_l\|_l=1, l=i_1+1,\cdots,n, l\neq i$. On the uniformization $p_{x_0}:\tilde U_{x_0}\to U_{x_0}$ we have coordinate system $(w^1,\cdots,w^n)$. Let $\Delta_\delta^*=\{w\in\mathbb{C}:0<|w|<\delta\}$ and $\Delta_\delta=\{w\in\mathbb{C}:|w|<\delta\}$, then we may identify $\tilde U_{x_0}$ with $(\Delta_\delta^*)^{i_1+k_1}\times(\Delta_\delta)^{n-i_1-k_1}$, where $\delta>0$ only depends on X_0, C_0, and D_0. We will construct quasi-coordinate on $\tilde U_{x_0}$ such that the pullback of g_0 has bounded geometry. The following two lemmas are well-known (see e.g. [TY, p. 602]).

Lemma 1.3. *The map $\rho_\theta:\Delta_\delta\to\Delta_\delta^*$ with $\rho_\theta(w)=\delta\exp\left(\frac{w+\delta}{w-\delta}+\sqrt{-1}\theta\right)$ is a universal covering map of Δ_δ^*, where $\theta\in[0,2\pi)$. The fundamental domain over $\Delta_\delta^*\setminus\{te^{\sqrt{-1}\theta}:0<t<\delta\}$ is $\{w\in\Delta_\delta:0<\delta\,\mathrm{Im}(w)<\pi|\delta-w|^2\}$.*

Lemma 1.4. *For $\eta\in(0,1)$, $\phi_\eta:\Delta_\delta\to\Delta_\delta$, with $\phi_\eta(w)=\delta\frac{w-\delta\eta}{\delta-\eta w}$, is an automorphism mapping $\eta\delta$ to the origin. Furthermore*

$$\{w\in\Delta_\delta:0<\delta Im(w)<\pi|\delta-w|^2,\ |w|^2+(-\log r)|\delta-w|^2<\delta^2\}$$
$$\subset\bigcup_{0<\eta<1}\phi_\eta^{-1}\left(\Delta_{\frac{1}{2}\delta}\right),$$

when r is chosen small enough.

If we shrink $\tilde U_{x_0}$ to $(\Delta_{r\delta}^*)^{i_1+k_1}\times(\Delta_{r\delta})^{n-i_1-k_1}$, then it is covered by

$$\bigcup_{\substack{0<\eta_l<1\\ \theta_l\in[0,2\pi)\\ l=1,\cdots,i_1+k_1}}\rho_{\theta_1}\circ\phi_{\eta_1}^{-1}\left(\Delta_{\frac{1}{2}\delta}\right)\times\cdots\times\rho_{\theta_{i_1+k_1}}\circ\phi_{\eta_{i_1+k_1}}^{-1}\left(\Delta_{\frac{1}{2}\delta}\right)\times\left(\Delta_{\frac{1}{2}\delta}\right)^{n-i_1-k_1}.$$

Consider immersion $F : \Delta^n_{\frac{1}{2}\delta} \to \tilde{U}_{x_0}$, with

$$F(v^1, \cdots, v^n) = \left(\rho_{\theta_1} \circ \phi^{-1}_{\eta_1}(v^1), \cdots, \rho_{\theta_{i_1+k_1}} \circ \phi^{-1}_{\eta_{i_1+k_1}}(v^{i_1+k_1}), v^{i_1+k_1+1}, \cdots, v^n\right).$$

Then by some simple computations, we get

$$w^l = \rho_{\theta_l} \circ \phi^{-1}_{\eta_l}(v^l) = \delta \exp\left(\frac{(1+\eta_l)(v^l+\delta)}{(1-\eta_l)(v^l-\delta)} + \sqrt{-1}\theta_l\right), \tag{1.1}$$

$$\frac{\partial}{\partial v^l} = -\frac{2\delta^2(1+\eta_l)}{(1-\eta_l)(v^l-\delta)^2} \exp\left(\frac{(1+\eta_l)(v^l+\delta)}{(1-\eta_l)(v^l-\delta)} + \sqrt{-1}\theta_l\right) \frac{\partial}{\partial w^l} \tag{1.2}$$

$$\log|w^l|^2 = 2\log\delta + 2\frac{(1+\eta_l)(|v^l|^2-\delta^2)}{(1-\eta_l)|v^l-\delta|^2} \tag{1.3}$$

for $l = 1, \cdots, i_1 + k_1$.

On $\tilde{U}_{x_0}$, we can write $\omega_0 = \omega_0^{'} + \omega_0^{''}$, where

$$\begin{aligned}\omega_0^{''} = & \sum_{k=1}^{k_1} \frac{\frac{\sqrt{-1}}{\pi} \cdot \partial \log \|v_k\|^2_{k,3} \wedge \bar{\partial} \log \|v_k\|^2_{k,3}}{(\log \varepsilon \|v_k\|^2_{k,3})^2} \\ & + \sum_{l=1}^{i_1} \frac{\frac{\sqrt{-1}}{\pi} \cdot \partial \log \|s_l\|^2_l) \wedge \bar{\partial} \log \|s_l\|^2_l}{(\log \varepsilon \|s_l\|^2_l)^2} \\ & + \frac{\frac{\sqrt{-1}}{\pi} \cdot \partial \log \prod_{l=1}^{i_1} \|s_l\|^2_l \wedge \bar{\partial} \log \prod_{l=1}^{i_1} \|s_l\|^2_l}{(\log \varepsilon \prod_{l=1}^{i_1} \|s_l\|^2_l)^2}\end{aligned}$$

$$\omega_0^{'} = \omega_0 - \omega_0^{''} = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n} b_{i\bar{j}} dw^i \wedge d\bar{w}^j.$$

Then $\omega_0^{'}$ is a positive (1,1)-form on $\tilde{U}_{x_0}$. By the choice of $(w^1, \cdots, w^n)$, we may assume $\|v_k\|^2_{k,3} = h_{i_1+k}|w^{i_1+k}|^2$ and $\|s_l\|_l = h_l|w^l|^2$ for some positive functions h_i on $\tilde{U}_{x_0}$, $i = 1, \cdots, i_1 + k_1$. Substituting (1.1), (1.2) and (1.3) into $F^*\omega_0$, we obtain

$$\begin{aligned}F^*\omega_0 = & \frac{\sqrt{-1}}{2\pi} \sum_{i,j \geq i_1+k_1+1} b_{i\bar{j}}(F(v^1, \cdots, v^n)) dv^i \wedge d\bar{v}^j \\ & + \frac{\sqrt{-1}}{\pi} \sum_{l \leq i_1+k_1} \Bigg[\frac{1}{\left(\delta^2 - |v^l|^2 - \frac{(1-\eta_l)|v^l-\delta|^2}{2(1+\eta_l)} \cdot \log(\varepsilon\delta^2 h_l)\right)^2}\end{aligned}$$

$$
\begin{aligned}
&\cdot \left(\delta^2 dv^l - \frac{(1-\eta_l)(v_l-\delta)^2}{2(1+\eta_l)} \cdot \partial \log h_l\right) \\
&\qquad \wedge \left(\delta^2 d\bar{v}^l - \frac{(1-\eta_l)(\bar{v}_l-\delta)^2}{2(1+\eta_l)} \cdot \bar{\partial} \log h_l\right)\Bigg] \\
&+ 4\delta^4 \sum_{i,j \le i_1+k_1} b_{i\bar{j}}(F(v^1, \cdots, v^n)) \frac{(1+\eta_i)(1+\eta_j)}{(1-\eta_i)(1-\eta_j)(v^i-\delta)^2(\bar{v}^j-\delta)^2} \\
&\cdot \exp\left(\frac{(1+\eta_i)(v^i+\delta)}{(1-\eta_i)(v^i-\delta)} + \sqrt{-1}\theta_i + \frac{(1+\eta_j)(\bar{v}^j+\delta)}{(1-\eta_j)(\bar{v}^j-\delta)} - \sqrt{-1}\theta_j\right) \\
&\cdot \frac{\sqrt{-1}}{2\pi} dv^i \wedge d\bar{v}^j \\
&- 4\delta^2 \operatorname{Re}\Bigg[\sum_{\substack{i \le i_1+k_1 \\ j \ge i_1+k_1+1}} b_{i\bar{j}}(F(v^1, \cdots, v^n) \frac{1+\eta_i}{(1-\eta_i)(v^i-\delta)^2} \\
&\qquad \cdot \exp\left(\frac{(1+\eta_i)(v^i+\delta)}{(1-\eta_i)(v^i-\delta)} + \sqrt{-1}\theta_i\right) \frac{\sqrt{-1}}{2\pi} dv^i \wedge d\bar{v}^j\Bigg] \\
&+ \frac{\sqrt{-1}}{\pi} \cdot \frac{1}{\left(\sum_{l \le i_1} \frac{2(1+\eta_l)(|v^l|^2-\delta^2)}{(1-\eta_l)|v^l-\delta|^2} + \log(\varepsilon \delta^{2i_1} \prod_{l \le i_1} h_l)\right)^2} \\
&\cdot \sum_{l \le i_1} \left(\frac{-2\delta^2(1+\eta_l)}{(1-\eta_l)(v^l-\delta)^2} dv^l + \partial \log h_l\right) \\
&\qquad \wedge \sum_{l \le i_1} \left(\frac{-2\delta^2(1+\eta_l)}{(1-\eta_l)(\bar{v}^l-\delta)^2} d\bar{v}^l + \bar{\partial} \log h_l\right)
\end{aligned} \tag{1.4}
$$

From the fact that $\lim x \to \infty x^p \exp(-x) = 0$ for any real number p, it is east to see that in (1.4) the first two terms are equivalent to a Euclidean metric on $\Delta^n_{\frac{1}{2}\delta}$, the next two terms are very small since we may choose η_l close to 1, the last term is positive and bounded. It is straight forward to check that $F^*\omega_0$ has bounded geometry.

(ii) We show that g_t has uniformly bounded curvature tensor for $t \neq 0$ small. It is known that outside a neighborhood of $\mathrm{Sing}(X_0)$, g_t has uniformly bounded geometry ([TY]). It suffices to bound curvature tensor $Rm(g_t)$ on some neighborhood of $\mathrm{Sing}(X_0)$. For any $x_0 \in \mathrm{Sing}(X_0)$, for simplicity, we assume that $x_0 \in (\cap_{l=1}^{i_1} X_{0l}) \cap (\cap_{j=1}^{j_1} C_{j0}) \cap (\cap_{k=1}^{k_1} D_{k0})$, $x_0 \notin X_{0l}, l > i_1$, $x_0 \notin C_{j0}, j > j_1$, and $x_0 \notin D_{k0}, k > k_1$. Take a neighborhood of x_0 with coordinate system $(z^1, \cdots, z^{n+1})$ in $\mathcal{X}$ such that X_{0l} is defined by

$$z^l = 0, l = n+2-i_1, \cdots, n+1,$$

D_k is defined by

$$z^k = 0, k = 1, \cdots, k_1,$$

and C_j is defined by

$$z^{k_1+j} = 0, j = 1, \cdots, j_1.$$

On the uniformization $p_{x_0} : \tilde{U}_{x_0} \to U_{x_0}$ we have coordinate system $(w^1, \cdots, w^{n+1})$. We may identify $\tilde{U}_{x_0}$ with $(\Delta_\delta^*)^{k_1} \times (\Delta_\delta)^{n-k_1}$, where $\delta > 0$ only depends on $\mathcal{X}$, $\mathcal{C}$, and $\mathcal{D}$. Then

$$\begin{aligned}&\tilde{U}_{x_0} \cap p_{x_0}^{-1}(X_t)\\ &= \{(w^1, \cdots, w^{n+1}) : w^{n+2-i_1} \cdots w^{n+1} = t, \ |w^l| < \delta, \ l = 1, \cdots, n+1\}.\end{aligned}$$

Assume $\|v_k\|_k = h_k|w^k|^2$ and $\|s_l\|_l = h_l|w^l|^2$ for $k = 1, \cdots, k_1$, $l = n+2-i_1, \cdots, n+1$. By the definition of metric g_t, we have (here we take $i = n+1$ in the definition of V_{it})

$$\begin{aligned}p_{x_0}^*\omega_t = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\Bigg[&\frac{b}{\prod_{k=1}^{k_1}\varepsilon|w^k|^{2\mu_k}(-\log(h_k|w^k|^2))^2}\\ &\cdot\frac{1}{\prod_{j=1}^{j_1}\varepsilon|w^{k_1+j}|^2\prod_{l=n+2-i_1}^{n}\varepsilon|w^l|^2}\\ &\cdot\prod_{l=n+2-i_1}^{n}\left(\frac{\pi}{\log\varepsilon|t|}csc\frac{\pi\log\varepsilon h_l|w^l|^2}{2\log\varepsilon|t|}\right)^2\\ &\cdot\left(\frac{\pi}{\log\varepsilon|t|}csc\frac{\pi\log(\varepsilon\prod_{l=n+2-i_1}^{n}h_l|w^l|^2)}{2\log\varepsilon|t|}\right)^2\Bigg]\end{aligned}$$

where b is a smooth function of

$$w^1, \bar{w}^1, \cdots, w^n, \bar{w}^n, \frac{t}{\prod_{l=n+2-i_1}^{n} w^l}, \frac{\bar{t}}{\prod_{l=n+2-i_1}^{n}\bar{w}^l}$$

with $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log b$ positive definite. By exchanging w^{n+1} with one of $w^{n+2-i_1}, \cdots, w^n$ if necessary, we may assume $|w^l| \geq \sqrt{|t|}$, $l = n+2-i_1, \cdots, n$. Then simple computations show

$$\omega_t = \frac{\sqrt{-1}}{2\pi}\sum_{\alpha,\beta=1}^{n+1} h_{\alpha\bar{\beta}}dw^\alpha \wedge d\bar{w}^\beta - \frac{\sqrt{-1}}{2\pi}\sum_{k=1}^{k_1}\partial\bar{\partial}\log(\log\varepsilon h_k|w^k|^2)^2$$

$$
\begin{aligned}
&+\frac{\sqrt{-1}}{2\pi}\sum_{l=n+2-i_1}^{n}\partial\bar\partial\log\left(\frac{\pi}{\log\varepsilon|t|}csc\frac{\pi\log\varepsilon h_l|w^l|^2}{2\log\varepsilon|t|}\right)^2\\
&+\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log\left(\frac{\pi}{\log\varepsilon|t|}csc\frac{\pi\log\left(\varepsilon\prod_{l=n+2-i_1}^{n}h_l|w^l|^2\right)}{2\log\varepsilon|t|}\right)^2\\
=&\frac{\sqrt{-1}}{2\pi}\sum_{\alpha,\beta=1}^{n+1}h_{\alpha\bar\beta}dw^\alpha\wedge d\bar w^\beta\\
&+\frac{\sqrt{-1}}{\pi}\sum_{k=1}^{k_1}\frac{\left(\frac{dw^k}{w^k}+\partial\log h_k\right)\wedge\left(\frac{d\bar w^k}{\bar w^k}+\partial\log h_k\right)}{(\log\varepsilon h_k|w^k|^2)^2}\\
&+\frac{\sqrt{-1}}{\pi}\sum_{k=1}^{k_1}\frac{\partial\bar\partial\log h_k|w^k|^2}{(-\log\varepsilon h_k|w^k|^2)}\\
&-\frac{\sqrt{-1}}{\pi}\sum_{l=n+2-i_1}^{n}\frac{\pi}{2\log\varepsilon|t|}\cdot ctg\frac{\pi\log\varepsilon h_l|w^l|^2}{2\log\varepsilon|t|}\cdot\frac{\partial^2\log h_l}{\partial w^\alpha\partial\bar w^\beta}dw^\alpha\wedge d\bar w^\beta\\
&+\frac{\sqrt{-1}}{\pi}\sum_{l=n+2-i_1}^{n}\left(\frac{\pi}{2\log\varepsilon|t|}csc\frac{\pi\log\varepsilon h_l|w^l|^2}{2\log\varepsilon|t|}\right)^2\\
&\quad\cdot\left(\frac{dw^l}{w^l}+\partial\log h_l\right)\wedge\left(\frac{d\bar w^l}{\bar w^l}+\bar\partial\log h_l\right)\\
&-\frac{\sqrt{-1}}{\pi}\cdot\frac{\pi}{2\log\varepsilon|t|}\cdot ctg\frac{\pi\log(\varepsilon\prod_{l=n+2-i_1}^{n}h_l|w^l|^2)}{2\log\varepsilon|t|}\\
&\quad\cdot\frac{\partial^2\log\prod_{l=n+2-i_1}^{n}h_l}{\partial w^\alpha\partial\bar w^\beta}dw^\alpha\wedge d\bar w^\beta\\
&+\frac{\sqrt{-1}}{\pi}\left(\frac{\pi}{2\log\varepsilon|t|}csc\frac{\pi\log(\varepsilon\prod_{l=n+2-i_1}^{n}h_l|w^l|^2)}{2\log\varepsilon|t|}\right)^2\\
&\quad\cdot\sum_{l=n+2-i_1}^{n}\left(\frac{dw^l}{w^l}+\partial\log h_l\right)\wedge\sum_{l=n+2-i_1}^{n}\left(\frac{d\bar w^l}{\bar w^l}+\bar\partial\log h_l\right),
\end{aligned}
$$

where $h_{\alpha\bar\beta}$ and $\frac{\partial^2\log h_i}{\partial w^\alpha\partial\bar w^\beta}$ are smooth functions of

$$
w^1,\bar w^1,\cdots,w^n,\bar w^n,\frac{t}{\prod_{l=n+2-i_1}^{n}w^l},\frac{\bar t}{\prod_{l=n+2-i_1}^{n}\bar w^l}
$$

and $h_{\alpha\bar\beta}$ is positive definite on $\tilde U_{x_0}$.

Define

$$(1.5)\qquad \begin{aligned} \omega_t' = {} & \frac{\sqrt{-1}}{2\pi}\sum_{\alpha,\beta=1}^{n-1} h_{\alpha\bar\beta}(w^1,\bar w^1,\cdots,w^n,\bar w^n,0,0)dw^\alpha\wedge d\bar w^\beta \\ & + \frac{\sqrt{-1}}{\pi}\sum_{k=1}^{k_1}\frac{dw^k\wedge d\bar w^k}{(|w^k|\cdot\log\varepsilon h_k|w^k|^2)^2} \\ & + \frac{\sqrt{-1}}{\pi}\sum_{l=n+2-i_1}^{n}\left(\frac{\pi}{2\log\varepsilon|t|}csc\frac{\pi\log\varepsilon h_l|w^l|^2}{2\log\varepsilon|t|}\right)^2\frac{dw^l\wedge d\bar w^l}{|w^l|^2}, \end{aligned}$$

and

$$(1.6)\qquad \begin{aligned} \omega_t'' = \omega_t' + {} & \frac{\sqrt{-1}}{\pi}\left(\frac{\pi}{2\log\varepsilon|t|}csc\frac{\pi\log(\varepsilon\prod_{l=n+2-i_1}^{n}h_l|w^l|^2)}{2\log\varepsilon|t|}\right)^2 \\ & \cdot\sum_{l=n+2-i_1}^{n}\frac{dw^l}{w^l}\wedge\sum_{l=n+2-i_1}^{n}\frac{d\bar w^l}{\bar w^l}. \end{aligned}$$

Let g_t' and g_t'' be the metric corresponding to ω_t' and ω_t'' respectively. Using the fact that $|w^l|\geq\sqrt{|t|}, l=n+2-i_1,\cdots,n$, g_t' has uniformly bounded curvature tensor. g_t'' is equivalent to g_t' uniformly in t, and their difference is bounded in C^3-norm defined by g_t'. So g_t'' has uniformly bounded curvature tensor. One can check that $\omega_t-\omega_t''$ are uniformly small in C^3-topology with respect to the metric g_t' when we choose ε small. So g_t has uniformly bounded curvature tensor on $\tilde U_{x_0}$. Lemma 1.2 is proved. □

Lemma 1.5. *For the metric g_t in Lemma 1.1, there is a smooth function f on $\mathcal{X}\setminus(\mathcal{C}\cup\mathcal{D}\cup\mathrm{Sing}(X_0))$ compatible with the V-manifold structure and bounded from above such that*

$$\mathrm{ric}(g_t)+\omega_t = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial f_t,$$
$$\mathrm{ric}(g_0)+\omega_0 = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial f_0,$$

where $f_t = f|_{X_t\setminus(C_t\cup D_t)}$ and $f_0 = f|_{X_0\setminus(C_0\cup D_0\cup\mathrm{Sing}(X_0))}$. Furthermore $-\Delta_{g_t}f_t\leq C$ for some constant C independent of t.

Proof. We define f by(t may be 0)

$$f_t = -\log\frac{\omega_t^n}{V_t}.$$

It suffices to show that f_t and $-\Delta_{g_t} f_t$ are uniformly bounded from above near $\mathcal{D} \cup \mathrm{Sing}(X_0)$. Here we only show that they are uniformly bounded from above near $\mathrm{Sing}(X_0)$, near $\mathcal{D}$ the proof is similar and easier. First we prove that f is bounded from above near $\mathrm{Sing}(X_0)$. Using the coordinate system $(w^1, \cdots, w^{n+1})$ in the proof of Lemma 1.2(ii), we have $\omega_t^n = (\omega_t^{''})^n(1+h)$, where h is a function with very small values. So we need to prove that $-\log \frac{(\omega_t^{''})^n}{V_t}$ is uniformly bounded from above.

Since $(\omega_1+\omega_2)^n \geq \omega_1^n$ when both ω_1 and ω_2 are non-negative (1,1)-forms, observing that last term in (1.6) is non-negative, we have

$$\begin{aligned} \frac{(\omega_t^{''})^n}{V_t} &\geq \frac{(\omega_t^{'})^n}{V_t} \\ &\geq C \cdot \prod_{k=1}^{k_1} \frac{1}{|w^k|^{2(1-\mu_k)}} \cdot \left(\frac{-\pi}{2\log \varepsilon |t|} csc \frac{\pi \log \varepsilon \prod_{l=n+2-i_1}^{n} h_l |w^l|^2}{2 \log \varepsilon |t|} \right)^{-2}, \end{aligned}$$

where C is a constant independent of t. Since $u \sin \frac{x}{u} \geq \frac{1}{2}x$ for $0 < x \leq u$, using the fact that $|w_l| \geq \sqrt{|t|}$, we have

$$\left(\frac{-\pi}{2\log \varepsilon |t|} csc \frac{\pi \log \varepsilon \prod_{l=n+2-i_1}^{n} h_l |w^l|^2}{2 \log \varepsilon |t|} \right)^{-2} \geq C \cdot \left(\sum_{l=n+2-i_1}^{n} \log \varepsilon h_l |w^l|^2 \right)^2.$$

So f is bounded from above by a positive constant independent of t.

Next we prove $-\Delta_{g_t} f_t \leq C$. Using the same coordinate system $(w^1, \cdots, w^{n+1})$ as above. Then by some computations we find

$$\begin{aligned} f_t &= -\log \frac{(\omega_t^{''})^n(1+h_t)}{V_t} \\ &= \log \prod_{k=1}^{k_1} |w^k|^{2(1-\mu_k)} - \log \left(\frac{\pi}{2\log \varepsilon |t|} csc \frac{\pi \log \varepsilon \prod_{l=n+2-i_1}^{n} h_l |w^l|^2}{2 \log \varepsilon |t|} \right)^{-2} + \tilde{h}_t, \end{aligned}$$

where $\tilde{h}$ has uniformly bounded C^3-norm with respect to metric g_t. However further computations show that both

$$-\Delta_{g_t} \log \prod_{k=1}^{k_1} |w^k|^{2(1-\mu_k)}$$

and

$$-\Delta_{g_t} \left[-\log \left(\frac{\pi}{2\log \varepsilon |t|} csc \frac{\pi \log \varepsilon \prod_{l=n+2-i_1}^{n} h_l |w^l|^2}{2 \log \varepsilon |t|} \right)^{-2} \right]$$

are bounded from above. So the lemma is proved. □

2. Proof of Theorem 0.1.

We adopt the notations of section 1. Now let $g_{E,t}$ be the Kähler-Einstein V-metrics on $(X_t \setminus D_t \cup \mathrm{Sing}(X_0))\left(\sum \frac{1}{m_j} C_{jt}\right)$ for each $t \in \Delta$. Then there are smooth V-functions φ_t on $(X_t \setminus D_t \cup \mathrm{Sing}(X_0))\left(\sum \frac{1}{m_j} C_{jt}\right)$ such that

$$\omega_{E,t} = \omega_t + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_t,$$

where $\omega_{E,t}$ are the Kähler forms associated with $g_{E,t}$. Furthermore, the following equation follows from $\mathrm{ric}(g_{E,t}) = -\omega_{E,t}$,

$$\omega_{E,t}^n = e^{f_t+\varphi_t}\omega_t^n,$$

where f_t is defined in Lemma 1.5.

Lemma 2.1. *There is a uniform constant C independent of t such that $sup|\varphi_t + f_t| \le C$ on $(X_t \setminus D_t \cup \mathrm{Sing}(X_0))\left(\sum \frac{1}{m_j} C_{jt}\right)$.*

Proof. It follows from maximal principle since by Lemma 1.2

$$(X_t \setminus D_t \cup \mathrm{Sing}(X_0))\left(\sum \frac{1}{m_j} C_{jt}\right)$$

has uniformly bounded geometry.(See [TY] or [B]) □

Lemma 2.2. (i) *There are two constants c and C both independent of t and x such that $e^{-c\varphi_t(x)}(n + \Delta_{g_t}\varphi_t(x)) \le C$.*

(ii) *For any compact set $K \subset \mathcal{X} \setminus (\mathcal{D} \cup \mathrm{Sing}(X_0))$, there is a uniform constant C_K depending on K but independent of t and x such that $n + \Delta_{g_t}\varphi_t \le C_K$.*

Proof. (i) Let x_0 be the point where $e^{-c\varphi_t(x)}(n+\Delta_{g_t}\varphi_t(x))$ attains its maximum. Note that in [Y1] this second-order derivative estimate of φ is bounded by a constant depending only on $supf$ and $sup(-\Delta_g f)$ for the complex

Monge-Apmére equation $(\omega_g + \partial\bar{\partial}\varphi)^n = e^{f+\varphi}\omega_g^n$. Using Lemma 1.2 and 1.5, we have the following estimate $n + \Delta_{g_t}\varphi_t(x_0) \leq C$ by the same computations as the second-order derivative estimate in [Y1] using quasi-coordinate system. On the other hand, from Lemma 2.1 and Lemma 1.5, we have $-\varphi_t(x) \leq f_t(x) + C \leq C' + C$. So $e^{-c\varphi_t(x_0)}(n + \Delta_{g_t}\varphi_t(x_0))$ is uniformly bounded.

(ii) From Lemma 2.1, for any compact set $K \subset \mathcal{X} \setminus (\mathcal{D} \cup \mathrm{Sing}(X_0))$, we can find C_K such that for any x in K, $|\varphi_t(x)| \leq C_K$. Now (ii) follows form (i). □

Corollary 2.1. *For any compact set $K \subset \mathcal{X} \setminus (\mathcal{D} \cup \mathrm{Sing}(X_0))$, there is a constant C_K depending on K but independent of t such that*

$$\sup_{X_t \cap K} \{|\varphi_t|, |\nabla^k_{g_t}\varphi_t| : 1 \leq k \leq 3\} \leq C_K.$$

Proof. By working in the quasi-coordinate system, we can show that $\varphi_t + f_t$ has uniformly bounded $C^{2,\alpha}$-norm on any compact set K(the proof is the same as the proof of Lemma 1.4 in [TY]). In the proof we need to use Lemma 2.1 above. The third derivative estimate of φ_t follows using Lemma 2.2 and the arguments in [Y1]. □

Now we conclude the proof of Theorem 0.1. For any sequence φ_{t_i}, by a diagonalizing argument using Corollary 2.1, we can find a subsequence which converges in the sense of Cheeger-Gromov under $C^{2,\frac{1}{2}}$-topology on $(X_0 \setminus D_0 \cup \mathrm{Sing}(X_0))\left(\sum \frac{1}{m_j} C_{j0}\right)$. Let φ_{t_i} be any convergent sequence and φ_∞ be the limit on $(X_0 \setminus D_0 \cup \mathrm{Sing}(X_0))\left(\sum \frac{1}{m_j} C_{j0}\right)$, then by Lemma 1.1 g_{E,t_i} will converge to a Kähler-Einstein V-metric $\tilde{g}_{E,0}$ outside $\mathrm{Sing}(X_0)$ in the sense of Cheeger-Gromov. We now prove that V-metric $\tilde{g}_{E,0}$ is complete. Fix a point P on $(X_0 \setminus D_0 \cup \mathrm{Sing}(X_0))\left(\sum \frac{1}{m_j} C_{j0}\right)$, let Q be another point close to $\mathrm{Sing}(X_0)$ and $\phi_{\beta,t}$ be defined as in Lemma 1.1. Then the distance between $\phi_{\beta,t_i}(P)$ and $\phi_{\beta,t_i}(Q)$ defined by metric g_{t_i} can be chosen arbitrary large if Q is close enough to $\mathrm{Sing}(X_0)$ and i is large enough. This is also true for the distance between $\phi_{\beta,t_i}(P)$ and $\phi_{\beta,t_i}(Q)$ defined by metric g_{E,t_i}. On the other hand if we choose i large enough, this distance approaches the distance between P and Q defined by metric $\tilde{g}_{E,0}$. So $\tilde{g}_{E,0}$ is complete ([Y2], p.474). However, the complete Kähler-Einstein V-metric on $(X_0 \setminus D_0 \cup \mathrm{Sing}(X_0))\left(\sum \frac{1}{m_j} C_{j0}\right)$ is unique([TY]), $\tilde{g}_{E,0} = g_{E,0}$.

This shows that φ_t converges to the unique smooth V-function φ_0 on $(X_0 \setminus D_0 \cup \mathrm{Sing}(X_0)) \left(\sum \frac{1}{m_j} C_{j0}\right)$ in the sense of Cheeger-Gromov, so $g_{E,t}$ converge to $g_{E,0}$ in the sense of Cheeger-Gromov.

Remark. In [T], Tian proved that for degeneration $\pi : \mathcal{X} \to \Delta$, the Peterson-Weil metrics bounded from above by $\frac{C \cdot |dz|^2}{|z|^2(-\log|z|^2)^3}$ on the punctured disc $\Delta \setminus \{0\}$ for some constant C. However we can not prove it for the degeneration family in Theorem 0.1 because we do not have $g_{E,t} \leq C \cdot g_t$ for some C independent of t.

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