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Harmonic and Quasi-Harmonic Spheres

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1. Introduction.

Let M, N be smooth compact Riemannian manifolds without boundary, m =dim$M$, and let $\phi : M \to N$ be a smooth map. Suppose that the sectional curvature of N is nonpositive, then Eells-Sampson proved in [ES] that ϕ is homotopic to a smooth harmonic map (such harmonic maps are unique except some special cases). The idea of the proof is to use the heat flow:

$$\partial_t u = \tau(u), \quad \text{in } M \times R_+, \tag{1.1}$$

$$u(x,0) = \psi(x), \ x \in M. \tag{1.2}$$

Here $\tau(u)$ is the stress tension-field of u so that $\tau(u) = 0$ if and only if u is a harmonic map. The key analytic estimate involved for the problem (1.1)-(1.2) is the following

$$\sup_{x \in M, t \geq t_0} |Du|^2(x,t) \leq C(t_0)E_0. \tag{1.3}$$

Here $t_0 > 0$, and $E_0 = \int_M |D\phi|^2(x)\,dx$. Note that (1.3) is always true for $t \leq t_0$, and t_0 is sufficiently small (depending on M,N and ϕ) and with $C(t_0)$ depending on $C^{1,\alpha}$ norm of ϕ.

The estimate (1.3) is derived from a Bochner-type identity and the fact that N is nonpositively curved. In particular, (1.3) is valid for every weakly harmonic map flow from M into N provided that N is nonpositively curved. (cf. Schoen[Sc])

One of the natural question is whether one may find some necessary and sufficient conditions for (1.3) to be valid. Or, for that matter, any other sufficient conditions (without referring to the curvature of N) that guaranttee (1.3) to hold.

It is, at least, the case for energy minimizing maps. Schoen-Uhlenbeck [SU] (Giaquinta-Giusti [GG] independently) proved that (1.3) is true for energy minimizing maps provided that there are no harmonic spheres S^l in N for $2 \leq l \leq m-1$. A smooth harmonic map from S^l to N is called a harmonic S^l, for $l \geq 2$, if it is not a constant map.

A year ago, the first author showed, see [L], that Schoen-Uhlenbeck's theorem remains to be true for stationary harmonic maps. In particular (1.3) is true for stationary harmonic maps whenever the universal cover $\tilde{N}$ of N supports a pointwise strictly convex function with quadratic growth. The later statement recovers essentially the result of Eells-Sampson [ES] for the static case.

The proofs in [L] seem to indicate that some more general statement may be true. In particular, the following

Conjecture. *Any weakly harmonic map of finite energy from M into N is smooth provided that there are no harmonic spheres S^l in N, for $2 \leq l \leq m-1$.*

Note that T. Rivieré [R] had constructed an example of a weakly harmonic map from B^3 into S^2 of finite energy, which is everywhere discontinuous. This, combines with a theorem of Evans [E] and Bethuel [B], implies that there are many exotic weakly harmonic maps into S^2 that are *definitely* not stationary.

One of the aims of the present work is to show another evidence (cf. Theorem A, Corollary B below) that the above (wild) conjecture may be true.

So far we have only discussed the static case. Is it possible also to recover the theorem of Eells-Sampson in the heat-flow case? The answer is yes for $m = 2$, see Struwe [S]. In general, Chen-Struwe [CS] have made an initial step. They proved the global existence of a partially regular weak-solution of (1.1)-(1.2) for any smooth, compact Riemannian manifold N. More precisely, they consider the gradient flows for the penalized energy:

$$I_\epsilon(u) = \int_M \left(\frac{1}{2}|Du|^2 + \frac{F(u)}{\epsilon^2}\right) dx, \tag{1.4}$$

where F is a smooth function of u such that

$$\begin{aligned} F(p) &= \mathrm{dist}^2(p, N), \text{ if } \mathrm{dist}(p, N) \leq \delta, \\ &= 4\delta^2, \text{ if } \mathrm{dist}(p, N) \geq 2\delta. \end{aligned}$$

Here we have viewed N as a submanifold of R^k(via Nash's embedding theorem), and δ is chosen so that $\mathrm{dist}^2(p, N)$ is smooth for

$$p \in \{p : \mathrm{dist}(p, N) \leq 2\delta\}.$$

For any $\epsilon > 0$, one can easily solve

$$\partial_t u_\epsilon - \Delta u_\epsilon - \frac{1}{\epsilon^2} f(u_\epsilon) = 0, \text{ in } M \times R_+, \tag{1.5}$$

$$u_\epsilon(x,0) = \phi(x), x \in M. \tag{1.6}$$

to find a global smooth solution. Here $f(u) = -\text{grad } F(u)$, Chen-Struwe [CS] then argued that, one may find a sequence $\epsilon_i \downarrow 0$ such that $u_{\epsilon_i} \rightharpoonup u$ in $H^1_{\text{loc}}(M \times R_+, N)$ and that u is a weak solution of (1.1)-(1.2). Moreover, the m-dimensional Hausdorff measure of the singular set of u, with respect to the parabolic metric, is locally finite. Later, Cheng [Ch] showed that $H^{m-2}(\text{sing } (u) \cap \{t = t_0\}) < \infty$, for any $t_0 > 0$.

The above conclusion seems, however, not strong enough to recover the main theorem of Eells-Sampson. Moreover, there are several rather natural questions which remain to be answered. For instance, is such u obtained in [CS] unique? (cf. Coron [C]) What is the relationship between critical points of $I_\epsilon(\cdot)$ and weakly harmonic maps from M into N?

In this paper we will use the gradient flow of $I_\epsilon(\cdot)$ to drive theorems similar to Schoen-Uhlenbeck [SU] and Lin [L1], and thus to recover the Eells-Sampson's theorem as a consequence. We should also establish some connections between critical points of $I_\epsilon(\cdot)$ and weakly harmonic maps from M into N.

Since our results and proofs are all local, we don't have to assume M to be compact. We can easily work with, say, a geodesic ball in M. For this reason and for the purpose of saving some notations, we shall simply work with the domain M being the unit ball in R^m.

Now let's state our main results, we start with the static case and consider solutions of

$$\Delta u_\epsilon + \frac{1}{\epsilon^2} f(u_\epsilon) = 0, \text{ in } B_1. \tag{1.7}$$

Theorem A. *Let $\epsilon_i \downarrow 0$, and u_{ϵ_i} be a sequence of solutions of (1.7) with $I_{\epsilon_i}(u_{\epsilon_i}) \le K < \infty$, and $u_{\epsilon_i} \to u$ weakly in $H^1(B_1)$. Suppose that there is no harmonic S^2 in N. Then*

$$e(u_{\epsilon_i})\, dx \equiv \left(\frac{1}{2}|Du_{\epsilon_i}|^2 + \frac{1}{\epsilon_i^2} F(u_{\epsilon_i})\right) dx \rightharpoonup \frac{1}{2}|Du|^2\, dx,$$

as Radon measures. In particular, $u_{\epsilon_i} \to u$ strongly in $H^1_{loc}(B_1)$, and $\int_{B_1} \frac{1}{\epsilon_i^2} F(u_{\epsilon_i})\, dx \to 0$.

Corollary B. *Under the assumption that there is no harmonic S^2 in N, the map u obtained in theorem A is a stationary harmonic map. In particular, the singular set of u has Hausdorff dimension at most $m-4$. If , in addition, N has no harmonic S^l for $3 \leq l \leq m-1$, then u is smooth and $u_{\epsilon_i} \to u$ in C^k norm, for any $k \geq 1$.*

Corollary B may be useful in order to study some "weakly stationary harmonic maps". We should not persuit this issue here.

Next we consider solutions of

$$\partial_t u_\epsilon - \Delta u_\epsilon - \frac{1}{\epsilon^2} f(u_\epsilon) = 0, \text{ in } B_1 \times (0,1). \tag{1.8}$$

Theorem C. *Let $\epsilon_i \downarrow 0$, u_{ϵ_i} be a sequence of solutions of (1.8) with*

$$\int_{B_1 \times [0,1]} (|\partial_t u_{\epsilon_i}|^2 + e(u_{\epsilon_i}))\, dx\, dt \leq K < \infty. \tag{1.9}$$

Suppose that there is no harmonic S^2 in N, and $u_{\epsilon_i} \to u$ weakly in $H^1(B_1 \times (0,1)$. Then

$$e(u_{\epsilon_i})\, dx\, dt \rightharpoonup \frac{1}{2}|Du|^2\, dxdt, \tag{1.10}$$

as Radon measures. In particular, $u_{\epsilon_i} \to u$ strongly in $H^1_{loc}(B_1 \times (0,1))$. The limit map u is a weak solution of (1.1), with $\mathcal{P}^m(sing(u)) = 0$, and u satisfies both energy inequality and monotonicity inequality (cf [CLL]). Here $\mathcal{P}^m$ denotes the m-dimensional Hausdorff measure with respect to the parabolic metric in R^{m+1}.

Remark. Naturally the solution u obtained in theorem C has the *small energy regularity* property (cf. [CS] or [CLL]). Moreover, it also satisfies the stationary condition introduced in [Fm]. One is then lead to the question as whether such weak solutions of (1.1) are unique (say, with respect to the Dirichlet boundary conditions), the answer remains open.

Theorem D. *Let $u \in H^1(M \times (0,1), N)$ be a solution of (1.1)-(1.2), which satisfies the energy monotonicity (3.11), the energy inequality (3.11"), and the small energy regularity (3.12) with $e(u_\epsilon)$ replaced by $\frac{1}{2}|Du|^2$. Then either u is smooth and*

$$\sup_{(x,t) \in M \times (0,1)} |Du|(x,t) < \infty, \tag{1.11}$$

or there is a harmoni S^l for some $l = 2, \cdots, m-1$, or there is a quasi-harmonic S^l for some $l = 3, \cdots, m$.

Here $\phi : R^l \to N$ is called a quasi-harmonic S^l, if ϕ is a nonconstant, smooth map from R^l to N such that it is a critical point of $\int_{R^l} |Dv|^2 e^{-\frac{|y|^2}{4}}\, dy$, i.e.,

$$\Delta\phi - \frac{1}{2} y \cdot D\phi + A(\phi)(D\phi, D\phi) = 0, \tag{1.12}$$

here A is the second fundamental form of N, and $\int_{R^l} |D\phi|^2 e^{-\frac{|y|^2}{4}}\, dy < \infty$. Note that, for such a ϕ, if one let $u(x,t) = \phi\left(\frac{x}{\sqrt{-t}}\right)$, then u is a self-similar solution of (1.1) from $R^l \times R_-$ into N.

From Ding-Lin [DL], we conclude that there are no quasi-harmonic S^l in N if the universal cover $\tilde{N}$ of N supports a pointwise strictly convex function with quadratic growth, Then Eells- Sampson's theorem follows from theorem D above.

We would like to point out for the special case that N is the unit circle in the complex plane, the compactness of solutions of (1.7) and (1.8) were discussed already in Lin [L2]. They are rather useful in the study of vortices, filaments, and codimension two submanifolds dynamics for Ginzburg-Landau type functionals.

Finally we would like to end the section with the following open questions.

Question. *For a compact, smooth Riemannian manifold N, are there any quasi-harmonic S^l, $l \geq 3$, of finite energy?*

In fact, besides a well-known theorem of Sacks-Uhlenbeck [SaU] which guarantees the existence of harmonic S^2, the authors are not aware of any general statement concerning the existence of harmonic S^l for $l \geq 3$.

2. Proof of Theorem A.

We will divide the proof into two cases.

Case 1. $m = 2$.

Define the concentration set Σ by

$$\Sigma = \bigcap_{r>0} \left\{ x \in B_1 : \liminf_{\epsilon \downarrow 0} \int_{B_r(x)} e(u_\epsilon) \geq \epsilon_0^2 \right\}$$

where ϵ_0 is the same constant as in Lemma 2.2 below. Then it is easy to see that Σ is closed and locally finite. Moreover, Lemma 2.2 implies that we can extract a subsequence of u_ϵ (denoted as itself) such that $u_\epsilon \to u$ in $C^1(B_1 \setminus \Sigma)$ locally and hence $u \in C^\infty(B_1, N)$ is a harmonic map, by the removable singularity theorem of [SaU].

Now we claim $\Sigma = \emptyset$. Suppose not. Then we choose $x_0 \in \Sigma$ and $r_0 > 0$ such that $B_{r_0}(x_0) \cap \Sigma = \{x_0\}$. Define (cf. [W])

$$Q_\epsilon(t) = \sup_{y \in B_{r_0}(x_0)} \int_{B_t(y)} e(u_\epsilon).$$

Then it is clear that there exist $t_\epsilon \downarrow 0$ and $x_\epsilon \to x_0$ such that

$$Q_\epsilon(t_\epsilon) = \int_{B_{t_\epsilon}(x_\epsilon)} e(u_\epsilon) = \frac{\epsilon_0^2}{2}.$$

Define rescaling maps $v_\epsilon : \Omega_\epsilon \to R^k$ by $v_\epsilon(x) = u_\epsilon(x_\epsilon + t_\epsilon x)$, we have

$$-\Delta v_\epsilon + \frac{1}{\left(\frac{\epsilon}{t_\epsilon}\right)^2} f(v_\epsilon) = 0, \text{ in } \Omega_\epsilon, \tag{2.1}$$

$$\int_{\Omega_\epsilon} \frac{1}{2}|Dv_\epsilon|^2 + \frac{1}{\left(\frac{\epsilon}{t_\epsilon}\right)^2} F(v_\epsilon) \le K < \infty, \tag{2.2}$$

and

$$\int_{B_1(y)} \frac{1}{2}|Dv_\epsilon|^2 + \frac{1}{\left(\frac{\epsilon}{t_\epsilon}\right)^2} F(v_\epsilon) \le \frac{\epsilon_0^2}{2}, \ \forall y \in \Omega_\epsilon, \tag{2.3}$$

with equality if $y = 0$. Here $\Omega_\epsilon = {t_\epsilon}^{-1}(B_{r_0}(x_0) \setminus \{x_\epsilon\})$. Therefore we may assume, by Lemma 2.2, that $v_\epsilon \to v$ in $H^1(R^2) \cap C^1(R^2)$ locally so that v satisfies

$$\epsilon_0^2 \le \int_{R^2} e(v) < \infty, \tag{2.4}$$

and either if $\epsilon / t_\epsilon \to 0$

$$\Delta v + A(v)(Dv, Dv) = 0, \text{ in } R^2, \tag{2.5}$$

or if $\epsilon/t_\epsilon \to \infty$

$$\Delta v = 0, \text{ in } R^2, \tag{2.6}$$

or if $\epsilon/t_\epsilon \to c > 0$

$$-\Delta v + \frac{1}{c^2} f(v) = 0, \text{ in } R^2. \tag{2.7}$$

Note that it is easy to see that v will be either nonconstant harmonic maps from S^2 into N or nonconstant harmonic function from S^2 in the cases of (2.5) and (2.6), which is impossible by (2.4), [SaU] and assumption on N. On the other hand, any v satisfying (2.7) and (2.4) must be constant. In fact, let $\phi \in C_0^\infty(B_2)$ be such that $\phi = 1$ on B_1, and define $\phi_n(x) = \phi(\frac{x}{n})$. Multiplying (2.7) by $\phi_n x \cdot Dv$, we get (as in the deriviation of the Pohozaev identity),

$$\int_{R^2} F(v)\phi_n \le C \int_{B_{2n}\setminus B_n} e(v) \to 0, \text{ as } n \to \infty.$$

Here we use (2.4). Therefore $F(v) \equiv 0$ and $\Delta v = 0$. Thus v is constant. □

Case 2. $m \ge 3$.

Let's first recall two key facts about u_ϵ as follows:

Lemma 2.1 (Energy Monotonicity Formula). *Let u_ϵ be as in theorem A. Then*

$$R^{2-m} \int_{B_R(x)} e(u_\epsilon) - r^{2-m} \int_{B_r(x)} e(u_\epsilon) = \int_{B_R(x)\setminus B_r(x)} |x|^{2-m} \left|\frac{\partial u_\epsilon}{\partial r}\right|^2 + 2\int_r^R \rho^{1-m} \int_{B_\rho(x)} \frac{F(u_\epsilon)}{\epsilon^2}, \tag{2.8}$$

for $\forall x \in B_1$ and $0 < r \le R < d(x, \partial B_1)$. In particular, $r^{2-m}\int_{B_r(x)} e(u_\epsilon)$ is monotonically non-decreasing with respect to r.

Lemma 2.2 (ϵ_0-Regularity Theorem). *Let u_ϵ be as in theorem A. Then there exist $\epsilon_0 > 0$ and $K_0 > 0$ such that if $R^{2-m}\int_{B_R(x)} e(u_\epsilon) \le \epsilon_0^2$, then*

$$\sup_{B_{\frac{R}{4}}(y)} e(u_\epsilon) \le K_0 R^{-m} \int_{B_R(x)} e(u_\epsilon), \tag{2.9}$$

for any $y \in B_{\frac{R}{2}}(x)$.

Now assume $u_\epsilon \rightharpoonup u_*$ weakly in $H^1(B_1)$, then $\mu_\epsilon \equiv e(u_\epsilon)\,dx \rightharpoonup \mu = \frac{1}{2}|Du_*|^2\,dx + \nu$ as Radon measures for some nonnegative Radon measure $\nu \geq 0$. Moreover, we define (cf. [Sc]),

$$\Sigma = \bigcap_{r>0} \left\{ x \in B_1 : \liminf_{\epsilon \downarrow 0} r^{2-m} \int_{B_r(x)} e(u_\epsilon)\,dx \geq \frac{\epsilon_0^2}{2} \right\}$$

Then (2.8) and (2.9) imply that Σ is closed and $H^{m-2}(\Sigma \cap B_R)$ is finite for any $R < 1$. Moreover, $u_\epsilon \to u_*$ in $C^1(B_1 \setminus \Sigma) \cap H^1(B_1 \setminus \Sigma)$ locally (after passing to subsequences, if needed) so that u_* is a weakly harmonic map on B_1, which is smooth away from Σ.

Claim 1. *$e(u_\epsilon) \to \frac{1}{2}|Du_*|^2$ in $B_1 \setminus \Sigma$ locally.*

To see this, we need to show $\nu(B_R) = 0$ for any ball $B_R \subset\subset B_1 \setminus \Sigma$. Letting $\epsilon \downarrow 0$, (2.8) implies

$$\text{(2.10)} \quad R^{2-m} \int_{B_R} \left(\frac{1}{2}|Du_*|^2\,dx + \nu\right) - r^{2-m} \int_{B_r} \left(\frac{1}{2}|Du_*|^2\,dx + \nu\right) = \int_{B_R \setminus B_r} |x|^{2-m} \left|\frac{\partial u_*}{\partial r}\right|^2 + 2\int_r^R \rho^{1-m}\nu(B_\rho).$$

Here we use that fact that $\frac{1}{\epsilon^2}F(u_\epsilon)\,dx \rightharpoonup \nu$ as Radon measures in $B_1 \setminus \Sigma$. On the other hand, since u_* is a smooth harmonic map on $B_1 \setminus \Sigma$, u_* satisfies

$$R^{2-m} \int_{B_R} \frac{1}{2}|Du_*|^2\,dx - r^{2-m} \int_{B_r} \frac{1}{2}|Du_*|^2\,dx = \int_{B_R \setminus B_r} |x|^{2-m} \left|\frac{\partial u_*}{\partial r}\right|^2,$$

hence

$$R^{2-m}\nu(B_R) - r^{2-m}\nu(B_r) = 2\int_r^R \rho^{1-m}\nu(B_\rho),$$

which implies, for $0 < r \leq R$,

$$\frac{d}{dr}(r^{-m}\nu(B_r)) = 0.$$

Hence $\nu(B_R) = 0$. □

Claim 1 implies that $\mathrm{sing}(u_*) \cup \mathrm{spt}(\nu) \subset \Sigma$. In fact,

Claim 2. *$\mathrm{sing}(u_*) \cup \mathrm{spt}(\nu) = \Sigma$.*

If $x_0 \notin \operatorname{sing}(u_*) \cup \operatorname{spt}(\nu)$, then u_* is smooth near x_0 and $\nu(B_{r_0}(x_0)) = 0$ for $r_0 > 0$ small so that

$$r_0^{2-m} \int_{B_{r_0}(x_0)} \left(\frac{1}{2}|Du_*|^2\, dx + \nu\right) \leq \frac{\epsilon_0^2}{4}.$$

Hence $r_0^{2-m} \int_{B_{r_0}(x_0)} e(u_\epsilon) < \frac{\epsilon_0^2}{2}$ for sufficiently small ϵ and $x_0 \notin \Sigma$ by Lemma 2.2. □

Lemma 2.1 also implies

$$R^{2-m}\mu(B_R(x)) \geq r^{2-m}\mu(B_r(x)), \tag{2.11}$$

for $0 < r \leq R < d(x, \partial B_1)$ and $\forall x \in B_1$. $\Theta^{m-2}(\mu, x) = \lim_{r\downarrow 0} r^{2-m}\mu(B_r(x))$ exists for all $x \in B_1$. Moreover,

$$\Theta^{m-2}(\nu, x) = \Theta^{m-2}(\mu, x)$$

for H^{m-2} a.e. $x \in \Sigma$, since $\lim_{r\downarrow 0} r^{2-m} \int_{B_r(x)} |Du_*|^2 = 0$ for H^{m-2} a.e. $x \in \Sigma$ (cf. Federer-Ziemer [FZ]). Note also, by definition of Σ and (2.8), that

$$\frac{\epsilon_0^2}{2} \leq \Theta^{m-2}(\mu, x) \leq C(K, \rho), \ \forall x \in \Sigma \cap B_\rho. \tag{2.12}$$

Suppose now that

$$e(u_\epsilon)\, dx \not\rightharpoonup \frac{1}{2}|Du_*|^2\, dx.$$

Then we must have

$$H^{m-2}(\Sigma) > 0, \ \text{ and } \nu(B_1) > 0.$$

From (2.11), one knows $\Theta^{m-2}(\mu, x)$ is upper semicontinuous. Therefore there exists $\bar\Sigma \subset \Sigma$, with $H^{m-2}(\bar\Sigma) = H^{m-2}(\Sigma) > 0$, such that $\Theta^{m-2}(\mu, x) (= \Theta^{m-2}(\nu, x))$ is approximately continuous for $x \in \bar\Sigma$, (cf.[F]).

For $\bar\Sigma$, we need the following geometric Lemma (cf. [L]),

Lemma 2.3. *There exists $E \subset \bar\Sigma$ with $H^{m-2}(E) > 0$ such that $\forall x \in E$ and $r_i \downarrow 0$ there are $\{x_i^j\}_{j=1}^{m-2} \subset \bar\Sigma \cap B_{r_i}(x)$ satisfying*

$$|x_i^j - x_i^k| \geq \delta r_i, 0 \leq j < k \leq m-2,$$
$$dist(x_i^j - x_i^0, span\{x_i^1 - x_i^0, \cdots, x_i^{j-1} - x_i^0\}) \geq \delta r_i, \ 2 \leq j \leq m-2,$$

for some uniform $\delta > 0$, here $x_i^0 = x$. □

Now pick a $x_0 \in E$, with $\Theta^{m-2}(\mu, x_0) > 0$, $\limsup_{r\downarrow 0} r^{2-m}H^{m-2}(\Sigma \cap B_r(x_0)) > 0$ and $\lim_{r\downarrow 0} r^{2-m}\int_{B_r(x_0)} |Du_*|^2 = 0$. Define the rescaling measures μ_i by letting $\mu_i(A) = r_i^{2-m}\mu(x_0 + r_i A)$ for any $A \subset R^m$. Then (2.11) gives $2^{m-2}\epsilon_0^2 \le \mu_i(B_2) \le C(x_0, K) < \infty$. One can then assume that $\mu_i \to \mu_*$ for some nonnegative Radon measure μ_*. By the diagonal process, one can see that

$$e(u_{\epsilon_i})\,dx \to \mu_*,\ u_{\epsilon_i} \to \text{constant weakly in } H^1,$$

here $\epsilon_i = r_i^{-1}\epsilon$. Then Σ_* , the support of μ_*, has $H^{m-2}(\Sigma_*) > 0$ and $\Theta^{m-2}(\mu_*, x) = \Theta^{m-2}(\mu, x_0)$ for any $x \in \Sigma_*$. Moreover, there exist $\{\xi^j\}_{j=1}^{n-2} \subset \Sigma_* \setminus B_\delta$ such that

$$\begin{gathered}|\xi^j - \xi^l| \ge \delta, 1 \le j < l \le m-2,\\ \mathrm{dist}(\xi^j, \mathrm{span}(\xi^1, \cdots, \xi^{j-1})) \ge \delta, 2 \le j \le m-2,\end{gathered}$$

with $\delta > 0$ is the same constant as in Lemma 2.3 . Denote $R^{m-2} = \mathrm{span}\{\xi^1, \cdots, \xi^{m-2}\} = \{(0, 0, x_3, \cdots, x_m) \in R^m\}$. Applying (2.8) with centers at $0, \xi^1, \cdots, \xi^{m-2}$, we can find a $\xi_0 \in \Sigma_*$ such that

$$\int_{B_{\frac{\delta}{4}}(\xi_0)} |D_T u_{\epsilon_i}|^2 + \epsilon_i^{-2}F(u_{\epsilon_i}) \to 0,$$

where T denotes vectors in R^{m-2}, the span of $\{\xi^i\}_{i=1}^{m-2}$. Hence

$$\int_{B^2_{\frac{\delta}{4}}(\xi_0)\times B^{m-2}_{\frac{\delta}{4}}(\xi_0)} \sum_{j=3}^{m} \left|\frac{\partial u_{\epsilon_i}}{\partial x_j}\right|^2 + \epsilon_i^{-2}F(u_{\epsilon_i}) \to 0, \tag{2.13}$$

which, combimes with the weak L^1 estimates of Hardy-Littlewood maximal functions (cf.[Se]), implies that there exists $A_i \subset B^{m-2}_{\frac{\delta}{4}}(\xi_0)$ with $H^{m-2}(A_i) > 0$ such that for any $p_i \in A_i$,

$$\sup_{r\in(0,\frac{\delta}{4})} r^{2-m}\int_{B_r^{m-2}(p_i)} f_i \to 0, \text{as } i \to \infty, \tag{2.14}$$

where $f_i = \int_{B^2_{\frac{\delta}{4}}(\xi_0)} \sum_{j=3}^m \left|\frac{\partial u_{\epsilon_i}}{\partial x_j}\right|^2 + \epsilon_i^{-2}F(u_{\epsilon_i})$. Now we have

Claim 3. $\mu_*(x_1, x_2, x_3, \cdots, x_m) = \Theta^{m-2}(\mu, x_0)H^{m-2}L\ (R^{m-2} \times \{\xi_0\})$.

To see this, let $\phi \in C_0^\infty(B^2_{\frac{\delta}{4}}(\xi_0))$, then for $3 \le j \le m$,

$$(2.15)\quad \frac{\partial}{\partial x_j}\int_{B^2_{\frac{\delta}{4}}(\xi_0)} \phi^2 e(u_{\epsilon_i}) = -2\int_{B^2_{\frac{\delta}{4}}(\xi_0)} \sum_{k=1}^{2} \phi \frac{\partial \phi}{\partial x_k}\frac{\partial u_{\epsilon_i}}{\partial x_k}\frac{\partial u_{\epsilon_i}}{\partial x_j} + \sum_{l=3}^{m} \frac{\partial}{\partial x_l}\int_{B^2_{\frac{\delta}{4}}(\xi_0)} \phi^2 \frac{\partial u_{\epsilon_i}}{\partial x_j}\frac{\partial u_{\epsilon_i}}{\partial x_l}.$$

Therefore (2.13)implies, for $3 \le j \le m$,

$$\frac{\partial}{\partial x_j}\int_{B^2_{\frac{\delta}{4}}(\xi_0)} \phi^2 e(u_{\epsilon_i}) \to 0, \text{ in } \mathcal{D}'(B^{m-2}_{\frac{\delta}{4}}(\xi_0)).$$

Which implies $\mu_*(x) = g(x_1, x_2)\, dx_3 \cdots dx_m$ for some Radon measure $g \ge 0$ whose support consists of finitely many points . For (x_1, x_2) $\in$spt$(g)\cap B^2_{\frac{\delta}{4}}(\xi_0)$, we have $g(x_1, x_2) = \Theta^{m-2}(\mu, x_0)$ and $\mu_* = \Theta^{m-2}(\mu, x_0) H^{m-2} \mathrm{L}\ (R^{m-2} \times \{\xi_0\})$. $\square$

Claim 3 clearly implies

$$e(u_{\epsilon_i})\, dx \to 0, \text{ in } B^m_{\frac{\delta}{4}}(\xi_0) \setminus R^{m-2} \times \{\xi_0\} \text{ locally.}$$

Now we start the bubble process as follows: choose $p_i \in A_i$, with $|p_i - \xi_0| \le \frac{\delta}{8}$, there exist $y_i \in B^2_{\frac{\delta}{8}}(\xi_0)$ and $\delta_i \downarrow 0$ such that

$$\int_{B^2_{\delta_i}(y_i)} e(u_{\epsilon_i})(y, p_i)\, dy = \max\left\{\int_{B^2_{\delta_i}(z)} e(u_{\epsilon_i})(y, p_i)\, dy : z \in B^2_{\frac{\delta}{4}}(\xi_0)\right\} = \frac{\epsilon_0^2}{C(m)},$$

for some large $C(m)$. Define the rescaling maps $v_i(x) = u_{\epsilon_i}((y_i, p_i) + \delta_i x)$, then

$$(2.16)\qquad -\Delta v_i + \frac{1}{(\delta_i^{-1}\epsilon_i)^2} f(v_i) = 0, \text{ in } B_{\frac{\delta}{8\delta_i}}(0).$$

(2.17)

$$r^{2-m}\int_{B_r^{m-2}(0)}\int_{B^2_{\frac{\delta}{8\delta_i}}(0)} \sum_{j=3}^{m}\left|\frac{\partial v_i}{\partial x_j}\right|^2 + \frac{1}{(\delta_i^{-1}\epsilon_i)^2} F(v_i) \to 0,$$

for $r < \frac{\delta}{8\delta_i}$.

$$(2.18)\quad \int_{B_1^2(0)} \frac{1}{2}|Dv_i|^2 + \frac{1}{(\delta_i^{-1}\epsilon_i)^2}F(v_i)(x_1,0)\,dx_1 = \frac{\epsilon_0^2}{C(m)}$$
$$\geq \int_{B_1^2(y)} \frac{1}{2}|Dv_i|^2 + \frac{1}{(\delta_i^{-1}\epsilon_i)^2}F(v_i)(x_1,0)\,dx_1, y \in \Omega_i.$$

Moreover,

$$(2.19)\quad 2^{2-m}\int_{B_2^{m-2}(0)\times B_2^2(z)} \left[\frac{1}{2}|Dv_i|^2 + \frac{1}{(\delta_i^{-1}\epsilon_i)^2}F(v_i)\right]dx \leq \epsilon_0^2,\ \forall z \in \Omega_i.$$

Here $\Omega_i = \delta_i^{-1}(B^2_{\frac{\delta}{4}}(\xi_0))$.

To get (2.19), one may apply the Allard's Strong Constancy Lemma ([A] (Page 3-5)) to (2.15), with $a = p_i$, $\frac{\delta}{8} \leq r \leq \frac{\delta}{4}$, m replaced by $m-2$, $u = \int_{B^2_{\delta_i}(y_i)} \phi^2 e(u_{\epsilon_i}) \in \mathcal{D}'(B_r^{m-2}(a))$, $f_l = -2\int_{B^2_{\frac{\delta}{4}}(y_i)} \sum_{k=1}^2 \phi \frac{\partial\phi}{\partial x_k}\frac{\partial u_{\epsilon_i}}{\partial x_k}\frac{\partial u_{\epsilon_i}}{\partial x_l}$, $X_l^n = \int_{B^2_{\frac{\delta}{4}}(y_i)} \phi^2 \frac{\partial u_{\epsilon_i}}{\partial x_l}\frac{\partial u_{\epsilon_i}}{\partial x_n}$ for $1 \leq l, n \leq m-2$, then conditions (a)-(e) in [A] are all satisfied (with small δ in (c)), so that for any small $\beta > 0$ there exists $0 \leq c < \infty$ such that, for large i,

$$(2.20)\quad r^{2-m}\left|\int_{B_r^{m-2}(p_i)\times B^2_{\frac{\delta}{4}}(y_i)} \eta\phi^2 e(u_{\epsilon_i}) - c\int_{B_r^{m-2}(p_i)} \eta\right|$$
$$\leq \beta \sup\{|\eta|(x) : x \in B_r^{m-2}(p_i)\},\ \frac{\delta}{8} \leq r \leq \frac{\delta}{4},$$

for any $\eta \in C_0^\infty(B_r^{m-2}(p_i))$ with support in $B^{m-2}_{\frac{r}{2}}(p_i)$. Moreover, using (2.18), we see $c \leq \frac{2\epsilon_0^2}{C(m)}$. Hence, $\forall z \in B^2_{\frac{\delta}{4}}(\xi_0)$,

$$(2\delta_i)^{2-m}\int_{B_{2\delta_i}^{m-2}(p_i)\times B_{2\delta_i}^2(z)} e(u_{\epsilon_i}) \leq \frac{\epsilon_0^2}{2},$$

if we choose β so small and $C(m)$ sufficiently large. This gives (2.19). Applying Lemma 2.2, we conclude that $v_i \to v = v(x_1, x_2)$ in $C^1(R^2 \times B_1^{m-2})$ locally so that if $\frac{\epsilon_i}{\delta_i} \downarrow c > 0$, $\Delta v + \frac{1}{c^2}f(v) = 0$, or if $\frac{\epsilon_i}{\delta_i} \downarrow 0$, $v : R^2 \to N$ is a harmonic map. Moreover, the strong convergence and energy monotonicity inequality of v_i implies

$$0 < \int_{R^2} e(v) < \infty,$$

which is impossible from step 1. This finishes the proof of Theorem A. $\square$

Proof of Corollary B.

Since u_{ϵ_i} are smooth solutions of (1.7), we have

$$\frac{d}{dt}|_{t=0}\int_{B_1^m}\frac{1}{2}|Du_{\epsilon_i,t}(x)|^2+\frac{1}{\epsilon_i^2}F(u_{\epsilon_i,t})\,dx=0, \tag{2.21}$$

where $u_{\epsilon_i,t}(x)=u_{\epsilon_i}(x+t\xi(x))$, and $\xi\in C_0^1(B_1,R^m)$. Hence

$$\int_{\mathbb{B},m}\left(\frac{1}{2}|Du_{\epsilon_i}|^2+\frac{1}{\epsilon_i^2}F(u_{\epsilon_i})\right)\mathrm{div}\xi-\frac{\partial u_{\epsilon_i}}{\partial x_k}\frac{\partial u_{\epsilon_i}}{\partial x_l}\frac{\partial\xi^k}{\partial x_l}=0. \tag{2.22}$$

Now it is easy to see that (2.22), with the help from Theorem A that $e(u_{\epsilon_i})\,dx\to\frac{1}{2}|Du|^2\,dx$, implies

$$\int|Du|^2\mathrm{div}\xi-2\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_l}\frac{\partial\xi^k}{\partial x_l}=0. \tag{2.23}$$

Which implies that u is a stationary harmonic map (cf. [B]). The rest of statements of Corollary B follow from Theorem D of [L].

3. Proof of Theorem C.

Now we turn our attention to theorem C. We still divide the proof into two parts.

Part I. $m=2$.

We may assume that $u_{\epsilon_i}\to u_*$ weakly in $H^1(B_1\times(0,1))$, $e(u_{\epsilon_i})\,dxdt\to\frac{1}{2}|Du_*|^2\,dxdt+\nu(x,t)$ as Radon measures for some nonnegative Radon measure ν. If the strong convergence fails, then $\nu(B_1\times(0,1))>0$. Denote $\Sigma=\mathrm{spt}\nu$. From Part II below, we have $\mathcal{P}^2(\Sigma)>0$ and can pick a point $z_0=(x_0,t_0)\in\Sigma$ such that $\bar{\Theta}^2(\Sigma,(x_0,t_0))=\overline{\lim}_{r\downarrow0}r^{-2}\mathcal{P}^2(\Sigma\cap P_r(z_0))\geq\frac{1}{4}$ and $\lim_{r\downarrow0}r^{-2}\int_{P_r((x_0,t_0))}|Du_*|^2=0$. For $r_i\downarrow0$, we define maps $\bar{u}_{\epsilon_i}(x,t)=u_{\epsilon_i}((x_0,t_0)+(r_ix,r_i^2t))$, then we have

$$\bar{u}_{\epsilon_i}\to\text{constant weakly but not strongly in }H^1,\ e(\bar{u}_{\epsilon_i})\,dxdt\to\nu_*(x,t),$$

for some Radon measure ν_* with $\nu_*(P_1(0))>0$, here $P_r(z)=B_r(z)\times(-r^2,r^2)$. Note $0\in\Sigma_*\equiv\mathrm{spt}(\nu_*)$, and $\mathcal{P}^2(\Sigma_*)>0$. Moreover $\Sigma_*^t=\Sigma_*\cap\{t\}$ is finite (cf. Part II). In fact, there exists $\rho_0>0$ such that $e(\bar{u}_{\epsilon_i})\,dx\not\to0$ for each $t \in (-\rho_0^2, 0)$, since, otherwise, $\int_{B_1} e(\bar{u}_{\epsilon_i})(\cdot, -\rho_0^2)\, dx$ is small for large i and Lemma 3.3 below implies $0 \notin \Sigma_*$. Therefore, there exists nonnegative Radon measures μ_t, with $\mu_t(B_1) > 0$, such that $e(\bar{u}_{\epsilon_i})\, dx \to \mu_t$ for each $t \in (-\rho_0^2, 0)$. From Part II, we also know that $\text{spt}\mu_t \subset \Sigma_*^t$ for each t, hence, on $B_1 \times (-\rho_0^2, 0)$, $\mu_t = \sum_{j=1}^{N(t)} C(t,j)\delta_{x_j^t}$ for some $x_j^t \in B_1$, $C(t,j) \geq \epsilon_0^2$ and $1 \leq N(t) \leq C(K) < \infty$. Now, we choose a $t_0 \in (-\rho_0^2, 0)$ such that

$$\lim_{\epsilon_i \downarrow 0} \int_{B_1} |\partial_t \bar{u}_{\epsilon_i}|^2(x, t_0)\, dx < \infty. \tag{3.1}$$

Now pick $x_j^{t_0} \in B_1$ for some $1 \leq j \leq N(t_0)$ and let $r_0 > 0$ be small enough. Then, at $x_j^{t_0}$, similar to the proof of theorem A, there exist $\delta_i \downarrow 0$ and $x_i \to x_j^{t_0}$ such that

(3.2)

$$\int_{B_{\delta_i}(x_i)} e(\bar{u}_{\epsilon_i})(\cdot, t_0) = \frac{\epsilon_0^2}{2} = \max\left\{\int_{B_{\delta_i}(x)} e(\bar{u}_{\epsilon_i})(\cdot, t_0) : x \in B_{r_0}(x_j^{t_0})\right\}.$$

Now consider $v_i(x) = \bar{u}_{\epsilon_i}(x_i + \delta_i x, t_0)$, on $\Omega_i \equiv \delta_i^{-1}(B_{r_0}(x_j^{t_0}) \setminus \{x_i\})$, we have

$$\Delta v_i + \frac{1}{\left(\frac{\epsilon_i}{\delta_i}\right)^2} f(v_i) = g_i, \text{ in } \Omega_i, \tag{3.3}$$

where $g_i(x) = \delta_i^2 \partial_t \bar{u}_{\epsilon_i}(x_j^{t_0} + \delta_i x)$. Hence $\|g_i\|_{L^2(\Omega_i)} \to 0$, and

$$\int_{B_1(0)} e(v_i)\, dx = \frac{\epsilon_0^2}{2} \geq \int_{B_1(x)} e(v_i), \forall x \in \Omega_i. \tag{3.4}$$

It is easy to see that $v_i \to v_\infty$ weakly in $H^1(R^2)$ locally. In fact, Lemma 3.1 below shows that the convergence is strong in $H^1(R^2)$ locally. Hence, either v_∞ is a nonconstant harmonic map from R^2 to N or a nonconstant solution to $\Delta v_\infty + \frac{1}{c_0^2} f(v_\infty) = 0$ in R^2 for some $c_0 > 0$, both are impossible by the assumption and theorem A. □

Lemma 3.1 (ϵ_0**-Compactness**). *Let $\epsilon_n \downarrow 0$, there exists $\epsilon_0 > 0$ such that if $u_n \in H^1(B_1, R^k)$ satisfy*

$$\Delta u_n + \frac{1}{\epsilon_n^2} f(u_n) = f_n, \tag{3.5}$$

with $\int_{B_1} e(u_n) \leq \epsilon_0^2$, $\|Du_n\|_{L^\infty} \leq \frac{C}{\epsilon_n}$, and $\|f_n\|_{L^2(B_1)} \to 0$. Then $u_n \to u$ in $H^1(B_1)$ locally and $u : B_1 \to N$ is a harmonic map, i.e.,

$$\Delta u + A(u)(Du, Du) = 0. \tag{3.6}$$

Proof. First note that, for any small $\delta_0 > 0$, we can choose ϵ_0 sufficiently small such that if $\int_{B_1} e(u_n) \le \epsilon_0^2$ and $|Du_n| \le \frac{C}{\epsilon_n}$, then $\mathrm{dist}(u_n, N) \le \delta_0$. For simplicity, we assume N to be a unit sphere S^k (The general case can be modified easily, cf. page 344–345 of [CL]). Therefore, $\rho_n = |u_n|$ and $\psi_n = \frac{u_n}{|u_n|}$ satisfy

$$\Delta\rho_n + \frac{1}{\epsilon_n^2}\rho_n(1-\rho_n^2) + \rho_n|D\psi_n|^2 = g_n, \tag{3.7}$$

$$\mathrm{div}(\rho_n^2 D\psi_n) + \rho_n|D\psi_n|^2\psi_n = h_n. \tag{3.8}$$

with $\|g_n\|_{L^2}, \|h_n\|_{L^2} \le \|f_n\|_{L^2}$. Since $\|\rho_n - 1\|_{L^\infty} \le \delta_0$ is small, the Calderon-Zygmund theory [Se] implies that there exists $C > 0$ such that

$$\int_{B_1^2} |D\psi_n|^4 \le C\int_{B_1^2}|D\psi_n|^2\left(\int_{B_1^2}|\mathrm{div}(\rho_n^2 D\psi_n)|^2 + \int_{B_1^2}|D\psi_n|^2\right). \tag{3.9}$$

Applying (3.9) to (3.8), we have

$$\begin{aligned}\int_{B_1^2}|\mathrm{div}(\rho_n^2 D\psi_n)|^2 &\le C\int_{B_1^2}|D\psi_n|^4 + \int_{B_1^2}|h_n|^2\\ &\le C\epsilon_0^2\int_{B_1^2}|\mathrm{div}(\rho_n^2 D\psi_n)|^2 + \int_{B_1^2}(|D\psi_n|^2 + |h_n|^2),\end{aligned}$$

hence if we choose ϵ_0 small enough then

$$\int_{B_1^2}|\mathrm{div}(\rho_n^2 D\psi_n)|^2 \le C\int_{B_1^2}(|D\psi_n|^2 + |f_n|^2).$$

Which implies that $D\psi_n$ is uniformly bounded locally in L^4 . Hence (3.7) becomes

$$\left(\Delta - \frac{c(x)}{\epsilon_n^2}\right)(1-\rho_n) = \bar{g}_n, \tag{3.10}$$

for some bounded nonnegative $c(x)$, here $\bar{g}_n$ is uniformly bounded in L^2. Hence, by the standard elliptic estimates, $1 - \rho_n \to 0$ in $L^\infty \cap H^1(B_1)$ locally. Therefore, $u_n \to u$ in $H^1(B_1)$ locally. □

Part II. $m \ge 3$

In order to deal with this case, we first recall some notations and two key facts about u_ϵ (cf. [S] [CS]). Let u_ϵ be a solution to (1.8) in $B_1 \times (0, 1)$.

Let $\mathcal{P}^m$ denote the m-dimensional Hausdorff measure in R^{m+1} with respect to the parabolic metric $\delta((x,t),(y,s)) = \max\{|x-y|, \sqrt{|t-s|}\}$, and H^{m-2} denote the $m-2$ dimensional Hausdorff measure in R^m with respect to the standard metric. For $z_0 = (x_0, t_0) \in B_1 \times (0,1)$, denote G_{z_0} as the fundamental solution to the (backward) heat equation

$$G_{z_0}(x,t) = [4\pi(t_0-t)]^{-\frac{m}{2}} \exp\left(-\frac{|x-x_0|^2}{4(t_0-t)}\right), x \in B_1, t < t_0.$$

Also

$$P_R(z_0) = \{z = (x,t) \in B_1 \times (0,1) : |x-x_0| < R, |t-t_0|^2 \leq R^2\}.$$
$$S_R(z_0) = \{z = (x,t) \in B_1 \times (0,1) : t = t_0 - R^2\}.$$
$$T_R(z_0) = \{z = (x,t) \in B_1 \times (0,1) : t_0 - 4R^2 < t < t_0 - R^2\}.$$

Define

$$\Psi(u_\epsilon, z_0, R) = \int_{T_R(z_0)} \eta^2(x) e(u_\epsilon)(x,t) G_{z_0}(x,t)\, dxdt,$$

$$\Phi(u_\epsilon, z_0, R) = R^2 \int_{S_R(z_0)} \eta^2(x) e(u_\epsilon)(x,t) G_{z_0}(x,t)\, dx,$$

for $0 < R < \frac{\sqrt{t_0}}{2}$. Here $\eta \in C_0^\infty(B_{r_0}(x_0))$ is such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ for $|x-x_0| \leq \frac{r_0}{2}$, $|D\eta| \leq \frac{2}{r_0}$, and $r_0 \leq 1 - |x_0|$. Then we have (cf. [S], [CS], [CL])

Lemma 3.2 (Energy Monotonicity Formula). *Let u_ϵ be as in theorem C. Then*

$$(3.11)\quad \Psi(u_\epsilon, z_0, R) + c\int_R^{R_0} \frac{e^{cr}}{r}\left(\int_{T_r(z_0)} \eta^2 \frac{|x \cdot Du_\epsilon + 2t\partial_t u_\epsilon|^2}{|t-t_0|} G_{z_0}\right) dr \leq e^{c(R_0-R)}\Psi(u_\epsilon, z_0, R_0) + CK(R_0-R).$$

$$(3.11')\qquad \Phi(u_\epsilon, z_0, R) \leq e^{c(R_0-R)}\Phi(u_\epsilon, z_0, R_0) + CK(R_0-R).$$

for some $c, C > 0$, and any $0 < R \leq R_0 \leq \min\{\frac{\sqrt{t_0}}{2}, r_0\}$. Here K is given by theorem C. □

For u_ϵ as above, note also that we have

$$(3.11'')\qquad \int_{P_R(z)} |\partial_t u_\epsilon|^2\, dxdt \leq CR^{-2} \int_{P_{2R}(z)} e(u_\epsilon)\, dxdt,$$

for any $P_{2R}(z) \subset B_1 \times (0,1)$.

Lemma 3.3 (**ϵ_0- Regularity Estimate**). *Let u_ϵ be as in theorem C. Then there exists $\epsilon_0 > 0$ such that if for $0 < R < min\{\frac{\sqrt{t_0}}{2}, r_0\}$,*

$$\Psi(u_\epsilon, z_0, R) \leq \epsilon_0^2,$$

then there holds

$$\sup_{P_{\delta R(z_0)}} e(u_\epsilon) \leq C(\delta R)^{-2}. \tag{3.12}$$

for some constant $C > 0$ and $\delta > 0$. □

Now assume $u_\epsilon \rightharpoonup u_*$ weakly in $H^1(B_1^m \times (0,1))$, then $e(u_\epsilon)\,dxdt \rightharpoonup \mu = \frac{1}{2}|Du_*|^2 + \nu$ as Radon measure for some Radon measure $\nu \geq 0$. Moreover we define (cf. [CS]),

$$\Sigma = \bigcap_{R>0} \{z \in B_1 \times (0,1) : \liminf_{R\downarrow 0} \int_{T_R(z)} \eta^2 e(u_\epsilon)(x,t) G_z(x,t)\,dxdt \geq \epsilon_0^2\},$$

where ϵ_0 is as in Lemma 3.3. Then (3.11) implies Σ is closed and $\mathcal{P}^m(\Sigma \cap P_R) < \infty$ for any $R < 1$. Lemma 3.3 implies that $u_\epsilon \to u_*$ in $C^1(B_1 \times (0,1) \setminus \Sigma) \cap H^1(B_1 \times (0,1) \setminus \Sigma)$ locally (after passing to subsequences, if needed) so that u_* satisfies (1.1) weakly and smooth away from Σ. If we define the slice concentration set $\Sigma^t = \Sigma \cap \{t\}$ for $0 < t < 1$, then it was proved by [Ch] that $H^{m-2}(\Sigma_t \cap K) < \infty$, for any $t \in (0,1)$ and compact $K \subset B_1$.

Claim 1. *$e(u_\epsilon)\,dxdt \to \frac{1}{2}|Du_*|^2\,dxdt$ locally in $B_1 \times (0,1) \setminus \Sigma$.*

To see this, one need to prove that $\nu(P_R(z_0)) = 0$ for any $P_R(z_0) \subset\subset B_1 \times (0,1) \setminus \Sigma$. Note that we actually have, for $0 < R \leq R_0 \leq \min\{\frac{\sqrt{t_0}}{2}, r_0\}$ (cf. [CS])

(3.13)

$$\begin{aligned}
&\Psi(u_\epsilon, z_0, R_0) = \Psi(u_\epsilon, z_0, R)\\
&+ \int_R^{R_0} r^{-1} \left(\int_{T_r(z_0)} \left[|s|^{-1}|y \cdot Du_\epsilon + 2s\partial_s u_\epsilon|^2 + \frac{2}{\epsilon^2} F(u_\epsilon) \right] \eta^2 G_{z_0}\,dyds \right) dr\\
&- \int_R^{R_0} r^{-1} \left(\int_{T_r(z_0)} Du_\epsilon(y \cdot Du_\epsilon + 2s\partial_s u_\epsilon) D\eta^2 G_{z_0}\,dyds \right) dr.
\end{aligned}$$

Taking $\epsilon \downarrow 0$, we get

$$\int_{T_{R_0}(z_0)} \left(\frac{1}{2}|Du_*|^2 + \nu\right) \eta^2 G_{z_0} = \int_{T_R(z_0)} \left(\frac{1}{2}|Du_*|^2 + \nu\right) \eta^2 G_{z_0}$$
$$+ \int_R^{R_0} r^{-1} \left(\int_{T_r(z_0)} \left[|s|^{-1}|y \cdot Du_* + 2s\partial_s u_*|^2 + 2\nu\right] \eta^2 G_{z_0}\, dyds \right) dr$$
$$- \int_R^{R_0} r^{-1} \left(\int_{T_r(z_0)} Du_*(y \cdot Du_* + 2s\partial_s u_*) D\eta^2 G_{z_0}\, dyds \right) dr.$$

Here we use the fact that $\frac{1}{\epsilon^2}F(u_\epsilon) \to \nu$ as Radon measure in $B_1 \times (0,1) \setminus \Sigma$. One the other hand, $u_* \in C^\infty(B_1 \times (0,1) \setminus \Sigma)$ satisfies (1.1) so that u_* satisfies (cf. [S]):

$$\int_{T_{R_0}(z_0)} \frac{1}{2}|Du_*|^2 \eta^2 G_{z_0} = \int_{T_R(z_0)} \frac{1}{2}|Du_*|^2 \eta^2 G_{z_0}$$
$$- \int_R^{R_0} r^{-1} \left(\int_{T_r(z_0)} Du_*(y \cdot Du_* + 2s\partial_s u_*) D\eta^2 G_{z_0}\, dyds \right) dr,$$
$$+ \int_R^{R_0} r^{-1} \left(\int_{T_r(z_0)} |s|^{-1}|y \cdot Du_* + 2s\partial_s u_*|^2 \eta^2 G_{z_0}\, dyds \right) dr.$$

Therefore, we have

$$\int_{T_{R_0}(z_0)} \eta^2 G_{z_0}\, d\nu = \int_{T_R(z_0)} \eta^2 G_{z_0}\, d\nu + 2\int_R^{R_0} r^{-1} \left(\int_{T_r(z_0)} \eta^2 G_{z_0}\, d\nu \right) dr.$$

Hence

$$\frac{d}{dr}\left(\int_{T_r(z_0)} \eta^2 G_{z_0}\, d\nu \right) = 2r^{-1} \int_{T_r(z_0)} \eta^2 G_{z_0}\, d\nu. \tag{3.14}$$

for $0 < r \leq R_0$, which implies

$$\int_{T_r(z_0)} \eta^2 G_{z_0}\, d\nu = \left(\frac{r}{R}\right)^2 \int_{T_R(z_0)} \eta^2 G_{z_0}\, d\nu,$$

therefore $\nu(P_R(z_0)) = 0$. □

Claim 1 implies that $\mathrm{sing}(u_*) \cup \mathrm{spt}(\nu) \subset \Sigma$. In fact,

Claim 2. *$sing(u_*) \cup spt(\nu) = \Sigma$.*

In fact, if $z_0 \notin \mathrm{sing}(u_*) \cup \mathrm{spt}(\nu)$, then there exists $\rho > 0$ such that $u_* \in C^\infty(P_\rho(z_0))$ and $\nu(P_\rho(z_0)) = 0$ so that

$$\rho^{-m} \int_{P_\rho(z_0)} \left(\frac{1}{2}|Du_*|^2 + \nu\right) \le \frac{\epsilon_0^2}{2},$$

and then $\rho^{-m} \int_{P_\rho(z_0)} e(u_\epsilon) \le \epsilon^2$ for sufficiently small ϵ and hence $z_0 \notin \Sigma$. □

For the measures μ and ν above, we define two density functions

$$\Theta^m(\mu, z) = \lim_{R\downarrow 0} \int_{T_R(z)} \eta^2 G_z \, d\mu,$$

and

$$\Theta^m(\nu, z) = \lim_{R\downarrow 0} \int_{T_R(z)} \eta^2 G_z \, d\nu,$$

for $z \in B_1 \times (0,1)$, if both of the limits exist. Then we have

Claim 3. (a) $\Theta^m(\mu, z)$ *exists for* $z \in B_1 \times (0,1)$ *and is upper-semicontinous;*

(b) $\epsilon_0^2 \le \Theta^m(\mu, z) \le C(K, r)$ *for any* $z \in \Sigma \cap P_r$.

(c) *For* $\mathcal{P}^m$ *a.e.* $z \in \Sigma$, $\Theta^m(\nu, z)$ *exists and* $\Theta^m(\nu, z) = \Theta^m(\mu, z)$.

From the monotonicity inequality of μ, we have, for $0 < R \le R_0$,

$$\int_{T_R(z)} \eta^2 G_z \, d\mu \le \int_{T_{R_0}(z)} \eta^2 G_z \, d\mu + CE_0(R_0 - R),$$

which implies $\Theta^m(\mu, z)$ exists for $z \in B_1 \times (0,1)$ and is upper-semicontinuous. Note that $\lim_{r\downarrow 0} r^{-m} \int_{P_r(z)} |Du_*|^2 = 0$ for $\mathcal{P}^m$ a.e. $z \in B_1 \times (0,1)$ (cf. [FZ]) (b) (c) then follows from the definition of Σ and (a). □

Now assume that $e(u_\epsilon)\, dxdt \not\rightharpoonup \frac{1}{2}|Du_*|^2 \, dxdt$, then one must have

$$\mathcal{P}^m(\Sigma) > 0, \text{ and } \nu(B_1 \times (0,1)) > 0.$$

Moreover Claim 4 shows that there exists $\bar\Sigma \subset \Sigma$ with $\mathcal{P}^m(\bar\Sigma) = \mathcal{P}^m(\Sigma) > 0$ such that $\Theta^m(\mu, z) = \Theta^m(\nu, z)$ is approximately continuous for $z \in \bar\Sigma$. Now, we can choose a $z_0 = (x_0, t_0) \in \bar\Sigma$ such that (i) $\limsup_{r\downarrow 0} r^{-m} \mathcal{P}^m(\Sigma \cap P_r(z_0)) > 0$; (ii) $\Theta^m(\mu, z)$ is approximately continuous at z_0; and (iii) $\lim_{r\downarrow 0} r^{-m} \int_{P_r(z_0)} |Du_*|^2 = 0$.

For $r_i \downarrow 0$, define the parabolic dilation D_{r_i} by

$$D_{r_i}(A) = \left\{z = (x,t) \in R^{m+1} : (x,t) = (r_i y, r_i^2 s) \text{ for some } (y,s) \in A\right\},$$

and the rescaling measures $\mu_i(A) = r_i^{-m}\mu(z_0 + D_{r_i}(A))$ for any $A \subset B_1 \times (0,1)$. Then we have $\epsilon_0^2 \le \mu_i(B_1 \times (0,1)) \le C(K)$. Hence we can assume that $\mu_i \to \mu_*$ for some Radon measure $\mu_* \ge 0$. By the diagonal process, one can extract subsequence $\epsilon_i \downarrow 0$.

$$e(u_{\epsilon_i})\, dxdt \to \mu_*, \text{ and } u_{\epsilon_i} \to \text{constant weakly in } H^1(B_1 \times (0,1)).$$

Note that Σ_*, the support of μ_*, is given by $\Sigma_* = \bigcup_{t\in(-1,1)} \Sigma_*^t$ and

$$\Sigma_*^t = \bigcap_{R>0} \left\{x \in B_1 : \liminf_{\epsilon_i \downarrow 0} \int_{T_R((x,t))} \eta^2 e(u_{\epsilon_i}) G_{(x,t)} \ge \epsilon_0^2\right\}.$$

So that $(0,0) \in \Sigma_*$, $\mathcal{P}^m(\Sigma_*) > 0$, and $\mu_*(B_1 \times (-1,1)) \ge \epsilon_0^2$.

Claim 4. *There exists $t_0 > 0$ such that $\Sigma_*^t \not\equiv \emptyset$ for any $t \in (-t_0, 0]$.*

Suppose not, for $t_0 > 0$, $\Sigma_*^{t_0} = \emptyset$. Then for any $x_0 \in B_1$, there exists $r_0 > 0$ such that

$$\liminf_{\epsilon_i \downarrow 0} \int_{T_{r_0}((x_0,t_0))} \eta^2 e(u_{\epsilon_i}) G_{(x_0,t_0)} < \epsilon_0^2,$$

so that Lemma 3.3 yields

$$\sup_{P_{\delta r_0}(x_0,t_0)} e(u_{\epsilon_i}) \le C(\delta r_0)^{-2},$$

for some $C > 0$ and $\delta > 0$. This implies that, for some $\bar{r} > 0$,

$$u_{\epsilon_i} \to \text{ constant in } C^2\left(B_{\frac{1}{2}} \times (t_0 - \bar{r}, t_0 + \bar{r})\right),$$

and

$$\nu\left(B_{\frac{1}{2}} \times (t_0 - \bar{r}, t_0 + \bar{r})\right) = 0,$$

which implies $(0,0) \notin \Sigma_*$ if we choose t_0 sufficiently small. Contradiction. □

From Claim 4, one see $e(u_{\epsilon_i})(x,t)\, dx \not\to 0$, for $t \in (-t_0, 0)$. On the other hand, there exist nonnegative Radon measures ν_t for $t \in (-t_0, 0)$ such that $e(u_{\epsilon_i})(x,t)\, dx \to \nu_t$, hence $\nu_t(B_1) > 0$ for $t \in (-t_0, 0)$. It is easy to see that $\mathrm{spt}\nu_t \subset \Sigma_*^t$ for $t \in (-t_0, 0)$. In fact,

Claim 5. *If $\nu_t(B_1) > 0$, then $H^{m-2}(spt(\nu_t)) > 0$.*

Suppose not. Then for any $\delta > 0$ there exists a covering $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$ of sptν_t, with $x_i \in$sptν_t, such that $\sum_{i=1}^{\infty} r_i^{m-2} < \delta$. Since

$$\nu_t(B_1 \setminus \bigcup_{i=1}^{\infty} B_{r_i}(x_i)) = 0,$$

we have

$$\int_{B_1 \setminus \cup_{i=1}^{\infty} B_{r_i}(x_i)} e(u_{\epsilon_j})(x,t)\,dx \to 0, \text{as } \epsilon_j \downarrow 0.$$

Moreover, by (3.11'),

$$\begin{aligned}
r_i^{2-m} \int_{B_{r_i}(x_i)} e(u_{\epsilon_j})(x,t)\,dx &\le e^{-1} r_i^2 \int_{(t+r_i^2)-r_i^2} \eta^2 e(u_{\epsilon_j})(y,t) G_{(x_i,t+r_i^2)}(y,t)\,dy \\
&= e^{-1}\Phi(u_{\epsilon_j}, r_i, (x_i, t+r_i^2)) \\
&\le e^{-1}\Phi(u_{\epsilon_j}, R, (x_i, t+r_i^2)) + CK(R - r_i) \\
&\le e^{-1} R^{2-m} \int_{t+r_i^2-R^2} e(u_{\epsilon_j}) \le C(R,K),
\end{aligned}$$

for some large $R > r_i$. Hence,

$$\begin{aligned}
\int_{\cup_{i=1}^{\infty} B_{r_i}(x_i)} e(u_{\epsilon_j})(x,t)\,dx &\le \sum_{i=1}^{\infty} \int_{B_{r_i}(x_i)} e(u_{\epsilon_j})(x,t)\,dx \\
&\le C \sum_{i=1}^{\infty} r_i^{m-2} \le C\delta,
\end{aligned}$$

so that if we choose $\delta < \frac{\nu_t(B_1)}{2C}$, then

$$\int_{B_1} e(u_{\epsilon_j})(x,t)\,dx \le \frac{3}{4}\nu_t(B_1),$$

for sufficiently small ϵ_j. Contradiction. □

Claim 5 gives $H^{m-2}(\Sigma_*^t) > 0$ for any $t \in (-t_0, 0)$. Now, we can pick another point $(x_1, t_1) \in \Sigma_*^{t_1}$, such that $H^{m-2}(\Sigma_*^{t_1}) > 0$ and $\bar{\Theta}^{m-2}(\Sigma_*^{t_1}, x_1) = \limsup_{r\downarrow 0} r^{2-m} H^{m-2}(\Sigma_*^{t_1} \cap B_r(x_1)) > 0$. Now we apply Lemma 2.3 to $\Sigma_*^{t_1}$ at x_1 to conclude that for $r_j \downarrow 0$ there exist $\{x_1^j, \cdots, x_{m-2}^j\} \subset \Sigma_*^{t_1}$ such that

$$|x_k^j - x_0^j| \ge \delta r_j, \forall 1 \le k \le m-2,$$

and

$$\operatorname{dist}(x_k^j - x_j^0, \operatorname{span}\{x_1^j - x_0^j, \cdots, x_{k-1}^j - x_0^j\}) \geq \delta r_j, \forall 1 \leq k \leq m-2,$$

where $x_0^j = x_1$. Let $\mu_{*,j}(A) = r_j^{-m}\mu_*((x_1,t_1) + D_{r_j}(A))$ for each j and define $v_{\epsilon_{ij}}(x,t) = u_{\epsilon_i}((x_1 + r_j x, t_1 + r_j^2 t))$. Then, by the diagonal process again, one can find a subsequence of ϵ_{ij} (denoted as ϵ_j) such that, as $\epsilon_j \downarrow 0$,

$$\mu_{*,j} \to \mu_{**}, e(u_{\epsilon_j})\, dxdt \to \mu_{**}.$$

Moreover, if we denote $\Sigma_{**} = \operatorname{spt}\mu_{**}$ and $\Sigma_{**}^t = \Sigma_{**} \cap \{t\}$, then

$$\operatorname{span}\{\xi_1, \cdots, \xi_{m-2}\} \subset \Sigma_{**}^0,$$

where $\xi_k = \lim_{j\to\infty} \frac{x_k^j - x_0^j}{r_j}$, for $1 \leq k \leq m-2$. Note that $\{\xi_1, \cdots, \xi_{m-2}\}$ spans a $m-2$ dimensional linear subspace of R^m. One also has $\mathcal{P}^m(\Sigma_{**}) > 0$, $u_{\epsilon_j} \to$constant weakly in H^1, and $\Theta^m(\mu_{**}, z)$ is constant for $z \in \Sigma_{**}$.

Applying (3.11) at centers $(0,0), (\xi_1, 0), \cdots, (\xi_{m-2}, 0)$ and using the fact that $\Theta^m(\mu_{**}, z)$ is constant for $z \in \Sigma_{**}$, we have for any $r > 0$,

$$\int_r^1 R\, dR \int_{T_1} |t|^{-1}\eta^2 |v_{j,R}^k|^2 G_{(\xi_k,0)}\, dxdt \to 0, \text{ as } j \to \infty, \tag{3.13}$$

for $0 \leq k \leq m-2$. Here $\xi_0 = (0,0)$ and $v_{j,R}^k = \frac{d}{dR} u_{\epsilon_j}((\xi_k, 0) + (Rx, R^2 t))$. Hence, by Fatou's Lemma, one has, for $0 \leq k \leq m-2$,

$$\lim_{\epsilon_j \downarrow 0} \int_{T_1} \eta^2 |v_{j,R}^k|^2 G_{(\xi_k,0)}\, dxdt = 0, \forall R \in (0,1). \tag{3.24}$$

Let $\{0\} \times R^{m-2}$ be the $\operatorname{span}\{\xi_1, \cdots, \xi_{m-2}\} = \{(0,0,y_3,\cdots,y_m) \in R^m\}$. Then, (3.14) implies

$$\lim_{\epsilon_j \downarrow 0} \int_{-t_1^2}^{-t_0^2} \int_{R^m} \eta^2 |D_T u_{\epsilon_j}|^2\, dxdt = 0, \tag{3.15}$$

for any $0 < t_0 < t_1 < \infty$. Here $T \in \{0\} \times R^{m-2}$, the $\operatorname{span}\{\xi_1, \cdots, \xi_{m-2}\} = \operatorname{span}\{\frac{\partial}{\partial y_3}, \cdots, \frac{\partial}{\partial y_m}\}$.

Claim 6.

$$\mu_{**}(x,y,t) = \Theta^m(\mu_{**}, (x,y,t))(H^{m-2} L\, (\{0\} \times R^{m-2}) \times \mathcal{P}^2 L\, \mathcal{S}).$$

Here $\mathcal{S} = \bigcup_{j=1}^l \{(x,t) \in R^2 \times R_- : x = \mathbf{c_j}\sqrt{-t}\}$ *for some* $1 \leq l < \infty$ *and* $\mathbf{c_j} \in R^2 \times \{0\}$. *Moreover, if* $(x,y,t) \in (\{0\} \times R^{m-2}) \times \mathcal{S}$ *then* $\Theta^m(\mu_{**}(x,y,t)) = \Theta^m(\mu, (x_0, t_0))$.

First we note that $\mathcal{P}^m(\Sigma_{**} \cap P_R) < \infty$ for any $R > 0$. Also, passing (3.11) to the limit, we see that

$$(D_r)_\#(\mu_{**}) = \mu_{**}, \forall r > 0,$$

therefore $\Sigma_{**} = D_r(\Sigma_{**})$ and we can write $\Sigma_{**} = \{(\mathbf{c}\sqrt{-t}, t) : \mathbf{c} \in \Sigma_{**}^{-1}, t \in R_-\}$. Now we need to show that $\Sigma_{**}^{-1} = \{0\} \times R^{m-2} \times \mathcal{S}$ with $\mathcal{S}$ as in the Claim. To do so, let $\phi \in C_0^\infty(R^2)$ and for $3 \leq k \leq m$, $0 < t_0 < t_1 < \infty$, we compute

(3.16)

$$\begin{aligned}
&\frac{\partial}{\partial y_k}\int_{-t_1^2}^{-t_0^2}\int_{R^2}\phi^2(x)e(u_{\epsilon_j})(x,y,t)\,dxdt \\
&= \int_{-t_1^2}^{-t_0^2}\int_{R^2}\phi^2\left[\frac{\partial u_{\epsilon_j}}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u_{\epsilon_j}}{\partial y_k}\right) + \frac{\partial u_{\epsilon_j}}{\partial y_l}\frac{\partial}{\partial y_l}\left(\frac{\partial u_{\epsilon_j}}{\partial y_k}\right) + \frac{1}{\epsilon_j^2}f(u_{\epsilon_j})\frac{\partial u_{\epsilon_j}}{\partial y_k}\right] \\
&= -\int_{-t_1^2}^{-t_0^2}\int_{R^2}\frac{\partial \phi^2}{\partial x}\frac{\partial u_{\epsilon_j}}{\partial x}\frac{\partial u_{\epsilon_j}}{\partial y_k} + \frac{\partial}{\partial y_l}\int_{-t_1^2}^{-t_0^2}\int_{R^2}\phi^2\frac{\partial u_{\epsilon_j}}{\partial y_k}\frac{\partial u_{\epsilon_j}}{\partial y_l} \\
&\quad + \int_{-t_1^2}^{-t_0^2}\int_{R^2}\phi^2\left(-\Delta u_{\epsilon_j} + \frac{1}{\epsilon_j^2}f(u_{\epsilon_j})\right)\frac{\partial u_{\epsilon_j}}{\partial y_k} \\
&= -\int_{-t_1^2}^{-t_0^2}\int_{R^2}\left(\frac{\partial \phi^2}{\partial x}\frac{\partial u_{\epsilon_j}}{\partial x} + \phi^2\frac{\partial u_{\epsilon_j}}{\partial t}\right)\frac{\partial u_{\epsilon_j}}{\partial y_k} \\
&\quad + \frac{\partial}{\partial y_l}\int_{-t_1^2}^{-t_0^2}\int_{R^2}\phi^2\frac{\partial u_{\epsilon_j}}{\partial y_k}\frac{\partial u_{\epsilon_j}}{\partial y_l}.
\end{aligned}$$

Which, combines with (3.15), implies, for $3 \leq k \leq m$,

$$\frac{\partial}{\partial y_k}\int_{-t_1^2}^{-t_0^2}\int_{R^2}\phi^2(x)\,d\mu_{**}(x,y,t) = 0, \tag{3.17}$$

in the sense of distribution for all $y \in \{0\} \times R^{m-2}$. Thus $\mu_{**}(x,y,t) = \nu_{**}(x,t)\,dy$. Hence if we denote $\bar{\Sigma}_{**} \subset R^2 \times \{0\} \times R_-$ as $\mathrm{spt}\nu_{**}$, then $\Sigma_{**} = \bar{\Sigma}_{**} \times (\{0\} \times R^{m-2})$ and $\bar{\Sigma}_{**} = \bigcup_{i=j}^{l}\{(\mathbf{c_j}\sqrt{-t}, t) : t \in R_-\}$ and $\mathbf{c_i} \in R^2 \times \{0\}$ for some $1 \leq l < \infty$. This finishes the proof of the Claim. □

From Claim 6, we may then assume that u_{ϵ_i} converges strongly to a constant in $H^1(R^m \times R_- \setminus (\{0\} \times R^{m-2}) \times \mathcal{S})$ locally.

Without loss of generality, we will assume $l = 1$ and denote $\mathbf{c_1} = \mathbf{c}$. From (3.15), we may apply the weak L^1-estimates of the local Hardy-Littlewood maximal function with respect to the parabolic distance in R^{m+1} (cf.[Se]) to conclude that there exists $A_j \subset (\{0\} \times R^{m-2}) \times \mathcal{S}$ with $\mathcal{P}^m(A_j) > 0$ such that for any $(\mathbf{c}\sqrt{-t_j}, y_j, t_j) \in A_j$

$$\sup_{r\in(0,\frac{1}{4})} r^{-m} \int_{P_r^{m-2}(y_j,t_j)} f_j \to 0, \text{as } j \to \infty, \tag{3.18}$$

where $f_j = \int_{B_1^2(\mathbf{c}\sqrt{-t_j})} \sum_{k=3}^m |\frac{\partial u_{\epsilon_j}}{\partial y_k}|^2\, dx$. Now, pick up $(y_j, t_j) \in A_j \cap (\{0\} \times R^{m-2} \times \mathcal{S})$ such that $|y_j| \le \frac{1}{2}$ and $-\frac{t_1^2}{2} \le |t_j| \le -\frac{t_0^2}{2}$ for some $0 < t_0 < t_1$.

Let $\delta_j \downarrow 0$ and $x_j \in B_{\frac{1}{4}}^2(\mathbf{c}\sqrt{-t_j})$ be such that

$$\begin{aligned}\delta_j^{-2} \int_{B_{\delta_j}^2(x_j)\times(t_j-\delta_j^2,t_j)} e(u_{\epsilon_j})(x, y_j, t)\, dxdt = \frac{\epsilon_0^2}{C(m)} \\ = \max\left\{\delta_j^{-2} \int_{B_{\delta_j}^2(z)\times(t_j-\delta_j^2,t_j)} e(u_{\epsilon_j})(\cdot, y_j, \cdot) : z \in B_{\frac{1}{2}}^2\left(\mathbf{c}\sqrt{-t_j}\right)\right\}.\end{aligned} \tag{3.19}$$

Define sequence of maps $v_j(x, y, t) = u_{\epsilon_j}((x_j, y_j, t_j) + (\delta_j(x, y), \delta_j^2 t))$, on $\Omega_j = \delta_j^{-1}((B_{\frac{1}{2}}^2(\mathbf{c}\sqrt{-t_j})) \times B_2^{m-2}) \times (-\delta_j^{-2}(-2t_1^2 + t_0^2), 0)$. Then v_j satisfies

$$\partial_t v_j - \Delta v_j - \frac{1}{\epsilon_j^2} f(v_j) = 0, \text{ in } \Omega_j. \tag{3.20}$$

$$\begin{aligned}\int_{B_1^2\times(-1,0)} e(v_j)(x, 0, t)\, dxdt = \frac{\epsilon_0^2}{C(m)} \\ = \max\left\{\int_{B_1^2(z)\times(-1,0)} e(v_j)(x, 0, t)\, dxdt : z \in \delta_j^{-1}\left(B_{\frac{1}{2}}^2\left(\mathbf{c}\sqrt{-t_j}\right)\right)\right\}.\end{aligned} \tag{3.21}$$

$$\sup_{r\in(0,\frac{1}{4\delta_j})} r^{-m} \int_{P_r(0)} \int_{B^2_{\frac{1}{2\delta_j}}(0)} \sum_{k=3}^m \left|\frac{\partial v_j}{\partial y_k}\right|^2 \to 0. \tag{3.22}$$

Moreover, by (3.16) one can apply Allard's Strong Constancy Lemma [A] (cf. the proof of theorem A) to conclude that

$$2^{-m} \int_{(B_2^2(z)\times B_2^{m-2}(0))\times(-1,0)} e(v_j)\, dxdydt \le \frac{2\epsilon_0^2}{C(m)}, \tag{3.23}$$

for all $z \in \delta_j^{-1}(B_{\frac{1}{2}}^2(\mathbf{c}\sqrt{-t_j}))$. In fact, we have

Claim 7. *For any* $z \in \delta_j^{-1}(B^2_{\frac{1}{2}}(\mathbf{c}\sqrt{-t_j}))$ *and* $t \in (-\infty, 0]$, *we have*

$$(3.14) \qquad 2^{-m} \int_{(B^2_2(z)\times B^{m-2}_2(0))\times(t-1,t)} e(v_j)\, dxdydt \le \frac{4\epsilon_0^2}{C(m)}.$$

To see this, one first note that the Fubini's theorem implies for each $z \in B^2_{\frac{1}{2}}(\mathbf{c}\sqrt{-t_j})$, there exists $t^j_z \in (t_j - \delta_j^2, t_j)$ such that

$$(3.25) \qquad \int_{B^2_{2\delta_j}(z)\times B^{m-2}_{2\delta_j}(y_j)} e(u_{\epsilon_j})(x,y,t^j_z)\, dxdy \le \frac{2\epsilon_0^2}{C(m)}\delta_j^{m-2}.$$

On the other hand, there exists sufficiently small $\beta_0 > 0$ such that if $|\frac{\bar{t}}{t^j_z}| \le 1+\beta_0$, then

$$(3.26) \qquad \int_{B^2_{2c^j_z\delta_j}(c^j_z z)\times B^{m-2}_{2c^j_z\delta_j}(y_j)} e(u_{\epsilon_j})(x,y,\bar{t})\, dxdy \le \frac{4\epsilon_0^2}{C(m)}\delta_j^{m-2},$$

where $c^j_z = \sqrt{\frac{\bar{t}}{t^j_z}}$. This follows from (3.14) , Claim 7, and (3.25). Rescaling (3.26), one see that, for any $z \in \delta_j^{-1}\left(B^2_{\frac{1}{2}}(\mathbf{c}\sqrt{-t_j})\right)$ and $t \in (-\delta_j^{-2}\beta_0, 0)$,

$$2^{-m} \int_{B^2_2(z)\times B^{m-2}_2(0)} e(v_j)(x,y,t)\, dxdy \le \frac{4\epsilon_0^2}{C(m)},$$

hence, integrating with respect to t,

$$2^{-m} \int_{(B^2_2(z)\times B^{m-2}_2(0))\times(t-1,t)} e(v_j)(x,y,s)\, dxdyds \le \frac{4\epsilon_0^2}{C(m)}$$

Therefore, by choosing sufficiently large $C(m)$, one has

$$(3.27) \qquad 2^{-m} \int_{(B^2_2(z)\times B^{m-2}_2(0))\times(t-2,t)} e(v_j)\, dxdyds \le \epsilon_0^2,$$

for $(z,t) \in \delta_j^{-2}(B^2_{\frac{1}{2}}(\mathbf{c}\sqrt{-t_j})) \times R_-$. From the local H^1 boundedness of v_j in $R^m \times R_-$, we may assume that $v_j \to v_\infty$ weakly in $H^1_{\text{loc}}(R^m \times R_-, R^k)$. Hence (3.15) implies

$$\int_{R^m\times R_-} \sum_{k=3}^m \left|\frac{\partial v_\infty}{\partial y_k}\right|^2 = 0,$$

which yields $v_\infty(x,y,t) = v_\infty(x,t)$ for $(x,y,t) \in R^m \times R_-$. On the other hand, from (3.24), we can apply Lemma 3.3 to get

$$v_j \to v_\infty, \text{ in } C^1_{\text{loc}}(R^2 \times (B_2^{m-2}) \times R_-, R^k).$$

Which, combines with (3.21), gives

$$\int_{B_1^2 \times (-1,0)} e(v_\infty)\, dxdt = \frac{\epsilon_0^2}{C(m)}.$$

Here $e(v_\infty)$ is either $\frac{1}{2}|Dv_\infty|^2 + \frac{1}{c^2}F(v_\infty)$ or $\frac{1}{2}|Dv_\infty|^2$. Hence v_∞ is nonconstant. Moreover, v_∞ satisfies either

$$\epsilon_j \downarrow c > 0,\ \partial_t v_\infty - \Delta v_\infty + \frac{1}{c^2} f(v_\infty) = 0, \text{ in } R^2 \times R_-,$$

or $\epsilon_j \downarrow 0$, $v(R^2 \times R_-) \subset N$, and $\partial_t v_\infty - \Delta v_\infty = A(v_\infty)(Dv_\infty, Dv_\infty)$. From the monotonicity inequality and energy inequality of v_j, we also know that

$$\sup_{t \in (-\infty,0)} \int_{R^2} e(v_\infty)(x,t)\, dx \le M < \infty,$$

and

$$\int_{R^2 \times R_-} |\partial_t v_\infty|^2 < \infty.$$

Now, we want to show that such v_∞ can not exist. In fact, we have

Claim 8. *Suppose $v : R^2 \times (-\infty, 0) \to R^k$ satisfies*

$$\sup_{t \in (-\infty,0)} \int_{R^2} e(v) \le M < \infty$$

$$\sup_{R^2 \times (-\infty,0)} e(v) \le M < \infty,$$

and either (i)

$$\partial_t v - \Delta v + \frac{1}{c^2} f(v) = 0, \tag{3.28}$$

for some $c > 0$ or (ii) $v \in C^2(R^2 \times (-\infty, 0), N)$ and

$$\partial_t v - \Delta v = A(v)(Dv, Dv). \tag{3.29}$$

Then v is constant.

To see this, one first note that v satisfies the energy inequality

$$\int_{-\infty}^{0} |\partial_t v|^2 + \int_{R^2} \frac{1}{2}|Dv|^2(\cdot,0) \leq \lim_{T\to-\infty} \int_{R^2} \frac{1}{2}|Dv|^2(\cdot,T) < \infty. \tag{3.30}$$

This can be obtained by multiplying the equations $\partial_t v\phi$ for suitable cut-off function ϕ. From the gradient bound of v, one also have

$$\|Dv\|_{C^k(R^2\times(-\infty,0))} \leq C(k,M). \tag{3.31}$$

Hence one may choose $t_n \downarrow -\infty$ such that

$$\int_{R^2} |\partial_t v|^2(\cdot,t_n) \to 0,$$

and

$$v(\cdot,t_n) \to v_\infty \text{ in } H^1(R^2)\cap C^2(R^2) \text{ locally }.$$

Hence v_∞ satisfies $\int_{R^2} e(v) < \infty$ and either

$$-\Delta v_\infty + \frac{1}{c^2} f(v_\infty) = 0,$$

or

$$-\Delta v_\infty = A(v_\infty)(Dv_\infty, Dv_\infty).$$

Which is necessarily constant (cf. Theorem A) so that

$$\lim_{T\to-\infty} \int_{R^2} \frac{1}{2}|Dv|^2(\cdot,T) = 0,$$

therefore v =constant and the proof of theorem C is complete. $\square$

4. Proof of Theorem D.

In this section, we will show a more general statement, which implies Theorem D as a consequence. Note that if N doesn't support harmonic S^2, then Theorem C tells that solutions u_ϵ to (1.5) satisfying (1.9), converges strongly in $H^1(B_1\times(0,1),R^k)$ to u, which is a weak solution to (1.1) and satisfies the energy monotonicity inequality, the energy inequality and the small energy regularity (cf. [CLL] or [F]). Moreover, arguments similar to Theorem C give the following.

Proposition 4.1. *Under the assumption that N doesn't support harmonic S^2. Let $\{u_n\} \subset H^1(B_1 \times (0,1), N)$ be a sequence of solutions of (1.1) with*

$$\int_{B_1\times(0,1)} (|\partial_t u_n|^2 + |Du_n|^2)\, dxdt \le K < \infty.$$

In addition, u_n satisfies (3.11), (3.11") and (3.12). Then u_n (after passing to possible subsequences) converges strongly in $H^1_{loc}(B_1 \times (0,1), N)$ to a map u, which satisfies weakly (1.1), (3.11), (3.11"), and (3.12). Hence there exists $\Sigma \subset B_1 \times (0,1)$ with $\mathcal{P}^m(\Sigma) = 0$ such that $u \in C^\infty(B_1 \times (0,1) \setminus \Sigma, N)$.

Remark. If $\{u_n\} \subset H^1(B_1 \times (0,1), N)$ are a sequence of smooth solutions of (1.1), then it is well-know (cf. [S]) that u_n satisfy (3.11), (3.11"), (3.12). Moreover, it was recently shown by Chen-Li-Lin [CLL], Feldman [Fm], and Chen-Wang [CW] that Lemma 3.3 (i.e., (3.12) holds for weak solutions of (1.1) which satisfy (3.11) and (3.12) (cf. [CLL]), provided that N is a round sphere or a Riemannian Homogeneous manifold. Although it is believed to be true, the small energy regularity estimates remain open for general Riemannian manifolds N.

Note that there doesn't exist quasi-harmonic S^2 to N in general. Hence Part I of the proof of Theorem C gives

Proposition 4.2. *Assume that N doesn't support harmonic S^2. Let $u \in H^1(B_1^2 \times (0,1), N)$ be any weak solution of (1.1), which satisfies the energy inequality:*
For $\eta \in C_0^\infty(B_1^2, R)$, $0 < t_1 \le t_2 < 1$,

$$\int_{B_1^2} \eta^2 |Du|^2(x,t_2)\, dx \le \int_{B_1^2} \eta^2 |Du|^2(x,t_1)\, dx + C\sqrt{t_2 - t_1}. \tag{4.1}$$

Then $u \in C^\infty(B_1^2 \times (0,1), N)$. Moreover

$$\sup_{B^2_{\frac{1}{2}} \times (\frac{1}{4},\frac{3}{4})} |Du|^2(x,t) \le C \int_{B_1^2\times(0,1)} |Du|^2. \tag{4.2}$$

Making use of the Proposition 4.1, we can now prove the following Theorem, which can be viewed as the parabolic analogous result of Federer's dimension reduction [F], one can refer to Schoen-Uhlenbeck [SU] for the energy minimizing harmonic map cases.

Theorem 4.3. *For $m \geq 3$. Assume that N doesn't support harmonic S^2. Let $u \in H^1(B_1 \times (0,1), N)$ be a weak solution of (1.1), which satisfies (3.11), (3.11"), (3.12). Then there exists a closed $\Sigma \subset B_1 \times (0,1)$, with parabolic Hausdorff dimension less than or equal $m-3$, such that $u \in C^\infty(B_1 \times (0,1) \setminus \Sigma, N)$. Moreover, Σ is discrete if $m = 3$.If, in addition, for some $2 \leq p \leq m-1$, N supports neither harmonic S^l for $2 \leq l \leq p$ nor quasi-harmonic S^k for $3 \leq k \leq p+1$, then the parabolic Hausdorff dimension of Σ is at most $m-p-2$.*

Proof. First note the singular set of u, $\Sigma \subset B_1 \times (0,1)$, is given by

$$\Sigma = \left\{ (x,t) \in B_1^m \times (0,1) : \lim_{r \downarrow 0} r^{-m} \int_{P_r(x,t)} |Du|^2(y,s)\, dyds \geq \epsilon_0^2 \right\},$$

where ϵ_0 is the small constant in (3.12). It is well-know that $\mathcal{P}^m(\Sigma) = 0$ (cf. [S]). Moreover, (3.11) implies Σ is closed. Therefore the parabolic Hausdorff dimension of Σ is less than or equal to m. Let $0 \leq s < m$ be such that $\mathcal{P}^s(\Sigma) > 0$. Then there exists $z_0 \in \Sigma$ such that (cf. [F])

$$\lim_{r_i \downarrow 0} r_i^{-s} \mathcal{P}^s(\Sigma \cap P_{r_i}(z_0)) > 0, \tag{4.3}$$

for a sequence $r_i \downarrow 0$. Look at maps $u_i(x,t) = u(z_0 + (r_i x, r_i^2 t), : P_1(0) \to N$. Then (3.11), (3.11"), and (3.12) imply that

$$\int_{P_1(0)} |\partial_t u_i|^2 + |Du_i|^2 \leq M < \infty,$$

hence u_i converges weakly in $H^1(P_1(0), N)$ to a map u_0 and hence strongly in $H^1_{\rm loc}(P_1(0), N)$ as well by Proposition 4.1. Hence u_0 is a weak solution of (1.1), and by (3.11),

$$\int_{T_r} |2t\partial_t u_0 + x \cdot Du_0|^2\, dxdt = 0, \forall r > 0.$$

which implies either $u_0(x,t) = u_0(\frac{x}{|x|}) : R^m \to N$ (i.e., u_0 is a homogeneous of degree zero harmonic map from R^m to N) or $u_0(x,t) = u_0(\frac{x}{\sqrt{-t}}) : R^m \times (-\infty, 0) \to N$ is a self-similar solution of (1.1). Since u_0 of the first type can be covered by those arguments of [L] for stationary harmonic maps, we will only consider the latter cases at each following step. Note that if Σ_i denotes the singular set of u_i in $P_1(0)$, we clearly have $\Sigma_i \cap P_{\frac{1}{2}}(0) = D_{r_i}(\Sigma \cap P_{\frac{r_i}{2}}(z_0))$ and hence $\mathcal{P}^s(\Sigma_i \cap P_{\frac{1}{2}}) = r_i^{-s}\mathcal{P}^s(\Sigma \cap P_{\frac{r_i}{2}}(z_0)$. Therefore, (4.3) implies

$$\lim_{i\to\infty} \mathcal{P}^s\left(\Sigma_i \cap P_{\frac{1}{2}}(0)\right) > 0.$$

On the other hand, if we denote Σ_0 as the singular set of u_0, then (3.12) implies for any $\delta_0 > 0$ there exists $i_0 >> 1$ such that $\mathrm{dist}(\Sigma_i, \Sigma_0) \leq \delta_0$ for $i \geq i_0$, here dist denote the parabolic distance. This, in particular, implies

$$\mathcal{P}^s\left(\Sigma_0 \cap P_{\frac{1}{2}}(0)\right) > 0.$$

Since u_0 is self-similar, we have $D_\lambda(\Sigma_0) \subset \Sigma_0$ for any $\lambda \geq 0$ and there are two possiblities: either we have $s \leq 0$, or we can choose a point $z_1 = (x_1, t_1) \in \Sigma_0 \cap \partial P_1(0)$, here $\partial P_1(0)$ denotes the parabolic boundary of $P_1(0)$, such that

$$\limsup_{r\to 0} r^{-s}\mathcal{P}^s(\Sigma_0 \cap P_r(z_1)) > 0.$$

Repeating the blowing-up argument of u_0 at the center z_1 we get a map $u_1 \in H^1(R^m \times R_-, N)$ with $\mathcal{P}^s(\Sigma_1 \cap P_1(0)) > 0$, which is easily seen to be independent of x_1 direction, i.e., $u_1((x_1, y, t)) = u_1(\frac{y}{\sqrt{-t}})$ for any $(x_1, y) \in R \times R^{m-1} = R^m$. If $s - 1 \leq 0$, we stop. Otherwise, there is a point $z_2 \in \Sigma_1 \cap (\partial P_1(0) \cap R^{m-1}) \times R_-$, and we repeat the argument at z_2. If we repeat the procedure n times , we get a map $u_n \in H^1_{\mathrm{loc}}(R^m \times R_-, N)$ which is a self-similar solution of (1.1) and satisfies $u_m(x_1, \cdots, x_n, y, t) = u_m(\frac{y}{\sqrt{-t}})$ for any $(x_1, \cdots, x_n, y) \in R^n \times R^{m-n} = R^m$ and $\mathcal{P}^s(\Sigma_n \cap P_1(0)) > 0$. We can repeat the argument utill $s - n \leq 0$. In order to have constructed u_n, we must have $s - n + 1 > 0$. Since $s < m$ and m is integer we then have $n \leq m - 1$. If $n \geq m - 2$, then we would have a map $u_n : R^m \times R_- :\rightarrow N$ such that $R^{m-2} \times R(t) \subset \Sigma_n$, here $R(t) \subset R^2 \times R_-$ is a self-similar curve passing through 0 and $R(t) \neq \{0\}$. Hence $\mathcal{P}^m(\Sigma_n) > 0$ contradicting the fact $\mathcal{P}^m(\Sigma_n) = 0$. Therefore $n \leq m - 3$. Since $\mathcal{P}^s(\Sigma_n) > 0$, we have $s \leq m - 3$, and since s can be any number smaller than $\mathrm{dim}\Sigma$ we have shown $\mathrm{dim}\Sigma \leq m - 3$. Suppose now that $m = 3$. Then Σ is of dimension 0. If Σ is not discrete, then there were a sequence $z_i \in \Sigma$ with $z_i \to z_0 \in \Sigma$, then we could choose $\lambda_i = 4\mathrm{dist}(z_i, z_0)$ and consider the scaled maps $u_{\lambda_i}(x, t) = u(z_0 + (\lambda_i x, \lambda_i^2 t))$ so that u_{λ_i} will converge strongly in $H^1_{\mathrm{loc}}(P_1(0), N)$ to a self-similar solution $u_0 : R^3 \times R_- \to N$ such that the singular set Σ_0 contains both 0 and a point at $\partial P_{\frac{1}{4}}(0)$, which implies $\mathcal{P}^2(\Sigma_0) > 0$. This contradicts the fact $\mathcal{P}^2(\Sigma_0) = 0$ again.

Under the additional assumptions as in the Theorem 4.3, we see if $n = m-3$, then we would have $u_n \in H^1_{\text{loc}}(R^m \times R_-, N)$ which is a self-similar solution of (1.1) such that $u_n(x,y,t) = \bar{u}_n(\frac{y}{\sqrt{-t}})$ for any $x \in R^{m-3}$ and $y \in R^3$ and $\bar{u}_n$ has an isolated singularity at $(y,t) = 0$. Therefore, $\bar{u}_n$ is a self-similar solution of (1.1) in $R^3 \times R_-$ with an isolated singularity at 0, which is trivial by assumption. Thus we had $n \leq m-4$. We can repeat the same reasoning for $n = m-4, \cdots, m-p-2$ and conclude that $n \leq m-p-2$ which then implies $\dim\Sigma \leq m-p-2$. This completes the proof of Theorem 4.3. □

Completion of Proof of Theorem D. Applying Theorem 4.3 with p replaced by $p = m-1$, we can conclude that the singular set Σ of u is empty. Then one can apply the small energy regularity estimates at each point in $B_{\frac{1}{2}} \times (\frac{1}{4}, \frac{3}{4})$ to get the gradient estimates (1.11).

Recently, we received a preprint by Digand Li, who refined Theorem D at time $T = +\infty$. □

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