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Hypercomplex varieties

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We give a number of equivalent definitions of hypercomplex varieties and construct a twistor space for a hypercomplex variety. We prove that our definition of a hypercomplex variety (used, e. g., in alg-geom 9612013) is equivalent to a definition proposed by Deligne and Simpson, who used twistor spaces. This gives a way to define hypercomplex spaces (to allow nilpotents in the structure sheaf). We give a self-contained proof of desingularization theorem for hypercomplex varieties: a normalization of a hypercomplex variety is smooth and hypercomplex.

Contents.

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1. Introduction.

Hypercomplex varieties are a very natural class of objects. This notion allows one to speak uniformly of a number of disparate examples coming from hyperkähler geometry and the theory of moduli spaces (Remark 4.4, Subsection 10.2).

A crucial property of hypercomplex varieties is that they can be desingularized in a functorial way, and this desingularization is a hypercomplex manifold. Moreover, this desingularization is achieved by taking normalization. This is a very useful result – see [A], [V4] for some of its uses.

This is why it is important to dwell on the definition of hypercomplex varieties. We give a number of equivalent definitions, in order to produce conditions which are easy to check in every context.

It is now possible to prove that something is *not* hypercomplex. In Proposition 10.3, we show that for a hypercomplex variety M, and a finite group G acting on M by automorphisms, the quotient M/G is *not* hypercomplex, unless G acts on M freely. More precisely, we show that, for $M_0 \subset M$ being the part of M where G acts freely, the natural hypercomplex structure on M_0/G cannot be extended to M/G.

1.1. An overview.

The two definitions of a hypercomplex variety, proposed in [V-d] and [V-d2], on one hand, and in [V3], [V1] on the other hand, are not identical. We show that these two definitions are in fact equivalent.

We give the weakest form of the definition of hypercomplex varieties (one used in [V3], [V1]), and give the proof of the desingularization theorem under these assumptions. This proof is essentially identical to the proof used in [V-d] and [V-d2], though our assumptions are weaker, and the arguments are more rigorous. Then we obtain the stronger version of the definition (used in [V-d] and [V-d2]) from the desingularization and a recent result of Kaledin [K].

An almost hypercomplex variety (Definition 4.1) is a real analytic variety M with a quaternionic action on its sheaf of real analytic differentials. For each $L \in \mathbb{H}$, $L^2 = -1$, L gives an almost complex structure on M. This almost complex structure is called *induced by the quaternionic action.* We say that M is *hypercomplex* (Definition 4.2) if there are induced almost complex structures $I, J \in \mathbb{H}$, such that $I \neq \pm J$, and I, J are integrable (Definition 2.11). We show then that *all* induced complex structures are integrable. This was *a definition* of the hypercomplex variety which we used in [V-d] and [V-d2].

For almost hypercomplex manifolds (i. e., smooth almost hypercomplex varieties), D. Kaledin [K] proved that integrability of two $I, J \in \mathbb{H}$, $I \neq \pm J$ implies integrability of all induced complex structures. We prove the desingularization theorem (Theorem 6.2) for hypercomplex varieties, and then apply Kaledin's result to obtain integrability of all induced complex structures (Theorem 6.3).

The last part of this paper deals with several other versions of the definition of hypercomplex variety, which are, as we show, equivalent.

We define the *twistor space* of a hypercomplex variety (Definition 7.2), constructed as an almost complex variety. Using Kaledin's theorem and desingularization, we prove that the almost complex structure on the twistors is integrable, i. e., the twistor space is a complex variety. We give a description of the hypercomplex structure in terms of the twistor space, following [HKLR], [S] and [De].

The twistor space Tw is a complex variety equipped with a holomorphic map $\pi : \mathrm{Tw} \longrightarrow \mathbb{C}P^1$ and an antilinear involution $\iota : \mathrm{Tw} \longrightarrow \mathrm{Tw}$. The original hypercomplex variety is identified with a fiber $\pi^{-1}(I)$, $I \in \mathbb{C}P^1$, and the data $(\mathrm{Tw}, \pi, \iota)$ are essentially sufficient to recover the hypercomplex structure on $M = \pi^{-1}(I)$ (see (7.1) for details and a precise statement).

It is possible to define a hypercomplex variety in terms of $(\mathrm{Tw}, \pi, \iota)$. This definition was proposed by Deligne and Simpson ([De], [S]). We show that their definition is equivalent to ours. Their definition has the significant advantage that it does not assume that M is reduced, and indeed might be used to define hypercomplex spaces, i. e., to allow nilpotents in the structure sheaf.

1.2. Twistor spaces: an introduction.

The twistor space of a hypercomplex variety is the following object.

Let M be a hypercomplex variety. Then the quaternion algebra $\mathbb{H}$ acts in $\Omega^1 M$ in such a way that for all $L \in \mathbb{H}$, $L^2 = -1$, the corresponding operator is an almost complex structure. Identifying the set $\{L \in \mathbb{H} \mid L^2 = -1\}$ with $\mathbb{C}P^1$, we obtain an almost complex structure $\mathcal{I}$ on $\mathbb{C}P^1 \times M$ acting on $T_{L,m}\mathbb{C}P^1 \times M = T_L \mathbb{C}P^1 \oplus T_m M$ as $I_{\mathbb{C}P^1} \oplus L$, where $L \in \mathbb{C}P^1$ acts on $T_m M$ as the corresponding quaternion, and $I_{\mathbb{C}P^1}$ is the complex structure operator on $\mathbb{C}P^1$. This almost complex structure is integrable (Claim 7.1; essentially, this is proven by D. Kaledin [K]). The corresponding variety is called *the twistor space of* M, denoted by Tw (Definition 7.2).

Consider the holomorphic projection map $\pi : \mathrm{Tw} \longrightarrow \mathbb{C}P^1$. Let $\iota_0 : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be the antilinear involution with no fixed points given by the central symmetry of $S^2 \subset \mathbb{R}^3$, and $\iota : \mathrm{Tw} \longrightarrow \mathrm{Tw}$ map (s, m) to $(\iota_0(s), m)$. Clearly, ι is also an antilinear involution.

The set of sections of the projection π is called *the space of twistor lines*, denoted by Sec. This space is equipped with complex structure, by Douady ([Do]).

Consider a twistor line $I \xrightarrow{s_m} (I \times m) \in \mathbb{C}P^1 \times M = \mathrm{Tw}$, where $m \in M$. Then s_m is called *a horisontal twistor line.* The variety of horisontal twistor lines is denoted by Hor. Clearly, the line $\mathrm{im}(s_m) \subset \mathrm{Tw}$ is fixed by the involution ι. One can show that a space of horisontal twistor lines is a connected component of the space Sec^ι of all twistor lines fixed by ι. Clearly, the real analytic variety Hor is identified with M, and the induced complex structures on M come from the natural isomorphisms between Hor and fibers of π. This shows how to recover the hypercomplex variety M from Tw, ι and π.

According to Deligne and Simpson ([De], [S]), singular hyperkähler varieties should be defined in terms of the following data (see (7.1) for details).

(i) Tw, π, ι

(ii) a component Hor of the variety of ι-invariant holomorphic sections of π.

satisfying the following conditions (see (7.2) for details).

(a) Through every point $x \in \mathrm{Tw}$ passes a unique line $s \in \mathrm{Hor}$

(b) Let $s \in Hor$. Consider the points x, y situated in a small neighbourhood U of $\mathrm{im}\, s \subset \mathrm{Tw}$, $\pi(x) \neq \pi(y)$. Assume that x and y belong to the same irreducible component of U. Then there exist a unique twistor line $s_{xy} \in \mathrm{Sec}$ passing through x, y.

The advantage of the definition of Deligne–Simpson (see Definition 7.7) is that it is easy to adopt for the scheme situation: one may allow nilpotents in structure sheaf. For the definition of a *hypercomplex space*, see Definition 10.1.

We show that Deligne–Simpson's definition of a hypercomplex variety is equivalent to ours (Theorem 9.1). The condition (b) in the above listing is very difficult to check. We replace it by an equivalent condition, which should be thought of as its linearization (see Definition 7.6 for details).

(b′) For each $s \in \mathrm{Hor}$, the conormal sheaf

$$\ker\left(\Omega^1(\mathrm{Tw})\Big|_{\mathrm{im}\, s} \longrightarrow \Omega^1(\mathrm{im}\, s)\right)$$

is isomorphic to $\oplus(\mathcal{O}(-1))$.

Theorem 8.1 shows that the above conditions (b) and (b′) are equivalent.

The condition (b′) is very easy to check, and hence is extremely useful. In Subsection 10.2, we use the condition (b) to give a new proof that a deformation space of a stable holomorphic bundle B over a hyperkähler manifold is a hyperkähler variety, provided that the Chern classes of B are $SU(2)$-invariant.

1.3. Desingularization of hypercomplex varieties.

Let M be a hypercomplex variety (4.2), I an integrable induced complex structure. Consider the complex variety (M, I), which is M with the complex structure defined by I. The Desingularization Theorem (Theorem 6.2) says that the normalization $\widetilde{(M, I)}$ of (M, I) is smooth and hypercomplex.

This desingularization is canonical and functorial: the hypercomplex manifold $\widetilde{(M,I)}$ is independent from the choice of induced complex structure I.

The proof of the Desingularization Theorem goes along the following lines. We define spaces with locally homogeneous singularities (5.2). A space with locally homogeneous singularities (SLHS) is an analytic space X such that for all $x \in X$, the x-completion of a local ring $\mathcal{O}_x X$ is isomorphic to an x-completion of the associated graded ring $(\mathcal{O}_x X)_{gr}$. We show that hypercomplex varieties are always SLHS (Theorem 5.14). This is proven using an elementary argument from commutative algebra.

We work with a complete local Noetherian ring A over $\mathbb{C}$, with a residual field $\mathbb{C}$. By definition, an automorphism $e: A \longrightarrow A$ is called *homogenizing* (5.5) if its differential acts as a dilatation on the Zariski tangent space of A, with dilatation coefficient $|\lambda| < 1$. As usual, by the Zariski tangent space we understand the space $(\mathfrak{m}_A/\mathfrak{m}_A^2)^*$, where $\mathfrak{m}_A$ is a maximal ideal of A. For a ring A equipped with a homogenizing automorphism $e: A \longrightarrow A$, we show that A has locally homogeneous singularities.

We complete the proof that all hypercomplex varieties are SLHS by constructing an explicit homogenizing automorphism in a local ring of germs of holomorphic functions on a complex variety underlying a given hypercomplex variety (Proposition 5.10).

The proof of Desingularization Theorem proceeds then with a study of a tangent cone of a hypercomplex variety. For every point of a hypercomplex variety, the corresponding Zariski tangent space $T_x M$ is equipped with a quaternionic action. This makes $T_x M$ into a hyperkähler manifold, with appropriately chosen metric. We show that the tangent cone $Z_x M$ of M, considered as a subvariety of $T_x M$, is trianalytic, i. e. analytic with respect to all induced complex structures (Proposition 4.6). It was proven in [V3] (see also Proposition 3.10), that trianalytic subvarieties are *totally geodesic*, i. e. all geodesics in such a subvariety remain geodesics in the ambient manifold. Since $T_x M$ is flat, its totally geodesic subvariety $Z_x M$ is a union of planes. Now from the local homogeneity of singularities of M it follows that M looks locally as its tangent cone, i. e. as a union of nonsingular hypercomplex varieties. This finishes the proof of Desingularization Theorem.

1.4. Contents.

The paper is organized as follows.

- The Introduction is independent from the rest of this paper.

- In Section 2, we recall some standard definitions and results from the theory of real analytic spaces. We define an almost complex real analytic space and show that the complex structure on a complex variety can be recovered from the corresponding almost complex structure on the underlying real analytic variety.

- Section 3 contains some well-known results and definitions from the hyperkähler geometry. We define trianalytic subvarieties of hyperkähler manifolds and show that trianalytic subvarieties naturally apppear in the complex geometry of hyperkähler manifolds.

- In Section 4, we define a hypercomplex variety. Examples of hypercomplex varieties include trianalytic subvarieties of hyperkähler manifolds (3.4) and moduli spaces of certain kinds of stable bundles over hyperkähler manifolds (Subsection 10.2). We study the tangent cone $Z_x M \subset T_x M$ of a hypercomplex variety, and show that it is a union of linear subspaces of the Zariski tanhgent space $T_x M$.

- Section 5 deals with spaces having locally homogeneous singularities (SLHS). Roughly speaking, these are analytic spaces M for which every point $x \in M$ has a system of coordinates $z_1, ..., z_n$ such that the corresponding epimorphism of formal completions

  $$\mathbb{C}[[z_1, ..., z_n]] \longrightarrow \hat{O}_x M$$

  has a homogeneous kernel (5.2, Claim 5.3).

  We show that a space has LHS if it is endowed with a system of infinitesimal automorphisms acting as dilatation on the tangent spaces (Proposition 5.6). We show that every hypercomplex variety is naturally equipped with such a system of automorphisms, thus proving that it is SLHS (Theorem 5.14).

- In Section 6, we prove the desingularization theorem for hypercomplex varieties: a normalization of a hypercomplex variety is smooth and hypercomplex (Theorem 6.2). This result is used to show that all induced complex structures on a hypercomplex variety are integrable (Theorem 6.3).

- In Section 7, we define a twistor space of a hypercomplex variety. We show how to reconstruct a hypercomplex variety from its twistor space. We axiomatize this situation, giving two sets of conditions which are satisfied for twistor spaces of hypercomplex varieties: we define *twistor spaces of hypercomplex type* (7.6) and *twistor spaces of Deligne-Simpson type* (7.7).

- In Section 8, we prove that these sets of conditions are equivalent: a variety is a twistor space of hypercomplex type if and only if it is a twistor space of Deligne–Simpson type.

- In Section 9, we show how to construct a hypercomplex variety starting from an arbitrary twistor space of hypercomplex (or, equivalently, Deligne-Simpson) type. This proves that a functor associating to each hypercomplex variety a twistor space of hypercomplex type is an equivalence of categories.

- In Section 10, we give some applications of the equivalence of categories constructed in Section 9. We define hypercomplex spaces (10.1), thus generalizing the definition of hypercomplex varieties to spaces with nilpotents. We show that a quotient of a hypercomplex variety by an action of a finite group G cannot be hypercomplex, unless G acts freely. Finally, we give another proof that the space of stable bundles over a compact hyperkähler manifold is hypercomplex, assuming its Chern classes are "suitable" (for the precise definition of suitability and the full statement, see Subsection 10.2; see also [V1]).

2. Real analytic varieties and spaces.

In this section, we follow [GMT].

2.1. Real analytic varieties and spaces: reduction, differentials.

Let I be an ideal sheaf in the ring of real analytic functions in an open ball B in $\mathbb{R}^n$. The set of common zeroes of I is equipped with a structure of ringed space, with $\mathcal{O}(B)/I$ as the structure sheaf. We denote this ringed space by $Spec(\mathcal{O}(B)/I)$.

Definition 2.1. By a *weak real analytic space* we understand a ringed space which is locally isomorphic to $Spec(\mathcal{O}(B)/I)$, for some ideal $I \subset \mathcal{O}(B)$. A *real analytic space* is a weak real analytic space for which the structure sheaf is coherent (i. e., locally of finite presentation).

For every real analytic variety X, there is a natural sheaf morphism of evaluation, $\mathcal{O}(X) \xrightarrow{ev} C(X)$, where $C(X)$ is the sheaf of complex valued continuous on X.

Definition 2.2. A *real analytic variety* is a weak real analytic space for which the natural sheaf morphism $\mathcal{O}(X) \longrightarrow C(X)$ is injective.

Let $(X, \mathcal{O}(X))$ be a real analytic space and $N(X) \subset \mathcal{O}(X)$ be the kernel of the natural sheaf morphism $\mathcal{O}(X) \longrightarrow C(X)$. Clearly, the ringed space $(X, \mathcal{O}(X)/N(X))$ is a real analytic variety. This variety is called *a reduction of* X, denoted X^r. The structure sheaf of X^r is not necessarily coherent, for examples see [GMT], III.2.15.

For an ideal $I \subset \mathcal{O}(B)$ we define the module of real analytic differentials on $\mathcal{O}(B)/I$ by

$$\Omega^1(\mathcal{O}(B)/I) = \Omega^1(\mathcal{O}(B)) \Big/ \Big(I \cdot \Omega^1(\mathcal{O}(B)) + dI \Big),$$

where B is an open ball in $\mathbb{R}^n$, and $\Omega^1(\mathcal{O}(B)) \cong \mathbb{R}^n \otimes \mathcal{O}(B)$ is the module of real analytic differentials on B. Patching this construction, we define the sheaf of real analytic differentials on any real analytic space. Likewise, one defines sheaves of analytic differentials for complex varieties and in other similar situations.

2.2. Real analytic spaces underlying complex analytic varieties.

Let X be a complex analytic variety. The *real analytic space underlying* X (denoted by $X_{\mathbb{R}}$) is the following object. By definition, $X_{\mathbb{R}}$ is a ringed space with the same topology as X, but with a different structure sheaf, denoted by $\mathcal{O}_{X_{\mathbb{R}}}$. Let $i : U \hookrightarrow B^n$ be a closed complex analytic embedding of an open subset $U \subset X$ to an open ball $B^n \subset \mathbb{C}^n$, and I be an ideal defining $i(U)$. Then

$$\mathcal{O}_{X_{\mathbb{R}}}\Big|_U := \mathcal{O}_{B^n_{\mathbb{R}}}/Re(I)$$

is a quotient sheaf of the sheaf of real analytic functions on B^n by the ideal $Re(I)$ generated by the real parts of the functions $f \in I$.

Note that, according to our definition of reduction, the real analytic space underlying X needs not be reduced, though it has no nilpotents in the structure sheaf.

Consider the sheaf $\mathcal{O}_X$ of holomorphic functions on X as a subsheaf of the sheaf $C(X, \mathbb{C})$ of continuous $\mathbb{C}$-valued functions on X. The sheaf $C(X, \mathbb{C})$ has a natural automorphism $f \longrightarrow \overline{f}$, where $\overline{f}$ is complex conjugation. By definition, the section f of $C(X, \mathbb{C})$ is called *antiholomorphic* if $\overline{f}$ is holomorphic. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions, and $\overline{\mathcal{O}}_X$ be the sheaf of antiholomorphic functions on X. Let $\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X \xrightarrow{i} \mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}$ be the natural multiplication map.

Claim 2.3. *Let X be a complex variety, $X_{\mathbb{R}}$ the underlying real analytic space. Then the natural sheaf homomorphism $i : \mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X \longrightarrow \mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}$ is injective. For each point $x \in X$, i induces an isomorphism on x-completions of $\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X$ and $\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}$.*

Proof. Clear from the definition. ☐

In the assumptions of Claim 2.3, let

$$\Omega^1(\mathcal{O}_{X_{\mathbb{R}}}), \quad \Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X), \quad \Omega^1(\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C})$$

be the sheaves of real analytic differentials associated with the corresponding sheaves of rings. There is a natural sheaf map

$$\Omega^1(\mathcal{O}_{X_{\mathbb{R}}}) \otimes \mathbb{C} = \Omega^1(\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}) \longrightarrow \Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X), \tag{2.1}$$

correspoding to the monomorphism

$$\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X \hookrightarrow \mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}.$$

Claim 2.4. *Tensoring both sides of (2.1) by $\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}$ produces an isomorphism*

$$\Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X) \bigotimes_{\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X} \left(\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}\right) = \Omega^1(\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}).$$

Proof. Clear. ☐

According to the general results about differentials (see, for example, [H], Chapter II, Ex. 8.3), the sheaf $\Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X)$ admits a canonical decomposition:

$$\Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X) = \Omega^1(\mathcal{O}_X) \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X \oplus \mathcal{O}_X \otimes_{\mathbb{C}} \Omega^1(\overline{\mathcal{O}}_X).$$

Let $\tilde{I}$ be an endomorphism of $\Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X)$ which acts as a multiplication by $\sqrt{-1}$ on

$$\Omega^1(\mathcal{O}_X) \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X \subset \Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X)$$

and as a multiplication by $-\sqrt{-1}$ on

$$\mathcal{O}_X \otimes_{\mathbb{C}} \Omega^1(\overline{\mathcal{O}}_X) \subset \Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X).$$

Let $\underline{I}$ be the corresponding $\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}$-linear endomorphism of

$$\Omega^1(\mathcal{O}_{X_{\mathbb{R}}}) \otimes \mathbb{C} = \Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X) \otimes_{\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X} \left(\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}\right).$$

A quick check shows that $\underline{I}$ is *real*, that is, comes from the $\mathcal{O}_{X_{\mathbb{R}}}$-linear endomorphism of $\Omega^1(\mathcal{O}_{X_{\mathbb{R}}})$. Denote this $\mathcal{O}_{X_{\mathbb{R}}}$-linear endomorphism by

$$I : \Omega^1(\mathcal{O}_{X_{\mathbb{R}}}) \longrightarrow \Omega^1(\mathcal{O}_{X_{\mathbb{R}}}),$$

$I^2 = -1$. The endomorphism I is called *the complex structure operator on the underlying real analytic space.* In the case when X is smooth, I coincides with the usual complex structure operator on the cotangent bundle.

Definition 2.5. Let M be a weak real analytic space, and

$$I : \Omega^1(\mathcal{O}_M) \longrightarrow \Omega^1(\mathcal{O}_M)$$

be an endomorphism satisfying $I^2 = -1$. Then I is called *an almost complex structure on M.*

2.3. Real analytic varieties and almost complex structures.

Let B be an open ball in $\mathbb{C}^n$, and $X \subset B$ a closed complex subvariety defined by an ideal $I \subset \mathcal{O}_B$. Let $X_{\mathbb{R}} \subset B_{\mathbb{R}}$ be the underlying real analytic space, and $X_{\mathbb{R}}^r \subset B_{\mathbb{R}}$ the underlying real analytic variety, with the respective ideal sheaves denoted by $I_{\mathbb{R}}$, $I_{\mathbb{R}}^r$. Consider the ideal $I_{\mathbb{R}}^r \otimes \mathbb{C} \subset \mathcal{O}_{B_{\mathbb{R}}} \otimes \mathbb{C}$.

Lemma 2.6. *In the above assumptions, the ideal $I_{\mathbb{R}}^r \otimes \mathbb{C}$ is generated by elements $f \cdot \overline{g}$, where $f, g \in \mathcal{O}_B$ are holomorphic functions on B satisfying $fg \in I$*

Proof. Clear. □

Let F be a sheaf of finitely generated $\mathcal{O}_Z$-modules over a real analytic variety Z. For all $z \in Z$, let $\mathfrak{m}_z \subset \mathcal{O}_Z$ be the ideal of all functions vanishing in z. The vector space $F/\mathfrak{m}_z \cdot F$ is called *the fiber of F in z*, denoted by $F\Big|_z$. We define the *value* for a section f of F as the corresponding element of the fiber $f\Big|_z \in F\Big|_z$.

Lemma 2.7. *Let F be a finite generated $\mathcal{O}_Z$-module over a real analytic variety Z, and f its section. Assume that $f\Big|_z = 0$ for all $z \in Z$. Then $f = 0$.*

Proof. Going to a closed subvariety if necessary, we may assume that $Sup(f) = Z$. Let $Z_0 \subset Z$ be an open subset such that $F\Big|_{Z_0}$ is free. Clearly, it suffices to show that when $f\Big|_{Z_0}$ is zero. Since $F\Big|_{Z_0}$ is free, $f\Big|_{Z_0}$ is an n-tuple of functions $(f_1, ...f_n)$, and $f\Big|_z = 0$ if and only if all f_i take value 0 at z. Applying the definition of real analytic variety, we obtain $f = 0$. □

Proposition 2.8. *Let $X_{\mathbb{R}}$ be a real analytic space underlying a complex variety X, $X_{\mathbb{R}}^r$ be its reduction, and $\Omega^1(X)$, $\Omega^1(X_{\mathbb{R}}^r)$ the corresponding sheaves of real analytic differentials. Consider the natural map*

$$\Omega^1(X_{\mathbb{R}}) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{\mathbb{R}}^r} \xrightarrow{\varphi_r} \Omega^1(X_{\mathbb{R}}^r). \tag{2.2}$$

Then φ_r is an isomorphism.

Proof. We work in notation introduced earlier in this section. Consider the closed embedding $X_{\mathbb{R}}^r \hookrightarrow X_{\mathbb{R}}$. Let $N \subset \mathcal{O}_{X_{\mathbb{R}}}$ be the ideal defining $X_{\mathbb{R}}^r$,

$$N = \ker\left(\mathcal{O}_{X_{\mathbb{R}}} \xrightarrow{ev} C(X)\right).$$

Clearly from the definitions,

$$\Omega^1(X_{\mathbb{R}}^r) = \Omega^1(X_{\mathbb{R}}) \otimes \mathcal{O}_{X_{\mathbb{R}}^r} \Big/ dN \otimes \mathcal{O}_{X_{\mathbb{R}}^r}.$$

To show that (2.2) is an isomorphism, it suffices to prove that the subsheaf $dN \subset \Omega^1(X_{\mathbb{R}})$, tensored by $\mathcal{O}_{X^r_{\mathbb{R}}}$, gives zero. By Lemma 2.7, it suffices to show that every section of

$$dN \otimes \mathcal{O}_{X^r_{\mathbb{R}}} \subset \Omega^1(X_{\mathbb{R}}) \otimes \mathcal{O}_{X^r_{\mathbb{R}}},$$

has value zero in x, for all points $x \in X$.

The fiber of $\Omega^1 X_{\mathbb{R}}$ in $x \in X$ is $\mathfrak{m}_x/\mathfrak{m}_x^2$, where $\mathfrak{m}_x \subset \mathcal{O}_{X_{\mathbb{R}}}$ is the ideal generated by all functions vanishing in x. For all $f \in \mathcal{O}_{X_{\mathbb{R}}}$ such that $f(x) = 0$, the value $df\Big|_x$ of $df \in \Omega^1 X_{\mathbb{R}}$ in x coincides with the class of f in $\mathfrak{m}_x/\mathfrak{m}_x^2$.

By Lemma 2.6, $N \otimes \mathbb{C}$ is generated by $f\overline{g}$, $f, g \in \mathcal{O}_B$, $fg \in I$. Therefore, $N \subset \mathfrak{m}_x^2 \otimes \mathbb{C}$ and by the above, the value $dN\Big|_x$ iz zero. This proves Proposition 2.8. □

From Proposition 2.8, it follows that a a real analytic variety underlying a given complex variety is equipped with a natural almost complex structure. The corresponding operator is called *the complex structure operator in the underlying real analytic variety.*

2.4. Integrability of almost complex structures.

Definition 2.9. Let X, Y be complex analytic varieties, and

$$f: \ X_{\mathbb{R}} \longrightarrow Y_{\mathbb{R}}$$

be a morphism of underlying real analytic varieties. Let $f^*\Omega^1_{Y_{\mathbb{R}}} \xrightarrow{P} \Omega^1_{X_{\mathbb{R}}}$ be the natural map of sheaves of differentials associated with f. Let

$$I_X: \ \Omega^1_{X_{\mathbb{R}}} \longrightarrow \Omega^1_{X_{\mathbb{R}}}, \quad I_Y: \ \Omega^1_{Y_{\mathbb{R}}} \longrightarrow \Omega^1_{Y_{\mathbb{R}}}$$

be the complex structure operators, and

$$f^*I_Y: \ f^*\Omega^1_{Y_{\mathbb{R}}} \longrightarrow f^*\Omega^1_{Y_{\mathbb{R}}}$$

be $\mathcal{O}_{X_{\mathbb{R}}}$-linear automorphism of $f^*\Omega^1_{Y_{\mathbb{R}}}$ defined as a pullback of I_Y. We say that f *commutes with the complex structure* if

$$P \circ f^*I_Y = I_X \circ P. \tag{2.3}$$

Theorem 2.10. *Let X, Y be complex analytic varieties, and*

$$f_{\mathbb{R}} : X_{\mathbb{R}} \longrightarrow Y_{\mathbb{R}}$$

be a morphism of underlying real analytic varieties which commutes with the complex structure. Then there exist a morphism $f : X \longrightarrow Y$ of complex analytic varieties, such that $f_{\mathbb{R}}$ is its underlying morphism.

Proof. By Corollary 9.4, [V3], the map f, defined on the sets of points of X and Y, is meromorphic; to prove Theorem 2.10, we need to show it is holomorphic. Let $\Gamma \subset X \times Y$ be the graph of f. Since f is meromorphic, Γ is a complex subvariety of $X \times Y$. It will suffice to show that the natural projections $\pi_1 : \Gamma \longrightarrow X$, $\pi_2 : \Gamma \longrightarrow Y$ are isomorphisms. By [V3], Lemma 9.12, the morphisms π_i are flat. Since $f_{\mathbb{R}}$ induces isomorphism of Zariski tangent spaces, same is true of π_i. Thus, π_i are unramified. Therefore, the maps π_i are etale. Since they are one-to-one on points, π_i etale implies π_i is an isomorphism. □

Definition 2.11. Let M be a real analytic variety, and

$$I : \Omega^1(\mathcal{O}_M) \longrightarrow \Omega^1(\mathcal{O}_M)$$

be an endomorphism satisfying $I^2 = -1$. Then I is called *an almost complex structure on M*. If there exist a structure $\mathfrak{C}$ of complex variety on M such that I appears as the complex structure operator associated with $\mathfrak{C}$, we say that I is *integrable.* Theorem 2.10 implies that this complex structure is unique if it exists.

3. Hyperkähler manifolds.

3.1. Definitions.

This subsection contains a compression of the basic definitions from hyperkähler geometry, found, for instance, in [Bes] or in [Bea].

Definition 3.1. ([Bes]) A *hyperkähler manifold* is a Riemannian manifold M endowed with three complex structures I, J and K, such that the following holds.

(i) the metric on M is Kähler with respect to these complex structures and

(ii) I, J and K, considered as endomorphisms of a real tangent bundle, satisfy the relation $I \circ J = -J \circ I = K$.

The notion of a hyperkähler manifold was introduced by E. Calabi ([C]).

Clearly, a hyperkähler manifold has the natural action of the quaternion algebra $\mathbb{H}$ on its real tangent bundle TM. Therefore its complex dimension is even. For each quaternion $L \in \mathbb{H}$, $L^2 = -1$, the corresponding automorphism of TM is an almost complex structure. It is easy to check that this almost complex structure is integrable ([Bes]).

Definition 3.2. Let M be a hyperkähler manifold, L a quaternion satisfying $L^2 = -1$. The corresponding complex structure on M is called *an induced complex structure*. The M considered as a complex manifold is denoted by (M, L).

Let M be a hyperkähler manifold. We identify the group $SU(2)$ with the group of unitary quaternions. This gives a canonical action of $SU(2)$ on the tangent bundle, and all its tensor powers. In particular, we obtain a natural action of $SU(2)$ on the bundle of differential forms.

Lemma 3.3. *The action of $SU(2)$ on differential forms commutes with the Laplacian.*

Proof. This is Proposition 1.1 of [V1]. □

Thus, for a compact M, we may speak of the natural action of $SU(2)$ in cohomology.

3.2. Trianalytic subvarieties in compact hyperkähler manifolds.

In this subsection, we give a definition and a few basic properties of trianalytic subvarieties of hyperkähler manifolds. We follow [V2].

Let M be a compact hyperkähler manifold, $\dim_{\mathbb{R}} M = 2m$.

Definition 3.4. Let $N \subset M$ be a closed subset of M. Then N is called *trianalytic* if N is a complex analytic subset of (M, L) for any induced complex structure L.

Let I be an induced complex structure on M, and $N \subset (M, I)$ be a closed analytic subvariety of (M, I), $dim_{\mathbb{C}} N = n$. Denote by $[N] \in H_{2n}(M)$ the homology class represented by N. Let $\langle N \rangle \in H^{2m-2n}(M)$ denote the Poincare dual cohomology class. Recall that the hyperkähler structure induces the action of the group $SU(2)$ on the space $H^{2m-2n}(M)$.

Theorem 3.5. *Assume that $\langle N \rangle \in H^{2m-2n}(M)$ is invariant with respect to the action of $SU(2)$ on $H^{2m-2n}(M)$. Then N is trianalytic.*

Proof. This is Theorem 4.1 of [V2]. □

Remark 3.6. Trianalytic subvarieties have an action of quaternion algebra in the tangent bundle. In particular, the real dimension of such subvarieties is divisible by 4.

Let M be a hyperkähler manifold, $\mathcal{R}$ the set of induced complex structures. The following result implies that for generic $I \in \mathcal{R}$, all complex subvarieties of (M, I) are trianalytic in M.

Definition 3.7. Let M be a compact hyperkähler manifold, $\mathcal{R}$ the set of induced complex structures. The complex structure $I \subset \mathcal{R}$ is called *generic with respect to the hyperkähler structure* if all rational (p,p)-classes

$$\omega \in \bigoplus_p H^{p,p}((M, I)) \cap H^{2p}(M, \mathbb{Z}) \subset H^*(M)$$

are $SU(2)$-invariant.

Proposition 3.8. *Let M be a compact hyperkähler manifold, $\mathcal{R}$ the set of induced complex structures. Then for all $I \in \mathcal{R}$ except a countable subset, I is generic.*

Proof. This is Proposition 2.2 from [V2]. □

3.3. Totally geodesic submanifolds.

Proposition 3.9. *Let $X \overset{\varphi}{\hookrightarrow} M$ be an embedding of Riemannian manifolds (not necessarily compact) compatible with the Riemannian structure. Then the following conditions are equivalent.*

(i) *Every geodesic line in X is geodesic in M.*

(ii) *Consider the Levi-Civita connection ∇ on TM, and restriction of ∇ to $TM\Big|_X$. Consider the orthogonal decomposition*

$$TM\Big|_X = TX \oplus TX^{\perp}. \tag{3.1}$$

Then, this decomposition is preserved by the connection ∇.

Proof. Well known; see, for instance, [Bes]. □

Proposition 3.10. *Let $X \subset M$ be a trianalytic submanifold of a hyperkähler manifold M, where M is not necessarily compact. Then X is totally geodesic.*

Proof. This is [V3], Corollary 5.4. □

4. Hypercomplex varieties.

4.1. Definition and examples.

Definition 4.1. Let M be a real analytic variety equipped with almost complex structures I, J and K, such that $I \circ J = -J \circ I = K$. Then M is called *an almost hypercomplex variety.*

An almost hypercomplex variety is equipped with an action of quaternion algebra on its sheaf of differentials. Each quaternion $L \in \mathbb{H}$, $L^2 = -1$ defines an almost complex structure on M. Such an almost complex structure is called *induced by the hypercomplex structure.*

Definition 4.2. Let M be an almost hypercomplex variety. We say that M is *hypercomplex* if there exist a pair of induced complex structures $I_1, I_2 \in \mathbb{H}$, $I_1 \neq \pm I_2$, such that I_1 and I_2 are integrable.

Caution. Not everything which looks hypercomplex satisfies the conditions of Definition 4.2. Take a quotient M/G of a hypercomplex manifold by an action of a finite group G, acting compatible with the hyperkähler structure. Then M/G is *not* hypercomplex, unless G acts freely (Proposition 10.3).

Claim 4.3. *Let M be a hyperkähler manifold. Then M is hypercomplex.*

Proof. Let I, J be induced complex structures. We need to identify $(M,I)_{\mathbb{R}}$ and $(M,J)_{\mathbb{R}}$ in a natural way. These varieties are canonically identified as C^∞-manifolds; we need only to show that this identification is real analytic. This is [V3], Proposition 6.5. □

Remark 4.4. Trianalytic subvarieties of hyperkähler manifolds are obviously hypercomplex. Define trianalytic subvarieties of hypercomplex varieties as subvarieties which are complex analytic with respect to all integrable induced complex structures. Clearly, trianalytic subvarieties of hypercomplex varieties are equipped with a natural hypercomplex structure. Another example of a hypercomplex variety is given in Subsection 10.2. For additional examples, see [V3].

4.2. Tangent cone of a hypercomplex variety.

Let M be a hypercomplex variety, I an integrable induced complex structure, $\widetilde{Z}_x(M,I)$ be the Zariski tangent cone to (M,I) in $x \in M$ and $Z_x(M,I)$ its reduction. Consider $Z_x(M,I)$ as a closed subvariety in the Zariski tangent space T_xM. The space T_xM has a natural quaternionic structure and admits a compatible metric. This makes T_xM into a hyperkähler manifold, isomorphic to $\mathbb{H}^n$.

Theorem 4.5. *Under these assumptions, the following assertions hold:*

(i) *The subvariety $Z_x(M,I) \subset T_xM$ is independent of the choice of integrable induced complex structure I.*

(ii) *Moreover, $Z_x(M,I)$ is a trianalytic subvariety of T_xM.*

Proof. Clearly, Theorem 4.5 (ii) follows from Theorem 4.5 (i). Theorem 4.5 (i) is directly implied by the following general result.

Proposition 4.6. *Let M be a complex variety, $x \in X$ a point, and $Z_xM \subset T_xM$ be the reduction of the Zariski tangent cone to M in x, considered as* *a closed subvariety of the Zariski tangent space* T_xM. *Let* $Z_xM_{\mathbb{R}} \subset T_xM_{\mathbb{R}}$ *be the Zariski tangent cone for the underlying real analytic space* $M_{\mathbb{R}}$. *Then* $(Z_xM)_{\mathbb{R}} \subset (T_xM)_{\mathbb{R}} = T_xM_{\mathbb{R}}$ *coincides with* $Z_xM_{\mathbb{R}}$.

Proof. For each $v \in T_xM$, the point v belongs to Z_xM if and only if there exist a real analytic path $\gamma: [0,1] \longrightarrow M$, $\gamma(0) = x$ satisfying $\frac{d\gamma}{dt} = v$. The same holds true for $Z_xM_{\mathbb{R}}$. Thus, $v \in Z_xM$ if and only if $v \in Z_xM_{\mathbb{R}}$. □

The following theorem shows that the Zariski tangent cone $Z_xM \subset T_xM$ is a union of planes $L_i \subset T_xM$.

Theorem 4.7. *Let* M *be a hypercomplex variety,* I *an induced complex structure and* $x \in M$ *a point. Consider the reduction of the Zariski tangent cone (denoted by* Z_xM*) as a subvariety of the quaternionic space* T_xM. *Let* $Z_x(M,I) = \cup L_i$ *be the irreducible decomposition of the complex variety* $Z_x(M,I)$. *Then*

(i) *The decomposition* $Z_x(M,I) = \cup L_i$ *is independent from the choice of induced complex structure* I.

(ii) *For every* i*, the variety* L_i *is a linear subspace of* T_xM*, invariant under the quaternion action.*

Proof. Let L_i be an irreducible component of $Z_x(M,I)$, $Z_x^{ns}(M,I)$ be the non-singular part of $Z_x(M,I)$, and $L_i^{ns} := Z_x^{ns}(M,I) \cap L_i$. Then L_i is a closure of L_i^{ns} in T_xM. In particular, L_i^{ns} is non-empty. Clearly from Theorem 4.5, $L_i^{ns}(M)$ is a hyperkähler submanifold in T_xM. By Proposition 3.10, L_i^{ns} is totally geodesic. A totally geodesic submanifold of a flat manifold is again flat. Therefore, L_i^{ns} is an open subset of a linear subspace $\widetilde{L}_i \subset T_xM$. Since L_i^{ns} is a hyperkähler submanifold, $\widetilde{L}_i$ is invariant with respect to quaternion action. The closure L_i of L_i^{ns} is a complex analytic subvariety of $T_x(M,I)$. Therefore, $\widetilde{L}_i = L_i$. This proves Theorem 4.7 (ii). From the above argument, it is clear that $Z_x^{ns}(M,I) = \coprod L_i^{ns}$ (disconnected sum). Taking connected components of $Z_x^{ns}M$ for each induced complex structure, we obtain the same decomposition $Z_x(M,I) = \cup L_i$, with L_i being closured of connected components. This proves Theorem 4.7 (ii). □

5. Hypercomplex varieties have locally homogeneous singularities.

This section follows [V-d2].

5.1. Spaces with locally homogeneous singularities.

Definition 5.1. (local rings with LHS) Let A be a local ring. Denote by $\mathfrak{m}$ its maximal ideal. Let A_{gr} be the corresponding associated graded ring for the $\mathfrak{m}$-adic filtration. Let $\hat{A}$, $\widehat{A_{gr}}$ be the $\mathfrak{m}$-adic completion of A, A_{gr}. Let $(\hat{A})_{gr}$, $(\widehat{A_{gr}})_{gr}$ be the associated graded rings, which are naturally isomorphic to A_{gr}. We say that A *has locally homogeneous singularities* (LHS) if there exists an isomorphism $\rho : \hat{A} \longrightarrow \widehat{A_{gr}}$ which induces the standard isomorphism $i : (\hat{A})_{gr} \longrightarrow (\widehat{A_{gr}})_{gr}$ on associated graded rings.

Definition 5.2. (SLHS) Let X be a complex or real analytic space. Then X is called *a space with locally homogeneous singularities* (SLHS) if for each $x \in X$, the local ring $\mathcal{O}_x X$ has locally homogeneous singularities.

The following claim might shed a light on the origin of the term "locally homogeneous singularities".

Claim 5.3. *Let A be a complete local Noetherian ring over $\mathbb{C}$, with a residual field $\mathbb{C}$. Then the following statements are equivalent*

(i) *A has locally homogeneous singularities*

(ii) *There exist a surjective ring homomorphism $\rho : \mathbb{C}[[x_1, ..., x_n]] \longrightarrow A$, where $\mathbb{C}[[x_1, ..., x_n]]$ is the ring of power series, and the ideal* $\ker\rho$ *is homogeneous in* $\mathbb{C}[[x_1, ..., x_n]]$.

Proof. Clear. □

Definition 5.4. Let M be a hypercomplex variety. Then M is called a space with locally homogeneous singularities (SLHS) if for all integrable induced complex structures I the (M, I) is SLHS.

5.2. Complete rings with automorphisms.

Definition 5.5. Let A be a local Noetherian ring over $\mathbb{C}$, with a residual field $\mathbb{C}$, equipped with an automorphism $e: A \longrightarrow A$. Let $\mathfrak{m}$ be a maximal ideal of A. Assume that e acts on $\mathfrak{m}/\mathfrak{m}^2$ as a multiplication by $\lambda \in \mathbb{C}$, $|\lambda| < 1$. Then e is called *a homogenizing automorphism of* A.

The aim of the present subsection is to prove the following statement.

Proposition 5.6. *Let A be a complete Noetherian ring over $\mathbb{C}$, with a residual field $\mathbb{C}$, equipped with a homogenizing authomorphism $e: A \longrightarrow A$. Then there exist a surjective ring homomorphism $\rho: \mathbb{C}[[x_1, ..., x_n]] \longrightarrow A$, such that the ideal $\ker\rho$ is homogeneous in $\mathbb{C}[[x_1, ..., x_n]]$. In particular, A has locally homogeneous singularities.*[2]

This statement is well known. A reader who knows its proof should skip the rest of this section.

Proposition 5.7. *Let A be a complete Noetherian ring over $\mathbb{C}$, with a residual field $\mathbb{C}$, equipped with a homogenizing authomorphism $e: A \longrightarrow A$. Then there exist a system of ring elements*

$$f_1, ..., f_m \in \mathfrak{m}, \quad m = \dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2,$$

which generate $\mathfrak{m}/\mathfrak{m}^2$, and such that $e(f_i) = \lambda f_i$.

Proof. Let $\underline{a} \in \mathfrak{m}/\mathfrak{m}^2$. Let $a \in \mathfrak{m}$ be a representative of $\underline{a}$ in $\mathfrak{m}$. To prove Proposition 5.7 it suffices to find $c \in \mathfrak{m}^2$, such that $e(a-c) = \lambda a - \lambda c$. Thus, we need to solve an equation

$$\lambda c - e(c) = e(a) - \lambda(a). \tag{5.1}$$

Let $r := e(a) - \lambda a$. Clearly, $r \in \mathfrak{m}^2$. A solution of (5.1) is provided by the following lemma.

Lemma 5.8. *In assumptions of Proposition 5.7, let $r \in \mathfrak{m}^2$. Then, the equation*

$$e(c) - \lambda c = r \tag{5.2}$$

has a unique solution $c \in \mathfrak{m}^2$.

[2]See Claim 5.3 for LHS property in terms of coordinate systems.

Proof. We need to show that the operator $P := (e-\lambda)\Big|_{\mathfrak{m}^2}$ is invertible. Consider the $\mathfrak{m}$-adic filtration $\mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \ldots$ on $\mathfrak{m}^2$. Clearly, P preserves this filtration. Since $\mathfrak{m}^2$ is complete with respect to the adic filtration, it suffices to show that P is invertible on the successive quotients. The quotient $\mathfrak{m}^2/\mathfrak{m}^i$ is finite-dimensional, so to show that P is invertible it suffices to calculate the eigenvalues. Since e is an automorphism, restriction of e to $\mathfrak{m}^i/\mathfrak{m}^{i-1}$ is a multiplication by λ^i. Thus, the eigenvalues of e on $\mathfrak{m}^2/\mathfrak{m}^i$ range from λ^2 to λ^{i-1}. Since $|\lambda| > |\lambda|^2$, all eigenvalues of $P\Big|_{\mathfrak{m}^2/\mathfrak{m}^i}$ are non-zero and the restriction of P to $\mathfrak{m}^2/\mathfrak{m}^i$ is invertible. This proves Lemma 5.8. □

The proof of Proposition 5.6. Consider the map

$$\rho: \mathbb{C}[[x_1, \ldots, x_m]] \longrightarrow A, \quad \rho(x_i) = f_i,$$

where $f_1, \ldots, f_m$ is the system of functions constructed in Proposition 5.7. By Nakayama's lemma, ρ is surjective.

Let $e_\lambda : \mathbb{C}[[x_1, \ldots, x_m]] \longrightarrow \mathbb{C}[[x_1, \ldots, x_m]]$ be the automorphism mapping x_i to λx_i. Then, the diagram

$$\begin{array}{ccc} \mathbb{C}[[x_1, \ldots, x_m]] & \xrightarrow{\ \rho\ } & A \\ {\scriptstyle e_\lambda}\downarrow & & \downarrow{\scriptstyle e} \\ \mathbb{C}[[x_1, \ldots, x_m]] & \xrightarrow{\ \rho\ } & A \end{array}$$

is by construction commutative. Therefore, the ideal $I = \ker \rho$ is preserved by e_λ. Clearly, every e_λ-preserved ideal $I \subset \mathbb{C}[[x_1, \ldots, x_m]]$ is homogeneous. Proposition 5.6 is proven. □

5.3. Automorphisms of local rings of hypercomplex varieties.

Let M be a hypercomplex variety, $x \in M$ a point, I an integrable induced complex structure. Let $A_I := \hat{\mathcal{O}}_x(M, I)$ be the adic completion of the local ring $\mathcal{O}_x(M, I)$ of x-germs of holomorphic functions on the complex variety (M, I). Clearly, the sheaf ring of the antiholomorphic functions on (M, I) coincides with $\mathcal{O}_x(M, -I)$. Thus, the corresponding completion ring is A_{-I}. The isomorphism of Claim 2.3 produces a natural epimorphism

$$A_I \widehat{\otimes_{\mathbb{C}}} A_{-I} \longrightarrow A_{\mathbb{R}}, \tag{5.3}$$

where

$$A_{\mathbb{R}} := \widehat{\mathcal{O}_x(M_{\mathbb{R}})} \otimes_{\mathbb{R}} \mathbb{C}$$

is the x-completion of the ring of germs of real analytic complex-valued functions on M. Consider the natural quotient map

$$p: \ A_{-I} \longrightarrow \mathbb{C}.$$

Consider the natural epimorphism of complete rings

$$A_I \widehat{\otimes_{\mathbb{C}} A}_{-I} \longrightarrow A_I, \quad a \otimes b \mapsto a \otimes p(b), \tag{5.4}$$

where $a \in A_I$, $b \in A_{-I}$, and

$$a \otimes b \in A_I \otimes_{\mathbb{C}} A_{-I}.$$

Lemma 5.9. *The kernel of* (5.4) *contains the kernel of* (5.3).

Proof. Consider an epimorphism $\varphi: \ B_x \longrightarrow A_I$ where

$$B_x = \mathbb{C}[[z_1, ..., z_n]].$$

Let $\mathfrak{I} \subset B_x$ be the kernel of φ. By Lemma 2.6, the ring $A_{\mathbb{R}}$ is naturally isomorphic to

$$(B_x)_{\mathbb{R}} = \mathbb{C}[[z_1, ..., z_n, \overline{z}_1, ..., \overline{z}_n]]/\mathfrak{I}_{\mathbb{R}},$$

where $\mathfrak{I}_{\mathbb{R}}$ is an ideal generated by all the products

$$f(z_1, ..., z_n) \cdot \overline{g}(\overline{z}_z, ..., \overline{z}_n),$$

such that $fg \in \mathfrak{I}$. Likewise, $A_I \widehat{\otimes_{\mathbb{C}} A}_{-I}$ is a quotient of

$$(B_x)_{\mathbb{R}} = \mathbb{C}[[z_1, ..., z_n, \overline{z}_1, ..., \overline{z}_n]]$$

by the ideal

$$\mathfrak{I} \cdot \mathbb{C}[[\overline{z}_1, ..., \overline{z}_n]] + \mathbb{C}[[z_1, ..., z_n] \cdot \overline{\mathfrak{I}}.$$

Slightly abusing the notation, we denote the corresponding quotient map by

$$\varphi: \ \mathbb{C}[[z_1, ..., z_n, \overline{z}_1, ..., \overline{z}_n]] \longrightarrow A_I \widehat{\otimes_{\mathbb{C}} A}_{-I}.$$

Let $a \in A_I \widehat{\otimes_{\mathbb{C}} A}_{-I}$ be an element which is mapped to zero by the map (5.3). Then a is a linear combination of $\varphi(f_i \overline{g}_i)$, for $f_i, g_i \in \mathbb{C}[[z_1, ..., z_n]]$, $fg \in \mathfrak{I}$. Therefore, it suffices to show that a lies in the kernel of (5.4) for

$a = \varphi(f\overline{g})$. Either g is invertible and $f \in \mathfrak{I}$, or $g(0, ..., 0) = 0$. In the first case, $f\overline{g} \in \mathfrak{I} \otimes \mathbb{C}[[\overline{z}_1, ..., \overline{z}_n]]$, so $\varphi(f\overline{g}) = 0$. In the second case, $p(\overline{g}) = 0$, so $1 \otimes p(a) = 0$ and a lies in the kernel of the map (5.4). This proves Lemma 5.9. $\square$

Consider the diagram

$$\begin{array}{ccc} A_I \widehat{\otimes_{\mathbb{C}} A}_{-I} & \longrightarrow & A_I \\ & \searrow & \\ & & A_{\mathbb{R}} \end{array}$$

formed from the arrows of (5.4) and (5.3). By Lemma 5.9, there exists an epimorphism $e_I : A_{\mathbb{R}} \longrightarrow A_I$, making this diagram commutative. Let $i_I : A_I \hookrightarrow A_{\mathbb{R}}$ be the natural embedding

$$a \mapsto a \otimes 1 \in A_I \widehat{\otimes_{\mathbb{C}} A}_{-I}.$$

For an integrable induced complex structure J, we define A_J, A_{-J}, i_J, e_J likewise.

Let $\Psi_{I,J} : A_I \longrightarrow A_I$ be the composition

$$A_I \xrightarrow{i_I} A_{\mathbb{R}} \xrightarrow{e_J} A_J \xrightarrow{i_J} A_{\mathbb{R}} \xrightarrow{e_I} A_I.$$

Clearly, for $I = J$, the ring morphism $\Psi_{I,J}$ is identity, and for $I = -J$, $\Psi_{I,J}$ is an augmentation map.

Proposition 5.10. *Let M be a hypercomplex variety, $x \in M$ a point, and I, J induced complex structures, such that $I \neq J$ and $I \neq -J$. Consider the map $\Psi_{I,J} : A_I \longrightarrow A_I$ defined as above. Then $\Psi_{I,J}$ is a homogenizing automorphism of A_I.*[3]

Proof. Let $d\Psi$ be the differential of $\Psi_{I,J}$, that is, the restriction of $\Psi_{I,J}$ to $(\mathfrak{m}/\mathfrak{m}^2)^*$, where $\mathfrak{m}$ is the maximal ideal of A_I. By Nakayama's lemma, to prove that $\Psi_{I,J}$ is an automorphism it suffices to show that $d\Psi$ is invertible. To prove that $\Psi_{I,J}$ is homogenizing, we have to show that $d\Psi$ is a multiplication by a complex number λ, $|\lambda| < 1$. As usually, we denote the real analytic variety underlying M by $M_{\mathbb{R}}$. Let T_I, T_J, $\underline{T}_{\mathbb{R}}$ be the Zariski tangent spaces to (M, I), (M, J) and $M_{\mathbb{R}}$, respectively, in $x \in M$. Consider the

[3]For the definition of a homogenizing automorphism, see Definition 5.5.

complexification $T_{\mathbb{R}} := \underline{T}_{\mathbb{R}} \otimes \mathbb{C}$, which is a Zariski tangent space to the local ring $A_{\mathbb{R}}$. To compute $d\Psi : T_I \longrightarrow T_I$, we need to compute the differentials of e_I, e_J, i_I, i_J, i. e., the restrictions of the homomorphisms e_I, e_J, i_I, i_J to the Zariski tangent spaces T_I, T_J, $T_{\mathbb{R}}$. Denote these differentials by de_I, de_J, di_I, di_J.

Lemma 5.11. *Let M be a hypercomplex variety, $M_{\mathbb{R}}$ the associated real analytic variety, $x \in M$ a point. Consider the space $T_{\mathbb{R}} := T_x(M_{\mathbb{R}}) \otimes \mathbb{C}$. For an induced complex structure I, consider the Hodge decomposition $T_{\mathbb{R}} = T_I^{1,0} \oplus T_I^{0,1}$. In our previous notation, $T_I^{1,0}$ is T_I. Then, di_I is the natural embedding of $T_I = T_I^{1,0}$ to $T_{\mathbb{R}}$, and de_I is the natural projection of $T_{\mathbb{R}} = T_I^{1,0} \oplus T_I^{0,1}$ to $T_I^{1,0} = T_I$.*

Proof. Clear. □

We are able now to describe the map $d\Psi : T_I \longrightarrow T_I$ in terms of the quaternion action. Recall that the space T_I is equipped with a natural $\mathbb{R}$-linear quaternionic action. For each quaternionic linear space $\underline{V}$ and each quaternion I, $I^2 = -1$, I defines a complex structure in $\underline{V}$. Such a complex structure is called *induced by the quaternionic structure.*

Lemma 5.12. *Let $\underline{V}$ be a space with quaternion action, and $V := \underline{V} \otimes \mathbb{C}$ its complexification. For each induced complex structure $I \in \mathbb{H}$, consider the Hodge decomposition $V := V_I^{1,0} \oplus V_I^{0,1}$. For induced complex structures $I, J \in \mathbb{H}$, let $\Phi_{I,J}(V)$ be the composition of the natural embeddings and projections*

$$V_I^{1,0} \longrightarrow V \longrightarrow V_J^{1,0} \longrightarrow V \longrightarrow V_I^{1,0}.$$

Using the natural identification $\underline{V} \cong V_I^{1,0}$, we consider $\Phi_{I,J}(V)$ as an $\mathbb{R}$-linear automorphism of the space $\underline{V}$. Then, applying the operator $\Phi_{I,J}(V)$ to the quaternionic space T_I, we obtain the operator $d\Psi$ defined above.

Proof. Follows from Lemma 5.11 □

As we have seen, to prove Proposition 5.10 it suffices to show that $d\Psi$ is a multiplication by a non-zero complex number λ, $|\lambda| < 1$. Thus, the proof of Proposition 5.10 is finished with the following lemma.

Lemma 5.13. *In the assumptions of Lemma* 5.12, *consider the map*

$$\Phi_{I,J}(V):\ V_I^{1,0} \longrightarrow V_I^{1,0}.$$

Then $\Phi_{I,J}(V)$ *is a multiplication by a complex number* λ. *Moreover,* λ *is a non-zero number unless* $I = -J$, *and* $|\lambda| < 1$ *unless* $I = J$.

Proof. Let $\underline{V} = \oplus \underline{V}_i$ be a decomposition of V into a direct sum of $\mathbb{H}$-linear spaces. Then, the operator $\Phi_{I,J}(V)$ can also be decomposed:

$$\Phi_{I,J}(V) = \oplus \Phi_{I,J}(V_i).$$

Thus, to prove Lemma 5.13 it suffices to assume that $\dim_{\mathbb{H}} \underline{V} = 1$. Therefore, we may identify $\underline{V}$ with the space $\mathbb{H}$, equipped with the right action of quaternion algebra on itself.

Consider the left action of $\mathbb{H}$ on $\underline{V} = \mathbb{H}$. This action commutes with the right action of $\mathbb{H}$ on $\underline{V}$. Consider the corresponding action

$$\rho:\ SU(2) \longrightarrow \operatorname{End}(\underline{V})$$

of the group of unitary quaternions $\mathbb{H}^{un} = SU(2) \subset \mathbb{H}$ on $\underline{V}$. Since ρ commutes with the quaternion action, ρ preserves $V_I^{1,0} \subset V$, for every induced complex structure I. In the same way, for each $g \in SU(2)$, the endomorphism $\rho(g) \in \operatorname{End}(V_I^{1,0})$ commutes with $\Phi_{I,J}(V)$.

Consider the 2-dimensional $\mathbb{C}$-vector space $V_I^{1,0}$ as a representation of $SU(2)$. Clearly, $V_I^{1,0}$ is an irreducible representation. Thus, by Schur's lemma, the automorphism $\Phi_{I,J}(V) \in \operatorname{End}(V_I^{1,0}))$ is a multiplication by a complex constant λ. The bounds $0 < |\lambda| < 1$ are implied by the following elementary argument. The composition $i_I \circ e_J$ applied to a vector $v \in V_I^{1,0}$ is a projection of v to $V_J^{1,0}$ along $V_J^{0,1}$. Consider the natural Euclidean metric on $V = \mathbb{H}$. Clearly, the decomposition $V = V_J^{1,0} \oplus V_J^{0,1}$ is orthogonal. Thus, the composition $i_I \circ e_J$ is an orthogonal projection of $v \in V_I^{1,0}$ to $V_J^{1,0}$. Similarly, the composition $i_J \circ e_I$ is an orthogonal projection of $v \in V_J^{1,0}$ to $V_I^{1,0}$. Thus, the map $\Phi_{I,J}(V)$ is an orthogonal projection from $V_I^{1,0}$ to $V_J^{1,0}$ and back to $V_I^{1,0}$. Such a composition always decreases a length of vectors, unless $V_I^{1,0}$ coincides with $V_J^{1,0}$, in which case $I = J$. Also, unless $V_I^{1,0} = V_J^{0,1}$, $\Phi_{I,J}(V)$ is non-zero; in the later case, $I = -J$. Proposition 5.10 is proven. □

From Proposition 5.10 and Proposition 5.6, we obtain the following theorem.

Theorem 5.14. *(the main result of this section) Let M be a hypercomplex variety. Then M is a space with locally homogeneous singularities (SLHS).* □

6. Desingularization of hypercomplex varieties.

6.1. The proof of desingularization theorem.

Proposition 6.1. *Let M be a hypercomplex variety, and I an integrable induced complex structure. Then the normalization of (M, I) is smooth.*

Proof. The normalization of $Z_x M$ is smooth by Theorem 4.7. The normalization is compatible with the adic completions ([M], Chapter 9, Proposition 24.E). Therefore, the integral closure of the completion of $\mathcal{O}_{Z_x M}$ is a regular ring. From Theorem 5.14 it follows that the integral closure of $\hat{\mathcal{O}}_x M$ is also a regular ring, where $\hat{\mathcal{O}}_x M$ is an adic completion of the local ring of holomorphic functions on (M, I) in a neighbourhood of x. Applying [M], Chapter 9, Proposition 24.E again, we obtain that the integral closure of $\mathcal{O}_x M$ is regular. This proves Proposition 6.1 □

Theorem 6.2. *(Desingularization theorem) Let M be a hypercomplex variety I an integrable induced complex structure. Let*

$$\widetilde{(M,I)} \xrightarrow{n} (M,I)$$

be the normalization of (M, I). Then $\widetilde{(M,I)}$ is smooth and has a natural hypercomplex structure $\mathcal{H}$, such that the associated map $n : \widetilde{(M,I)} \longrightarrow (M,I)$ agrees with $\mathcal{H}$. Moreover, the hypercomplex manifold $\widetilde{M} := \widetilde{(M,I)}$ is independent of the choice of induced complex structure I.

Proof. The variety $\widetilde{(M,I)}$ is smooth by Proposition 6.1. Let $x \in M$, and $U \subset M$ be a neighbourhood of x. Let $\mathfrak{R}_x(U)$ be the set of irreducible components of U which contain x. There is a natural map $\tau : \mathfrak{R}_x(U) \longrightarrow Irr(Spec\hat{\mathcal{O}}_x M)$, where $Irr(Spec\hat{\mathcal{O}}_x M)$ is a set of irreducible components of $Spec\hat{\mathcal{O}}_x M$, where $\hat{\mathcal{O}}_x M$ is a completion of $\mathcal{O}_x M$ in x. Since

$\mathcal{O}_x M$ is Henselian ([R], VII.4), there exist a neighbourhood U of x such that $\tau : \mathfrak{R}_x(U) \longrightarrow Irr(Spec\hat{\mathcal{O}}_x M)$ is a bijection. Fix such an U. Since M is a space with locally homogeneous singularities, the irreducible decomposition of U coincides with the irreducible decomposition of the tangent cone $Z_x M$.

Let $\coprod U_i \stackrel{u}{\longrightarrow} U$ be the morphism mapping a disjoint union of irreducible components of U to U. By Theorem 4.7, the x-completion of $\mathcal{O}_{U_i}$ is regular. Shrinking U_i if necessary, we may assume that U_i is smooth. Then, the morphism u coincides with the normalization of U.

For each variety X, we denote by $X^{ns} \subset X$ the set of non-singular points of X. Clearly, $u(U_i) \cap U^{ns}$ is a connected component of U^{ns}. Therefore, $u(U_i)$ is trianalytic in U. By Remark 4.4, U_i has a natural hypercomplex structure, which agrees with the map u. This gives a hypercomplex structure on the normalization $\widetilde{U} := \coprod U_i$. Gluing these hypercomplex structures, we obtain a hypercomplex structure $\mathcal{H}$ on the smooth manifold $\widetilde{(M,I)}$. Consider the normalization map $n : \widetilde{(M,I)} \longrightarrow M$, and let $\widetilde{M}^n := n^{-1}(M^{ns})$. Then, $n\Big|_{\widetilde{M}^n} : \widetilde{M}^n \longrightarrow M^{ns}$ is an etale finite covering which is compatible with the hypercomplex structure. Thus, $\mathcal{H}\Big|_{\widetilde{M}^n}$ can be obtained as a pullback from M. Clearly, a hypercomplex structure on a manifold is uniquely defined by its restriction to an open dense subset. We obtain that $\mathcal{H}$ is independent from the choice of I. □

6.2. Integrability of induced complex structures.

Theorem 6.3. *Let M be a hypercomplex variety, I an induced complex structure. Then I is integrable.*

Proof. Kaledin has proven Theorem 6.3 for smooth M ([K]). Let $\widetilde{M}$ be a desingularization of M, which is hypercomplex. Then I induces an integrable almost complex structure on $\widetilde{M}$. From the local structure of the singularities of M, it is clear that M is obtained from $\widetilde{M}$ by gluing in pairs certain trianalytic subvarieties $X_i \subset \widetilde{M}$. Since I induces an integrable complex structure on $\widetilde{M}$, the X_i are complex subvarieties of $(\widetilde{M}, I)$, and the identification procedures are complex analytic with respect to I. Performing these identification morphisms on $(\widetilde{M}, I)$, we obtain a complex variety M' such that (M, I) is the underlying almost complex variety. This proves Theorem 6.3. □

7. Twistor spaces of hypercomplex varieties.

Let M be a hypercomplex variety, $M_{\mathbb{R}}$ the underlying real analytic variety. Consider the variety $\operatorname{Tw}_{\mathbb{R}} := M_{\mathbb{R}} \times S^2$, where S^2 is the 2-dimensional sphere identified with the real variety underlying $\mathbb{C}P^1$. We endow $\operatorname{Tw}_{\mathbb{R}}$ with an almost complex structure as follows. We have a decomposition

$$\Omega^1 \operatorname{Tw}_{\mathbb{R}} = \pi^*\Omega^1 S^2 \oplus \sigma^*\Omega^1 M_{\mathbb{R}},$$

where $\pi : \operatorname{Tw}_{\mathbb{R}} \longrightarrow S^2$, $\sigma : \operatorname{Tw}_{\mathbb{R}} \longrightarrow M$ are the natural projection maps. Let C be the natural map $\Omega^1 M_{\mathbb{R}} \otimes \mathbb{H} \xrightarrow{C} \Omega^1 M_{\mathbb{R}}$ arising from the quaternionic structure. As usual we identify the points of S^2 with induced complex structures on M, which are quaternions $L \in \mathbb{H}$, $L^2 = -1$. This gives a natural real analytic map $i : S^2 \longrightarrow \mathbb{H}$. A composition of C and i gives an endomorphism $\mathcal{I}_0 : \sigma^*\Omega^1 M_{\mathbb{R}} \longrightarrow \sigma^*\Omega^1 M_{\mathbb{R}}$. In terms of the fibers, the endomorphism $\mathcal{I}_0$ can be described as follows. For $(s,m) \in S^2 \times M_{\mathbb{R}} = \operatorname{Tw}_{\mathbb{R}}$, $\mathcal{I}_0$ acts on $\sigma^*\Omega^1 M_{\mathbb{R}}\Big|_{(s,m)} = \Omega^1 M_{\mathbb{R}}\Big|_m$ by I_s, where $I_s = i(s) \in \mathbb{H}$ is the induced complex structure corresponding to $s \in S^2$. Since S^2 is identified with $\mathbb{C}P^1$, this space has a natural complex structure $I_{\mathbb{C}P^1} : \Omega^1 S^2 \longrightarrow \Omega^1 S^2$. Let $\mathcal{I}$ be an almost complex structure $\mathcal{I} : \Omega^1 \operatorname{Tw}_{\mathbb{R}} \longrightarrow \Omega^1 \operatorname{Tw}_{\mathbb{R}}$ acting as $\pi^* I_{\mathbb{C}P^1}$ on $\pi^*\Omega^1 S^2$ and as $\mathcal{I}_0$ on $\sigma^*\Omega^1 M_{\mathbb{R}}$.

Claim 7.1. *The constructed above almost complex structure on* $\operatorname{Tw}_{\mathbb{R}}$ *is integrable.*

Proof. For M non-singular, this is proven by D. Kaledin [K]. For M singular, the proof essentially repeats the proof of Theorem 6.3: we apply the desingularization theorem (Theorem 6.2), and then Kaledin's result. □

Definition 7.2. Let M be a hypercomplex variety. Consider the complex variety $(\operatorname{Tw}, \mathcal{I})$ obtained in Claim 7.1. Then Tw is called *a twistor space* of M.

It is possible to characterize the hypercomplex varieties in terms of the twistor spaces. This characterization is the main purpose of the present paper.

Consider the unique antilinear involution $\iota_0 : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ with no fixed points. This involution is obtained by central symmetry with center in 0 if we identify $\mathbb{C}P^1$ with a unit sphere in $\mathbb{R}^3$. Let $\iota : \operatorname{Tw} \longrightarrow \operatorname{Tw}$ be an involution of the twistor space mapping $(s, m) \in S^2 \times M = \mathrm{Tw}$ to $(\iota_0(s), m)$. Clearly, ι is antilinear.

Definition 7.3. Let $s: \mathbb{C}P^1 \longrightarrow \mathrm{Tw}$ be a section of the natural holomorphic projection $\pi: \mathrm{Tw} \longrightarrow \mathbb{C}P^1$, $s \circ \pi = Id\big|_{\mathbb{C}P^1}$. Then s is called *a twistor line.* The space Sec of twistor lines is finite-dimensional and equipped with a natural complex structure, as follows from deformation theory ([Do]).

Remark 7.4. Sometimes in this paper it is assumed implicitly that Sec is a reduced complex analytic space. Since Sec is defined as the Douady space of certain curves of the twistor family this is not immediately obvious. There are two ways around that – one can define Sec as the Barlett space (so it is automatically reduced) or just show directly by deformation theory that the Douady space of sections of a twistor family is always reduced and is in fact isomorphic to the Barlett space.

Let Sec^ι be the space of all lines $s \in \mathrm{Sec}$ which are fixed by ι. The space Sec^ι is equipped with a structure of a real analytic space. We have a natural map $\tau: M_{\mathbb{R}} \longrightarrow \mathrm{Sec}^\iota$ associating to $m \in M$ the line $s: \mathbb{C}P^1 \longrightarrow \mathrm{Tw}$, $s(x) = (x, m) \in S^2 \times M = \mathrm{Tw}$. Such twistor lines are called *horizontal twistor lines.* Denote the set of horizontal twistor lines by $\mathrm{Hor} \subset \mathrm{Sec}$.

Lemma 7.5. *Let M be a hypercomplex variety,* Tw *its twistor space and $\tau: M_{\mathbb{R}} \longrightarrow \mathrm{Sec}^\iota$ the real analytic map constructed above. Then τ is a closed embedding identifying M with one of connected components of* Sec^ι.

Proof. By the Desingularization Theorem (Theorem 6.2), we may assume that M is smooth. For smooth M, Lemma 7.5 is a well-known statement which can be easily deduced from the deformation theory. For details, the reader is referred to [HKLR]. □

The following data suffice to recover the hypercomplex variety M:

(7.1)

- A complex analytic variety Tw, equipped with a morphism $\pi: \mathrm{Tw} \longrightarrow \mathbb{C}P^1$.
- An antilinear involution $\iota: \mathrm{Tw} \longrightarrow \mathrm{Tw}$ such that $\iota \circ \pi = \pi \circ \iota_0$
- A choice of connected component Hor of $\mathrm{Sec}^\iota \subset \mathrm{Sec}$.

The hypercomplex variety M is reconstructed as follows. The real analytic structure on $M = \text{Hor}$ comes from Sec^ι. For $I \in \mathbb{C}P^1$, consider the map $p_I : \text{Hor} \longrightarrow \pi^{-1}(I) \subset \text{Tw}$, $s \in \text{Hor} \longrightarrow s(I) \in \text{Tw}$. This identifies Hor with $\pi^{-1}(I) \subset \text{Tw}$. We obtained a complex structure I on M, for each $I \in \mathbb{C}P^1$. Identifying $\mathbb{C}P^1$ with a subset of quaternions, we recover the original quaternion action on $\Omega^1 M$. The data (7.1) satisfies the following properties (condition (ii) is implicit in the quaternionic action).

(7.2)

(i) For each point $x \in \text{Tw}$, there is a unique line $s \in \text{Hor} \subset \text{Sec}^\iota$, passing through x.

For every $s \in \text{Sec}$, *we identify the image of* s, $\text{im}\, s \subset \text{Tw}$, *with* $\mathbb{C}P^1$.

(ii) For every line $s \in \text{Hor} \subset \text{Sec}^\iota$, the conormal sheaf

$$N_s^* = \ker \left(\Omega^1 \text{Tw} \Big|_{\text{im}\, s} \xrightarrow{s^*} \Omega^1(\text{im}\, s) \right)$$

of $\text{im}\, s$ is isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus ... \oplus \mathcal{O}(-1)$.

Definition 7.6. The data (7.1) satisfying the conditions (i), (ii) of (7.2) are called *a twistor space of hypercomplex type.* We have shown how to associate a twistor space of hypercomplex type to every hypercomplex variety.

Denote the corresponding functor by $\mathcal{F}$.

Condition (ii) of (7.2) can be replaced by the following condition.

(7.2)

(ii′) For every line $s \in \text{Hor} \subset \text{Sec}^\iota$, there exists an open neighbourhood $U \subset \text{Tw}$ of $\text{im}\, s$, such that for every $x, y \in U$, $\pi(x) \neq \pi(y)$, there exists a unique twistor line $s_{x,y}$ passing through x and y, provided that x and y belong to the same irreducible component of U.

Condition (ii) of (7.2) should be thought of as a linearization of (7.2) (ii′). In the subsequent section, we shall see that these conditions are equivalent.

Definition 7.7. The data (7.1) satisfying the conditions (i), (ii′) of (7.2) are called *a twistor space of Deligne-Simpson type.* These conditions were proposed by Deligne and Simpson ([S], [De]) in order to define singular hyperkähler manifolds.

8. Twistor spaces of Deligne-Simpson type.

The main result of this section is the following theorem.

Theorem 8.1. *Let* $(\mathrm{Tw}, \pi, \iota, \mathrm{Hor})$ *be the data of (7.1) satisfying condition (i) of (7.2). Then the conditions (ii) and (ii′) are equivalent. In other words,* $(\mathrm{Tw}, \pi, \iota, \mathrm{Hor})$ *is a twistor space of hypercomplex type if and only if* $(\mathrm{Tw}, \pi, \iota, \mathrm{Hor})$ *is a twistor space of Deligne–Simpson type.*

The proof of Theorem 8.1 takes the rest of this section.

Under assumptions of (7.1), (7.2) (i) consider the map $\sigma : \mathrm{Tw} \longrightarrow \mathrm{Hor}$ associating to a point $x \in \mathrm{Tw}$ the unique horizontal line passing through this point. This map is continuous, and induces a homeomorphism $\sigma \times \pi : \mathrm{Tw} \longrightarrow \mathrm{Hor} \times \mathbb{C}P^1$.

Lemma 8.2. *Let* $(\mathrm{Tw}, \pi, \iota, \mathrm{Hor})$ *be the data of (7.1) satisfying condition (i) of (7.2), and* $U \subset \mathrm{Hor}$ *an arbitrary open subset. Then* $\sigma^{-1}(U)$ *is preserved by* ι.

Proof. Let $s \in U \subset \mathrm{Hor}$. Then ι preserves a line $\mathrm{im}\, s \subset \mathrm{Tw}$, and thus, $\iota(\mathrm{im}\, s) \subset \sigma^{-1}(s) \subset \sigma^{-1}(U)$. This proves Lemma 8.2. □

We prove the implication

"$(\mathrm{Tw}, \pi, \iota, \mathrm{Hor})$ of hypercomplex type"

$$\Rightarrow$$

"$(\mathrm{Tw}, \pi, \iota, \mathrm{Hor})$ of Deligne–Simpson type".

The statement of Theorem 8.1 is local in Hor, as follows from Lemma 8.2. Consider the evaluation maps

$$p_I : \mathrm{Hor} \longrightarrow \pi^{-1}(I), \quad s \longrightarrow s(I),$$

defined for all $I \in \mathbb{C}P^1$. The map σ gives a homeomorphism $\pi^{-1}(I) \xrightarrow{\sigma} \mathrm{Hor}$. A homeomorphism preserves the dimension of the variety. Thus, for a smooth point $x \in \mathrm{Tw}$, the point $\sigma(x) \in \mathrm{Hor}$ is equidimensional [4] in Hor. Using the homeomorphism $\sigma \times \pi : \mathrm{Tw} \longrightarrow \mathrm{Hor} \times \mathbb{C}P^1$, we find that for a

[4] Equidimensional point of X is a point where all irreducible components of X have the same dimension.

smooth point $x \in \mathrm{Tw}$ and $y \in \mathrm{Tw}$ satisfying $\sigma(y) = \sigma(x)$, the point y is equidimensional in Tw.

The real dimensions of the varieties Hor and $\pi^{-1}(I)$ are equal. On the other hand, for every $m \in \mathrm{Hor}$, the dimension of the tangent space $T_{p_I(m)}\pi^{-1}(I)$ is equal to the dimension of $N^*_m\Big|_{p_I(m)}$. Since $N^*_m\Big|_{p_I(m)}$ is a bundle, the dimension of $T_{p_I(m)}\pi^{-1}(I)$ is the same for all $I \in \mathbb{C}P^1$. The local ring is regular if and only if the dimension of its tangent space is equal to the dimension of the ring. The dimensions of the varieties $\pi^{-1}(J)$ and $\pi^{-1}(I)$, and the dimensions of the corresponding tangent spaces are equal. Thus, for a smooth point $m \in \pi^{-1}(I)$, and every $J \in \mathbb{C}P^1$, the points $p_J(\sigma(m)) \in \pi^{-1}(J)$ are smooth in $\pi^{-1}(J)$.

We obtained the following result.

Claim 8.3. *Let* $(\mathrm{Tw}, \pi, \iota, \mathrm{Hor})$ *be a twistor space of hypercomplex type, and* $m \in \mathrm{Tw}$ *a smooth point. Let* σ *denote the natural continuous map* $\sigma : \mathrm{Tw} \longrightarrow \mathrm{Hor}$. *Then, for every point* $m' \in \mathrm{Tw}$ *such that* $\sigma(m) = \sigma(m')$, m' *is smooth.* □

Lemma 8.4. *Let* $(\mathrm{Tw}, \pi, \iota, \mathrm{Hor})$ *be a twistor space of hypercomplex type, and* $m \in \mathrm{Tw}$ *a smooth point. Then*

(i) $\sigma(m)$ *is a smooth point of* Hor.

(ii) *Moreover, for a smooth neighbourhood* U *of* $m \in Tw$, $(\sigma^{-1}(\sigma(U)) \subset \mathrm{Tw}, \pi, \iota, \sigma(U))$ *is a twistor space of Deligne-Simpson type.*

Proof. Let $s = \operatorname{im} m \subset \mathrm{Tw}$ be the horisontal twistor line corresponding to m. From the deformation theory we know that the deformations of a smooth curve s are classified by the sections of the normal bundle $\Gamma(Ns)$, with obstructions corresponding to $H^1(Ns)$. The cohomology space $H^1(Ns)$ vanishes, because $Ns = \bigoplus \mathcal{O}(1)$ is ample. Thus, $\Gamma(Ns)$ has a dimension $2(\dim \mathrm{Tw} - 1)$. For a small deformations s' of s, $Ns' = \bigoplus \mathcal{O}(1)$, since $\bigoplus \mathcal{O}(1)$ is semistable. Thus, $T_{s'} \mathrm{Sec}$ is constant in a neighborhood of s, where Sec is the space of sections of $\pi : \mathrm{Tw} \longrightarrow \mathbb{C}P^1$. This implies that s is a smooth point of Sec, and hence, s is a smooth point of Hor. Lemma 8.4 (i) is proven.

To prove (ii), let I, J be distinct points in $\mathbb{C}P^1$ and consider the map $p_{IJ} : \mathrm{Sec} \longrightarrow \pi^{-1}(I) \times \pi^{-1}(J)$, $\gamma \longrightarrow (\gamma(I), \gamma(J))$. We have to prove that p_{IJ} has invertible differential in s. The tangent space $T_s \mathrm{Sec}$ is, as we have seen, $\Gamma(Ns)$. The differential of the map $p_I : \mathrm{Sec} \longrightarrow \pi^{-1}(I), \gamma \longrightarrow \gamma(I)$ coincides

with the restriction map $r_I : \Gamma(Ns) \longrightarrow Ns\Big|_I$. Since Ns is $\bigoplus \mathcal{O}(1)$, the differential $dp_{IJ} = r_I \times r_J$ is an isomorphism (a section of $\bigoplus \mathcal{O}(1)$ is uniquely determined by its value in two distinct points). This proves Lemma 8.4 (ii).
□

We return to the proof of an implication "$(\mathrm{Tw}, \pi, \iota, \mathrm{Hor})$ of hypercomplex type" $\Rightarrow$ "$(\mathrm{Tw}, \pi, \iota, \mathrm{Hor})$ of Deligne–Simpson type". Let Tw^{ns} be the set of non-singular points of Tw, I, J be two distinct points of $\mathbb{C}P^1$. Let $W \subset \mathrm{Sec}$ be an open neighbourhood of a horisontal line $s \in \mathrm{Sec}$, such that its closure $\overline{W}$ is compact and $\overline{V}_{IJ}$ the set of all triples $(x, y, s_{xy}) \in \pi^{-1}(I) \times \pi^{-1}(J) \times \mathrm{Sec}$ such that $s_{xy} \in \overline{W}$, and s_{xy} is a twistor line passing through x and y. Let $V_{IJ} \subset \overline{V}_{IJ}$ be the set of the triples $(x, y, s_{xy}) \in \overline{V}_{IJ}$, for which the corresponding twistor line s_{xy} belongs to the non-singular part of Tw.

Lemma 8.5. *In the above assumptions, consider the forgetful map* $p : \overline{V}_{IJ} \longrightarrow \pi^{-1}(I) \times \pi^{-1}(J)$. *Then* $p\left(\overline{V}_{IJ}\right)$ *is a closure of* $p\left(V_{IJ}\right)$.

Proof. The space $\overline{V}_{IJ}$ is compact, and hence its image is compact and thus closed. It remains to show that V_{IJ} is dense in $\overline{V}_{IJ}$. By Claim 8.3, for each smooth point $m \in \mathrm{Tw}$, a neighbourhoor of $\sigma(m) \in \mathrm{Sec}$ belings to V_{IJ}. Since the set of smooth points of Tw is dense in Tw, for $\overline{V}_{IJ}$ sufficiently small, V_{IJ} is dense in $\overline{V}_{IJ}$. □

Let U be an open subset of Tw, and $X_U \subset \pi^{-1}(I) \times \pi^{-1}(J)$ be the set of all $(x, y) \in \pi^{-1}(I) \times \pi^{-1}(J)$ belonging to the same irreducible component of $U \subset \mathrm{Tw}$. Consider the forgetful map $p : \overline{V}_{IJ} \longrightarrow \pi^{-1}(I) \times \pi^{-1}(J)$. Clearly, the image of p intersected with $U \times U$ lies in X_U, so we may assume that p maps $\overline{V}_{IJ}$ to X_U. Computing the differential of p as in the proof of Lemma 8.4, we find that dp is locally injective for s_{xy} in a neighbourhood of Hor. To prove the condition of Deligne and Simpson, it remains to show that p is locally a surjection onto X_U, for sufficiently small U. By Lemma 8.4, the image of $p\Big|_{V_{IJ}}$ is dense in $X_U \cap \mathrm{Tw}^{ns} \times \mathrm{Tw}^{ns}$, for U sufficiently small. On the other hand, the closure of $\operatorname{im} p\Big|_{V_{IJ}}$ is $\operatorname{im} p\Big|_{\overline{V}_{IJ}}$ by Lemma 8.5, so p is locally a surjection. We proved that the Deligne-Simpson condition holds for all twistor spaces of hypercomplex type.

Assume now that $(\mathrm{Tw}, \pi, \iota, \mathrm{Hor})$ is a twistor space of Deligne–Simpson type. Let $s \in \mathrm{Hor}$. Consider the evaluation map $p_{IJ} : \mathrm{Sec} \longrightarrow \pi^{-1}(I) \times \pi^{-1}(J)$, $I \neq J \in \mathbb{C}P^1$. By Deligne-Simpson's condition, p_{IJ} induces an isomorphism

$$dp_{IJ}: T_s \operatorname{Sec} \longrightarrow T_{s(I)}\pi^{-1}(I) \times T_{s(I)}\pi^{-1}(I) \tag{8.1}$$

of the tangent spaces. Thus, dimension of $T_{s(I)}\pi^{-1}(I)$ is independent from the choice of I. We obtain that the conormal sheaf N^*s is a bundle, and it makes sense to speak of the normal bundle Ns.

Through each point in a neighbourhood of s passes a deformation of s. Thus, $T_s \operatorname{Sec} = \Gamma(Ns)$; there is no first order obstructions to the deformation. Since $T_s \operatorname{Sec} = Ns$, the map (8.1) can be interpreted as

$$dp_{IJ}: \Gamma(Ns) \longrightarrow Ns\Big|_I \times Ns\Big|_J$$

Since (8.1) is an isomorphism, Ns is a bundle which is isomorphic to $\bigoplus \mathcal{O}(1)$. Theorem 8.1 is proven. □

9. Hypercomplex varieties and twistor spaces of hypercomplex type.

The main result of this paper is the following theorem.

Theorem 9.1. *Consider the functor $\mathcal{F}$ of Definition 7.6, associating to a hypercomplex variety the corresponding twistor space of hypercomplex type. Then $\mathcal{F}$ is an equivalence of categories.*

Proof. We have shown how to recover the hypercomplex structure from the twistor space. This proves that $\mathcal{F}$ is full and faithful. It remains to show that each twistor space of hypercomplex type is obtained from a hypercomplex variety. Thus, Theorem 9.1 is implied by the following statement.

Theorem 9.1′. *Let*

$$(\operatorname{Tw}, \pi, \iota, \operatorname{Hor})$$

be a twistor space of hypercomplex type. Then Hor *admits a hypercomplex structure $\mathcal{H}$, such that* $(\operatorname{Tw}, \pi, \iota, \operatorname{Hor})$ *is a twistor space of* $(\operatorname{Hor}, \mathcal{H})$.

The rest of this section is devoted to the proof of Theorem 9.1′.

Lemma 9.2. *Let* Hor $= \bigcup H_i$ *be an irreducible decomposition of the variety* Hor. *Let* $\mathrm{Tw}_i := \sigma^{-1}(H_i)$, *where* $\sigma : \mathrm{Tw} \longrightarrow \mathrm{Hor}$ *is the standard continuous map. Then* Tw_i *is preserved by* ι, *and*

$$\left(Tw_i, \pi\Big|_{\mathrm{Tw}_i}, \iota\Big|_{\mathrm{Tw}_i}, H_i\right)$$

is a twistor space of hypercomplex type.

Proof. It is clear from the definition that Tw_i is preserved by ι and that for every $m \in \mathrm{Tw}_i$ there is a unique horisontal line $s \in H_i$ passing through m. It remains to prove the condition (ii) of (7.2), or, equivalently, condition (ii)′ of (7.2). But, since Tw satisfies the condition (ii)′ of (7.2), Tw_i satisfies this condition automatically. □

Claim 9.3. *Let* $f : X \longrightarrow Y$ *be a homeomorphism of complex analytic varieties. Then* f *maps irreducible components of* X *to irreducible components of* Y.

Proof. Clear. □

Lemma 9.4. *Let* $(\mathrm{Tw}, \pi, \iota, \mathrm{Hor})$ *be a twistor space of hypercomplex type, and* $I \in \mathbb{C}P^1$. *Consider the map* $p_I : \mathrm{Hor} \longrightarrow \pi^{-1}(I)$, $s \longrightarrow s(I)$. *Let* $p : \mathrm{Hor} \times \mathbb{C}P^1 \longrightarrow \mathrm{Tw}$ *map* (s, I) *to* $s(I)$. *Then* p *and* p_I *induce isomorphisms of corresponding real analytic varieties.*

Proof. Using Claim 9.3, Lemma 9.2 and Lemma 8.2, we may assume that the fiber $\pi^{-1}(I)$ is locally irreducible and thus, equidimensional. The real dimensions of the varieties Hor, $\pi^{-1}(I)$ are clearly equal. Thus, to show that p, p_I induce isomorphisms, it suffices to show that corresponging maps of local rings are surjective. By Nakayama, for this we need to show that p, p_I induce surjection on the Zariski tangent spaces. The differential of the evaluation map $ev_I : \mathrm{Sec} \longrightarrow \pi^{-1}(I)$, $s \longrightarrow s(I)$ is the standard restriction map $r_I : \gamma \longrightarrow \gamma\Big|_I$, where $\gamma \in T_s \mathrm{Sec} = \Gamma(Ns)$, and $\gamma\Big|_I \in Ns\Big|_I = T_{s(I)}\pi^{-1}(I)$. Thus, dp_I is a composition of r_I and the embedding $T_s \mathrm{Hor} \hookrightarrow T_s \mathrm{Sec}$. The image of $T_s \mathrm{Hor} \hookrightarrow T_s \mathrm{Sec}$ coincides with the set of fixed points of the involution $d\iota : T_s \mathrm{Sec} \longrightarrow T_s \mathrm{Sec}$. Thus, dp_I is an isomorphism by the following trivial result of linear algebra.

Lemma 9.5. *Let V be a vector space of complex dimension $2n$, W a vector space of complex dimension n, and $\varphi : V \longrightarrow W$ an epimorphism. Let $V' \subset V$ be a totally real subspace of real dimension $2n$. Then $\varphi\big|_{V'} : V' \longrightarrow W$ is an isomorphism.* □

A similar argument proves that p is also an isomorphism. Lemma 9.4 is proven. □

We obtained that the real analytic variety Hor is isomorphic to one underlying $\pi^{-1}(I)$, for all $I \in \mathbb{C}P^1$. This gives a set of integrable almost complex structures on Hor, parametrized by $\mathbb{C}P^1$. The following linear algebraic argument shows that these complex structures satisfy quaternionic relations. This finishes the proof of Theorem 9.1.

Let F be an n-dimensional holomorphic vector bundle on $\mathbb{C}P^1$, $F \cong \oplus\mathcal{O}(1)$. Consider the restriction maps $r_I : \Gamma(F) \longrightarrow F\big|_I$, defined for each $I \in \mathbb{C}P^1$. Let $W \subset \Gamma(F)$ be a totally real subspace of real dimension $2n$. By Lemma 9.5, r_I induces an isomorphism between $F\big|_I$ and W. This gives a set of complex structures on W, parametrized by $\mathbb{C}P^1$.

Claim 9.6. *These complex structures satisfy quaternionic relations.*

Proof. Clear. □

10. Some applications.

10.1. Hypercomplex spaces.

Using Theorem 9.1, it is possible to generalize the definition of hypercomplex varieties, allowing nilpotents, in such a way that a reduction of a hypercomplex space is a hypercomplex variety.

Consider the antilinear involution $\iota_0 : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ defined in 7.

Definition 10.1. (Hypercomplex spaces) Let Tw be a complex analytic space, $\pi : \mathrm{Tw} \longrightarrow \mathbb{C}P^1$ a holomorphic map, and $\iota : \mathrm{Tw} \longrightarrow \mathrm{Tw}$ an antilinear automorphism, such that $\iota \circ \pi = \pi \circ \iota_0$. Let Sec be the space of sections of π equipped with a structure of a complex analytic space, and Sec^ι be the real analytic space of sections s of π satisfying $s \circ \iota_0 = \iota \circ s$. Let Hor be a connected component of Sec^ι. Then $(\mathrm{Tw}, \pi, \iota, \mathrm{Hor})$ is called *a hypercomplex space* if

(i) For each point $x \in \mathrm{Tw}^r$, there exist a unique line $s \in \mathrm{Hor}^r$ passing through x, where Tw^r, Hor^r is a reduction of Tw, Hor.

(ii) Let $s \in \mathrm{Hor}$, and $U \subset \mathrm{Tw}$ be a neighbourhood of s such that an irreducible decomposition of U coincides with the irreducible decomposition of Tw in a neighbourhood of $s \subset \mathrm{Tw}^r$. Let

$$\overline{X} := \pi^{-1}(I) \times \pi^{-1}(J) \cap U \times U,$$

where I, J distinct points of $\mathbb{C}P^1$. Let $p_{IJ} : U \longrightarrow \overline{X} \subset \pi^{-1}(I) \times \pi^{-1}(J)$ be the evaluation map, $s \longrightarrow (s(I), s(J))$. Then there exist a closed subspace $X \subset \overline{X}$, obtained as a union of some of irreductible components of $\overline{X}$, and an open neighbourhood $V \subset \mathrm{Sec}$ of $s \in \mathrm{Sec}$, such that p_{IJ} is an open embedding of V to X.

10.2. Stable bundles over hyperkähler manifolds.

Let M be a compact hyperkähler manifold, I an induced complex structure and B a stable holomorphic bundle over (M, I), such that the first two Chern classes $c_1(B)$, $c_2(B)$ are $SU(2)$-invariant, with respect to the natural action of the group $SU(2)$ on the cohomology of M (Lemma 3.3).

Recall that $SU(2)$ acts on the space of differential forms on M. This allows us to speak of $SU(2)$-invariant differential forms, for instance of connections with $SU(2)$-invariant curvature. In [V1], the following theorem was proven.

Theorem 10.2. *There exist a unique Hermitian connection ∇ on B such that its curvature Θ is $SU(2)$-invariant. Conversely, if such a connection exists on a holomorphic bundle B over (M, I), then B is a direct sum of stable bundles with $SU(2)$-invariant Chern classes.* □

We show that the moduli space $\mathrm{Def}(B)$ of deformations of B is hypercomplex. Consider the twistor space Tw of M, and a standard real analytic map $\sigma : \mathrm{Tw} \longrightarrow M$. Let $\sigma^* B$ be the pullback of B equipped with the connection which is trivial along the fibers of σ. Due to the standard properties of the twistor transform ([KV], Section 5) it is clear that $\sigma^* B$ is holomorphic.

Restricting σ^*B to the fibers of $\pi : \operatorname{Tw} \longrightarrow \mathbb{C}P^1$, we obtain holomorphic bundles B_J on $\pi^{-1}(J) = (M, J)$. By Theorem 10.2, B_J is stable. Let $\hat{\operatorname{Tw}}$ be the moduli space of sheaves of type $i^J_* F$, where $i^J : (M, J) = \pi^{-1}(J) \hookrightarrow \operatorname{Tw}$ is the natural embedding, and F a stable holomorphic bundle which is a deformation of B_J (i. e. belongs in the same deformation class). Then $\hat{\operatorname{Tw}}$ is equipped with a holomorphic fibration $\hat\pi : \hat{\operatorname{Tw}} \longrightarrow \mathbb{C}P^1$, $i^J_* F \longrightarrow J$. Mapping F to $\iota^* F$, we obtain an antilinear involution $\hat\iota$ of $\hat{\operatorname{Tw}}$. The operation $(B, J) \longrightarrow B_J$ gives an $\hat\iota$-invariant section of $\hat\pi$, parametrized by $\operatorname{Def}(B)$. To show that thus obtained quadruple $(\hat{\operatorname{Tw}}, \hat\pi, \hat\iota, \operatorname{Def}(B))$ is hypercomplex, it suffices to prove the condition (ii) of (7.2). Equivalently, we may prove (ii′) of Definition 7.7. On the other hand, Proposition 2.19 of [KV] implies (ii′). This gives another proof that the space of stable deformations of B is hypercomplex, in addition to that given in [V1].

10.3. Quotients of hypercomplex varieties by an action of a finite group.

Let M be a hypercomplex variety and G a finite group acting on M, generically free. Assume that G preserves the hypercomplex structure, and acts freely outside of nonempty finite set of fixed points, denoted by Γ. Clearly, $\Big(M\backslash\{\Gamma\}\Big)/G$ has a natural structure of a hypercomplex variety.

Proposition 10.3. *The hypercomplex structure on*

$$\Big(M\backslash\{\Gamma\}\Big)/G \subset M/G$$

cannot be extended to M/G.

Proof. Consider the space Tw/G fibered over $\mathbb{C}P^1$, with corresponding action of ι. Then Hor/G gives an open subset in the space of ι-invariant sections of $\pi : \operatorname{Tw}/G \longrightarrow \mathbb{C}P^1$. Let $p : M \longrightarrow M/G$, $p : \operatorname{Tw} \longrightarrow \operatorname{Tw}/G$ be the natural quotient maps. If the hypercomplex structure on $M\backslash\{\Gamma\}/G \subset M/G$ were extendable to M/G, the space of horisontal sections would have been $p(\operatorname{Hor})$. Applying Theorem 9.1, we obtain the following assertion.

Claim 10.4. *The hypercomplex structure on $M\backslash\{\Gamma\}$ can be extended to M/G if an only if for all $s : \mathbb{C}P^1 \longrightarrow \operatorname{Tw}/G$, $s \in p(\operatorname{Hor})$, the conormal sheaf of* $\operatorname{im} s$ *in* Tw/G *is* $\oplus\mathcal{O}(-1)$. □

Let $s \in \mathrm{Hor}$ be a horisontal twistor line in Tw which passes through fixed point of G-action. Consider its formal neighbourhood in Tw. Let $\mathcal{O}(s)$ be the corresponding complete ring over $\mathbb{C}P^1$ and $\mathcal{O}(s)_{gr}$ be the associated graded ring. Then the ring $\mathcal{O}(s)_{gr}$ is isomorphic to $\oplus S^*(\mathcal{O}(-1))/D$, where D is a graded ideal lying in

$$\bigoplus_{k=2}^{\infty} S^k(\mathcal{O}(-1))$$

Let $\mathcal{O}(\hat{s})_{gr} \subset \mathcal{O}(s)_{gr}$ be the sheaf of G-invariant sections of $\oplus S^*(\mathcal{O}(-1))/D$. Clearly, $\mathcal{O}(\hat{s})_{gr}$ is the graded ring of the formal neighbourhood of $\hat{s} \subset \mathrm{Tw}/G$. Since x is an isolated fixed point of G action, the group acts on the Zariski tangent space T_xM without invariants. Therefore, $\mathcal{O}(\hat{s})_{gr}$ lies in

$$\bigoplus_{k=2}^{\infty} S^k(\mathcal{O}(-1))/D \subset \oplus S^*(\mathcal{O}(-1))/D = \mathcal{O}(s)_{gr}.$$

Thus, the conormal sheaf of $\mathcal{O}(\hat{s})_{gr}$ is isomorphic to $\mathcal{O}(i_1) \oplus \mathcal{O}(i_2) \oplus ...$, where $i_1, ..., i_k < -2$. Therefore, M/G cannot be hypercomplex. Proposition 10.3 is proven. □

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References.

[A] D. Arapura, *Geometry of cohomology support loci II: integrability of Hitchin's map*, alg-geom/9701014, 17 pages, AMS LaTeX.

[Bea] A. Beauville, *Varietes Kähleriennes dont la première classe de Chern est nulle*, J. Diff. Geom. **18** (1983), 755–782.

[Bes] A. Besse, *Einstein Manifolds*, Springer-Verlag, New York (1987).

[C] E. Calabi, *Metriques kählleriennes et fibrès holomorphes*, Ann. Ecol. Norm. Sup. **12** (1979), 269–294.

[De] P. Deligne, *A letter to Carlos Simpson*, referred to in [S].

[Do] A. Douady, *Le probleme des modules pour les sous-espaces analitiques compactes d'un espace analitique donné*, Ann. Inst. Fourier, **16** (1966), 1–95.

[GMT] F. Guaraldo, P. Macri and A. Tancredi, *Topics on real analytic spaces*, Advanced lectures in mathematics, Braunschweig: F. Vieweg, 1986.

[H] R. Hartshorne, *Algebraic geometry*, Graduate texts in mathematics, vol. 52, New York : Springer-Verlag, 1977.

[HKLR] N.J. Hitchin, A. Karlhede, U. Lindström and M. Roček, *Hyperkähler metrics and supersymmetry*, Comm. Math. Phys (1987).

[K] D. Kaledin, *Integrability of the twistor space for a hypercomplex manifold*, Selecta Math. New Series, **4** (1998), 271–278. Also preprint alg-geom 9612016, 1996, LaTeX, 9 pages.

[KV] D. Kaledin and M. Verbitsky, *Non-Hermitian Yang-Mills connections*, Selecta Math. New Series, 1998. Also preprint alg-geom 9606019, 1996, LaTeX, 48 pages.

[M] H. Matsumura, *Commutative algebra,* Mathematics lecture note series, vol. 56, Benjamin/Cummings Pub. Co., Reading, Mass., 1980.

[R] M. Raynaud, *Anneaux Locaux Henséliens*, Springer LNM 169, 1970.

[S] C.T. Simpson, *Nonabelian Hodge theory*. Proceedings of the International Congress of Mathematicians, (Kyoto, 1990), 747–756, Math. Soc. Japan, Tokyo, 1991.

[V1] M. Verbitsky, *Hyperholomorphic bundles over a hyperkähler manifold*, alg-geom electronic preprint 9307008 (1993), 43 pages, LaTeX, also published in: Journ. of Alg. Geom., **5** no. 4 (1996), 633–669.

[V2] M. Verbitsky, *Hyperkähler embeddings and holomorphic symplectic geometry II,* alg-geom electronic preprint 9403006 (1994), 14 pages, LaTeX, also published in: GAFA **5** no. 1 (1995), 92–104.

[V3] M. Verbitsky, *Deformations of trianalytic subvarieties of hyperkähler manifolds*, alg-geom electronic preprint 9610010 (1996), 51 pages, LaTeX2e.

[V4] M. Verbitsky, *New examples of compact hyperkähler manifolds*, in preparation.

[V-d] M. Verbitsky, *Desingularization of singular hyperkähler varieties I,* electronic preprint 9611015 (1996), 13 pages, LaTeX 2e (to appear in Math. Res. Let.)

[V-d2] M. Verbitsky, *Desingularization of singular hyperkähler varieties II,* electronic preprint 9612013 (1996), 15 pages, LaTeX 2e.

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