COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 7, Number 2, 347-353, 1999

On manifolds with non-negative Ricci curvature and Sobolev inequalities

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Let M be a complete n-dimensional Riemanian manifold with non-negative Ricci curvature in which one of the Sobolev inequalities $\left(\int |f|^p dv\right)^{1/p} \leq C\left(\int |\nabla f|^q dv\right)^{1/q}$, $f \in C_0^\infty(M)$, $1 \leq q < n$, $1/p = 1/q - 1/n$, is satisfied with C the optimal constant of this inequality in $\mathbb{R}^n$. Then M is isometric to $\mathbb{R}^n$.

Let M be a complete Riemannian manifold of dimension $n \geq 2$. Denote by dv the Riemannian volume element on M and by ∇ the gradient operator.

In this note, we are concerned with manifolds M in which a Sobolev inequality of the type

$$\left(\int |f|^p dv\right)^{1/p} \leq C\left(\int |\nabla f|^q dv\right)^{1/q}, \tag{1}$$

$1 \leq q < n$, $1/p = 1/q - 1/n$, holds for some constant C and all C^∞ compactly supported functions f on M. The best constants $C = K(n,q)$ for which (1) holds in $\mathbb{R}^n$ are known and were described by Th. Aubin [Au] and G. Talenti [Ta]. Namely, $K(n,1) = n^{-1}\omega_n^{-1/n}$ where ω_n is the volume of the Euclidean unit ball in $\mathbb{R}^n$, while

$$K(n,q) = \frac{1}{n}\left(\frac{n(q-1)}{n-q}\right)^{(q-1)/q}\left(\frac{\Gamma(n+1)}{n\omega_n\Gamma(n/q)\Gamma(n+1-n/q)}\right)^{1/n}$$

if $q > 1$. Moreover, for $q > 1$, the equality in (1) is attained by the functions $(\lambda + |x|^{q/(q-1)})^{1-(n/q)}$, $\lambda > 0$, where $|x|$ is the Euclidean length of the vector x in $\mathbb{R}^n$. We are actually interested here in the geometry of those manifolds M for which one of the Sobolev inequalities (1) is satisfied with the best constant $C = K(n,q)$ of $\mathbb{R}^n$. The result of this note is the following theorem.

Theorem. *Let M be a complete n-dimensional Riemannian manifold with non-negative Ricci curvature. If one of the Sobolev inequalities (1) is satisfied with $C = K(n,q)$, then M is isometric to $\mathbb{R}^n$.*

The particular case $q = 1$ ($p = n/(n-1)$) is of course well-known. In this case indeed, the Sobolev inequality is equivalent to the isoperimetric inequality

$$(\mathrm{vol}_n(\Omega))^{(n-1)/n} \le K(n,1)\mathrm{vol}_{n-1}(\partial\Omega)$$

where $\partial\Omega$ is the boundary of a smooth bounded open set Ω in M. If we let $V(x_0,s) = V(s)$ be the volume of the geodesic ball $B(x_0,s) = B(s)$ with center x_0 and radius s in M, we have

$$\frac{d}{ds}\mathrm{vol}_n\left(B(s)\right) = \mathrm{vol}_{n-1}\left(\partial B(s)\right).$$

Hence, setting $\Omega = B(s)$ in the isoperimetric inequality, we get

$$V(s)^{(n-1)/n} \le K(n,1)V'(s)$$

for all s. Integrating yields $V(s) \ge (nK(n,1))^{-n}s^n$, and since $K(n,1) = n^{-1}\omega_n^{-1/n}$, for every s,

$$V(s) \ge V_0(s) \tag{2}$$

where $V_0(s) = \omega_n s^n$ is the volume of the Euclidean ball of radius s in $\mathbb{R}^n$. If M has non-negative Ricci curvature, by Bishop's comparison theorem (cf. e.g. [Ch]) $V(s) \le V_0(s)$ for every s, and by (2) and the case of equality, M is isometric to $\mathbb{R}^n$. The main interest of the Theorem therefore lies in the case $q > 1$. As usual, the classical value $q = 2$ (and $p = 2n/(n-2)$) is of particular interest (see below). It should be noticed that known results already imply that the scalar curvature of M is zero in this case (cf. [He], Prop. 4.10).

Proof of the Theorem. It is inspired by the technique developed in the recent work [B-L] where a sharp bound on the diameter of a compact Riemannian manifold satisfying a Sobolev inequality is obtained, extending the classical Myers theorem.

We thus assume that the Sobolev inequality (1) is satisfied with $C = K(n,q)$ for some $q > 1$. Recall first that the extremal functions of this inequality in $\mathbb{R}^n$ are the functions $(\lambda + |x|^{q'})^{1-(n/q)}$, $\lambda > 0$, where $q' = q/(q-1)$. Let now x_0 be a fixed point in M and let $\theta > 1$. Set $f = \theta^{-1}d(\cdot, x_0)$ where d is the distance function on M. The idea is then to apply the Sobolev inequality (1), with $C = K(n,q)$, to $(\lambda + f^{q'})^{1-(n/q)}$, for every $\lambda > 0$ to deduce a differential inequality whose solutions may be compared to the extremal Euclidean case. Set, for every $\lambda > 0$,

$$F(\lambda) = \frac{1}{n-1}\int \frac{1}{(\lambda + f^{q'})^{n-1}}\,dv.$$

Note first that F is well defined and continuously differentiable in λ. Indeed, by Fubini's theorem, for every $\lambda > 0$,

$$F(\lambda) = q' \int_0^\infty V(\theta s)\, \frac{s^{q'-1}}{(\lambda + s^{q'})^n}\, ds$$

(where we recall that $V(s) = V(x_0, s)$ is the volume of the ball with center x_0 and radius s). By Bishop's comparison theorem, $V(s) \le V_0(s)$ for every s. It follows that $0 \le F(\lambda) < \infty$ and that F is differentiable.

Together with a simple approximation procedure, apply now the Sobolev inequality (1) with $C = K(n,q)$ to $(\lambda + f^{q'})^{1-(n/q)}$ for every $\lambda > 0$. Since $|\nabla f| \le 1$ and $1/p = 1/q - 1/n$, we get

$$\left(\int \frac{1}{(\lambda + f^{q'})^n}\, dv \right)^{1/p} \le K(n,q) \left(\frac{n-q}{q-1} \right) \left(\int \frac{f^{q'}}{(\lambda + f^{q'})^n}\, dv \right)^{1/q}.$$

In other words, setting

$$\alpha = \left(K(n,q) \left(\frac{n-q}{q-1} \right) \right)^{-q},$$

for every $\lambda > 0$,

$$\alpha \left(-F'(\lambda) \right)^{q/p} - \lambda F'(\lambda) \le (n-1) F(\lambda). \tag{3}$$

We now compare the solutions of the differential inequality (3) to the solutions H of the differential equality

$$\alpha \left(-H'(\lambda) \right)^{q/p} - \lambda H'(\lambda) = (n-1) H(\lambda), \quad \lambda > 0. \tag{4}$$

It is plain that a particular solution H_0 of (4) is given by the extremal functions of the Sobolev inequality in $\mathbb{R}^n$, namely

$$H_0(\lambda) = \frac{1}{n-1} \int_{\mathbb{R}^n} \frac{1}{(\lambda + |x|^{q'})^{n-1}}\, dx = \frac{A}{\lambda^{(n/q)-1}}$$

where

$$A = H_0(1) = \frac{1}{n-1} \int_{\mathbb{R}^n} \frac{1}{(1 + |x|^{q'})^{n-1}}\, dx = \frac{q}{n-q} \left(\frac{\alpha(n-q)}{n(q-1)} \right)^{p/(p-q)}$$

(as a solution of (4)).

We claim that if $F(\lambda_0) < H_0(\lambda_0)$ for some $\lambda_0 > 0$, then $F(\lambda) < H_0(\lambda)$ for every $\lambda \leq \lambda_0$. Indeed, if this is not the case, let λ_1 be defined by

$$\lambda_1 = \sup\{\lambda < \lambda_0; F(\lambda) = H_0(\lambda)\}.$$

Now, for every $\lambda > 0$, $\varphi_\lambda(X) = \alpha X^{q/p} + \lambda X$ is strictly increasing in $X \geq 0$ so that (3) reads as

$$-F'(\lambda) \leq \varphi_\lambda^{-1}\left((n-1)F(\lambda)\right)$$

while, by (4),

$$-H_0'(\lambda) = \varphi_\lambda^{-1}\left((n-1)H_0(\lambda)\right).$$

Therefore

$$(F - H_0)'(\lambda) \geq \varphi_\lambda^{-1}\left((n-1)H_0(\lambda)\right) - \varphi_\lambda^{-1}\left((n-1)F(\lambda)\right) \geq 0$$

on the set $\{F \leq H_0\}$. Hence $(F - H_0)' \geq 0$ on the interval $[\lambda_1, \lambda_0]$ so that $F - H_0$ is non-decreasing on this interval. But then, in particular,

$$0 = (F - H_0)(\lambda_1) \leq (F - H_0)(\lambda_0) < 0$$

which is a contradiction.

Recall now, $\lambda > 0$,

$$F(\lambda) = \frac{1}{n-1}\int \frac{1}{(\lambda + f^{q'})^{n-1}}\,dv = q'\int_0^\infty V(\theta s)\,\frac{s^{q'-1}}{(\lambda + s^{q'})^n}\,ds$$

while

$$\begin{aligned} H_0(\lambda) &= \frac{1}{n-1}\int_{\mathbb{R}^n} \frac{1}{(\lambda + |x|^{q'})^{n-1}}\,dx = q'\int_0^\infty V_0(s)\,\frac{s^{q'-1}}{(\lambda + s^{q'})^n}\,ds \\ &= \frac{A}{\lambda^{(n/q)-1}}. \end{aligned}$$

The local geometry indicates that

$$\liminf_{\lambda \to 0} \frac{F(\lambda)}{H_0(\lambda)} \geq \theta^n > 1. \tag{5}$$

Indeed, write

$$F(\lambda) = q'\theta^{(n-1)q'}\int_0^\infty V(s)\,\frac{s^{q'-1}}{(\theta^{q'}\lambda + s^{q'})^n}\,ds.$$

As $V(s) \sim V_0(s)$ when $s \to 0$, for every $\varepsilon > 0$, there is $\delta > 0$ such that, for every $\lambda > 0$,

$$\begin{aligned}\int_0^\infty V(s)\,\frac{s^{q'-1}}{(\theta^{q'}\lambda + s^{q'})^n}\,ds &\geq (1-\varepsilon)\int_0^\delta V_0(s)\,\frac{s^{q'-1}}{(\theta^{q'}\lambda + s^{q'})^n}\,ds \\ &\geq \frac{(1-\varepsilon)}{\theta^{q'((n/q)-1)}\lambda^{(n/q)-1}}\int_0^{\delta/\theta\lambda^{1/q'}} V_0(s)\frac{s^{q'-1}}{(1+s^{q'})^n}\,ds.\end{aligned}$$

Hence, for every $\lambda > 0$,

$$\frac{F(\lambda)}{H_0(\lambda)} \geq \theta^n\,\frac{(1-\varepsilon)\int_0^{\delta/\theta\lambda^{1/q'}} V_0(s)\frac{s^{q'-1}}{(1+s^{q'})^n}\,ds}{\int_0^\infty V_0(s)\frac{s^{q'-1}}{(1+s^{q'})^n}\,ds},$$

from which (5) follows as $\lambda \to 0$.

We can now conclude the proof of the Theorem. By the claim and (5), we have that $F(\lambda) \geq H_0(\lambda)$ for every $\lambda > 0$, that is

$$\int_0^\infty [V(\theta s) - V_0(s)]\,\frac{s^{q'-1}}{(\lambda + s^{q'})^n}\,ds \geq 0.$$

Letting $\theta \to 1$,

$$\int_0^\infty [V(s) - V_0(s)]\,\frac{s^{q'-1}}{(\lambda + s^{q'})^n}\,ds \geq 0$$

for every $\lambda > 0$. Since by Bishop's theorem $V(s) \leq V_0(s)$ for every s when M has non-negative curvature, it must be that $V(s) = V_0(s)$ for almost every s, and thus every s by continuity. By the case of equality in Bishop's theorem, M is isometric to $\mathbb{R}^n$. The proof of the Theorem is complete. □

It is natural to conjecture that the Theorem may actually be turned into a volume comparison statement as it is the case for $q = 1$. That is, in a manifold M satisfying the Sobolev inequality (1) with the constant $K(n,q)$ for some $q > 1$, and without any curvature assumption, for every x_0 and every s,

$$V(x_0, s) \geq V_0(s).$$

This is well-known up to a constant (depending only on n and q) but the preceding proof does not seem to be able to yield such a conclusion.

To conclude this note, we comment some related comparison theorem. The Sobolev inequality (1) belongs to a general family of inequalities of the type

$$\left(\int |f|^r dv\right)^{1/r} \leq C\left(\int |f|^s dv\right)^{\theta/s} \left(\int |\nabla f|^q dv\right)^{(1-\theta)/q}, \quad f \in C_0^\infty(M),$$

with

$$\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{p}$$

(cf. [B-C-L-SC]). Inequality (1) corresponds to $\theta = 0$. When $q = 2$, the classical value, and $r = 2$, other choices of interest are $\theta = 2/(n+2)$ which corresponds to the Nash inequality

$$(6) \quad \left(\int |f|^2 dv\right)^{1+(2/n)} \leq C\left(\int |f| dv\right)^{4/n} \int |\nabla f|^2 dv, \quad f \in C_0^\infty(M),$$

and the limiting value $\theta = 1$ which corresponds to the logarithmic Sobolev or entropy-energy inequality

$$(7) \quad \int f^2 \log f^2 dv \leq \frac{n}{2} \log\left(C\int |\nabla f|^2 dv\right), \quad f \in C_0^\infty(M), \quad \int f^2 dv = 1$$

(cf. [Da]). As for the Sobolev inequality (1), the optimal constants for these two inequalities (6) and (7) in $\mathbb{R}^n$ are known ([C-L] and [Ca] respectively), so that one may ask for a statement analogous to the Theorem in case of these inequalities. As a result, it was shown in [B-C-L] that this is indeed the case for the logarithmic Sobolev inequality (7), that is, a n-dimensional Riemannian manifold with non-negative Ricci curvature satisfying (7) with the best constant of $\mathbb{R}^n$ is isometric to $\mathbb{R}^n$. The proof there relies on optimal heat kernel bounds in manifolds satisfying the logarithmic Sobolev inequality (7) with the best constant of $\mathbb{R}^n$. Namely, if $p_t(x,y)$ denotes the heat kernel on M, then, for every $t > 0$,

$$\sup_{x,y\in M} p_t(x,y) \leq \frac{1}{(4\pi t)^{n/2}} = \sup_{x,y\in\mathbb{R}^n} p_t^0(x,y)$$

where $p_t^0(x,y)$ is the heat kernel on $\mathbb{R}^n$. One then concludes with the results of P. Li [Li] relating an optimal large time heat kernel decay to the maximal volume growth of balls in manifolds with non-negative Ricci curvature. The analogous results for the Nash inequality (6) are so far open.

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RECEIVED MARCH 6, 1997.

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