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Remarks on the existence of branch bubbles on the blowup analysis of equation $-\triangle u = e^{2u}$ in dimension two

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It is well known that for a sequence of metrics in a fixed conformal class with constant scalar curvature on a manifold with dimension 3 (see [4] for further reference) or higher (see [2] for further reference), the blowingup set must be finite and simple (each bubbling point carry exactly one sphere). The corresponding statement in dimension 2 is widely expected to hold. The author learned this problem from a joint paper of YanYan Li and Tai Shafrir [3] when he studied a different but related problem [5], [6]. The purpose of this paper is to construct an example of a sequence of metrics in a unit disk with constant curvature 1 and uniformly bounded area which develops branch bubbles at the center of the disk, thereby providing an counter example to the analogous statements in dimension two. Interested readers are referred to [3] for detailed history of this problem and further references.

Very briefly, Brezis and Merle [1] studied a sequence of solutions $\{u_n\}$ in an open unit disk B_1 satisfying the equation:

$$-\triangle u_n = e^{2u_n} \tag{1}$$

where $\int_{B_1} e^{2u_n}|d\,z|^2 \leq C_1$ for some constant C_1. Then [1] concludes that there exists a subsequence $\{u_{n_k}\}$ satisfying one of the following three alternatives (mutually exclusive):

1. $\{u_{n_k}\}$ is bounded in $L^\infty_{loc}(B_1)$.
2. $u_{n_k} \to -\infty$ uniformly on any compact subsets of B_1.
3. there exists at most a finite blowing up set $S = \{p_1, p_2, \cdots, p_m\}$ such that
$$e^{2u_{n_k}} \to \sum_{i=1}^{i=m} \alpha_i \cdot \delta_{p_i}$$
in the sense of measure with concentrated mass $\alpha_i \geq 2\pi$, $\forall i$.

It is conjectured by Brezis and Merle that the concentrated mass α_i appearing in alternative (3) should be of the form $\alpha_i = 4\pi \cdot l_i$ with l_i being some positive integer. This conjecture was proved by YanYan Li and Tai Shafrir [3]. However, it is widely believed and also expected by [3] that l_i must be 1 for all i. In this paper, we want to show that this belief is false by constructing explicitly a couter example. Thus gives an interesting and rather surprising answer to the question raised in [3]. We always use $B_r(z)$ to denote the Euclidean ball of radius r center at z, while B_r denotes a ball of radius r centered at $z = 0$.

Remark 1. Both [1] and [3] prove some more general results than what we quote here. Readers are referred to their papers for details.

Definition 1. A sequence $\{u_n\}$ of metrics (i.e.,$\{e^{2u_n}|d\,z|^2\}$) in a domain B_1 is called a multiple m branch bubbling sequence if $e^{2u_n}\,|d\,z|^2$ has constant curvature 1 (satisfies the curvature equation (1)) and

$$e^{2u_n} \longrightarrow 4\pi\, m \cdot \delta_{\{0\}},$$

in the sense of measure.

The conjecture of [3] states that there is no multiple m branch bubbling sequence for any $m > 1$. We will disprove this conjecture by explicitly construct a multiple m branch bubbling sequence of metrics in the unit disk (thus in any open disk) for any positive integer m.

Definition 2. A sequence $\{u_n\}$ is called a degree m pre-branch bubbling sequence if

1. $e^{2u_n}\,|d\,z|^2$ is defined in a domain B_{r_n} ($r_n \to \infty$) with constant curvature 1 (satisfies the curvature equation (1)) and a uniformly bounded area
$$\int_{B_{r_n}} e^{2u_n}|d\,z|^2 \leq C$$
for some constant C independent of n.

2. There exist m distinct bubble points $z = z_1, z_2, \cdots, z_m$ such that e^{2u_n} converges to 0 uniformly in any compact domain away from these bubbling points and
$$e^{2u_n} \longrightarrow 4\pi \sum_{i=0}^{m} l_i \cdot \delta_{\{z_i\}},$$

in the sense of measure. If $l_i = 1$ for all $1 \leq i \leq m$, then we call this sequence a degree m simple pre-branch-bubbling sequence.

Essentially, multiple m branch bubbling sequence and degree m pre-branch-bubbling sequence are equivalent.

Proposition 1. *For each degree m simple pre-branch bubbling sequence, there exists a corresponding multiple m branch bubbling sequence.*

Proof. Let $\{u_n\}$ in $B_{r_n}(r_n \to \infty)$ be a degree m simple pre-branch bubbling sequence. Using the reverse blowing up procedure, we have:

$$\tilde{u}_n(z) = u_n(r_n \cdot z) + \ln |r_n|.$$

It is easy to show that $\{\tilde{u}_n\}$ is a multiple m branch bubbling sequence by a direct computation. □

Remark 2. The converse of this proposition is essentially true. For any multiple m branch bubbling sequence, by a careful rescaling argument, one can obtain a degree $k\,(2 \leq k \leq m)$ simple pre-branch bubbling sequence (which may require blowup several times). We omit this part of the argument because it is not essential to the construction of the example. The important point to keep in mind is: the goal of blowing up is not to obtain sphere, but to separate the bubble points. Interested readers are encouraged to read [5], where this type of blowing up argument has been written down in detail.

The following is a well known fact about Riemannian geometry in dimension 2:

Proposition 2. *For any holomorphic function $f(z)$ in a domain Ω where $f'(z)$ never vanishes in Ω, then the metric*

$$\frac{4|f'(z)|^2}{(1+|f|^2)^2} \cdot |d\,z|^2$$

has constant curvature 1 on Ω.

We will prove the main theorem first by referring it to two technical theorems which will be proved immediately afterwards. From now on, any point where a function vanishes is called a zero point of that function.

Theorem 1. *For any integer m, there exists a multiple m branch bubbling sequence.*

Proof. According to Theorem 3 on p.300, there exists an entire function which has exactly m simple zero points and whose derivative nowhere vanishes. Following Theorem 2 on p.298, such an entire function leads to the existence of a degree m simple pre-branch bubbling sequence. Therefore there exists a multiple m branch bubbling sequence according to Proposition 1. □

Remark 3. The set of all the multiple m branch bubbling sequence is at least as large as the set of all entire functions which have exactly m distinct zero points and their derivatives nowhere vanish. The remaining problem is if the two sets are equivalent in some sense.

Theorem 2. *For any entire function f such that it has only m distinct simple zero points and f' vanishes at nowhere, then there exists $\lambda_n \to \infty$ such that*

$$\frac{4\lambda_n^2 \cdot |f'(z)|^2}{(1+\lambda_n^2 \cdot |f|^2)^2} \cdot |d\,z|^2 \qquad \text{on } B_n$$

is a degree m simple pre-branch-bubbling sequence.

Remark 4. For any entire function f which has more than one zero points and f' never vanishes, the metric $\frac{4|f'(z)|^2}{(1+|f|^2)^2} \cdot |d\,z|^2$ has infinite area in the complex plane. Thus $\{\lambda_n\}$ must be chosen carefully in order to have a uniformly bounded area in a sequence of increased disks which exhausts the entire complex plane.

Proof. Let f be an entire function with a finite number of simple zero points at $z = z_1, z_2, \cdots, z_m$ such that $f'(z)$ has no zero in the entire complex plane. Suppose that

$$\mu_i = |f'(z_i)| > 0, \forall\, 1 \le i \le m.$$

For n large enough such that B_n contains all zero points, we claim (fixing n)

$$\lim_{\lambda\to\infty} \int_{B_n} \frac{4 \cdot \lambda^2 \cdot |f'(z)|^2}{\left(1+\lambda^2 \cdot |f|^2\right)^2} |d\,z|^2 = m \cdot 4\pi.$$

Fixing $\epsilon > 0$, then there exists a $\delta > 0$ such that

$$(1-\epsilon)\,\mu_i\,|z - z_i| \le |f(z)| \le (1+\epsilon)\,\mu_i\,|z - z_i|, \forall\, z \in B_\delta(z_i),\ \forall 1 \le i \le m$$

and

$$(1-\epsilon)\,\mu_i \le |f'(z)| \le (1+\epsilon)\,\mu_i, \forall\, z \in B_\delta(z_i),\ \forall 1 \le i \le m.$$

In $B_n \setminus \bigcup_i B_\delta(z_i)$, we have

$$\lim_{\lambda\to\infty} \frac{4\cdot\lambda^2\cdot|f'(z)|^2}{(1+\lambda^2\cdot|f|^2)^2} = 0$$

since $\inf_{z\in B_n\setminus\bigcup_i B_\delta(z_i)} |f(z)| > 0$. In each $B_\delta(z_i)$, we have

$$\begin{aligned}\frac{(1-\epsilon)^2}{(1+\epsilon)^2}\cdot\frac{4\cdot\lambda^2\cdot(1+\epsilon)^2\cdot\mu_i^2}{(1+\lambda^2\cdot(1+\epsilon)^2\cdot\mu_i^2\,|z|^2)^2} &\le \frac{4\cdot\lambda^2\cdot|f'(z)|^2}{(1+\lambda^2\cdot|f|^2)^2}\\ &\le \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot\frac{4\cdot\lambda^2\cdot(1-\epsilon)^2\cdot\mu_i^2}{(1+\lambda^2\cdot(1-\epsilon)^2\cdot\mu_i^2\cdot|z|^2)^2}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{(1-\epsilon)^2}{(1+\epsilon)^2}\cdot\int_{|z-z_i|\le(1+\epsilon)\cdot\lambda\cdot\mu_i\cdot\delta}\frac{4}{(1+|z|^2)^2}|d\,z|^2 &\le \int_{B_\delta(z_i)}\frac{4\cdot\lambda^2\cdot|f'(z)|^2}{(1+\lambda^2\cdot|f|^2)^2}|d\,z|^2\\ &\le \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot\int_{|z-z_i|\le(1-\epsilon)\cdot\lambda\cdot\mu_i\cdot\delta}\frac{4}{(1+|z|^2)^2}|d\,z|^2.\end{aligned}$$

Observe that

$$\int_{B_n}\frac{4\cdot\lambda^2\cdot|f'(z)|^2}{(1+\lambda^2\cdot|f|^2)^2}|d\,z|^2 = \int_{B_n\setminus\bigcup_i B_\delta(z_i)} + \sum_{i=1}^m\int_{B_\delta(z_i)}\frac{4\cdot\lambda^2\cdot|f'(z)|^2}{(1+\lambda^2\cdot|f|^2)^2}|d\,z|^2$$

and

$$\lim_{\lambda\to\infty}\int_{|z-z_i|\le(1\pm\epsilon)\cdot\lambda\cdot\mu_i\cdot\delta}\frac{4}{(1+|z|^2)^2}|d\,z|^2 = 4\pi.$$

Therefore,

$$\begin{aligned}o(\frac{1}{\lambda}) + \frac{(1-\epsilon)^2}{(1+\epsilon)^2}\cdot m\cdot 4\pi &\le \int_{B_n}\frac{4\cdot\lambda^2\cdot|f'(z)|^2}{(1+\lambda^2\cdot|f|^2)^2}|d\,z|^2\\ &\le o(\frac{1}{\lambda}) + \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot m\cdot 4\pi,\end{aligned}$$

where $\lim_{\lambda\to\infty} o(\frac{1}{\lambda}) = 0$. Let $\lambda\to\infty$ first, we have

$$\begin{aligned}\frac{(1-\epsilon)^2}{(1+\epsilon)^2}\cdot m\cdot 4\pi &\le \liminf_{\lambda\to\infty}\int_{B_n}\frac{4\cdot\lambda^2\cdot|f'(z)|^2}{(1+\lambda^2\cdot|f|^2)^2}|d\,z|^2\\ &\le \limsup_{\lambda\to\infty}\int_{B_n}\frac{4\cdot\lambda^2\cdot|f'(z)|^2}{(1+\lambda^2\cdot|f|^2)^2}|d\,z|^2\\ &\le \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot m\cdot 4\pi.\end{aligned}$$

Let $\epsilon \longrightarrow 0$, then

$$\lim_{\lambda\to\infty}\int_{B_n}\frac{4\cdot\lambda^2\cdot|f'(z)|^2}{(1+\lambda^2\cdot|f|^2)^2}|d\,z|^2 = m\cdot 4\pi.$$

Thus the claim is true.

Again fix a small number $\epsilon > 0$. For each n, chose λ_n such that

$$\int_{B_n}\frac{4\cdot\lambda_n^2\cdot|f'(z)|^2}{(1+\lambda_n^2\cdot|f|^2)^2}|d\,z|^2 = m\cdot 4\pi + o(\frac{1}{n}),$$

and

$$\max_{z\in B_n\setminus\cup_{i=1}^m B_\epsilon(z_i)}\frac{4\cdot\lambda_n^2\cdot|f'(z)|^2}{(1+\lambda_n^2\cdot|f|^2)^2} = o(\frac{1}{n}),$$

where $\lim_{n\to\infty} o(\frac{1}{n}) = 0$. It is straightforward to verify that

$$\frac{4\cdot\lambda_n^2\cdot|f'(z)|^2}{(1+\lambda_n^2\cdot|f|^2)^2}|d\,z|^2$$

on B_n is a degree m simple pre-branch bubbling sequence. □

Theorem 3. *For each positive integer $m > 0$, there exists an entire function in the complex plane such that it has exactly m simple zero points and its derivative vanishes at nowhere.*

Proof. For $m = 1$, choose $f(z) = a\cdot z + b$. For $m = 2$, may assume the two zero points are $z = 0, z = 1$. Consider

$$f(z) = e^{g(z)}\cdot z(z-1), \qquad \text{where}\, g'(z) = \frac{e^{\pi\sqrt{-1}(1-z)} + 1 - 2z}{z(z-1)}.$$

Clearly, that $g'(z)$ is a well defined entire function, so is $g(z)$. Thus $f(z)$ has only two zero points. Now

$$f'(z) = e^{g(z)}\cdot e^{\pi\sqrt{-1}(1-z)}$$

has no zero points at all in the entire complex plane.

For generic $m > 1$, let $z_1, z_2, \cdots, z_m$ be m distinct designated zero points, define a polynomial

$$P(z) = \prod_{i=1}^m (z - z_i).$$

Let w_i be defined such that $e^{w_i} = P'(z_i)$. Define a (m-1) degree polynomial $Q(z)$ such that it takes value w_i at z_i:

$$Q(z) = \prod_{i=1}^{m}(z - z_i) \cdot \sum_{i=1}^{m} \frac{w_i}{z - z_i}$$

Let $g(z)$ be defined as

$$g'(z) = \frac{e^{Q(z)} - P'(z)}{P(z)}.$$

Clearly, $g'(z)$ is an entire function. Then

$$f(z) = e^{g(z)} \cdot P(z)$$

has exactly m zero points and its derivatives has no zero points since

$$f'(z) = e^{g(z)}(g'(z) \cdot P(z) + P'(z)) = e^{g(z)} \cdot e^{Q(z)}.$$

□

Open Problem. *Is our method of constructing a multiple m branching bubbling sequence unique ? In other words: is the converse of theorem 2 true?*

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