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The heat equation with inhomogeneous Dirichlet boundary conditions

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We establish the existence of an asymptotic expansion for the heat content asymptotics with inhomogeneous Dirichlet boundary conditions and compute the first 5 coefficients in the asymptotic expansion.

0. Introduction.

Let M be a smooth compact Riemannian manifold of dimension m with smooth boundary ∂M. For $\phi \in C^\infty(\partial M)$, let $\mathcal{E}(\phi)(t)$ be the total heat energy content of M where the initial temperature is 0 and where the boundary of M is kept at temperature ϕ; see §1 for a more precise definition. Let Φ_ϕ be the harmonic function with boundary value ϕ. It is well known that for large time the temperature profile of M approaches Φ_ϕ, and that

$$\lim_{t\to\infty} \mathcal{E}(\phi)(t) = \int_M \Phi_\phi.$$

This is a globally defined invariant which is not locally computable. For short time the total heat content $\mathcal{E}(\phi)(t)$ has an asymptotic expansion. It is somewhat surprising in contrast to the large time behaviour that the coefficients in that expansion are locally computable.

Theorem 0.1. *As $t \downarrow 0$, $\mathcal{E}(\phi)(t) \cong \sum_{n\geq 1} \mathcal{B}_n(\phi)t^{n/2}$. There exist locally defined geometric invariants B_n on the boundary so that $\mathcal{B}_n(\phi) = \int_{\partial M} \phi\cdot B_n$.*

The coefficients $\mathcal{B}_n(\phi)$ express the net heat flow into and out of the manifold over the boundary ∂M. The case $\phi = 1$ has particular geometrical significance since the coefficients are then invariants of the boundary of M.

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The classical analytical tools to study these inhomogeneous problems rely on explicit formulas for the Dirichlet heat kernel, and are available in a few special cases only. See, for example, Carslaw and Jaeger [4, §9.11-9.12] for the ball in $\mathbb{R}^3$. In Theorem 0.2 below, we give the first five asymptotic coefficients; we will use methods of invariance theory to derive these formulas. Let L be the second fundamental form on the boundary, let R be the Riemann curvature tensor of M and let $\rho_{ij} := R_{ikkj}$ be the Ricci tensor. Let ';' and ':' denote covariant differentiation with respect to the Levi-Civita connections of M and of ∂M respectively. We choose a local orthonormal frame $\{e_1, ..., e_m\}$ for the tangent bundle of M restricted to the boundary so that e_m is the inward unit normal. Let indices a, b, c etc. range from 1 through $m-1$. We adopt the Einstein convention and sum over repeated indices.

Theorem 0.2. *The geometric invariants B_1, ..., B_4 in Theorem 0.1 are given by*

(1) $$B_1 = \frac{2}{\sqrt{\pi}}.$$

(2) $$B_2 = -\frac{1}{2}L_{aa}.$$

(3) $$B_3 = \frac{1}{6\sqrt{\pi}}\{L_{aa}L_{bb} - 2L_{ab}L_{ab} - 2\rho_{mm}\}.$$

(4) $$B_4 = \frac{1}{32}\{2L_{ab}L_{ab}L_{cc} - 4L_{ab}L_{ac}L_{bc} + 2R_{ambm}L_{ab} - 2R_{abcb}L_{ac} - \rho_{ii;m} - 2L_{ab:ab}\}.$$

Moreover

(5)
$$\begin{aligned}\mathcal{B}_5(1) = -\frac{1}{240\sqrt{\pi}}\int_{\partial M}&\{8\rho_{mm;mm} - 8L_{aa}\rho_{mm;m} + 16L_{ab}R_{ammb;m}\\
&- 4\rho_{mm}^2 + 16R_{ammb}R_{ammb} - 4L_{aa}L_{bb}\rho_{mm} - 8L_{ab}L_{ab}\rho_{mm}\\
&+ 64L_{ab}L_{ac}R_{mbcm} - 16L_{aa}L_{bc}R_{mbcm} - 8L_{ab}L_{ac}R_{bddc}\\
&- 8L_{ab}L_{cd}R_{acbd} + 4R_{abcm}R_{abcm} + 8R_{abbm}R_{accm} - 16L_{aa:b}R_{bccm}\\
&- 8L_{ab:c}L_{ab:c} + L_{aa}L_{bb}L_{cc}L_{dd} - 4L_{aa}L_{bb}L_{cd}L_{cd} + 4L_{ab}L_{ab}L_{cd}L_{cd}\\
&- 24L_{aa}L_{bc}L_{cd}L_{db} + 48L_{ab}L_{bc}L_{cd}L_{da}\}.\end{aligned}$$

Here is a brief guide to the paper. In §1, we prove Theorem 0.1. We first establish the existence of the asymptotic expansion and then prove the coefficients are locally computable. In §2, we derive Theorem 0.2 (1)-(4) from previously known results for the homogeneous case. The remainder of the paper is devoted to the proof of Theorem 0.2 (5). In §3, we establish some product formulas. In §4, we use these formulas to compute $\mathcal{B}_5$ and complete the proof of Theorem 0.2.

1. Heat Content Asymptotics.

We begin by giving rigorous definitions of the heat content functions with which we shall be working. Let D be a scalar operator of Laplace type. For $\Phi \in C^\infty(M)$ and $\phi \in C^\infty(\partial M)$, let $u_\Phi(x;t)$ and $u_\phi(x;t)$ mapping $M \times [0,\infty)$ to $\mathbb{R}$ be the unique solutions of the equations:

$$\begin{aligned} &\partial_t u_\Phi = -Du_\Phi,\ u_\Phi(x;0) = \Phi(x) \text{ for } x \in M, \\ &u_\Phi(y;t) = 0 \text{ for } y \in \partial M,\ t > 0 \end{aligned} \tag{1.1a}$$

and

$$\begin{aligned} &\partial_t u_\phi = -Du_\phi,\ u_\phi(x;0) = 0 \text{ for } x \in M, \\ &u_\phi(y;t) = \phi(y) \text{ for } y \in \partial M,\ t > 0. \end{aligned} \tag{1.1b}$$

The function u_Φ solves the heat equation with initial temperature Φ and zero (Dirichlet) boundary condition and the function u_ϕ solves the heat equation with initial temperature 0 and inhomogeneous boundary condition defined by ϕ. Let $\tilde{\Phi}$ be an auxiliary test function. Let

$$E(\Phi,\tilde{\Phi},D)(t) := \int_M u_\Phi \tilde{\Phi} \text{ and } \mathcal{E}(\phi,\tilde{\Phi},D)(t) := \int_M u_\phi \tilde{\Phi}. \tag{1.2}$$

Let Δ be the scalar Laplacian. We recover the function $\mathcal{E}(\phi)$ by setting $D = \Delta$ and $\tilde{\Phi} = 1$ in equation (1.2).

Lemma 1.1. *(1) As $t \downarrow 0$, $E(\Phi,\tilde{\Phi},D)(t) \cong \sum_{n\geq 0} \beta_n(\Phi,\tilde{\Phi},D)t^{n/2}$. There exist local invariants $\beta_n^{\partial M}(\Phi,\tilde{\Phi},D) \in C^\infty(\partial M)$ so that*

$$\beta_{2k-1}(\Phi,\tilde{\Phi},D) = \int_{\partial M} \beta_{2k-1}^{\partial M}(\Phi,\tilde{\Phi},D)$$

and so

$$\beta_{2k}(\Phi,\tilde{\Phi},D) = \int_{\partial M} \beta_{2k}^{\partial M}(\Phi,\tilde{\Phi},D) + (-1)^k \int_M D^k\Phi(x)\tilde{\Phi}(x)dx/k!.$$

(2) $\beta_0(\Phi,\tilde\Phi,D)=\int_M\Phi(x)\tilde\Phi(x)dx$ *and* $\beta_1(\Phi,\tilde\Phi,D)=-\frac{2}{\sqrt{\pi}}\int_{\partial M}\Phi\tilde\Phi.$

(3) *If* $\Phi|_{\partial M}=0$*, then* $\beta_n(\Phi,\tilde\Phi,D)=-\frac{2}{n}\beta_{n-2}(D\Phi,\tilde\Phi,D).$

(4) *If* $\tilde D$ *is the formal adjoint of* D*, then* $\beta_n(\Phi,\tilde\Phi,D)=\beta_n(\tilde\Phi,\Phi,\tilde D).$

(5) $\partial_\epsilon|_{\epsilon=0}\beta_n(\Phi,\tilde\Phi,D-\epsilon)=\beta_{n-2}(\Phi,\tilde\Phi,D).$

(6) *Let* M *be the unit disk in* $\mathbb{R}^m$ *for* $m\geq 2$*. Then*

$$\beta_5(1,1,\Delta)=\frac{1}{240\sqrt{\pi}}(m-1)(m-3)(m+3)(m-7)vol(S^{m-1}).$$

Proof. We refer to [2, Lemma 1.3] and to [5, equation 2.17] for the proof of (1), we refer to [2, Theorem 1.1] for the proof of (2), we refer to [2, Lemma 3.2] for the proof of (3, 4), we refer to [5, Lemma 2.2] for the proof of (5), and we refer to [2, Theorem 4.2] for the proof of (6). □

We can now establish the existence of the asymptotic series given in Theorem 0.1 for inhomogeneous boundary conditions. Let $\Phi=\Phi_\phi$ be a harmonic function on M so that $\Phi|_{\partial M}=\phi$. Then $\Phi-u_\Phi$ satisfies the equations defining u_ϕ so $\Phi-u_\Phi=u_\phi$.
Consequently $\mathcal{E}(\phi,1,\Delta)(t)=\int_M\Phi-E(\Phi,1,\Delta)(t)$. Thus the asymptotic series for $E(\Phi,1,\Delta)(t)$ given in Lemma 1.1 shows there is a corresponding asymptotic series for $\mathcal{E}$ where

$$\mathcal{B}_n(\phi)=-\beta_n(\Phi)\text{ for }n\geq 1. \tag{1.3}$$

The constant term in the asymptotic expansion β_0 is cancelled by $-\int_M\Phi$ so the asymptotic expansion for $\mathcal{E}$ begins with $n=1$ and not $n=0$.

The formalism which expresses $\mathcal{B}_n(\phi)=-\beta_n(\Phi)$ a priori permits the normal derivatives of Φ to enter into the formulas; this would involve the Dirichlet to Neumann operator which is not local. Thus to complete the proof of Theorem 0.1, we must show that the normal derivatives of Φ do not appear.

We now recall some local geometry of operators of Laplace type. Let indices ν,μ range from 1 through m and index a local coordinate frame on the interior of M. Let Γ be the Christoffel symbols of the Levi-Civita connection of M. Let $D = -(g^{\nu\mu}\partial_\nu\partial_\mu + P^\nu\partial_\nu + Q)$ be an operator of Laplace type on $C^\infty(M)$. Let Γ be the Christoffel symbols. We refer to [6, Theorem 2.1] for the proof of the following Lemma.

Lemma 1.2. *There exists a unique connection ∇ and smooth potential $\mathcal{P}$ so that $D\Phi = -g^{\nu\mu}\Phi_{;\nu\mu} - \mathcal{P}$. We have*

$$\mathcal{P} = Q - g^{\nu\mu}(\partial_\mu\omega_\nu + \omega_\nu\omega_\mu - \omega_\sigma\Gamma_{\nu\mu}{}^\sigma),$$

where $\omega_\nu = \frac{1}{2}g_{\nu\mu}(P^\mu + g^{\sigma\gamma}\Gamma_{\sigma\gamma}{}^\mu)$ is the connection 1 form of ∇.

We begin the proof that the invariants $\mathcal{B}_n$ are locally computable by constructing a 4-fold decomposition of the invariants $\beta_n^{\partial M}$ of Lemma 1.1 using the parity of the number of normal derivatives of Φ and $\tilde{\Phi}$. We have to be careful with the decomposition since we could use the identity $\Phi_{;aa} = \Phi_{:aa} - L_{aa}\Phi_{;m}$ to mix even and odd parities. We consider bilinear partial differential operators $A(\Phi, \tilde{\Phi})$ which are invariantly defined and which only involve tangential covariant derivatives of Φ and $\tilde{\Phi}$ using the restriction of the connections ∇ and $\tilde{\nabla}$ to the boundary of M and using the Levi-Civita connection of the boundary. Thus, for example, we would permit expressions of the form $\Phi_{:a}\tilde{\Phi}_{:a}$ or $L_{aa}\Phi\tilde{\Phi}$ but would not permit expressions $\Phi_{;a}\tilde{\Phi}_{;a}$ or $L_{aa}\Phi_{;m}\tilde{\Phi}$. Let $\mathcal{A}$ be the vector space of all such operators. Let $\Phi^{(0)} := \Phi$ and $\Phi^{(1)} := \Phi_{;m}$. For $p = 0, 1$ and $q = 0, 1$, let $\mathcal{V}_{p,q}$ be the vector space of all operators of the form

$$B_{p,q}(\Phi, \tilde{\Phi}, D) = \sum_{a,b} A_{a,b}((D^a\Phi)^{(p)}, (\tilde{D}^b\tilde{\Phi})^{(q)})$$

for $A_{a,b} \in \mathcal{A}$. Instead of using the variables $\Phi_{;m\cdots m}$, we use the variables $D^a\Phi$ and $(D^a\Phi)_{;m}$. This permits us to decompose $\beta_n^{\partial M} = \sum_{p,q} B_{n,p,q}$ for $B_{n,p,q} \in \mathcal{V}_{p,q}$.

Lemma 1.3. *We have* $\int_{\partial M} B_{n,1,0}(\Phi, \tilde{\Phi}, D) = 0.$

Proof. We proceed by induction on n; the cases $n = 0$ and $n = 1$ follow from Lemma 1.1. Choose F and $\tilde{F}$ so that

$$D^aF|_{\partial M} = 0, \qquad (D^aF)_{;m}|_{\partial M} = (D^a\Phi)_{;m}|_{\partial M},$$

$$\tilde{D}^a\tilde{F}|_{\partial M} = \tilde{D}^a\tilde{\Phi}|_{\partial M} \quad \text{and} \quad (\tilde{D}^a\tilde{F})_{;m}|_{\partial M} = 0$$

for all a. Then $B_{u,v}(F,\tilde F,D)=0$ if $(u,v)\neq(1,0)$ so

$$\int_{\partial M}\beta_{n-2}^{\partial M}(DF,\tilde F,D)=\int_{\partial M}B_{n-2,1,0}(DF,\tilde F,D)=0.$$

If $n=2k$, we use Lemma 1.1 to see

$$\begin{aligned}\beta_n(F,\tilde F,D)&=\frac{(-1)^k}{k!}\int_M D^kF\cdot\tilde F+\int_{\partial M}B_{n,1,0}(F,\tilde F,D)\\&=-\frac{1}{k}\beta_{n-2}(F,\tilde F,D)=\frac{(-1)^k}{k!}\int_M D^{k-1}(DF)\cdot\tilde F.\end{aligned}$$

Thus

$$\int_{\partial M}B_{n,1,0}(\Phi,\tilde\Phi,D)=\int_{\partial M}B_{n,1,0}(F,\tilde F,D)=0.$$

The argument is similar if n is odd; the interior integrals are not present. □

We can now complete the proof of Theorem 0.1. Let $\phi\in C^\infty(\partial M)$. Choose Φ harmonic with $\Phi|_{\partial M}=\phi$. We apply equation (1.3). We have $B_{n,p,1}(\Phi,1)=0$ and $A_{a,b}(\Delta^a\Phi,\Delta^b 1)=0$ for $(a,b)\neq(0,0)$. By Lemma 1.3, $\int_{\partial M}B_{n,1,0}(\Phi,1)=0$. The interior integral if $n\geq 2$ is even vanishes so

$$\mathcal{B}_n(\phi)=-\beta_n(\Phi)=-\int_{\partial M}A_{0,0}(\Phi,1,\Delta).$$

We integrate by parts tangentially to eliminate any tangential covariant derivatives of Φ and express

$$\int_{\partial M}-A_{0,0}(\Phi,1,\Delta)=\int_{\partial M}B_n(g)\Phi=\int_{\partial M}B_n(g)\phi.\square$$

2. The heat content asymptotics $\mathcal{B}_n$ for $n\leq 4$.

In this section, we prove Theorem 0.2 (1)-(4). We begin by recalling some results for the homogeneous case. Let Ω be the curvature of the connection defined by D. We refer to [1, 2] for the proof of the following result; see also [3, 7, 8].

Lemma 2.1.

(1) $\quad\beta_0(\Phi,\tilde\Phi,D)=\int_M\Phi\tilde\Phi.$

(2) $\beta_1(\Phi,\tilde\Phi,D) = -\frac{2}{\sqrt{\pi}}\int_{\partial M}\Phi\tilde\Phi.$

(3) $\beta_2(\Phi,\tilde\Phi,D) = -\int_M(D\Phi\cdot\tilde\Phi) + \int_{\partial M}\{\frac{1}{2}L_{aa}\Phi\tilde\Phi - \Phi\cdot\tilde\Phi_{;m}\}.$

(4)
$$\beta_3(\Phi,\tilde\Phi,D) = -\frac{2}{\sqrt{\pi}}\int_{\partial M}\Big\{\frac{2}{3}(\Phi_{;mm}\tilde\Phi + \Phi\tilde\Phi_{;mm} - L_{aa}\Phi_{;m}\tilde\Phi - L_{aa}\Phi\tilde\Phi_{;m}) - \Phi_{:a}\tilde\Phi_{:a} + \mathcal{P}\Phi\tilde\Phi + \frac{1}{12}(L_{aa}L_{bb} - 2L_{ab}L_{ab} - 2\rho_{mm})\Phi\tilde\Phi\Big\}.$$

(5)
$$\begin{aligned}\beta_4(\Phi,\tilde\Phi,D) = {}& \frac{1}{2}\int_M(D\Phi\cdot\tilde D\tilde\Phi) + \frac{1}{32}\int_{\partial M}\{16(D\Phi)_{;m}\tilde\Phi\\ &+ 16\Phi(\tilde D\Phi)_{;m} - 8L_{aa}D\Phi\cdot\tilde\Phi - 8L_{aa}\Phi\tilde D\Phi\\ &+ (4\mathcal{P}_{;m} - 2L_{ab}L_{ab}L_{cc} + 4L_{ab}L_{ac}L_{bc}\\ &- 2R_{ambm}L_{ab} + 2R_{abcb}L_{ac} + \rho_{ii;m} + 2L_{ab:ab})\Phi\tilde\Phi\\ &- 8L_{ab}\Phi_{:a}\tilde\Phi_{:b} - 4\Omega_{am}\Phi_{:a}\tilde\Phi + 4\Omega_{am}\Phi\cdot\tilde\Phi_{:a}\}.\end{aligned}$$

We let $D=\Delta$ be the scalar Laplacian. Then the potential $\mathcal{P}=0$ and the connection ∇ is flat so $\Omega=0$. Let Φ be a harmonic function with $\Phi|_{\partial M}=1$. Set $\tilde\Phi=1$. We use equation (1.3) to derive assertions (1), (2), and (4) of Theorem 0.2 from the corresponding assertions of Lemma 2.1. We may use the identity:

$$0 = \Phi_{;ii} = \Phi_{;mm} + \Phi_{:aa} - L_{aa}\Phi_{;m}$$

to see

$$\Phi_{;mm} - L_{aa}\Phi_{;m} = \Delta_{\partial M}\Phi = \Delta_{\partial M}\phi = 0.$$

Assertion (3) of Theorem 0.2 now follows. Note that $\Phi_{;m}$ does not appear in the formula! □

3. Product formulas.

Let $T^k := S^1\times\ldots S^1$ be the k dimensional torus with the usual periodic parameters. Let $M_1 := (r_0,r_1)\times T^k$ and $M_2 := T^\ell$. We use coordinates (r,u) for $u=(u_i)$ on M_1 and coordinates $v=(v_j)$ on M_2. We consider a diagonal warped product metric on $M := M_1\times M_2$ of the form $ds^2 = dr^2 + du_i^2 + g_j dv_j^2$. We shall assume $g_j(r,u,v)$ is identically equal to 1 near $r=r_1$ so only the boundary component $\{r_0\}\times T^{k+\ell}$ is relevant. Let $g=\Pi_j g_j^{1/2}$. Then $\Delta_M = D_1 + D_2$ for $D_1 = -g^{-1}\partial_r g\partial_r - g^{-1}\partial_i^u g\partial_i^u$ and $D_2 = -g^{-1}\partial_j^v g g_j^{-1}\partial_j^v$.

Lemma 3.1. *If $g = h_1(r,u)h_2(v)$, then*

$$\beta_n(1,1,\Delta_M) = \beta_n(1,h_1,D_1)\int_{M_2} h_2(v)dv.$$

Proof. The operator D_1 is independent of v by assumption. Let $u_1(r,u)$ solve equation (1.1a) for the operator D_1 on M_1 with initial condition $\Phi = 1$. Then $u_M(r,u,v) := u_1(r,u)$ solves equation (1.1a) for the operator Δ_M on M with the same initial condition. Since $dvol_M = gdrdudv$,

$$\begin{aligned} E(1,1,\Delta_M)(t) &= \int_{M_1} u_1(r,u;t)h_1(r,u)drdu \cdot \int_{M_2} h_2(v)dv \\ &= E(1,h_1,D_1)(t)\int_{M_2} h_2(v)dv. \square \end{aligned}$$

The operator D_1 of Lemma 3.1 is not a general operator of Laplace type. It is defined on a flat manifold with totally geodesic boundary with flat associated connection; the curvature of ∇ and $\tilde{\nabla}$ vanishes. We compute β_5 for such operators in the following Lemma.

Lemma 3.2. *Let $M = (r_0,r_1)\times T^k$ with the flat product metric.*

(1) *Let D be an operator of Laplace type on M defining a flat connection. Then*

$$\begin{aligned} \beta_5(\Phi,\tilde\Phi,D) = \frac{1}{15\sqrt{\pi}}\int_{\partial M} \{ & \mathcal{P}_{;mm}\Phi\tilde\Phi - 4\mathcal{P}D\Phi\cdot\tilde\Phi - 4\mathcal{P}\Phi\tilde D\tilde\Phi + \mathcal{P}^2\Phi\tilde\Phi \\ & - 8D^2\Phi\cdot\tilde\Phi - 8\Phi\tilde D^2\tilde\Phi - 8D\Phi\cdot\tilde D\tilde\Phi + 5\mathcal{P}(\Phi_{;aa}\tilde\Phi + \Phi\tilde\Phi_{;aa}) \\ & + 9\Phi_{;aa}\tilde\Phi_{;bb} + 4(\Phi_{;mm}\tilde\Phi_{;aa} + \Phi_{;aa}\tilde\Phi_{;mm})\}. \end{aligned}$$

(2) *Let $\tilde g = e^h$ and $D = -\tilde g^{-1}(\partial_r\tilde g\partial_r + \sum_{a\le k}\partial_{x_k}\tilde g\partial_{x_k})$. Then*

a) *$\mathcal{P} = \mathcal{P}_r + \mathcal{P}_t$ for*

$$\mathcal{P}_r = -\frac{1}{2}\partial_r^2 h - \frac{1}{4}(\partial_r h)^2$$

and

$$\mathcal{P}_t = -\sum_i\left(\frac{1}{2}\partial_i^{u2}h + \frac{1}{4}(\partial_i^u h)^2\right).$$

b) $$\beta_5(1,\tilde g,D) = \frac{1}{240\sqrt{\pi}}\int_{\partial M}\{(16\mathcal{P}_{r;mm} + 16\mathcal{P}_{t;mm} + 16\mathcal{P}_r^2)\tilde g\}.$$

Proof. One can use dimensional analysis to see that the local formulas $\beta_n^{\partial M}$ have a certain homogeneity property; see the discussion in [2, pp53–54]. Using the symmetry property $\beta_5(\Phi,\tilde\Phi,D)=\beta_5(\tilde\Phi,\Phi,\tilde D)$, we see there exist constants so

(3.1)
$$\begin{aligned}\beta_5(\Phi,\tilde\Phi,D)=\frac{1}{15\sqrt{\pi}}\int_{\partial M}&\{a_1\mathcal{P}_{;mm}\Phi\tilde\Phi+a_2\mathcal{P}_{;m}(\Phi_{;m}\tilde\Phi+\Phi\tilde\Phi_{;m})\\&+a_3\mathcal{P}(D\Phi\cdot\tilde\Phi+\Phi\tilde D\tilde\Phi)+a_4\mathcal{P}\Phi_{;m}\tilde\Phi_{;m}+a_5\mathcal{P}^2\Phi\tilde\Phi\\&+a_6(D^2\Phi\cdot\tilde\Phi+\Phi\tilde D^2\Phi)+a_7\{(D\Phi)_{;m}\tilde\Phi_{;m}\\&+\Phi_{;m}(\tilde D\Phi)_{;m}\}+a_8D\Phi\cdot\tilde D\Phi+a_9\mathcal{P}(\Phi_{;aa}\tilde\Phi+\Phi\tilde\Phi_{;aa})\\&+a_{10}\mathcal{P}\Phi_{;a}\tilde\Phi_{;a}+a_{11}\Phi_{;aa}\tilde\Phi_{;bb}+a_{12}(\Phi_{;mm}\tilde\Phi_{;aa}\\&+\Phi_{;aa}\tilde\Phi_{;mm})+a_{13}\Phi_{;ma}\tilde\Phi_{;ma}\}.\end{aligned}$$

Product formulas then show the constants a_i are independent of the dimension m.

We first take $k=0$ so $M=[r_0,r_1]$. Since $D\Phi=-\Phi_{;mm}-\mathcal{P}\Phi$, by Lemma 2.1

$$\beta_3(\Phi,\tilde\Phi,D)=\frac{1}{3\sqrt{\pi}}\int_{\partial M}\{4D\Phi\cdot\tilde\Phi+4\Phi\tilde D\tilde\Phi+2\mathcal{P}\Phi\tilde\Phi\}. \tag{3.2}$$

Suppose $\Phi|_{\partial M}=0$. By Lemma 1.1, $\beta_5(\Phi,\tilde\Phi,D)=-\frac{2}{5}\beta_3(D\Phi,\tilde\Phi,D)$. Thus we have

$$\beta_5(\Phi,\tilde\Phi,D)=\frac{1}{15\sqrt{\pi}}\int_{\partial M}\{-8D^2\Phi\cdot\tilde\Phi-8D\Phi\cdot\tilde D\tilde\Phi-4\mathcal{P}D\Phi\cdot\tilde\Phi\}$$

and hence $a_2=0$, $a_3=-4$, $a_4=0$, $a_6=-8$, $a_7=0$, and $a_8=-8$. Let Φ be general. We see that $a_5=1$ by using Lemma 1.1 and equation (3.2):

$$\begin{aligned}\beta_3(\Phi,\tilde\Phi,D)&=\partial_\epsilon|_{\epsilon=0}\beta_5(\Phi,\tilde\Phi,D-\epsilon I)\\&=\frac{1}{15\sqrt{\pi}}\int_{\partial M}\{20\Phi\cdot\tilde\Phi+20\Phi\tilde D\tilde\Phi+8\mathcal{P}\Phi\tilde\Phi+2a_5\mathcal{P}\}.\end{aligned}$$

We have $ds_e^2=dr^2+r^2ds_\theta^2$ and $\Delta_e:=-\partial_r^2-(m-1)r^{-1}\partial_r+r^{-2}\Delta_\theta$ are the flat metric and Euclidean Laplacian in polar coordinates.
Let $M_1:=(r_0,r_1)\subset\mathbb{R}$ and let $M:=\{x:r_0<|x|<r_1\}\subset\mathbb{R}^m$. On $C^\infty(M_1)$, let $D:=-\partial_r^2-(m-1)r^{-1}\partial_r$. By Lemma 1.1 and Lemma 3.1,

$$\begin{aligned}\beta_5(1,1,\Delta_e)&=\beta_5(1,r^{m-1},D)\ \mathrm{vol}(S^{m-1})\\&=\frac{1}{240\sqrt{\pi}}(m-1)(m-3)(m+3)(m-7)(r_0^{m-5}+r_1^{m-5})\mathrm{vol}(S^{m-1}).\end{aligned}$$

By Lemma 1.2, $\omega = \frac{1}{2}(m-1)r^{-1}$ and $\mathcal{P} = -\frac{1}{4}(m-1)(m-3)r^{-2}$. Note $D(1) = 0$ and $\tilde{D}(r^{m-1}) = 0$. We show $a_1 = 1$ by equating coefficients in the equation

$$\frac{1}{15\sqrt{\pi}}\{-\frac{3}{2}a_1(m-1)(m-3) + \frac{1}{16}(m-1)^2(m-3)^2\}$$
$$= \frac{1}{240\sqrt{\pi}}(m-1)(m-3)(m+3)(m-7).$$

To evaluate the remaining coefficients, we take $k = 1$ so $M = [r_0, r_1] \times S^1$. Let $D = D_r + D_u$ for $D_r = -\partial_r^2$ and $D_u = -(\partial_u^2 + \mathcal{P}(u))$. Let $\Phi(r,u) = \phi(r)\psi(u)$ and $\tilde{\Phi}(r,u) = \tilde{\phi}(r)\tilde{\psi}(u)$. Because $u_\Phi = u_\phi u_\psi$, we have

$$\beta_n(\Phi, \tilde{\Phi}, D) = \sum_{p+q=n} \beta_p(\phi, \tilde{\phi}, D_r)\beta_q(\psi, \tilde{\psi}, D_u). \tag{3.3}$$

Since the boundary of the circle is empty,

$$\beta_{2k}(\psi, \tilde{\psi}, D_u) = (-1)^k \int_{S^1} (D_u^k\psi \cdot \tilde{\psi})/k!$$

and $\beta_{2k+1}(\psi, \tilde{\psi}, D_t) = 0$. We use equations (3.1), (3.3), and (3.4). By equating coefficients of the invariants $\mathcal{P}\Phi \cdot \tilde{\Phi}_{:aa} + \mathcal{P}\Phi_{;aa}\tilde{\Phi}$, $\mathcal{P}\Phi_{;a}\tilde{\Phi}_{;a}$, $\Phi_{;aa}\tilde{\Phi}_{;bb}$, $\Phi_{;ma}\tilde{\Phi}_{;ma}$, and $\Phi_{;mm}\tilde{\Phi}_{;aa} + \Phi_{;aa}\tilde{\Phi}_{;mm}$, we show that

$$\frac{1}{15\sqrt{\pi}}(a_9 - 20) = -\frac{2}{\sqrt{\pi}}\frac{1}{2}, \quad \frac{1}{15\sqrt{\pi}}a_{10} = 0, \quad \frac{1}{15\sqrt{\pi}}(a_{11} - 24) = -\frac{2}{\sqrt{\pi}}\frac{1}{2},$$
$$\frac{1}{15\sqrt{\pi}}a_{13} = 0, \quad \text{and} \quad \frac{1}{15\sqrt{\pi}}(a_{12} - 24) = -\frac{2}{\sqrt{\pi}}\frac{2}{3}.$$

We complete the proof of the assertion (1) by solving these equations to see $a_9 = 5$, $a_{10} = 0$, $a_{11} = 9$, $a_{12} = 4$, and that $a_{13} = 0$. Assertion (2a) follows directly from Lemma 1.2. Assertion (2b) follows from assertion (1) since $1_{;aa} = \tilde{g}_{;aa} = -\mathcal{P}_u$ and $1_{;mm} = \tilde{g}_{;mm} = -\mathcal{P}_r$. □

4. The computation of $\mathcal{B}_5$.

In this section, we complete the proof of Theorem 0.2 (5); by equation (1.3) it suffices to compute $\beta_5(1, 1, \Delta)$.

Lemma 4.1. (1) *There exist universal constants so that*

$$\begin{aligned}\beta_5(1,1,\Delta_M) = \frac{1}{240\sqrt{\pi}} \int_{\partial M} \{ & b_1\rho_{mm;mm} + b_2 L_{aa}\rho_{mm;m} \\ & + b_3 L_{ab}R_{ammb;m} + b_4\rho_{mm}^2 + b_5 R_{ammb}R_{ammb} \\ & + b_6 L_{aa}L_{bb}\rho_{mm} + b_7 L_{ab}L_{ab}\rho_{mm} + b_8 L_{ab}L_{ac}R_{mbcm} \\ & + b_9 L_{aa}L_{bc}R_{mbcm} + b_{10}R_{ammb}R_{accb} + b_{11}L_{aa}L_{bc}R_{bddc} \\ & + b_{12}L_{ab}L_{ac}R_{bddc} + b_{13}L_{ab}L_{cd}R_{acbd} + b_{14}R_{abcm}R_{abcm} \\ & + b_{15}R_{abbm}R_{accm} + b_{16}L_{aa:b}R_{bccm} + e_1 L_{ab:c}L_{ab:c} \\ & + e_2 L_{aa}L_{bb}L_{cc}L_{dd} + e_3 L_{aa}L_{bb}L_{cd}L_{cd} + e_4 L_{ab}L_{ab}L_{cd}L_{cd} \\ & + e_5 L_{aa}L_{bc}L_{cd}L_{db} + e_6 L_{ab}L_{bc}L_{cd}L_{da}\}.\end{aligned}$$

(2) $e_2 = 1$, $e_3 = -4$, $e_4 + e_5 = -20$, $e_6 = 48$.

(3) $b_1 = 8$, $b_2 = -8$, $b_3 = 16$, $b_4 = -4$, $b_5 = 16$, $b_6 = -4$, $b_7 = -8$, $b_8 = 64$, $b_9 = -16$, $b_{10} = 0$, $b_{11} = 0$, $b_{12} = b_{13}$, $e_4 = -4 - b_{12}$, $e_5 = -16 + b_{12}$.

(4) $b_{12} = -8$, $b_{13} = -8$, $b_{14} = 4$, $b_{15} = 8$, $b_{16} = -16$, $e_1 = -8$.

Proof. We use the Weyl calculus to write down a basis of invariants. We omit the invariants $\{\rho_{ii:aa}, \rho_{mm:aa}\}$ since they integrate to zero. We omit $\rho_{ii;mm}$ and $\rho_{ii;m}L_{aa}$ since we may express $\int_{\partial M}\rho_{ii;mm}$ and $\int_{\partial M}\rho_{ii;m}L_{aa}$ in terms of $\int_{\partial M}\rho_{mm;mm}$ and $\int_{\partial M}\rho_{mm;m}L_{aa}$ plus lower order terms. Since $R_{abcm} = L_{bc:a} - L_{ac:b}$, we may symmetrize the covariant derivatives of L modulo this tensor. We integrate by parts to avoid expressions in the second tangential covariant derivatives $L_{ab:cd}$ of L. We use product formulas to see the coefficients of the invariants ρ_{ii}^2 ,$|\rho|^2$, $|R|^2$, $\rho_{mm}\rho_{ii}$, $L_{aa}L_{bb}\rho_{ii}$, and $L_{ab}L_{ab}\rho_{ii}$ vanish. This proves the first assertion. If M is the unit disk in $\mathbb{R}^m$, then by Lemma 1.1

$$\begin{aligned} e_2(m-1)^4 + e_3(m-1)^3 + (e_4+e_5)(m-1)^2 + e_6(m-1) \\ = (m^3 - 7m^2 - 9m + 63)(m-1).\end{aligned}$$

The second assertion now follows.

We use the results of §3 to prove the remaining assertions. We take $k = 0$ and consider the metric $ds^2 = dr^2 + e^{2f_i(r,v)}dv_i^2$ on $[r_0, r_1] \times T^\ell$ as in Lemma

3.1. Let '/' denote partial differentiation. Let m denote ∂_r; the remaining indices range from 1 through ℓ and index the ∂_a^v derivatives. Then

$$
\begin{aligned}
R_{mabm;m} &= -\delta_{ab}(f_{a/mm} + f_{a/m}^2)_{/m}, \\
R_{mabm} &= -\delta_{ab}(f_{a/mm} + f_{a/mm}^2), \\
R_{mabm;mm} &= -\delta_{ab}(f_{a/mm} + f_{a/m}^2)_{/mm}, \\
L_{ab} &= -\delta_{ab} f_{a/m}, \qquad \text{and} \\
R_{abcd} &= L_{ac}L_{bd} - L_{ad}L_{bc} + R^{\partial M}_{abcd}
\end{aligned} \tag{4.1}
$$

We use equation (4.1) and sum over indices a, b, etc:

$$
\begin{aligned}
b_1\rho_{mm;mm} &= b_1(-f_{a/mmmm} - 2f_{a/mmm}f_{a/m} - 2f_{a/mm}^2), \\
b_2L_{aa}\rho_{mm;m} &= b_2f_{a/m}(f_{b/mmm} + 2f_{b/mm}f_{b/m}), \\
b_3L_{ab}R_{ammb;m} &= b_3(f_{a/mmm}f_{a/m} + 2f_{a/mm}f_{a/m}^2), \\
b_4\rho_{mm}^2 &= b_4(f_{a/mm} + f_{a/m}^2)(f_{b/mm} + f_{b/m}^2), \\
b_5R_{ammb}R_{ammb} &= b_5(f_{a/mm} + f_{a/m}^2)^2, \\
b_6L_{aa}L_{bb}\rho_{mm} &= -b_6f_{a/m}f_{b/m}(f_{c/mm} + f_{c/m}^2), \\
b_7L_{ab}L_{ab}\rho_{mm} &= -b_7f_{a/m}^2(f_{b/mm} + f_{b/m}^2), \\
b_8L_{ab}L_{ac}R_{mbcm} &= -b_8f_{a/m}^2(f_{a/mm} + f_{a/m}^2) \\
b_9L_{aa}L_{bc}R_{mbcm} &= -b_9f_{a/m}f_{b/m}(f_{b/mm} + f_{b/m}^2) \\
b_{10}R_{ammb}R_{accb} &= b_{10}(f_{a/mm} + f_{a/m}f_{a/m})(f_{a/m}f_{a/m} + (-f_{a/m}f_{c/m} \\
&\quad + R^{\partial M}_{acca})) \\
b_{11}L_{aa}L_{bc}R_{bddc} &= b_{11}(f_{a/m}f_{b/m}^3 - f_{a/m}f_{b/m}^2f_{d/m} \\
&\quad + f_{a/m}f_{b/m}R^{\partial M}_{bddb}) \\
b_{12}L_{ab}L_{ac}R_{bddc} &= b_{12}(f_{a/m}^4 - f_{a/m}^3f_{d/m} + f_{a/m}^2R^{\partial M}_{adda}) \\
b_{13}L_{ab}L_{cd}R_{acbd} &= b_{13}(f_{a/m}^2f_{b/m}^2 - f_{a/m}^4 + f_{a/m}f_{b/m}R^{\partial M}_{abab}) \\
e_2L_{aa}L_{bb}L_{cc}L_{dd} &= e_2f_{a/m}f_{b/m}f_{c/m}f_{d/m} \\
e_3L_{aa}L_{bb}L_{cd}L_{cd} &= e_3f_{a/m}f_{b/m}f_{c/m}^2 \\
e_4L_{ab}L_{ab}L_{cd}L_{cd} &= e_4f_{a/m}^2f_{b/m}^2 \\
e_5L_{aa}L_{bc}L_{cd}L_{dc} &= e_5f_{a/m}f_{b/m}^3, \qquad \text{and} \\
e_6L_{ab}L_{bc}L_{cd}L_{da} &= e_6f_{a/m}^4.
\end{aligned} \tag{4.2}
$$

The terms $b_{14}R_{abcm}R_{abcm}$, $b_{15}R_{abbm}R_{accm}$, $b_{16}L_{aa:b}R_{bccm}$, and $e_1L_{ab:c}L_{ab:c}$ will not play a role in this analysis. Let $\omega_r = \frac{1}{2}f_{a/m}$, $D = -(\partial_r^2 + \omega_r\partial_r)$, and let $\mathcal{P} = -\frac{1}{4}(2f_{a/mm} + \sum_{a,b} f_{a/m}f_{b/m})$. We compute

$$\begin{aligned}16\mathcal{P}_{;mm} + 16\mathcal{P}^2 &= -8f_{a/mmmm} + (-8f_{a/mmm}f_{b/m} - 4f_{a/mm}f_{b/mm}) \\ &\quad + 4f_{a/mm}f_{b/m}f_{c/m} + f_{a/m}f_{b/m}f_{c/m}f_{d/m}.\end{aligned} \tag{4.3}$$

Let $\omega_r = w_r(r)$. We use the results of §3 to equate the coefficients of the expressions

$$\begin{array}{llll} f_{a/mm}f_{a/m}f_{b/m}, & f_{a/mmmm}, & f_{a/mm}f_{a/m}^2, & \mathcal{H}f_{a/mmm}, \\ \mathcal{H}f_{a/m}^3, & \mathcal{H}^2f_{a/mm}, & f_{a/mm}f_{b/m}^2, & f_{a/mm}f_{b/mm}, \\ \mathcal{H}^2f_{a/m}^2, & f_{a/mm}^2, & (f_{a/m}f_{b/m})^2, & f_{a/mm}R_{acca}^{\partial M}, \\ f_{a/m}^4, & \mathcal{H}^4, & f_{a/mmm}f_{a/m} & \end{array}$$

in the sum of terms from displays (4.2) and (4.3) to derive the equations

$$\begin{aligned} 2b_2 - b_9 - b_{10} &= 0 \\ -b_1 &= -8 \\ 2b_3 + 2b_5 - b_8 + b_{10} &= 0, \\ b_2 &= -8, \\ -b_9 - b_{10} - b_{11} - b_{12} + e_5 &= 0, \\ -b_6 &= 4, \\ 2b_4 - b_7 &= 0, \\ b_4 &= -4, \\ -b_6 - b_{11} + e_3 &= 0, \\ -2b_1 + b_5 &= 0, \\ b_4 - b_7 + b_{13} + e_4 &= 0, \\ b_{10} &= 0, \\ b_5 - b_8 + b_{10} + b_{12} - b_{13} + e_6 &= 0, \\ e_2 &= 1, \qquad \text{and} \\ -2b_1 + b_3 &= 0. \end{aligned}$$

Assertion (3) now follows. As a scholium to our computations we see

$$\mathcal{P}_r = \frac{1}{4}\{-2f_{a/mm} - f_{a/m}f_{b/m}\}, \qquad \text{and} \tag{4.4}$$

(4.5) $$\beta_5(1,1,\Delta_M) - \frac{1}{240\sqrt{\pi}}\int_{S^1}\{(16\mathcal{P}_{r;mm}+16\mathcal{P}_r^2)\tilde{g}\}$$
$$= \frac{1}{240\sqrt{\pi}}\int_{\partial M}\{b_{12}L_{ab}L_{ac}R^{\partial M}_{bddc} + b_{12}L_{ab}L_{cd}R^{\partial M}_{acbd} + b_{14}R_{abcm}R_{abcm}$$
$$+ b_{15}R_{abbm}R_{accm} + b_{16}L_{aa:b}R_{bccm} + e_1 L_{ab:c}L_{ab:c}\}.$$

Next we take a metric of the form $ds^2_M = dr^2 + dx^2 + e^{2f_v(r,x)}dv_j^2$. Let m denote ∂_r and 1 denote ∂_x. The remaining indices u, v will range from 2 through $m-1$ and index the ∂_v^j derivatives. We sum over u, v, etc. and compute

$$\begin{aligned}
\mathcal{P}_u &= -\frac{1}{2}f_{u/mm} - \frac{1}{4}f_{u/m}f_{v/m}\\
R_{u1mv} &= \delta_{uv}(-f_{u/1m} - f_{u/1}f_{u/m})\\
R_{u11v} &= \delta_{uv}(-f_{u/11} - f^2_{u/1})\\
R^{\partial M}_{uvvu} &= \delta_{uv}f^2_{u/1} - f_{u/1}f_{v/1}\\
b_{12}L_{ab}L_{ac}R^{\partial M}_{bddc} &= b_{12}f^2_{u/m}(-f_{u/11} - f^2_{u/1}) + f^2_{u/m}(\delta_{uv}f^2_{u/1} - f_{u/1}f_{v/1})\\
b_{13}L_{ab}L_{cd}R^{\partial M}_{acbd} &= b_{13}f_{u/m}f_{v/m}(f_{u/1}f_{v/1} - \delta_{uv}f^2_{u/1})\\
b_{14}R_{abcm}R_{abcm} &= 2b_{14}R^2_{1uum} = 2b_{14}(f_{u/1m} + f_{u/1}f_{u/m})^2\\
b_{15}R_{abbm}R_{accm} &= b_{15}(f_{u/1m} + f_{u/1}f_{u/m})(f_{v/1m} + f_{v/1}f_{v/m})\\
b_{16}L_{aa:b}R_{bccm} &= b_{16}f_{u/1m}(f_{v/1m} + f_{v/1}f_{v/m})\\
e_1 L_{ab:c}L_{ab:c} &= (f^2_{u/1m}f_{u/1m} + f^2_{u/m}f^2_{u/1})\\
16\mathcal{P}_{u;mm} &= -8f_{u/11mm} - 8(f_{u/1m}f_{v/1m} + f_{u/1mm}f_{v/1})
\end{aligned}$$

$$\int_{r_0\times S^1}\{16g\mathcal{P}_{u;mm}\} = \int_{\partial M}\{b_{12}L_{ab}L_{ac}R^{\partial M}_{bddc} + b_{12}L_{ab}L_{cd}R^{\partial M}_{acbd}$$
$$+ b_{14}R_{abcm}R_{abcm} + b_{15}R_{abbm}R_{accm} + b_{16}L_{aa:b}R_{bccm} + e_1 L_{ab:c}L_{ab:c}\}$$

$$\int_{\partial M}\{-b_{12}f^2_{u/m}f_{u/11} - b_{12}f^2_{u/m}f_{u/1}f_{v/1}\} = \int_{\partial M}2b_{12}f_{u/1m}f_{u/1}f_{u/m},$$

and

$$\int_{r_0\times S^1}\{e^{f_u}(-8f_{u/11mm} - 8f_{u/11m}f_v)\} = 0.$$

There are no other integral identities among the monomials involved. We equate coefficients of $f_{u/1m}f_{v/1}f_{v/m}$, $f_{u/1m}f_{v/1m}$, $f_{u/1m}f_{u/1}f_{u/m}$, $f^2_{u/1m}$,

$f_{u/m}f_{v/m}f_{u/1}f_{v/1}$, and $f^2_{u/m}f^2_{u/1}$ to derive the following equations:

$$\begin{aligned} &2b_{15}+b_{16}=0, \quad b_{15}+b_{16}=-8, \qquad 2b_{12}+4b_{14}=0,\\ &2b_{14}+e_1=0, \quad b_{13}+b_{15}=0, \text{ and } \; -b_{13}+2b_{14}+2e_1=0. \end{aligned}$$

We use these equations and the preceding assertions to complete the proof. □

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