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On the generic eigenvalue flow of a family of metrics and its application

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We define a generic eigenvalue flow of a parameter family of metrics, which the corresponding eigenfunction is continuous in parameters. Then we apply the result to the study of polynomial growth harmonic functions on a complete manifold. Under the assumption that the manifold has some cone structures at infinity which is not necessarily unique, we obtain a uniform growth estimate.

1. Introduction.

It is well-known that eigenvalues are continuous dependent of parameters in some nice space. We are interested in the continuity of eigenfunctions for one-parameter family of elliptic operators. It was proved by K. Uhlenbeck that in the generic sense eigenvalues have one-dimensional eigenspace. Clearly, if all eigenvalues are of simple multiplicity, then the eigenfunctions for those operators considered are continuous in the parameter. But unfortunately, eigenspaces are not always one-dimensional. As we show in an example in next section, eigenvalues do often intersect when the parameter varies. Moreover, those intersections are stable in perturbations.

On another hand, when the eigenfunction is continuous in parameter, such as in our applications, we can utilize it to construct almost harmonic functions even when the metric tensor are changing slowly in parameters. Actually the initial motivation are from the problem of polynomial growth harmonic functions on certain complete Riemanannian manifold. Indeed, there are cone structures at infinity for complete Riemanannian manifolds with nonnegative Ricci curvature, quadratic curvature decay and Euclidean volume growth ([CT]) or with Ricci curvature lower bounded and Euclidean volume growth ([CC2]). The cone structures are not unique in general, but they form a compact set, that suggests us to study a family of operators

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with parameters varying in a compact set. In particular, let us consider for a family of operators

$$\Delta_{g(t)} = \frac{1}{\sqrt{G(t)}}\frac{\partial}{\partial x_i}\left(g^{ij}(t)\sqrt{G(t)}\frac{\partial}{\partial x_j}\right) \tag{1.1}$$

with the metric $g(t) \in C^2(\tilde{X}, T(M) \bigotimes T(M))$ and $\tilde{X}$ is a compact set of a manifold.

We first apply the transversality theory to define eigenfunction flows, which eigenfunctions are continuous in the parameter t. In section 2, we prove Theorem 2.8. Then we apply it to the study of polynomial growth harmonic functions on a complete manifold in section 3. There are many related works on this problem (see [CM]) and the reference there. Here we make use of the so called three annuli lemma, inspired by an idea of J. Cheeger. The three annuli lemma was used before by L. Simon [S], Cheeger and Tian [CT], etc.. We obtain an a priori estimate for the growth of harmonic functions on a complete manifold. Then as a corollary, we obtain an estimate on the dimension of polynomial growth harmonic functions under the assumption that the manifold has some cone structures at infinity. This was motivated by a recent paper of Colding and Minicozzi [CM]. In that paper, in particular, they proved that the dimension of harmonic functions with polynomial growth is finite for any complete manifold with Ricci curvature nonnegative, as Yau conjectured.

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2. Generic Eigenfunction Flow.

In this section, we consider eigenfunctions defined on $(M, g(t))$ where M is compact with $\partial M = \emptyset$ and the metric tensor $g(t)$ depends on the parameter t twice continuously differentiable. For simplicity we first consider the case that the parameter is in $[0, 2\pi]$. Then we know that eigenvalues are continuous differentiable in t and

$$0 = \lambda_0(t) < \lambda_1(t) \leq \lambda_2(t) \leq \cdots$$

Let $G = \{g(t)|g(t)$ is the metric of M, $g(t) \in C^2([0,2\pi], T(M)\bigotimes T(M))\}$. We want to show that we can define the eigenfunction flow $\phi_i(t)$ for generic $g(t)$ in G which continuously depends on t.

Let $S_k = \{u \in H_k^2(M),\ \int_M u^2 = 1\}$, $\mathcal{M}_2 = \{g|g \in C^2(T(M)\bigotimes T(M))$ is the metric of M $\}$. We consider the map

$$\phi : S_k \times R \times \mathcal{M}_2 \mapsto H_{k-2}(M)$$

given by

$$\phi(u,\lambda,g) = \Delta_g u + \lambda u = L_g u. \tag{2.1}$$

By the study of the regular value of ϕ, K. Uhlenbeck obtained the following result in [U]

Lemma 2.1. *The set $\{g \in \mathcal{M}_2|\Delta_g$ has one-dimensional eigenfunction $\}$ is a residual set in $\mathcal{M}_2$.*

Therefore for $g(t) \in G$ we may assume that $\Delta_{g(0)}$ has one-dimensional eigenfunctions. Consider eigenvalues of $\Delta_{g(t)}$ as t varying. First we know by the following example, they may have intersections.

Example 2.2. Consider S^2 with the following metric

$$d\theta^2 + g(t,\theta)d\phi^2$$

where $g(t,\theta) = a^2(t)\sin^2\theta$ and $\frac{1}{2} \le a(t) \le 1$. Then the eigenvalue problem becomes

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial\theta}\left(\sqrt{g}\frac{\partial u}{\partial\theta}\right) + \frac{1}{g}\frac{\partial^2 u}{\partial\phi^2} + \lambda u = 0. \tag{2.2}$$

Let $u = F(\theta)H(\phi)$, then (2.2) becomes

$$\frac{\sqrt{g}\frac{\partial}{\partial\theta}(\sqrt{g}F')}{F} + \frac{H''}{H} + \lambda g = 0. \tag{2.3}$$

Put $-H''/H = n^2$, for $n = 1,2,\cdots$. Now we solve

$$\frac{\sqrt{g}\frac{\partial}{\partial\theta}(\sqrt{g}F')}{F} + \lambda g - n^2 = 0, \qquad 0 \le \theta \le \pi. \tag{2.4}$$

Let $\lambda_{n,i}$ be its eigenvalue, then

$$\lambda_{n,i} = \inf_{F\in\mathcal{F}} \frac{\int_0^\pi \sqrt{g}F'^2 + \frac{n^2}{\sqrt{g}}F^2}{\int_0^\pi \sqrt{g}F^2}, \qquad \int_0^\pi \sqrt{g}FF_{n,j} = 0, \quad j < i. \tag{2.5}$$

where $F_{n,j}$ is the eigenfunction corresponding to $\lambda_{n,j}$ and $\mathcal{F}$ is the weighted sobolev space with

$$\|u\|^2_{W^{2,1}} = \int_0^\pi \sqrt{g}(u'^2 + u^2).$$

For $g = a^2 \sin^2\theta$, it is easy to see that $\lambda_{n,i}$ continuously depends on a. When $a = 1$, (2.4) becomes

$$(2.6) \qquad (\sin\theta F')' - \frac{n^2}{\sin\theta} F + \sin\theta F = 0, \qquad 0 \le \theta \le \pi.$$

Let $\mu = \cos\theta$, then (2.6) deduce

$$(2.7) \qquad \frac{d}{d\mu}\left((1-\mu^2)\frac{dF}{d\mu}\right) - \frac{n^2}{1-\mu^2} F + \lambda F = 0, \qquad -1 \le \mu \le 1.$$

In this case, $\lambda_{n,i} = (n+i-1)(n+i)$, $i = 1, 2, \cdots$. The corresponding eigenfunction is the so called Legendre function.

Similarly we find for $a = 1/2$, the eigenvalue of (2.5) is $\lambda_{n,i} = (2n+i-1)(2n+i)$. In conclusion, we have

$$(2.8) \qquad \lambda_{n,i}\left(\frac{1}{2}\right) = (2n+i-1)(2n+i), \quad n = 0, 1, \cdots, \quad i = 1, 2, \cdots.$$

$$(2.9) \qquad \lambda_{n,i} = (n+i-1)(n+i), \quad n = 0, 1, \cdots, \quad i = 1, 2, \cdots.$$

All eigenvalues of (2.2) for $a = 1/2$ and $a = 1$ are given by (2.8) and (2.9) respectively. Note that $\lambda_{n,i}$ is continuous dependent of a, it is easy to check that when a is varying from 1/2 to 1, eigenvalues of (2.5) must have infinite intersections.

Remark 2.3. We can choose $a^2 = a(t,\theta)^2$, such that $a(t,0) = a(t,\pi) = 1$ and $Ricc(S^2) \ge 1$.

Now we back to our problems. For a fixed C_0, we consider those eigenvalues of $\Delta_{g(t)}$ with $\lambda_k(t) < C_0$. Let t_0 be a point where

$$(2.10) \qquad \cdots < \lambda_{k-1}(t_0) < \lambda_k(t_0) = \cdots = \lambda_{k+l}(t_0) < \lambda_{k+l+1}(t_0) < C_0.$$

Moreover, we may assume that those eigenvalues which are less than C_0 are one-dimensional in a small neighborhood of t_0 and $t \ne t_0$. Otherwise we can replace $g(t)$ by a small perturbation $g_\epsilon(t)$. Since we assume $\Delta_{g(0)}$ has one-dimensional eigenfunctions, then the normalized $\phi_k(t)$ with $\lambda_k(t) < C_0$ is in

$C(M \times [0,\epsilon))$ for some small $\epsilon > 0$. And $\phi_k(t)$ is a continuous flow, if $\lambda_k(t)$ does not intersect with other eigenvalues. We want to show that it can be continuously defined at the intersection point, up to a small perturbation. In fact, we have the following main result of this section.

Lemma 2.4. *Let t_0 be as above, then for a residual set of $g(t) \in G$,*

$$\lim_{t \to t_0} (\phi_k(t), \phi_{k+1}(t), \cdots \phi_{k+l}(t)) \tag{2.11}$$

exists by a properly chosen of the sign of those eigenfunctions.

Proof. Let $\psi_1, \cdots, \psi_l$ be normalized orthogonal eigenfunctions corresponding to $\lambda_k(t_0)$. For any sequence $\{t_i\}$, there exists a subsequence, still denoted by $\{t_i\}$ such that

$$\lim_{t \to t_0} \phi_k(t_i) = c_1\psi_1 + c_2\psi_2 + \cdots + c_l\psi_l. \tag{2.12}$$

And here $\sum_{i=1}^{l} {c_i}^2 = 1$. Since
(2.13)
$$(\Delta_{g(t_i)} - \Delta_{g(t_0)})\phi_k(t_i) + (\lambda_k(t_i) - \lambda_k(t_0))\phi_k(t_i) + \Delta_{g(t_0)}\phi_k(t_i) + \lambda_k(t_0)\phi_k(t_i) = 0.$$

Therefore, for $j = 1, 2, \cdots l$

$$\int \psi_j[(\Delta_{g(t_i)} - \Delta_{g(t_0)})\phi_k(t_i) + (\lambda_k(t_i) - \lambda_k(t_0))\phi_k(t_i)]dv_{g(t_0)} = 0. \tag{2.14}$$

Let $t_i \to t_0$, we have

$$\int \psi_j \left[\dot{\Delta}_{g(t_0)} \left(\sum_{i=1}^{l} c_i\psi_i \right) + \dot{\lambda}(t_0) \left(\sum_{i=1}^{l} c_i\psi_i \right) \right] = 0, \tag{2.15}$$

for $j = 1, 2, \cdots l$. Since (2.15) has non-trivial solution $(c_1, c_2, \cdots c_l)$, we deduce

$$\begin{vmatrix} a_{11} + \dot{\lambda}(t_0) & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} + \dot{\lambda}(t_0) & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} + \dot{\lambda}(t_0) \end{vmatrix} = 0, \tag{2.16}$$

where $a_{ij} = \int (\dot{\Delta}_{g(t_0)}\psi_j)\psi_i dv_{g(t_0)}$.

Let $h = g'_{\alpha\beta}(t_0)g^{\beta\alpha}(t_0)$. By a simple calculation, we have

$$(2.17)\quad a_{ij} = \frac{\lambda_k(t_0)}{2}\int h\psi_i\psi_j + \int (g^{\alpha\beta}(t_0))'\frac{\partial\psi_i}{\partial x_\alpha}\frac{\partial\psi_j}{\partial x_\beta} - \frac{1}{2}\int h\left\langle\nabla\psi_i,\nabla\psi_j\right\rangle.$$

We only need to show that up to a generic small perturbation of $g'(t_0)$ with $g(t_0)$ fixed, the matrix $(a_{ij})_{l\times l}$ has l different eigenvectors. If so, then $\lim_{t_i\to t_0}\phi_k(t)$ has only $2l$ possible choice. By our assumption, $\phi_k(t)$ is continuous dependent of t in a small neighborhood of t_0 with $t \neq t_0$. Then by a simple argument and a properly chosen of the sign of $\phi_k(t)$, $\lim_{t\to t_0}\phi_k(t)$ exists. Therefore $\phi_k(t)$ can be defined to be a continuous function of t at $t = t_0$.

We claim that those $g'(t_0)$ such that $(a_{ij})_{l\times l}$ has one-dimensional eigenfunction is a second category set in $\mathcal{M}_2$. As in [U], we consider the map

$$\Phi:\quad S^l\times R\times\mathcal{M}_2\mapsto R^l$$

given by $\Phi(\xi,\lambda,g'(t_0)) = A(g'(t_0))\xi+\lambda\xi$, where $S^l = \{\xi\in R^l|\sum_{i=1}^l {\xi_i}^2 = 1\}$, $A(g'(t_0)) = (a_{ij})_{l\times l}$.

Let D_2 be the differential of Φ in $\mathcal{M}_2$ and D_1 denote the differential of Φ in the direction of $S^l\times R$. We only need to show that (see [U]) $D_2\Phi$ has dense image in R^l. If this is not true, then there exist $\xi,\eta\in S^l$ such that for all $g'(t_0)\in C^2(T(M)\otimes T(M))$, we have

$$(2.18)\qquad \left\langle A(g'(t_0))\xi,\eta\right\rangle = 0,$$

which implies that for some normalized eigenfunctions u, v, we have

$$(2.19)\quad \frac{\lambda_k(t)}{2}\int g'_{\alpha\beta}g^{\beta\alpha}uv - \frac{1}{2}g'_{\alpha\beta}g^{\beta\alpha}g^{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + g^{\alpha i}g'_{ij}g^{j\beta}\frac{\partial u}{\partial x_\alpha}\frac{\partial v}{\partial x_\beta} = 0.$$

In particular, let $g'_{\alpha\beta} = fg_{\alpha\beta}$ for some $f\in C^1(M)$, then

$$(2.20)\qquad \frac{n}{2}\lambda_k(t_0)\int fuv + \left(1-\frac{n}{2}\right)\int fg^{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} = 0.$$

Since f can be arbitrary, we obtain

$$(2.21)\qquad n\lambda_k uv = (n-2)g^{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}.$$

As proved in [U], (2.21) is impossible. Here we give a short argument.

For $n = 2$, obviously it is impossible.

For $n > 2$, consider critical points of uv on M, we have $\bigtriangledown u = -\frac{u}{v} \bigtriangledown v$, if $v \neq 0$ at this point. Therefore (2.21) deduce at this point

$$-|\bigtriangledown v|^2 = \frac{n}{n-2}\lambda v^2. \tag{2.22}$$

Since $\lambda > 0$, we deduce at the critical point of uv, $u = v = 0$. Then $uv \equiv 0$ in M, a contradiction. Then we have finished the proof of Lemma 2.3.

Remark 2.5. For a residual set of $g(t) \in G$, we can define the continuous eigenfunction flow $\phi_1(t), \phi_2(t), \cdots \phi_k(t)$ whenever the corresponding eigenvalue is less than C_0. But after the intersection points, those eigenvalues may not be in the right order. We can also choose a sequence of the bound $C_0(k)$ and $C_0(k) \to \infty$. And then any eigenfunction at $t = 0$ defines an eigenfunction flow for generic $g(t) \in G$.

The eigenfunction flow has a very important property.

Lemma 2.6. *If $g(t)$ is periodic in t of period 2π, then for a residual set of $g(t) \in G$ the eigenfunction flow is a periodic function of period 2π.*

Proof. First we note

$$\lambda_k(t) \int \phi_k(t)^2 = -\int \phi_k(t)\dot{\Delta}_{g(t)}\phi_k(t). \tag{2.23}$$

Then we have, for $0 \leq t \leq 2\pi$

$$\dot{\lambda}_k(t) \leq C_2(|g(t)|_{C^1})\lambda_k(t). \tag{2.24}$$

Here $|g(t)|_{C^1}$ is the norm of the two tensor. For this given $g(t)$, we may assume $|g(t)|_{C^1} \leq C_1$. Then (2.24) deduce for $0 \leq t \leq 2\pi$ for some constant C_4

$$\lambda_k(t) \leq C_4\lambda_k(0). \tag{2.25}$$

Now we choose a special metric $g_1(t) = (2 + \sin t)g_0$, where g_0 is a metric in $\mathcal{M}_2$ which is independent of t and Δ_{g_0} has one-dimensional eigenfunctions. Then we can see easily that the eigenfunction flow for the metric $g_1(t)$ is 2π periodic. For the given $g(t)$, we can consider a $h(s,t)$ in G, such that $h(t,0) = g_0(t)$ and $h(t,1) = g(t)$, moreover for all $0 \leq s \leq 1$, $h(t,s)$ can define a eigenfunction flow. We note that for $0 \leq s \leq 1$, the eigenfunction flow continuously depends on s. Then it is 2π periodic for all $0 \leq s \leq 1$. Then we have proved the lemma.

Remark 2.7. From the proof of Lemma 2.5, it is easy to see the eigenfunction flow is stable for the generic perturbation in G. We sometimes also call it eigenvalue flow, but what is important in this flow is that the corresponding eigenfunction is continuous in t.

Now we consider more general case where $t \in \tilde{X}$, $\tilde{X}$ is a compact set of manifold. Our main result is

Theorem 2.8. *Let $\tilde{X}$ be a compact, connected subset of a manifold, $g(t) \in C^2(\tilde{X}, T(M) \otimes T(M)) = \tilde{\mathcal{M}_2}$, then for a residual set of $g(t) \in \tilde{\mathcal{M}_2}$, we can define an eigenvalue flow $\lambda_k(t)$ which is continuous in t and satisfies* (2.24), *moreover the corresponding eigenfunctions is also continuous in t. Therefore for any given $d > 0$, there exist finite number $k_1, \cdots k_j$, such that*

$$spec\{\lambda_k(t) | t \in \tilde{X}, k > 1\} \setminus \{\lambda_{k_1}(t), \cdots \lambda_{k_j}(t)\}$$

has gaps at $\lambda = d$ for all $t \in \tilde{X}$.

Proof. Let $E_{\lambda(t)}$ be the eigenspace corresponding to $\lambda(t)$, $P_{\lambda(t)}$ be the projection to $E_{\lambda(t)}$ in the sense of L^2 expansion. Let $e_1(t), \cdots e_m(t)$ be an orthonormal basis of $T(\tilde{X})$. Consider the map

$$\tilde{\Phi} : S_2 \times R \times \tilde{\mathcal{M}_2} \mapsto E_\lambda(t)$$

given by

$$\tilde{\Phi}(v, g(t), \mu) = P_{\lambda(t)}(\nabla_{e_i} \Delta_{g(t)} v + \mu v). \tag{2.26}$$

As in Lemma 2.4, we can prove that for generic $g(t) \in \tilde{\mathcal{M}_2}$, zero is a regular value of $\tilde{\Phi}$. That is, the eigenspace of the operator $P_{\lambda(t)}(\nabla_{e_i} \Delta_{g(t)})$ is one dimensional for generic $g(t)$ and $i = 1, \cdots m$.

Therefore at the point t, $\lambda(t)$ and $\nabla_{e_i}\lambda(t)$ is uniquely determined. At the point $t \in \tilde{X}$, either $\lambda(t)$ has one dimensional eigenspace or $\nabla_{e_i}\lambda(t)$ has different derivatives and the corresponding eigenfunction can be continuously defined for multiple eigenvalues. Then we can define continuously the eigenvalues flow locally at this point.

Now we fix a base point $t_0 \in \tilde{X}$. And we may assume that $\Delta_{g(t_0)}$ has one dimensional eigenspace, $\lambda_1, \lambda_2, \cdots \lambda_k \cdots$. Consider a generic curve $c : (I, \partial I) \mapsto (\tilde{X}, t_0)$, $I = [0, 2\pi]$. Then we know that the eigenvalue flow $\lambda_k(c(s))$ is a continuous function of s. We only need to show $\lambda_k(c(0)) = \lambda_k(c(2\pi))$. This can done by the cobordism argument as in Lemma 2.6. Then $\lambda_k(t)$ and its corresponding eigenfunction are continuous for all $t \in \tilde{X}$ for generic $g(t) \in \tilde{\mathcal{M}_2}$. Then our theorem follows easily.

Remark 2.9. Notice (2.24), (2.25) and eigenvalues continuously depend on parameters, eigenvalues are well defined and continuous it t for any $g(t) \in C^1(\tilde{X}, T(M) \otimes T(M))$, but after taking limit the corresponding eigenfunctions may not continuous in t. So we can only define the eigenfunction flow for generic $g(t)$.

3. Application in the growth Estimates.

In this section, we study the harmonic function with polynomial growth by using the three annuli lemma. This technique was used before (cf. [S], [CT], [QT]). Let M be an open manifold with lower bounded Ricc curvature, which is our basic assumption in this paper. Then we know, from the Cheeger-Gromov compactness theorem, that for $p \in M$ and any sequence $\{r_k\}, r_k \to \infty$, the sequence of pointed rescaled manifolds, $(M, p, r_k^{-2}g)$, has a subsequence which converges in the pointed Gromov-Hausdorff topology to a length space, M_∞. And M_∞, the so called tangent cone at infinity, might a priori depend on the sequence $\{r_k\}$. If $Ricc(M) \geq 0$ and M has maximal volume growth, then M_∞ is a metric cone [CC1], that is, $M_\infty = (0, \infty) \times_r X$, where X is a length space. Under some additional assumptions on the curvature decay, M_∞ is in fact a smooth manifold (cf. [CT]). In general, the regularity of the cross section X, need more detailed study.

In this paper, we always assume that

$$M_\infty = (0, \infty) \times_r (N, g(t)), \tag{3.1}$$

where $(N, g(t))$ is a family of compact manifold without boundary, $g(t) \in C^1(\tilde{X}, T(M) \otimes T(M))$, $\tilde{X}$ is a compact, connected subset of a manifold.

For simplicity, in the following discussion we only prove our result in a simple case. That is, we assume that $g(t)$ is continuously differentiable and 2π-periodic in t, where t depends on the subsequence of $\{r_k\}$. We may assume that $(N, g(t))$ has a continuous eigenfunction flow. In fact, after a generic perturbation we can choose $(N, g_\epsilon(t))$ such that $(N, g_\epsilon(t))$ has a continuous eigenfunction flow and $g_\epsilon(t) \to g(t)$ as $\epsilon \to 0$. The following discussions are still valid for ϵ sufficiently small. Let $A_{a,b}$ be the geodesic annulus on M with inner radius a and outer radius b. From our assumption on M, we know that for any $\epsilon > 0$, there exist $\{r_{k,i}\}$, $\{\Omega_{k,i}\}$ and k_0, I_0 with $\Omega_{k,i} r_{k,i} = r_{k,i+1}$, $\Omega_{k,I+1} r_{k,I+1} = r_{k+1,1}$ such that for $k > k_0$, $I > I_0$

$$d_{GH}(A_{r_{k,1}, \Omega_{k,1} r_{k,1}}, (r_{k,1}, \Omega_{k,1} r_{k,1}) \times_r (N, g(t_1))) < \epsilon r_{k,1}$$
$$d_{GH}(A_{r_{k,2}, \Omega_{k,2} r_{k,2}}, (r_{k,2}, \Omega_{k,2} r_{k,2}) \times_r (N, g(t_2))) < \epsilon r_{k,2}$$

$$\cdots$$

$$d_{GH}(A_{r_{k,i},\Omega_{k,i}r_{k,i}},(r_{k,i},\Omega_{k,i}r_{k,i})\times_r(N,g(t_i)))<\epsilon r_{k,i}$$

$$\cdots$$

where $0=t_1<\cdots<t_i<\cdots<t_{I+1}=2\pi$ is a division.

We consider harmonic functions defined on M with $|u|\leq C(1+r^d)$, which is called at most d order growth harmonic functions. Note the eigenvalue $\lambda_j(t)$ is a 2π periodic function. Let $\phi_{j0}(t)$, $\phi_{j1}(t)$, $\cdots$, $\phi_{jm_1}(t)$ be all those eigenfunctions whose eigenvalue intersect with a number λ_{d_2} which will be given later with some $t\in[0,2\pi]$. We stress that there are only finite number of such $\lambda_{j_k}(t)$ because $\lambda_j(t)$ is periodic.

For technique reason, we assume the condition (S): there exists a bilipschitz homeomorphic map

$$\Phi_{k,i}:\quad (r_{k,i},\Omega_{k,i}r_{k,i})\times_r(N,g(t_i))\mapsto A_{r_{k,i},\Omega_{k,i}r_{k,i}}$$

For i fixed, let $A^k_{1,\Omega_{k,i}}$ be the rescaled $A_{r_{k,i},\Omega_{k,i}r_{k,i}}$ under the metric $\frac{1}{r^2_{k,i}}g$. Then for any $\epsilon>0$, there exist k_0, I_0 and $r_{k,i}$ as before, such that for $k>k_0$, $I>I_0$

$$d_{GH}(A^k_{1,\Omega_{k,i}},(1,\Omega_{k,i})\times_r(N,g(t_i))<\epsilon. \tag{3.2}$$

We may assume the homeomorphic map $\Phi_{k,i}$ satisfies, for any $x,y\in(1,\Omega_{k,i})\times_r(N,g(t_i)$

$$|d_{M_\infty}(x,y)-d_M(\Phi_{k,i}(x),\Phi_{k,i}(y))|<\epsilon. \tag{3.3}$$

Now we let

$$\overline{\phi_j}(t_i)=\phi_j(t_i)\circ\Phi^{-1}_{k,i}, \tag{3.4}$$

where $\phi_j(t_i)$ is an eigenfunction on $(N,g(t_i))$. We define a new function v_j by

$$v_j=\left[\fint_{\partial B_r}u\overline{\phi_j}(t_i)\right]\overline{\phi_j}(t_i), \tag{3.5}$$

for $r_{k,i}\leq r\leq\Omega_{k,i}r_{k,i}$, where $r=d(p,\cdot)$, $\fint_{\partial B_r}=\frac{1}{vol(\partial B_r)}\int_{\partial B_r}$. It is easy to see that $v_j\in L^2_{loc}(M)$. We note that $a_jr^{p_j(t_i)}\phi_j(t_i)$ is a harmonic function on the metric cone $(0,\infty)\times_r(N,g(t_i))$, where $\lambda_j(t_i)=p_j(t_i)(p_j(t_i)+n-2)$, $p_j(t_i)>0$. We shall show that for i fixed v_j is near $a_jr^{p_j(t_i)}\phi_j(t_i)$ in some sense.

Given $L > 1$ and a family of annuli $\{A_{a_k, La_k}\}$. Let $A^k_{1,L}$ be the rescaled annulus under the metric $\frac{1}{a_k^2} g$. Then we know there exists a subsequence of $\{a_k\}$, still denoted by a_k, and a metric $g(\bar{t})$, such that

$$\lim_{k\to\infty} d_{GH}(A^k_{1,L}, (1, L) \times_r (N, g(\bar{t})) = 0.$$

For k large, let $\Phi_k : A^k_{1,L} \mapsto (1, L) \times_r (N, g(\bar{t}))$ be a homeomorphic map. Put $u_k = u \circ \Phi_k^{-1}$, where u is a given harmonic function on M.

Lemma 3.1. *There exist a subsequence of* $\left\{ \frac{u_k}{|u|_{L^\infty\left(A^k_{\frac{1}{2},2L}\right)}} \right\}$, *still denoted by itself, such that*

$$\frac{u_k}{|u|_{L^\infty\left(A^k_{\frac{1}{2},2L}\right)}} \to u_0 \quad \text{in} \quad C^0((1, L) \times_r (N, g(\bar{t}))). \tag{3.6}$$

Furthermore, u_0 is a harmonic function on the tangent cone at infinity.

Proof. First we know, from Cheng-Yau's gradient estimates $\frac{\nabla u_k}{|u|_{L^\infty\left(A^k_{\frac{1}{2},2L}\right)}}$ is uniformly bounded on $A_{1,L}$. Therefore $\frac{u_k}{|u|_{L^\infty\left(A^k_{\frac{1}{2},2L}\right)}}$ is a compact sequence, then there exists a subsequence such that (3.6) is true. Obviously u_0 is a harmonic function.

Now, we let $\lambda^* = \inf_{t\in[2,2\pi)} \lambda_1(t) > 0$, $d_0 < 0$ and $d_0(d_0 + n - 2) < \lambda^*$. Then we have the following three annuli lemma.

Lemma 3.2. *Given $L > 1$, there exists a k_0, for $k > k_0$*

$$\fint_{A_{L^k, L^{k+1}}} u^2 < \frac{1}{2}\left(L^{2d_0} \fint_{A_{L^{k-1}, L^k}} u^2 + L^{-2d_0} \fint_{A_{L^{k+1}, L^{k+2}}} u^2 \right), \tag{3.7}$$

where $\fint_{A_{L^k, L^{k+1}}} = \frac{1}{vol(A_{L^k, L^{k+1}})} \fint_{A_{L^k, L^{k+1}}}$. *So if*

$$\fint_{A_{L^k, L^{k+1}}} u^2 > L^{2d_0} \fint_{A_{L^{k-1}, L^k}} u^2,$$

then

$$\fint_{A_{L^{k+1}, L^{k+2}}} u^2 > L^{2d_0} \fint_{A_{L^k, L^{k+1}}} u^2.$$

Proof. We first prove (3.7) on a metric cone $(0,\infty)\times_r (N, g(\bar{t}))$. A harmonic function in a metric cone has the following expansions

$$u = \sum a_j r^{p_j}\phi_j + \sum b_j r^{-(n-2)-p_j}\phi_j, \tag{3.8}$$

where $p_j \geq 0$ and $p_j(p_j+n-2)=\lambda_j$, λ_j, ϕ_j is eigenvalue, eigenfunction in $(N, g(\bar{t}))$. Then we only need to show that

$$\frac{r^{2p_j+n}|_{L^k}^{L^{k+1}}}{r^n|_{L^k}^{L^{k+1}}} < \frac{1}{2}\left(L^{2d_0}\frac{r^{2p_j+n}|_{L^{k-1}}^{L^k}}{r^n|_{L^{k-1}}^{L^k}} + L^{-2d_0}\frac{r^{2p_j+n}|_{L^{k+1}}^{L^{k+2}}}{r^n|_{L^{k+1}}^{L^{k+2}}}\right), \tag{3.9}$$

or

$$\frac{r^{2q_j+n}|_{L^{k+1}}^{L^k}}{r^n|_{L^k}^{L^{k+1}}} < \frac{1}{2}\left(L^{2d_0}\frac{r^{2q_j+n}|_{L^k}^{L^{k-1}}}{r^n|_{L^{k-1}}^{L^k}} + L^{-2d_0}\frac{r^{2q_j+n}|_{L^{k+2}}^{L^{k+1}}}{r^n|_{L^{k+1}}^{L^{k+2}}}\right), \tag{3.10}$$

for $q_j = -(n-2)-p_j < \frac{n}{2}$. Or for some p_j with $2(n-2)+2p_j = n$

$$\frac{\ln r|_{L^k}^{L^{k+1}}}{r^n|_{L^k}^{L^{k+1}}} < \frac{1}{2}\left(L^{2d_0}\frac{\ln r|_{L^{k-1}}^{L^k}}{r^n|_{L^{k-1}}^{L^k}} + L^{-2d_0}\frac{\ln r|_{L^{k+1}}^{L^{k+2}}}{r^n|_{L^{k+1}}^{L^{k+2}}}\right). \tag{3.11}$$

(3.9)-(3.11) can be reduced to

$$1 < \frac{1}{2}(L^{2d_0-2p_j} + L^{-2d_0+2p_j}), \tag{3.12}$$

or

$$1 < \frac{1}{2}(L^{2d_0-2q_j} + L^{-2d_0+2q_j}). \tag{3.13}$$

From our assumption we know $d_0 \neq q_j$, $d_0 \neq p_j$, then (3.12) and (3.13) are true.

Now we prove Lemma 3.2 by contradictions. Suppose there exists a subsequence $\{l_k\}$, such that on annuli $A_{L^{l_k-1},L^{l_k}}$, $A_{L^{l_k},L^{l_k+1}}$ and $A_{L^{l_k+1},L^{l_k+2}}$, inequality (3.7) is not true. Then we apply Lemma 3.1 on $A_{L^{l_k-1},L^{l_k+2}}$, we deduce a contradiction by taking limit.

Lemma 3.3. *The limit u_0 in Lemma 3.1 has the following expansion in the metric cone $(1, L)\times_r (N, g(\bar{t}))$*

$$u_0 = \sum a_j r^{p_j(\bar{t})}\phi_j(\bar{t}),$$

where $p_j(\bar{t}) \geq 0$ and $p_j(\bar{t})(p_j(\bar{t})+n-2) = \lambda_j(\bar{t})$.

Proof. From Lemma 3.2, we deduce that there exists k_1 for $k > k_1$

$$\fint_{A_{L^k,L^{k+1}}} u^2 > L^{2d_0} \fint_{A_{L^{k-1},L^k}} u^2. \tag{3.14}$$

Otherwise we have for all $k > k_0$

$$\fint_{A_{L^k,L^{k+1}}} u^2 < L^{2d_0} \fint_{A_{L^{k-1},L^k}} u^2. \tag{3.15}$$

We note that the convergence in (3.6) is true in $C^0((\delta, \delta^{-1}) \times_r (N, g(\bar{t}))$ where δ is any small positive constant. Then (3.15) is also true for u_0 on any two annuli in $(\delta, \delta^{-1}) \times_r (N, g(\bar{t})$. It follows that

$$u_0 = \sum b_j r^{-(n-2)-p_j} \phi_j. \tag{3.16}$$

Therefore there exists a ball B_{L^k} in M where u achieves its maximum in the interior of B_{L^k}. This contradicts the maximum principle. Then we apply (3.14) to its limit, Lemma 3.3 follows easily.

Now we go back to the function v_j. We have the following growth estimate.

Lemma 3.4. *Given $L > 1$, $\delta > 0$ and m_1 , let $p_{m^*} = \inf_{t\in[0,2\pi]} p_j(t)$, $p_M = \sup_{t\in[0,2\pi]} p_j(t)$, there exist R and a division I_0, for $a > R$, and if*

$$\fint_{A_{a,La}} v_j^2 \geq \frac{1}{3m_1} \fint_{A_{a,La}} u^2, \tag{3.17}$$

then

$$\fint_{A_{La,L^2a}} v_j^2 \geq L^{2p_{m^*}-\delta} \fint_{A_{a,La}} v_j^2. \tag{3.18}$$

$$\fint_{A_{La,L^2a}} v_j^2 \leq L^{2p_M+\delta} \fint_{A_{a,La}} v_j^2. \tag{3.19}$$

Proof. we prove the lemma again by contradictions. Suppose there exists a $\delta > 0$, $L > 1$ and $\{a_k\}$, $a_k \to \infty$, for any fixed division I, we have

$$\fint_{A_{La_k,L^2a_k}} v_j^2 \leq L^{2p_{m^*}-\delta} \fint_{A_{a_k,La_k}} v_j^2 \tag{3.20}$$

and

$$\fint_{A_{La_k,L^2a_k}} v_j^2 \geq \frac{1}{3m_1} \fint_{A_{a_k,La_k}} u \tag{3.21}$$

Furthermore, we may assume that there exists a $\bar{t}$, $t_{i-1} < \bar{t} \leq t_i$, such that $\lim_{k\to\infty} d_{GH}(A^k_{1,L}, (1,L) \times_r (N, g(\bar{t})) = 0$, where $A^k_{1,L}$ is the rescaled annulus of A_{a_k,La_k}. By Lemma 3.1, we may assume $\frac{u}{|u|_{L^\infty(A^k_{1,L})}} \to u_0$ and u_0 satisfies (3.13). If the division I is large enough so that

$$|\phi_j(t_l) - \phi_j(\bar{t})|_{L^\infty(N)} \leq \frac{\epsilon}{vol(N)}, \qquad l = i-1, i.$$

And then

$$\left| \fint_{(N,g(\bar{t}))} \phi_j(t_l)\phi_j(\bar{t}) - 1 \right| < \epsilon, \qquad l = i-1, i,$$

$$\left| \fint_{(N,g(\bar{t}))} \phi_j(t_l)\phi_\alpha(\bar{t}) \right| < \epsilon, \qquad l = i-1, i, \quad \alpha \neq j.$$

Then (3.20) deduce

$$\begin{aligned} &\frac{1}{vol(A_{L,L^2})} \int_L^{L^2} a_j^2 r^{2p_j(\bar{t})+n-1} dr - \epsilon \fint_{A_{L,L^2}} u_0^2 \\ &\leq L^{2p_{m^*}-\delta} \left[\frac{1}{vol(A_{1,L})} \int_1^L a_j^2 r^{2p_j(\bar{t})+n-1} dr + \epsilon \fint_{A_{1,L}} u_0^2 \right]. \end{aligned} \tag{3.22}$$

And (3.21) deduce

$$\frac{1}{vol(A_{1,L})} \int_1^L a_j^2 r^{2p_j(\bar{t})+n-1} dr \geq \left(\frac{1}{3m_1} - \epsilon \right) \fint_{A_{1,L}} u_0^2. \tag{3.23}$$

Since $p_j(\bar{t}) \geq p_{m^*}$ for ϵ sufficiently small, (3.22) and (3.23) give a contradiction. Similarly we can prove (3.19).

Now let $\lambda_d = d(d+n-2)$ and j_{m_0} be the first integer satisfying

$$\min_{j > j_{m_0}, 0 \leq t \leq 2\pi} \lambda_j(t) > \lambda_d + \frac{\delta}{2}. \tag{3.24}$$

Let $\lambda_{d_1} = \max_{j \leq j_{m_0}, 0 \leq t \leq 2\pi} \lambda_{j_{m_0}}(t)$, $\lambda_{d_1} = d_1(d_1+n-2)$ with $d_1 > 0$. Choose $\lambda_{d_2} = \lambda_{d_1} + \delta$ and d_2 satisfying

$$\lambda_{d_2} = d_2(d_2+n-2) \qquad d_2 > 0. \tag{3.25}$$

As we mentioned before, let $\lambda_{j_0}(t), \lambda_{j_1}(t), \cdots \lambda_{j_{m_1}}(t)$ be all the eigenvalues which intersects with λ_{d_2} for some $t \in [0, 2\pi]$. Obviously $j_0 > j_{m_0}$ and then

$$\min_{0 \le l \le m_1, 0 \le t \le 2\pi} \lambda_{j_l}(t) > \lambda_d.$$

Put $v = \sum_{j_0 \le j \le j_m} v_j$. It is easy to check that any $\epsilon > 0$ there exists R_0 for $a > R$

$$\left| \fint_{A_{a,La}} u^2 - \fint_{A_{a,La}} (u-v)^2 - \fint_{A_{a,La}} v^2 \right| \le \epsilon \fint_{A_{a,La}} u^2. \tag{3.26}$$

Since u is at most order d growth, from Lemma 3.4 and (3.26) we deduce that there exists R_1 for $a > R_1$

$$\fint_{A_{a,La}} (u-v)^2 > \fint_{A_{a,La}} u^2. \tag{3.27}$$

For $u - v$ we have the following growth estimate which is also called the three annulus lemma.

Lemma 3.5. *Let $L > 1$ and d_2 be given as before, then there exist k_0 and a division I_0 for $k > k_0$*

$$\begin{aligned} \fint_{A_{L^k,L^{k+1}}} & (u-v)^2 \\ & < \frac{1}{2}\left[L^{2d_2} \fint_{A_{L^{k-1},L^k}} (u-v)^2 + L^{-2d_2} \fint_{A_{L^{k+1},L^{k+2}}} (u-v)^2 \right], \end{aligned} \tag{3.28}$$

therefore if

$$\fint_{A_{L^k,L^{k+1}}} (u-v)^2 > L^{2d_2} \fint_{A_{L^{k-1},L^k}} (u-v)^2,$$

then

$$\fint_{A_{L^{k+1},L^{k+2}}} (u-v)^2 > L^{2d_2} \fint_{A_{L^k,L^{k+1}}} (u-v)^2.$$

Proof. We prove this lemma again by contradictions. Suppose that there exists a sequence $\{k_l\}$, $k_l \to \infty$, such that (3.28) is not true for any given I_0. As in the proof of Lemma 3.4, we may assume

$$\lim_{k \to \infty} d_{GH}(A^{k_l}_{1,L}, (1, L) \times_r (N, g(\bar{t})) = 0,$$

with $t_{i-1} < \overline{t} \le t_i$, and $\frac{u}{|u|_{L^\infty(A^{k_l}_{1,L^3})}} \to u_0$, u_0 satisfying (3.13). If I is large enough so that

$$|\phi_j(t_l) - \phi_j(\overline{t})|_{L^\infty(N)} \le \frac{\epsilon}{m_1 vol(N)}, \qquad l = i-1, i, \quad j = j_0, j_1, \cdots j_{m_1},$$

and then we deduce

$$\begin{aligned} (3.29) \qquad & \sum_{j \ne j_0, \cdots, j_{m_1}} a_j^2 \fint_{A_{L,L^2}{}^\infty} r^{2p_j(\overline{t})+n-1} \phi_j^2(\overline{t}) - \epsilon \fint_{A^\infty_{L,L^2}} u_0^2 \\ & \ge \frac{1}{2} \Bigg[L^{2d_2} \Bigg(\sum_{j \ne j_0, \cdots, j_{m_1}} a_j^2 \fint_{A_{1,L}{}^\infty} r^{2p_j(\overline{t})+n-1} \phi_j^2(\overline{t}) + \epsilon \fint_{A^\infty_{1,L}} u_0^2 \Bigg) \\ & \quad + L^{-2d_2} \Bigg(\sum_{j \ne j_0, \cdots, j_{m_1}} a_j^2 \fint_{A_{L^2,L^3}{}^\infty} r^{2p_j(\overline{t})+n-1} \phi_j^2(\overline{t}) + \epsilon \fint_{A^\infty_{L^2,L^3}} u_0^2 \Bigg) \Bigg], \end{aligned}$$

where $A^\infty_{1,L} = (1, L) \times_r (N, g(\overline{t}))$. Since $d_2 \ne p_j(\overline{t})$ for $j \ne j_0, \cdots j_{m_1}$, for ϵ small (3.29) is impossible.

From Lemma 3.4 and Lemma 3.5, we can have the growth estimate for u.

Lemma 3.6. *If u is a harmonic function with growth order at most d, then for $L > 1$ fixed, there exists k_0, for $k > k_0$*

$$(3.30) \qquad \fint_{A_{L^k,L^{k+1}}} u^2 \le L^{2d_2} \fint_{A_{L^{k-1},L^k}} u^2.$$

Proof. By Lemma 3.4, we only need to show that (3.30) is true for $k > k_0$ and some large I for $u - v$. Form Lemma 3.5, if (3.30) is not true for $u - v$, then for $k > k_0$

$$(3.31) \qquad \fint_{A_{L^k,L^{k+1}}} (u-v)^2 > L^{2d_2} \fint_{A_{L^{k-1},L^k}} (u-v)^2.$$

By induction, we have

$$(3.32) \qquad \fint_{A_{L^k,L^{k+1}}} (u-v)^2 \ge L^{2d_2(k-k_0-2)} \fint_{A_{L^{k_0+1},L^{k_0+2}}} (u-v)^2.$$

We note $d_2 > d$, (3.26) and (3.27), then (3.32) is impossible.

We actually proved the following growth estimates for harmonic functions.

Theorem 3.7. *Under the assumptions of* (3.1), (S), *and Ricc*(M) *is bounded from below by* $-c/r^2$, *where* c *is a positive constant and* r *is the distance function from some fixed point,* $A_{L^k,L^{k+1}}$ *is the annulus with* $L^k \leq r \leq L^{k+1}$, *then there exist* $C = C(M)$, $k_0 = k_0(M)$ *such that for any at most order* d *growth harmonic function* u, *we have*

$$\fint_{A_{L^k,L^{k+1}}} u^2 \leq L^{2Cd(k-k_0-2)} \fint_{A_{L^{k_0+1},L^{k_0+2}}} u^2,$$

for $k > k_0$.

Now we have the following corollary.

Theorem 3.8. *Under the assumptions of* (3.1), (S), *and Ricc*(M) *is bounded from below by* $-c/r^2$, *where* c *is a positive constant and* r *is the distance function from some fixed point, then the dimension of at most order* d *growth harmonic functions is not more than the dimension of at most order* Cd *growth harmonic functions on its tangent cone at infinity. Here* $C = C(M)$ *is given in (2.25).*

Proof. Let u, w be two linear independent harmonic functions on M with growth order at most d.

Put $w_k = \gamma_k w$, $u_k = \alpha_k u - \beta_k w$ such that

$$\fint_{A_{L^{k-1},L^{k+2}}} {w_k}^2 = 1,$$

$$\fint_{A_{L^{k-1},L^{k+2}}} {u_k}^2 = 1,$$

$$\fint_{A_{L^{k-1},L^{k+2}}} w_k u_k = 0.$$

We may assume as before, there exists a subsequence $\{k_l\}$ so that

$$\lim_{l\to\infty} d_{GH}(A^{k_l}_{L^{-1},L^2}, (L^{-1}, L^2) \times_r (N, g(\bar{t})) = 0. \tag{3.33}$$

And both u_{k_l} and w_{k_l} has a limit $\overline{u}$ and $\overline{w}$ respectively. By lemma 3.3 and lemma 3.6, we can deduce as before

$$\overline{u} = \sum_{0\leq p_j\leq d_2} a_j r^{p_j} \phi_j. \tag{3.34}$$

$$\overline{w} = \sum_{0 \le p_j \le d_2} c_j r^{p_j} \phi_j. \tag{3.35}$$

Moreover

$$\fint_{A^{\infty}_{L^{-1},L^2}} \overline{uw} = 0. \tag{3.36}$$

Then our theorem follows easily from (3.34)-(3.36).

Corollary 3.9. *In addition to the assumption of theorem 3.8, we assume that $Ricc(N) \ge 0$, then the dimension of at most order d growth harmonic functions is not more than $C[d(d+n-3)]^{(n-1)/2}$ where C is a constant dependent of M.*

In fact by a result of Li and Yau [LY2], we have

$$\lambda_k \ge \frac{C_1(n)}{diam(N)} k^{\frac{2}{n-1}}. \tag{3.37}$$

And $\lambda_k(t) \le C_2(M)\lambda_k(0)$, then we deduce our corollary easily.

In general, if the Ricc curvature of the cross section is only lower bounded, there are also some estimates on the eigenvalues, (see [SY]) so is the estimate of the dimension of the harmonic functions on M.

Remark 3.10. In the discussion of theorem 3.8, in fact, the assumption that $Ricc(M)$ is lower bounded is not necessary. Our argument, however, can be also applied to the case when the cross section of M has singularities which will be discussed later.

References.

[CC1] J. Cheeger and T.H. Colding, *Lower bounds on the Ricci curvature and the almost rigidity if warped products*, preprint.

[CC2] J. Cheeger and T.H. Colding, *Almost rigidity of warped products and the structure of space with Ricci curvature bounded below*, C. R. Acad. Sci. Paris **320**, serie 1 (1995), 353–357.

[CCM] J. Cheeger, T.H. Colding, and W.P. Minicozzi II, *Linear growth harmonic functions on complete manifolds with nonegative Ricci curvature*, GAFA.

[CCT] J. Cheeger, T.H. Colding, and G, Tian, *On the singularities of space with Ricci curvature bounded.*

[CT] J. Cheeger and G. Tian, *On the cone structure at infinity of Ricci flat manifolds with Euclidean volume growth and quadratic curvature decay*, Invent. Math. **118** (1994), 493–571.

[CY] S.Y. Cheng and S.T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975), 333–354.

[C] T.H. Colding, *Ricci curvature and volume convergence*, preprint.

[CM] T.H. Colding and W.P. Minicozzi II, *Harmonic functions with polynomial growth*, preprint.

[G] M. Gromov, *Paul Levy's isoperimetric inequality.*

[L] F.H. Lin, *Asymptotically conic elliptic operators and Liouville type theorems*, preprint.

[LT1] P. Li and L-F. Tam, *Linear growth harmonic functions on a complete manifold*, J. Diff. Geom. **29** (1989), 421–425.

[LT2] P. Li and L-F. Tam, *Complete surface with finite total curvature*, J. Diff. Geom. **33** (1991), 139–168.

[LY1] P. Li and S.T. Yau, *On the parabolic kernel of the Schordinger operator*, Acta. Math. **156** (1986), 153-201.

[LY2] P. Li and S.T. Yau, *Eigenvalues of a compact Riemannian manifold*, AMS Symposium on Geometry of the Laplace Operator, XXXVI, Hawaii (1979), 205-240.

[P] G. Perelman, *Manifolds of positive Ricci curvature with almost maximal volume*, JAMS, **7** (1994), 299–305.

[QT] J. Qing and G. Tian, *Necks between bubbles*, preprint.

[S] L. Simon, *Asymptotics for a class of non-linear evolution equations with applications to geometric problems*, Annals of Math. **118** (1983), 525–571.

[SY] R. Schoen and S.T.Yau, *Differential Geometry*, Sci. Acd. press (1988) (in chinese).

[TY] G. Tian and S.T.Yau, *Complete Kahler manifolds with zero Ricci curvature II*, Invent. Math. **106** (1991), 27–60.

[Y1] S.T.Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure and Appl. Math. **28** (1975), 201–228.

[Y2] S.T. Yau, *Open problems in geometry*, Proceeding of Symposia in Pure Math. **54**, Part 1, Ed R. Greene and S.T. Yau.

[Y3] S.T. Yau, *Open problems in geometry*, *Chern– A Great Geometer of the Twentieth Century*, Ed. S.T. Yau.

[U] K. Uhlenbeck, *Generic properties of eigenfunctions*, Amer. J. of Math. **9** (1976), 1059–1078.

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