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Gradient Estimation on Navier-Stokes Equations

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In this note, we present a prior uniform gradient estimates on solutions to the 3-dimensional Navier-Stokes equations. It is shown that the gradient of the velocity field is locally uniformly bounded in L^∞-norm provided that either the scaled local L^2-norm of the vorticity or the scaled local total energy is small. In particular, our results imply that the smooth solutions to 3-dimensional Navier-Stokes equations cannot develop finite time singularity and suitable weak solutions are in fact regular if either the scaled local L^2-norm of the vorticity or the scaled local energy is small.

1. Introduction.

The study of the incompressible Navier-Stokes equations in three space dimensions has a long history. In the pioneering works [Le], [Ho], Leray and Hopf proved the existence of its weak solutions with initial and boundary conditions. However, we do not know yet whether or not the solution develops singularities in finite time even if all the data, such as initial and boundary conditions, are C^∞-smooth.

In [Sch], V. Scheffer began to study the partial regularity theory of the Navier-Stokes equations. Deeper results were obtained by L. Caffarelli, R. Kohn and L. Nirenberg in [CKN]. They proved a local partial regularity theorem for a particular class of weak solutions. They showed that, for any such weak solution, the singular set has one-dimensional Hausdorff measure zero. In particular, their local regularity theorem implies that there is an $\epsilon > 0$ satisfying: for any suitable weak solution u of Navier-Stokes equations, if

$$\overline{\lim}_{r\to 0} r^{-1} \int_{|y-x|<r, |s-t|<r^2} |\nabla u|^2 \leq \epsilon, \tag{1.1}$$

then u is regular near (x,t). Recently, F.H. Lin and Liu gave a simplified proof of the main results in [CKN], see [LL].

This note grows out of our efforts in understanding [CKN]. Here we present some new necessary and sufficient conditions for the regularity of the solutions to Navier-Stokes system. One of our observations is that instead of the condition (1.1), the local behavior of the solutions to the Navier-Stokes equations is dominated by the scaled local L^2-norm of the vorticity. More precisely, first we will show that there is a small positive number ε such that, for any smooth solution u of the Navier-Stokes, if

$$\sup_{r\leq r_0}\left(r^{-1}\int_{\mathbb{B}_r(x_0,t_0)}|\operatorname{curl} u|^2dxdt\right)<\varepsilon \tag{1.2}$$

where $\mathbb{B}_r(x_0,t_0)$ denotes the parabolic ball with radius r and center at (x_0,t_0) (cf. section 3), then $r^2|\nabla u|$ is uniformly bounded in $\mathbb{B}_{r/2}(x_0,t_0)$ for $r \leq r_1$. Another main observation of this note is that instead of the smallness assumption (1.1), the regularity of the solution is guaranteed by the requirements that either $r^{-1}\iint_{\mathbb{B}_r(x_0,t_0)}|\nabla u|^2dxdt$ or $\sup_{t_0-r^2\leq t<t_0} r^{-1}\int_{B_r(x_0)}|u(x,t)|^2dx$ is uniformly bounded and the scaled local energy is small, i.e.,

$$\sup_{r\leq r_0}\left(r^{-3}\int_{\mathbb{B}_r(x_0,t_0)}|u|^2dxdt\right)<\varepsilon. \tag{1.3}$$

Finally, we mention that our estimate also leads to the following observation that any suitable weak solution u of the Navier-Stokes equation will be regular at (x_0,t_0) if the local scaled L^3-norm of the velocity is suitably small, i.e.

$$\overline{\lim}_{r\to 0^+}r^{-2}\iint_{\mathbb{B}_r(x_0,t_0)}|u|^3dxdt\leq\varepsilon \tag{1.4}$$

for a uniform small positive number ε.

Our proof seems to work for generalized Navier-Stokes equations of any space dimensions. As an example, we will show a local partial regularity theorem for stationary Navier-Stokes equations of any dimensions (cf. section 2). The regularity theorem of this sort was previously proved by M. Struwe for the stationary Navier-Stokes equations of dimension five [Str1]. As shown later by M. Struwe [Str2], such regularity results can be used to construct smooth solutions of the stationary Navier-Stokes equations in $\mathbb{R}^5$ or with space periodic boundary conditions in the dimension 5. It should be noted that there are a lot of literatures on the studies of solutions to the stationary Navier-Stokes system in higher space dimensions. In particular, Frehse and Ruzicka have proved the existence of smooth solutions to the stationary Navier-Stokes system with space preriodic conditions for dimensions up to 15, see [FR1]. There are also various partinal regularity results available, see [FR2] and the references therein.

In the following, we will first study the stationary Navier-Stokes equation in section 2 and then the incompressible Navier-Stokes equations for three space dimensions in section 3.

2. Gradient Estimate on Stationary Navier-Stokes Equations.

In this section we are interested in the local behavior of solutions to stationary Navier-Stokes equations in a smooth open domain in $\mathbb{R}^n$. Thus let Ω be a domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and u and p be smooth solutions to the equations

$$-\Delta u + (u \cdot \nabla)u + \nabla p = 0 \qquad \text{in } \Omega \tag{2.1}$$

$$\operatorname{div} u = 0 \qquad \text{in } \Omega \tag{2.2}$$

with the bound

$$\int_\Omega |u(x)|^2 \, dx \leq M_0 < +\infty \tag{2.3}$$

where M_0 is an absolute positive constant. For $x \in \mathbb{R}^n$ we denote $B_R(x) = \{y \in \mathbb{R}^n, |y-x| < R\}$. In the case of no confusion, we will skip the center of the ball from the notations and write simply B_R. Denote the vorticity by $w(x) =^* du(x)$, where the notation is that of exterior calculus. The main result of this section is the following a priori estimate on the solution $u(x)$.

Theorem 2.1. *There exists an absolute constant $\varepsilon_0 > 0$ such that the following statement is true: Let $u(x)$ and $p(x)$ be smooth solutions to the stationary Navier-Stokes equation* (2.1)–(2.2) *satisfying* (2.3) *and assume that there is a R_0 such that on a small neighborhood*

$$R^{-(n-4)} \int_{B_R} |{}^*du(x)|^2 \, dx \leq \varepsilon_0 \qquad \textit{for all } R \leq R_0 . \tag{2.4}$$

Then there exists a constant $R_1 < R_0$ such that

$$\sup_{B_{R/2}} |\nabla u| \leq C R^{-2} \quad \textit{for all} \quad R \leq R_1, \tag{2.5}$$

with C an absolute constant.

The key step in the proof of Theorem 2.1 is the following proposition.

Proposition 2.2. *For any $\varepsilon_1 > 0$, there exists as absolute constant δ such that if* (2.4) *holds with $\varepsilon_0 \le \delta$ for some $R_0 > 0$, then there exists a positive constant $R_1 = R_1(R_0) < R_0$ such that*

$$h(R) \equiv \frac{1}{R^{n-2}} \int_{B_R} |u(x)|^2 dx \le \varepsilon_1 \quad \textit{for all} \quad R \le R_1. \tag{2.6}$$

Proposition 2.2 will be a simple consequence of the following lemma.

Lemma 2.3. *Assume* (2.4) *holds. Then for any $\lambda \in \left(0, \frac{1}{2}\right]$ and $\rho \le R_0$, the following inequality holds*

$$h(R) \equiv h(\lambda\rho) \le 2^{n+2}\lambda^2 h(\rho) + c(\lambda)\varepsilon_0 \tag{2.7}$$

with $c(\lambda)$ being a positive constant given by

$$c(\lambda) = \frac{4\lambda^2}{(1-\lambda)^n} \frac{4}{n^2} + \frac{2}{n^2}\left(1 + \frac{1}{\lambda}\right)^2. \tag{2.8}$$

Proof of Proposition 2.2. It should be clear that the Proposition 2.2 follows from Lemma 2.3 by iteration. Indeed, first we fix a $\lambda \in \left(0, \frac{1}{2}\right]$ as that $\mu = 2^{n+2}\lambda^2 < 1$, and iterate inequality (2.7) k times to obtain

$$h(\lambda^k R_0) \le \mu^k h(R_0) + \frac{1-\mu^k}{1-\mu} c(\lambda)\varepsilon_0 \quad \text{for} \quad k = 1, 2, \dots . \tag{2.9}$$

Now for any given $\varepsilon_1 > 0$, we choose ε_0 so that

$$\frac{1}{1-\mu} c(\lambda)\varepsilon_0 < \frac{1}{2}\varepsilon_1 \lambda^{n-2}. \tag{2.10}$$

Next, we choose an integer K_0 so that

$$\mu^{K_0} h(R_0) = \frac{\mu^{K_0}}{R_0^{n-2}} M_0 < \frac{1}{2}\varepsilon_1 \lambda^{n-2}. \tag{2.11}$$

We define $R_1 = \lambda^{K_0} R_0$. Now, for any $0 < R \le R_1$, there exists a $k \le K_0$ so that $\lambda^{k+1} R_0 \le R \le \lambda^k R_0$. Thus

$$\begin{aligned} h(R) < \frac{1}{\lambda^{n-2}} h(\lambda^k R_0) &\le \left[\mu^k h(R_0) + \frac{1-\mu^k}{1-\mu} c(\lambda)\varepsilon_0\right] \frac{1}{\lambda^{n-2}} \\ &\le \left[\mu^{k-K_0}\mu^{K_0} h(R_0) + \frac{1}{1-\mu} c(\lambda)\varepsilon_0\right] \frac{1}{\lambda^{n-2}} < \varepsilon_1 \end{aligned}$$

which completes the proof of Proposition 2.2. □

It thus suffices to prove Lemma 2.3. Due to the translation invariance property of the Navier-Stokes equations, one can assume that the balls are centered at origin, and for simplicity of presentation, $\rho \equiv 1$, and

$$\frac{1}{R^{n-4}} \int_{B_R} |{}^*du|^2 \, dx \leq \varepsilon_0 \qquad \text{for all } R \leq 1 \,. \tag{2.12}$$

(The general case follows a similar estimate).It follows from the Biot-Sawart law that

(2.13)

$$u^*(x) = \int_{B_1} \nabla\Gamma(x-y) \wedge w(y) \, dy + H(x) \qquad \text{for all } x \in B_1 \,,$$

where $\Gamma(x)$ is the standard normalized fundamental solution of Laplace's equation in $\mathbb{R}^n$, and H is a harmonic function in $B_1(0)$. Now set $R = \lambda$, $\lambda \in \left(0, \frac{1}{2}\right]$. By the mean value property of a harmonic function, one can show easily that

(2.14)

$$\frac{1}{R^{n-2}} \int_{B_R(0)} |H(x)|^2 \, dx \leq \frac{\lambda^2}{(1-\lambda)^n} \int_{B_1(0)} |H(x)|^2 \, dx \,.$$

To estimate the integral on the right hand side of (2.14), we first derive a bound on $A(x) \equiv \int_{B_1(0)} \nabla\Gamma(x-y) \wedge w(y) \, dy$ in terms of L_2-norm of the vorticity. It follows from the standard argument for convolution operator that

$$\int_{B_1(0)} |A(x)|^2 \, dx \leq \left(\int_{B_2(0)} |\nabla\Gamma(z)| \, dz \right)^2 \left(\int_{B_1(0)} |w(\xi)|^2 \, dz \right) .$$

But

$$\int_{B_2(0)} |\nabla\Gamma(z)| \, dz \leq \frac{2}{n} ,$$

and so

$$\int_{B_1(0)} |A(x)|^2 \, dx \leq \frac{4}{n^2} \int_{B_1(0)} |w(z)|^2 \, dz \leq \frac{4}{n^2} \varepsilon_0 \,. \tag{2.15}$$

As a consequence of (2.13–2.14), we obtain that

$$(2.16)\quad \frac{1}{R^{n-2}}\int_{B_R(0)}|H(x)|^2\,dx$$
$$\le \frac{2\lambda^2}{(1-\lambda)^n}\int_{B_1(0)}|u(x)|^2\,dx+\frac{2\lambda^2}{(1-\lambda)^n}\cdot\int_{B_1(0)}|A(x)|^2\,dx$$
$$\le \frac{2\lambda^2}{(1-\lambda)^n}\int_{B_1(0)}|u(x)|^2\,dx+\frac{2\lambda^2}{(1-\lambda)^n}\frac{4}{n^2}\varepsilon_0\,.$$

Next, we estimate $R^{-(n-2)}\int_{B_R(0)}|A(x)|^2dx$. A simple estimate shows

$$\int_{B_R(0)}|A(x)|^2\,dx\le\left(\int_{B_{1+R}(0)}|\nabla\Gamma(z)|\,dz\right)^2\cdot\int_{B_R(0)}|w(x)|^2\,dx$$
$$\le\frac{1}{n^2}(1+R)^2\int_{B_R(0)}|w(x)|^2\,dx$$
$$\le\frac{1}{n^2}(1+R)^2R^{n-4}\varepsilon_0\,,$$

where we have used the assumption (2.12). Thus

$$(2.17)\qquad \frac{1}{R^{n-2}}\int_{B_R(0)}|A(x)|^2\,dx\le\frac{1}{n^2}\left(1+\frac{1}{\lambda}\right)^2\varepsilon_0\,.$$

It follows from (2.13), (2.16) and (2.17) that

$$(2.18)\quad \frac{1}{R^{n-2}}\int_{B_R(0)}|u(x)|^2\,dx$$
$$\le\frac{2}{R^{n-2}}\int_{B_R(0)}|A(x)|^2\,dx+\frac{2}{R^{n-2}}\int_{B_R(0)}|H(x)|^2\,dx$$
$$\le C_0\lambda^2\int_{B_1(0)}|u(x)|^2\,dx+c(\lambda)\varepsilon_0$$

with C_0 and $c(\lambda)$ given by

(2.19)
$$C_0=\frac{4}{(1-\lambda)^n}\,,\quad\text{and}\quad c(\lambda)=\frac{4\lambda^2}{(1-\lambda)^n}\frac{4}{n^2}+\frac{2}{n^2}\left(1+\frac{1}{\lambda}\right)^2.$$

This completes the proof of Lemma 2.3.

We now turn to the proof of Theorem 2.1. Due to the scaling property of the Navier-Stokes equations, one needs only to show Theorem 2.1 with $R=1$. Next, we observe that it suffices to bound the vorticity, i.e.

Lemma 2.4. *Assume (2.3). Then*

$$\sup_{x\in B_{\frac{1}{2}}(0)} |\nabla u(x)| \le C_1$$

if $\sup_{x\in B_1(0)} |{*du}(x)| \le C_2$ *for two absolute constants* C_1 *and* C_2.

Proof. Recall the notation that $w(x) =^* du(x)$. This lemma follows from

$$w(x) = \left(\nabla\Gamma * \left((n-1)w\wedge u\right)\right)(x) + H_1(x) \quad \text{for all } x\in B_{\frac{1}{2}}(0) \tag{2.20}$$

with the convolution integral over $B_1(0)$ and H_1 being a harmonic function on $B_1(0)$, and the representation formula (2.13) with $B_1(0)$ by the standard elliptic regularity argument (see [Se1] and [Mo]). We just sketch it here for completeness. Indeed, since $w\in L^\infty(B_{\frac{1}{2}})$ and $u\in L^2(B_{\frac{1}{2}})$ (Proposition 2.2), one gets that $(n-1)w\wedge u\equiv g_1\in L^2(B_{\frac{1}{2}})$. This, $w\in L^2(B_{\frac{1}{2}})$ and (2.20) give that $H_1\in L^2(B_{\frac{1}{2}}(0))$. Thus, $H_1\in L^\infty(B_{\frac{1}{2}})$ since it is harmonic. Next, similar argument using (2.13) shows that $H\in L^\infty(B_{\frac{1}{2}})$. It follows from this, (2.13), and $w\in L^\infty(B_{\frac{1}{2}})$ that $u^*\in L^\infty(B_{\frac{1}{2}})$. Consequently $g_1=(n-1)w\wedge u\in L^\infty(B_{\frac{1}{2}})$. Then (2.20) again implies that w is Hölder continuous, and (2.13) in turn shows that ∇u is also Hölder continuous. In particular, ∇u is uniformly bounded. Writing out all the estimates corresponding to these statements gives the desired Lemma 2.4. □

We are now in the position to present the main argument to conclude Theorem 2.1.

Proposition 2.5. *Under the same assumptions as in Theorem* 2.1, *there exists a positive constant* C_3 *such that*

$$\sup_{x\in B_{\frac{R}{2}}(0)} |^*\, du(x)| \le C_3 R^{-2} \quad \textit{for all} \quad R\le R_1 \tag{2.21}$$

Proof of Proposition 2.5. It suffices to prove (2.21) for $R=1$. We will use a similar argument as in the study of partial regularity of harmonic maps [Sc]. Define $x_1\in \overset{\circ}{B}_1(0)$ (interior of the ball $B_1(0)$) and e_1 as follows

$$\sup_{x\in B_1(0)}(1-|x|)^2\,|w(x)| = (1-|x_1|)^2 e_1\,, \tag{2.22}$$

so $e_1 \equiv |w(x_1)|$. By definition, one has

$$|w(x)| \leq (2(1-|x_1|))^2 e_1 \qquad \forall x \in B_{\frac{1}{2}}(0)\,, \tag{2.23}$$

and

$$|w(x)| \leq 4e_1 \qquad \forall x \in B_{\frac{1}{2}(1-|x_1|)}(x_1) \tag{2.24}$$

The solution $u(x)$ on the ball $B_{\frac{1}{2}(1-|x_1|)}(x_1)$ can be rescaled as follows

$$v(y) = \frac{1}{\sqrt{e_1}} u\left(x_1 + \frac{y}{\sqrt{e_1}}\right) \qquad \forall y \in B_{\frac{\sqrt{e_1}}{2}(1-|x_1|)}(0) \tag{2.25}$$

Then $v(y)$ is a solution to the stationary Navier-Stokes equations, and furthermore, its vorticity

(2.26)
$$\tilde{w}(y) \equiv^* dv(y) = \frac{1}{e_1} w\left(x_1 + \frac{y}{\sqrt{e_1}}\right) \qquad \forall y \in B_{\frac{\sqrt{e_1}}{2}(1-|x_1|)}(0)$$

satisfies the following equations (see [Se1])

$$\begin{cases} -\nabla^2 \tilde{w} & = \operatorname{div} g \\ \operatorname{div} v & = 0 \end{cases} \qquad \text{in } B_{\frac{\sqrt{e_1}}{2}(1-|x_1|)}(0)\,, \tag{2.27}$$

where $g = (n-1)\tilde{w} \wedge v$. We then claim:

Lemma 2.6. *There exists an absolute constant C_4 such that*

$$\sqrt{e_1}(1-|x_1|) \leq C_4\,. \tag{2.28}$$

Assuming (2.28) for a moment, we conclude from (2.23) and (2.28) that

$$|w(x)| \leq (2C_4)^2 \equiv C_3 \qquad \forall x \in B_{\frac{1}{2}}(0)\,.$$

which completes the proof of Proposition 2.5. □

It remains to show Lemma 2.6.

Proof of Lemma 2.6. If (2.28) is not true, one can assume that

$$\sqrt{e_1}(1-|x_1|) > 4\,.$$

and so

$$B_2(0) \subset B_{\frac{\sqrt{e_1}}{2}(1-|x_1|)}(0)\,. \tag{2.29}$$

It follows from this, (2.24), and (2.28) that

$$|\tilde{w}(y)| \le 4 \qquad \text{for all } y \in B_2(0) \tag{2.30}$$

Note that (2.27) now holds on any open domain $D \subset B_2(0)$. It follows that the following representations hold (see [Se1])

$$\tilde{w}(y) = (\nabla\Gamma * g)(y) + H_2(y)\,, \qquad \forall y \in D, \tag{2.31}$$

$$v^*(y) = \int_D \nabla\Gamma(y-\xi) \wedge \tilde{w}(\xi) d\xi + H_3(y)\,, \forall y \in D, \tag{2.32}$$

where both $H_2(y)$ and H_3 are harmonic on D. We now derive the desired contradiction by following the two steps:

Step 1 **L_∞-estimate on v**

Take D to be $\overset{\circ}{B}_2(0)$ and rewrite (2.32) as

$$\begin{aligned} v^*(y) &= \int_{\overset{\circ}{B}_2(0)} \nabla\Gamma(y-\xi) \wedge \tilde{w}(\xi)\, d\xi + H_3(y) \\ &\equiv A_3(y) + H_3(y) \\ &\qquad\qquad \forall y \in B_1(0)\,. \end{aligned} \tag{2.33}$$

We first estimate the harmonic part H_3. For any $y \in B_1(0)$,

$$\begin{aligned} |H_3(y)| &= \frac{1}{w_n}\left|\int_{B_1(y)} H_3(z)\, dz\right| \le \left(\frac{1}{w_n}\int_{B_1(y)} |H_3(z)|^2\, dz\right)^{\frac{1}{2}} \\ &\le \left(\frac{1}{w_n}\int_{B_1(y)} |v(z)|^2\, dz\right)^{\frac{1}{2}} + \left(\frac{1}{w_n}\int_{B_1(y)} |A_3(z)|^2\, dz\right)^{\frac{1}{2}} \\ &\equiv I_1 + I_2\,. \end{aligned}$$

Simple calculation shows

$$I_1 \equiv \left(\frac{1}{w_n}\int_{B_1(y)} |v(z)|^2\, dz\right)^{\frac{1}{2}}$$
$$\leq \left(\frac{1}{w_n}\int_{B_2(0)} |v(z)|^2\, dz\right)^{\frac{1}{2}}$$
$$= \frac{2^{\frac{n}{2}}}{w_n^{\frac{1}{2}}} h^{\frac{1}{2}} \left(\frac{2}{\sqrt{e_1}}\right),$$

while I_2 can be estimated in a same way as in (2.15):

$$I_2 \equiv \left(\frac{1}{w_n}\int_{B_1(y)} |A_3(z)|^2\, dz\right)^{\frac{1}{2}}$$
$$\leq \left(\frac{1}{w_n^{\frac{1}{2}}}\int_{B_2(0)} |\nabla\Gamma(z)|\, dz\right)\left(\int_{B_2(0)} |\tilde{w}(\xi)|^2\, d\xi\right)^{\frac{1}{2}}$$
$$\leq \frac{2}{n w_n^{\frac{1}{2}}}\varepsilon_0^{\frac{1}{2}}.$$

Consequently,

$$||H_3(\cdot)||_{L^\infty(B_1(0))} \leq \left(\frac{2^n}{w_n} h\left(\frac{2}{\sqrt{e_1}}\right)\right)^{\frac{1}{2}} + \left(\frac{2^2}{n w_n}\varepsilon_0\right)^{\frac{1}{2}}.$$

Next, choose $p \in (1, \frac{n}{n-1})$, and $q(>1)$ such that $p^{-1}+q^{-1}=1$. One has by Hölder inequality that for all $y \in B_1(0)$

$$|A_3(y)| = \left|\int_{B_2(0)} [\nabla\Gamma(y-\xi)\wedge\tilde{w}(\xi)]\, d\xi\right|$$
$$\leq \left(\int_{B_2(0)} |\nabla\Gamma(y-\xi)|^p\, d\xi\right)^{\frac{1}{p}}\left(\int_{B_2(0)} |\tilde{w}(\xi)|^q\, d\xi\right)^{\frac{1}{q}}$$
$$\leq ||\nabla\Gamma(\cdot)||_{L^p(B_3(0))}\, 4^{\frac{q-2}{q}}\, ||\tilde{w}(\cdot)||_{L^2(B_1(0))}^{\frac{2}{q}}$$
$$= 4^{1-\frac{2}{q}}\, ||\nabla\Gamma(\cdot)||_{L^p(B_3)}\, \varepsilon_0^{\frac{2}{q}},$$

where we have used (2.30). Thus,

$$||A_3(\cdot)||_{L^\infty(B_1(0))} \leq 4^{1-\frac{2}{q}}\, ||\nabla\Gamma(\cdot)||_{L^p(B_3)}\, \varepsilon_0^{\frac{2}{q}}.$$

We arrive at

(2.34)
$$||v(\cdot)||_{L^\infty(B_1(0))} \leq C_5\varepsilon_0^{\frac{2}{q}} + C_6\varepsilon_0^{\frac{1}{2}} + C_7\left(h\left(\frac{2}{\sqrt{e_1}}\right)\right)^{\frac{1}{2}}$$

with absolute constants C_5, C_6, and C_7 derived above.

Step 2 L_∞**-estimate of** $\tilde{w}$

This can be achieved in a similar way as in Step 1. Rewrite (2.31) as

(2.35)
$$\tilde{w}(y) \equiv A_2(y) + H_2(y) \qquad \text{for all } y \in B \equiv \overset{\circ}{B}_1(0)\,,$$

with

$$A_2(y) = \int_{B_1(0)} \nabla\Gamma(y-\xi)g(\xi)\,d\xi\,, \quad g \equiv (n-1)\tilde{w}\wedge v\,.$$

First, for $y \in B_{\frac{1}{2}}(0)$,

$$\begin{aligned}
||H_2(\cdot)||_{L^2(B_{\frac{1}{2}}(y))} &\leq ||\tilde{w}(\cdot)||_{L^2(B_{\frac{1}{2}}(y))} + ||A_2(\cdot)||_{L^2(B_{\frac{1}{2}}(y))} \\
&\leq \varepsilon_0^{\frac{1}{2}} + ||\nabla\Gamma(\cdot)||_{L^1(B_2(0))}\,||g(\cdot)||_{L^2(B_{\frac{1}{2}}(y))} \\
&\leq \varepsilon_0^{\frac{1}{2}} + (n-1)\,||v||_{L^\infty(B_1(0))}\,||\tilde{w}(\cdot)||_{L^2(B_{\frac{1}{2}}(y))} \\
&\leq C_8\varepsilon_0^{\frac{1}{2}}\,,
\end{aligned}$$

where C_8 is an absolute constant. Since H_2 is harmonic, so for $y \in B_{\frac{1}{2}}(0)$ one has

$$\begin{aligned}
|H_2(y)| &= \left|\frac{2^n}{w_n}\int_{B_{\frac{1}{2}}(y)} H_2(z)\,dz\right| \leq \frac{2^{\frac{n}{2}}}{w_n^{\frac{1}{2}}}\left(\int_{B_{\frac{1}{2}}(y)} |H_2(z)|^2\,dz\right)^{\frac{1}{2}} \\
&\leq \left(\frac{2^n}{w_n}\right)^{\frac{1}{2}} C_8\varepsilon_0^{\frac{1}{2}} \equiv C_9\varepsilon_0^{\frac{1}{2}}\,.
\end{aligned}$$

Thus

(2.36)
$$||H_2(\cdot)||_{L^\infty(B_{\frac{1}{2}}(0))} \leq \left(\frac{2^n}{w_n}\right)^{\frac{1}{2}} C_8\varepsilon_0^{\frac{1}{2}}\,.$$

On the other hand, for p and q as in Step 1, one gets

$$
\begin{aligned}
&||A_2(\cdot)||_{L^\infty(B_1(0))} \\
&\qquad \le ||\nabla\Gamma(\cdot)||_{L^p(B_2)}\, ||g(\cdot)||_{L^q(B_1(0))} \\
&\qquad \le ||\nabla\Gamma(\cdot)||_{L^p(B_2)}\, (n-1)\, ||v||_{L^\infty(B_1)}\, ||\tilde{w}(\cdot)||_{L^q(B_1)} \\
&\qquad \le (n-1)4^{1-\frac{2}{q}}\, ||v||_{L^\infty(B_1)}\, ||\nabla\Gamma(\cdot)||_{L^p(B_2)}\, \varepsilon_0^{\frac{q}{2}} \\
&\qquad \le C_{10}\, ||v||_{L^\infty(B_1)}\, \varepsilon_0^{\frac{q}{2}} .
\end{aligned}
$$

It follows that

$$
||\tilde{w}(\cdot)||_{L^\infty(B_{\frac{1}{2}}(0))} \le C_{10}\, ||v||_{L^\infty(B_1)}\, \varepsilon_0^{\frac{q}{2}} + C_9\varepsilon_0^{\frac{1}{2}} . \tag{2.37}
$$

In particular, $1 = |\tilde{w}(0)| \le C_{10}\, ||v||_{L^\infty(B_1)}\, \varepsilon_0^{\frac{q}{2}} + C_9\varepsilon_0^{\frac{1}{2}}$, which yields the desired contradiction due to Proposition 2.2. Thus Lemma 2.6 is proved.

□

3. Estimates on Navier-Stokes Equations.

In this section we intend to present some spatial gradient estimates on solutions of the time dependent Navier-Stokes equations provided that either the scaled local total energy or the scaled local total vorticity is suitably small. Though the analysis can be carried out for arbitrary number of spatial dimensions, we will concentrate on the case of three spatial dimension.

Let Ω be a domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, and T be any fixed positive constant. Set $D = \Omega \times [0, T]$ (here Ω may be unbounded). The 3-D Navier-Stokes equations may be written in the form

$$
\partial_t u - \Delta u + \operatorname{div}(u \otimes u) + \nabla p = 0 \qquad \text{in } D\,, \tag{3.1}
$$

$$
\operatorname{div} u = 0\,, \tag{3.2}
$$

which are accompanied by the following initial and boundary conditions (for example)

$$
u|_{t=0} = u_0\,, \qquad u|_{\partial\Omega\times[0,T]} = 0\,. \tag{3.3}
$$

We will consider any smooth solution (u,p) to (3.1)–(3.2). which satisfies the following bounds:

$$\sup_{0\le t\le T}||u(\cdot,t)||^2 + \iint_D |\nabla u|^2 (x,t)\, dx\, dt \le M_0\,, \tag{3.4}$$

$$\iint_D |p(x,t)|^{5/4} dxdt \le M_1\,, \tag{3.5}$$

where M_0 and M_1 are two absolute constants. We remark that the bounds (3.4) and (3.5) are natural conditions since they are satisfied for suitable weak solutions (see [CKN]).

We shall use the following notations. Any point $(x,t) \in D$ will be denoted by Q, i.e. $Q = (x,t)$. The parabolic ball centered at a point Q with radius R will be denoted as $\mathbb{B}_R(Q) = B_R(x) \times (t - R^2, t)$. In the case there is no danger of confusion, we will omit the mention of the center of the ball and simply write $\mathbb{B}_R$. For a given solution (u,p) to the Navier-Stokes equations (3.1) and (3.2), the scaled total energy, the scaled vorticity, and other scaled quantities on the ball $\mathbb{B}_R(Q)$ are defined to be the following dimensionless quantities

$$\begin{aligned}
E(R) &\equiv \frac{1}{R^3}\iint_{\mathbb{B}_R} |u(x,t)|^2 dxdt,\\
W(R) &\equiv \frac{1}{R}\iint_{\mathbb{B}_R} |\operatorname{curl} u(x,t)|^2 dxdt,\\
E_1(R) &= \sup_{R^2 \le t<0} \int_{B_R} |u(x,t)|^2 dxdt,\\
E_2(R) &= \frac{1}{R}\iint_{\mathbb{B}_R} |\nabla u(x,t)|^2 dxdt,\\
E_3(R) &= \frac{1}{R^2}\iint_{\mathbb{B}_R} |u(x,t)|^3 dxdt.
\end{aligned} \tag{3.6}$$

One of the main results of this section asserts that the local behavior of the solution to the Navier-Stokes equations (3.1) and (3.2) can be dominated by the above sealed quantities in (3.6). More precisely we have

Theorem 3.1. *There exists an absolute constant* $\varepsilon > 0$ *with the following property. Let* $(u,p)(x,t)$ *be a smooth solution to* (3.1)–(3.2) *satisfying the bonds in* (3.4)–(3.5). *Assume that there exists a* $R_0 > 0$ *such that one of the following three conditions hold*

(1) *Either* $\sup_{0<R\leq R_0} E_1(R) < +\infty$ *or* $\sup_{0<R\leq R_0} E_2(R) < +\infty$, *and*

(3.7)
$$E(R) \equiv \frac{1}{R^3}\iint_{\mathbb{B}_R} |u(x,t)|^2 dxdt \leq \varepsilon \quad \textit{for all} \quad R \leq R_0,$$

(2)

$$\sup_{0<R\leq R_0} W(R) \leq \varepsilon, \tag{3.8}$$

(3)

$$\sup_{0<R\leq R_0} E_3(R) \leq \varepsilon, \tag{3.9}$$

then

$$\sup_{\mathbb{B}_{R/2}} |\nabla u| \leq CR^{-2} \quad \textit{for} \quad R \leq R_1 \tag{3.10}$$

for some $R_1 < R_0$ *with* C *being an absolute constant.*

We note that Theorem 3.1 improves somewhat the results implied in [CKN] and [NRS]. As in the previous section, Theorem 3.1 will follow the following uniform estimates.

Theorem 3.2. *For any given* $\delta > 0$, *there exists positive absolute constants* ε *and* R_0 *with the properties that*

(1) *if either the condition (1),* (3.7), *or condition (2),* (3.8), *in Theorem* 3.1 *holds, then there exists a constant* $R_1 = R_1(R_0)$, $0 < R_1 \leq R_0$, *such that for all* $R \leq R_1$,

$$\begin{aligned} E(R) &+ E_1(R) + E_2(R) \\ &+ R^{-26/5}\left(\int_{-R^2}^{0}\left(\int_{B_R}|p(y,t)|dy\right)^{5/4} dt\right)^{8/5} \leq \delta, \end{aligned} \tag{3.11}$$

(2) *if* $E_3(R) \leq \varepsilon$ *for all* $R \leq R_0$, *then there exists a constant* $R_1 = R_1(R_0)$, $0 < R_1 \leq R_0$, *such that for all* $R \leq R_1$,

$$E_1(R) + E_2(R) + R^{-2}\iint_{\mathbb{B}_R} |p|^{3/2} dxdt \leq \delta. \tag{3.12}$$

Remark 1. It should be emphasized that there is no smoothness requirement for the solution (u, p) in the Theorem 3.2. Indeed, the conclusion (3.9) holds for both smooth solutions and the suitable weak solutions defined in [CKN]. See the proof of Theorem 3.2 which will be given later. In particular, the suitable weak solution will be regular at the center of the ball $\mathbb{B}_R$ under any of the conditions Theorem 3.1 on suitable weak solutions.

Remark 2. With Theorem 3.2 at hand, the Theorem 3.1 can be proved by modifying slightly the argument in [CKN]. However we will present in the next subsection a different approach by using a similar technique as in the previous section. This argument is simple and clear, but requires a slightly stronger assumption that conditions (3.7) or (3.8) hold on a small neighborhood.

Remark 3. It will be clear from our analysis in the next section that the condition $\sup_{0<R\leq R_0} E_2(R) < \infty$ in (3.7) of Theorem 3.1 can be replaced by $\sup_{0<R\leq R_0} W(R) < +\infty$.

The rest of this section is devoted to the proof of these theorems. First, we give a proof of Theorem 3.1.

3.1. Proof of Theorem 3.1.

We will first assume that Theorem 3.2 holds. Then Theorem 3.1 can be proved in a similar spirit as for Theorem 2.1. In fact, by assuming Theorem 3.2, on can apply directly the Proposition 2 in [CKN] to conclude Theorem 3.1. However, we prefer to give a different approach by using a similar technique as used in Proposition 2.5. We first observe that due to the scaling property and translation invariance of the Navier-Stokes equations (3.1)–(3.2), it suffices to prove (3.10) for the ball centered at the origin $(0, 0) \equiv 0$ with radius $R \equiv 1$. The second observation is that arguing in a similar way as for Lemma 2.4 by using Theorem 3.2 instead of Proposition 2.2, one can conclude (3.10) as long as one can show that the vorticity $w(x, t) \equiv \text{curl} u(x, t)$ is uniformly bounded, i.e.,

$$\sup_{\mathbb{B}_{\frac{1}{2}}(Q_0)} |w(x, t)| \leq C \tag{3.13}$$

The rest of this subsection is devoted to the verification of (3.13). Define the "parabolic" distance between two points $Q_2 = (x_2, t_2)$ and $Q_1 = (x_1, t_1)$

$(t_2 \le t_1)$ by

$$d(Q_2, Q_1) \equiv \max\left(|x_2, -x_1|, \sqrt{t_1 - t_2}\right), \quad t_2 \le t_1 . \tag{3.14}$$

Then

$$\mathbb{B}_R(Q_0) = \{Q = (x,t) \mid d(Q, Q_0) < R\},$$

and

$$\partial\mathbb{B}_R = \{Q \mid d(Q, Q_0) = R\}.$$

Define $Q_1 \in \mathbb{B}_1(0)$ and $e_1 (= |w(Q_1)|)$ by

$$\sup_{\mathbb{B}_1(0)} \left[(1 - d(Q,0))^2 |w(Q)|\right] = (1 - d(Q_1, 0))^2 e_1 , \tag{3.15}$$

so that

$$|w(Q)| \le \left(\frac{1 - d(Q_1,0)}{1 - d(Q,0)}\right)^2 e_1 \qquad \forall Q \in \mathbb{B}_1(0) . \tag{3.16}$$

In particular, one has that

$$|w(Q)| \le (2(1 - d(Q_1, 0)))^2 e_1 \qquad \forall Q \in \mathbb{B}_{\frac{1}{2}}(0) , \tag{3.17}$$

and

$$|w(Q)| \le 2^2 e_1 , \qquad \forall Q \in \bar{\mathbb{B}}_{\frac{1}{2}(1-d(Q_1,0))}(Q_1) . \tag{3.18}$$

Using (3.17), one can conclude (3.13) provided that the following claim holds:

Claim. *There exists an absolutely constant C such that*

$$\sqrt{e_1}(1 - d(Q_1, 0)) \le C . \tag{3.19}$$

As before, the proof of this claim is given by contradiction. If (3.19) is not true, one may assume that $\sqrt{e_1}(1 - d(Q_1, 0)) > 4$ so that

$$\mathbb{B}_2(0) \subset \mathbb{B}_{\frac{\sqrt{e_1}}{2}(1-d(Q_1,0))}(0) . \tag{3.20}$$

We can now rescale the solution $u(x,t)$ near Q_1 as follows:

$$\text{(3.21)} \quad \text{Set} \quad v(y,s) = \frac{1}{\sqrt{e_1}} u\left(x_1 + \frac{y}{\sqrt{e_1}}, t_1 + \frac{s}{e_1}\right)$$
$$\text{for all } (y,s) \in \mathbb{B}_{\frac{\sqrt{e_1}}{2}(1-d(Q_1,0))}(0) .$$

Then $v(y,s)$ is a solution to the Navier-Stokes equations, and its corresponding vorticity

$$\tilde{w}(y,s) \equiv \operatorname{curl}_y v(y,s) = \frac{1}{e_1} w\left(x_1 + \frac{y}{\sqrt{e_1}}\,, t_1 + \frac{s}{e_1}\right) \tag{3.22}$$

satisfies

(3.23)
$$\begin{cases} \partial_s \tilde{w}^i - \Delta_y \tilde{w}^i + \frac{\partial}{\partial y_j}\left(v^i \tilde{w}^j - \tilde{w}^i v^j\right) = 0 & \text{in } \mathbb{B}_2(0) \\ \operatorname{div}_y v = 0 & \text{in } \mathbb{B}_2(0)\,. \end{cases}$$

It follows from (3.20)–(3.22), and (3.18) that

$$\sup\nolimits_{\mathbb{B}_2(0)} |\tilde{w}(y,s)| \le 4\,. \tag{3.24}$$

Furthermore, Theorem 3.2 implies that there is an absolute constant $C > 0$ so that

(3.25)
$$\sup\nolimits_{-4\le\tau\le 0} \int_{B_2} |v(y,\tau)|^2\, dy + \iint_{\mathbb{B}_2(0)} |\nabla_y v|^2 (y,\tau)\, dy\, d\tau \le c\delta\,,$$

where δ can be small if

$$\iint_{\mathbb{B}_2(0)} |v(y,s)|^2\, dy\, ds = 2^3 E(\sqrt{e_1}) \tag{3.26}$$

is suitably small. The desired contradiction is derived by using (3.23)–(3.26) and the fact that $|\tilde{w}(0)| = e_1^{-1}|w(Q_1)| = 1$ as follows. We first note that the value of $\tilde{w}$ at $(y,s) = (0,0)$ can be represented through an integral using the heat kernel. Indeed, let $K(y,s)$ be the backward heat kernel with the dirac mass at $(y,s) = (0,0)$, i.e.,

$$K(y,s) = \frac{1}{\left(\sqrt{4\pi(-s)}\right)^3} \exp\left\{-\frac{y^2}{4(-s)}\right\} \quad s < 0\,, \tag{3.27}$$

and $\psi(y,s)$ be a smooth cut-off function defined by

$$\psi(y,s) = \begin{cases} 0 & \frac{1}{2} \le d\left((y,s),(0,0)\right) \le 1, \\ 1 & 0 \le d\left((y,s),(0,0)\right) \le \frac{1}{4}, \\ C^\infty & \frac{1}{4} \le d\left((y,s),(0,0)\right) \le \frac{1}{2}. \end{cases} \tag{3.28}$$

Now multiplying the first equation in (3.23) by $\psi(y,s)K(y,s)$, integrating the resulting expressions over $\mathbb{B}_1(0)$, and after several integration by parts and taking limit, we can derive

$$
\begin{aligned}
\tilde{w}(0,0) = & \iint_{\mathbb{B}_1(0)} [(\partial_s\psi + \Delta\psi)K + 2\nabla\psi\cdot\nabla K]\,\tilde{w}(y,s)\,dy\,ds \\
& + \iint_{\mathbb{B}_1(0)} (K\nabla_y\psi + \psi\nabla_y K)g(y,s)\,dy\,ds\,,
\end{aligned}
\tag{3.29}
$$

where $g = \tilde{w}\otimes v - v\otimes\tilde{w}$. It follows from (3.27)–(3.28) and (3.25) that

$$
\begin{aligned}
& \left|\iint_{\mathbb{B}_1(0)} [(\partial_s\psi + \Delta\psi)K + 2\nabla\psi\cdot\nabla K]\,\tilde{w}(y,s)\,dy\,ds\right| \\
& \le \left(\iint_{\mathbb{B}_1(0)} |[(\partial_s\psi + \Delta\psi)K + 2\nabla\psi\cdot\nabla K]\,(y,s)|^2\,dy\,ds\right)^{\frac{1}{2}} \\
& \quad \cdot\left(\iint_{\mathbb{B}_1(0)} |\tilde{w}(y,s)|^2\,dy\,ds\right)^{\frac{1}{2}} \\
& \le C\delta^{\frac{1}{2}}\,,
\end{aligned}
\tag{3.30}
$$

where we have used the fact that $(\partial_s\psi+\Delta\psi)K+2\nabla\psi K$ is uniformly bounded on $\mathbb{B}_1(0)$ due to (3.27) and (3.28). Next, using the definition of g, one can obtain

$$
\begin{aligned}
& \left|\iint_{\mathbb{B}_1(0)} (K\nabla_y\psi)g(y,s)\,dy\,ds\right| \\
& \le \left(\iint_{\mathbb{B}_1(0)} |K(\nabla_y\psi)|^2\,dy\,ds\right)^{\frac{1}{2}}\left(\iint_{\mathbb{B}_1(0)} |g(y,s)|^2\,dy\,ds\right)^{\frac{1}{2}} \\
& \le C\left(\iint_{\mathbb{B}_1(0)} |g(y,s)|^2\,dy\,ds\right)^{\frac{1}{2}} \\
& \le C\left(\sup\nolimits_{\mathbb{B}_1(0)}|\tilde{w}|\right)\left(\iint_{\mathbb{B}_1(0)} |v(y,s)|^2\,dy\,ds\right)^{\frac{1}{2}} \\
& \le C\sqrt{E(\sqrt{e_1})}\,,
\end{aligned}
\tag{3.31}
$$

where we have used (3.25)–(3.30). Finally, we estimate the last integral on the right hand side of (3.29). Using Hölder and Sobolev's inequalities and

(3.24)–(3.28), one has

$$
\begin{aligned}
(3.32)\quad & \left| \iint_{\mathbb{B}_1(0)} (\psi \nabla_y K) g(y,s)\, dy\, ds \right| \\
\leq & \left(\iint_{\mathbb{B}_1(0)} |g(y,s)|^6\, dy \right)^{\frac{1}{6}} \left(\iint_{\mathbb{B}_1(0)} |\psi \nabla_y K|^{\frac{6}{5}}\, dy\, ds \right)^{\frac{5}{6}} \\
\leq & C \left(\int_{-1}^{0} s^{-\frac{9}{10}}\, ds \right)^{\frac{5}{6}} \left(\int_{-1}^{0} ds \int_{B_1(0)} |v(y,s)|^6\, dy \right)^{\frac{1}{6}} \sup_{\mathbb{B}_1(0)} |\tilde{w}| \\
\leq & C \left[\left(\int_{-1}^{0} \left(\int_{B_1(0)} |\nabla_y v|^2\, dy \right)^3 ds \right)^{\frac{1}{6}} \right. \\
& \left. + \left(\int_{-1}^{0} \left(\int_{B_1(0)} |v|^2\, dy \right)^3 ds \right)^{\frac{1}{6}} \right] \\
\leq & C \left[\left(\sup_{-1 \leq s \leq 0} \int_{B_1(0)} |\nabla_y v(y,s)|^2\, dy \right)^{\frac{1}{3}} \left(\iint_{\mathbb{B}_1(0)} |\nabla_y v|^2\, dy\, ds \right)^{\frac{1}{6}} \right. \\
& \left. + \left(\sup_{-1 \leq s \leq 0} \int_{B_1(0)} |v(y,s)|^2\, dy \right)^{\frac{1}{3}} \left(\iint_{\mathbb{B}_1(0)} |v|^2\, dy\, ds \right)^{\frac{1}{6}} \right] \\
\leq & C \left(\delta^{\frac{1}{2}} + \sqrt{E(\sqrt{e_1})} \right),
\end{aligned}
$$

where we have used the following estimate

$$
\begin{aligned}
(3.33)\quad \sup_{-1 \leq s \leq 0} \int_{B_1(0)} & |\nabla_y v(y,s)|^2\, dy \\
& \leq C \sup_{-1 \leq s \leq 0} \int_{B_1(0)} |v(y,s)|^2\, dy + C \iint_{\mathbb{B}_2(0)} |\tilde{w}|^2\, dy\, ds \\
& \quad + C \iint_{\mathbb{B}_2(0)} |\bar{v}(y,s)|^2\, dy\, ds\,.
\end{aligned}
$$

Collecting (3.29)–(3.32) yields the desired contradiction

$$
(3.34)\qquad 1 = |\tilde{w}(0,0)| \leq C \left(\sqrt{\delta} + \sqrt{E(\sqrt{e_1})} \right).
$$

It remains to prove (3.33). First, it is noted that (3.23) and (3.24) give

$$\text{(3.35)} \quad \sup_{-1\le s\le 0} \int_{B_1(0)} |\tilde{w}(y,s)|^2\, dy \le C \iint_{\mathbb{B}_2(0)} |\tilde{w}|^2\, dy\, ds + c \iint_{\mathbb{B}_2(0)} |v|^2\, dy\, ds$$

as follows by multiplying (3.23) with $\tilde{w}^i(y,s)\phi(y,s)$, here ϕ is a smooth "cut-off" function which is one on $\mathbb{B}_1(0)$ and vanishes on $\partial\mathbb{B}_2(0)$, integrating over $B_2(0)\times[-4,s]$, and using standard manipulations. Next, one has that for all $(y,s)\in\mathbb{B}_1(0)$

$$\text{(3.36)} \qquad v^*(y,s) = \int_{B_2(0)} \nabla\Gamma(y-z)\wedge\tilde{w}(z,s)\, dz + H(z,s)$$

with $H(z,s)$ being a harmonic function on $\mathbb{B}_2(0)$ for each $s\in(-4,0)$. It thus follows from the standard elliptic regularity argument (see the proof of Lemma 2.4) that

$$\text{(3.37)} \quad \sup_{-1\le s\le 0} \int_{B_1(0)} |\nabla v(y,s)|^2\, dy \le C\left(\sup_{-1\le s\le 0} \int_{B_1(0)} |v(y,s)|^2\, dy + \sup_{-1\le s\le 0} \int_{B_1(0)} |\tilde{w}(y,s)|^2\, dy\right).$$

Consequently, we conclude (3.33) from (3.35) and (3.37). So the proof of the Claim (3.19) is completed. □

Finally, we turn to the main estimates in Theorem 3.2.

3.2. Proof of Theorem 3.2.

Throughout this section C_i $(i = 0,1,2,\dots)$, C, and $0(1)$ will denote generic positive absolute constants unless stated otherwise. For the simplicity of presentation, we will also use the following notations

(3.38)

$$P_1(R) = R^{-26/5}\left(\int_{-R^2}^{0}\left(\int_{B_R} |p(y,t)|dy\right)^{5/4} dt\right)^{8/5},$$

(3.39)

$$P_2(R) = R^{-2}\iint_{\mathbb{B}_R} |p(y,t)|^{3/2} dx dt.$$

We also set

$$\phi_1(R) = E(R) + E_1(R) + E_2(R) + P_1(R), \tag{3.40}$$

$$\phi_2(R) = E_1(R) + E_2(R) + P_2(R). \tag{3.41}$$

Due to the assumptions (3.4) and (3.5), one can verify easily by using Sobolev-Poincare inequality and the classical Calderon-Zygmund estimate that there exist two absolute positive constants r_0 and M_2 such that

$$\phi_1(r_0) \le M_2 \quad \text{and} \quad \phi_2(r_0) \le M_2. \tag{3.42}$$

Our goal is to show that $\phi_1(R)$ and $\varphi_2(R)$ can be small under the corresponding conditions in Theorem 3.2.

First, we prove the part (1) of the Theorem 3.2. To this need, we need the following lemma.

Lemma 3.3. (i) *There exist absolute constants C_0 and C_1 such that for any $\lambda \in \left(0, \frac{1}{4}\right]$, $r = \lambda\rho$, and $\rho \le r_0$, one has*

$$\begin{aligned}\phi_1(r) \le C_0\lambda^{4/5}\phi_1(\rho) + C_1\{&\lambda^{-5/2}E_1^{1/4}(\rho)E^{1/4}(\rho)E_2^{1/2}(\rho)\\ &+ \lambda^{-2}E_1^{1/2}(\rho)E^{1/4}(\rho)E_2^{3/4}(\rho)\\ &+ \lambda^{-9/5}E_1^{3/5}(\rho)E^{1/5}(\rho)E_2^{1/5}(\rho)\\ &+ \lambda^{-5/2}(E_1(\rho)E_2(\rho)E(\rho))^{1/2}\\ &+ \lambda^{-7/4}E_1^{1/2}(\rho)E^{1/8}(\rho)E_2^{7/8}(\rho)\\ &+ \lambda^{-16/5}E_1^{2/5}(\rho)E^{3/5}(\rho)E_2(\rho)\}\end{aligned} \tag{3.43}$$

(ii) *There exists an absolute constant C_2 such that for any $\lambda \in \left(0, \frac{1}{2}\right]$, $r = \lambda\rho$, and $\rho \le r_0$, one has*

$$E_2(r) \le 72\lambda^2 E_2(\rho) + C_2\left(8\lambda^2 + \frac{1}{\lambda}\right)W(\rho). \tag{3.44}$$

Let us assume Lemma 3.3 for a moment and continue the proof of the part (1) of Theorem 3.2. There are several cases.

Case 1.

(3.45)

$$\sup\nolimits_{0<R\le R_0} E_1(R) = M_3 \quad \text{and} \quad \sup\nolimits_{0<R\le R_0} E(R) \le \varepsilon$$

where M_3 is a finite absolute constant. In this case, it follows from (3.43) in Lemma 3.3 that

(3.46)
$$\phi_1(r) \leq C_3\lambda^{4/5}\phi_1(\rho) + C_4\lambda^{-16/5}\varepsilon^{3/5}\phi_1(\rho) + C_5\lambda^{-98/5}\varepsilon^{1/2}$$

where we have assumed $R_0 \leq r_0$, $\varepsilon \leq 1$, $r = \lambda\rho$, and $\lambda \in \left(0, \frac{1}{4}\right]$. We now fix λ in (3.46) so that $2C_3\lambda^{4/5} = \mu < 1$. Then for $\varepsilon < \left(\frac{C_3}{C_4}\lambda^4\right)^{5/3}$, (3.46) becomes

$$\phi_1(r) \leq \mu\phi_1(\rho) + C_5\lambda^{-98/5}\varepsilon^{1/4}, \qquad r = \lambda\rho, \rho \leq R_0. \tag{3.47}$$

Now (3.11) in the Theorem 3.2 follows from (3.47) by iteration in the same way as in the proof of Proposition 2.2 from Lemma 2.3.

Case 2.

(3.48)
$$\sup_{0<R\leq R_0} E_2(R) = M_4 \quad \text{and} \quad \sup_{0<R\leq R_0} E(R) \leq \varepsilon$$

where M_4 is a finite absolute constant.

In this case, Lemma 3.3 yields that for $r = \lambda\rho$, $\lambda \in \left(0, \frac{1}{4}\right]$, and $\rho \leq R_0 \leq r_0$,

$$\phi_1(r) \leq C_6\lambda^{4/5}\phi_1(\rho) + C_7\lambda^{-98/5}\varepsilon^{1/2}. \tag{3.49}$$

Now we can proceed as in the previous case. Theorem 3.2 is proved in this case.

Case 3. The condition (3.8) holds, i.e.

$$\sup_{0<R\leq R_0} W(R) \leq \varepsilon. \tag{3.50}$$

It follows from (3.50) and (3.44) in Lemma 3.3 that

$$E_2(r) \leq 72\lambda^2 E_2(\rho) + C_2\left(8\lambda^2 + \frac{1}{\lambda}\right)\varepsilon \tag{3.51}$$

for all $\lambda \in \left(0, \frac{1}{2}\right], r = \lambda\rho$, $\rho \leq R_0 \leq r_0$. A simple iteration shows

$$E_2(\lambda^k\rho) \leq (72\lambda^2)^k E_2(\rho) + \frac{1-(72\lambda^2)^k}{1-(72\lambda^2)} C_2\left(8\lambda^2 + \frac{1}{\lambda}\right)\varepsilon \tag{3.52}$$

for $k = 1, 2, \ldots$.

As a consequence of (3.52), one shows as before that for any $\delta_0 > 0$ there exists $\varepsilon_0 > 0$ such that for $\varepsilon \le \varepsilon_0$, (3.50) implies that

$$\sup_{0<r\le r_1} E_2(r) \le \delta_0 \tag{3.53}$$

for some r_1 depending only on R_0, $r_1 \le R_0$. Combining (3.53) with (3.43) in Lemma 3.3 shows that for $\lambda \in \left(0, \frac{1}{4}\right]$, $r = \lambda\rho$, and $\rho \le r_1$,

$$\begin{aligned}\phi_1(r) &\le C_0\lambda^{4/5}\phi_1(\rho) + C_1\{\lambda^{-5/2}\phi_1^{1/2}(\rho)\delta_0 + \lambda^{-1}\phi_1^{3/4}(\rho)\delta_0^{3/4} \\ &\quad + \lambda^{-5/2}\phi_1(\rho)\delta_0^{1/2} + \lambda^{-7/4}\phi_1^{3/8}(\rho)\delta_0^{7/8} + \lambda^{-16/5}\phi_1(\rho)\delta_0\} \\ &\le C_8\lambda^{4/5}\phi_1(\rho) + C_9\lambda^{-16/5}\delta_0^{1/2}\phi_1(\rho) + C_{10}\lambda^{-32/5}\delta^{7/5}.\end{aligned} \tag{3.54}$$

Now the desired estimate (3.11) in Theorem 3.2 can be proved in the same way as in case 1.

Thus the proof of the first part of the Theorem 3.2 is considered complete. It remains to prove Lemma 3.3. The analysis will be based on the following basis identities

$$\begin{aligned}\int_{B_\rho}(\theta|u|^2)(x,t)dx &- \iint_{\mathbb{B}_\rho^t}(\partial_t\theta|u|^2)(x,s)dxds + 2\iint_{\mathbb{B}_\rho^t}(\theta|\nabla u|^2)(x,s)dxds \\ &\le 2\iint_{\mathbb{B}_\rho^t}|\nabla\theta|\,|u|\,|\nabla u|dxds + 2\iint_{\mathbb{B}_\rho^t}(\theta u)\cdot(u\cdot\nabla u)dxds \\ &\quad - 2\iint_{\mathbb{B}_\rho^t}\theta u\cdot\nabla p\,dxds\end{aligned} \tag{3.55}$$

where $\theta(x,s)$ is a smooth test function vanishing on $\partial\mathbb{B}_\rho$, and $\mathbb{B}_\rho^t = \mathbb{B}_\rho \cap \{s \le t\}$ for any $\rho > 0$, and

$$-\Delta p(x,s) = \partial_i\partial_j(u_iu_j)(x,s) \text{ on } \mathbb{B}_\rho. \tag{3.56}$$

It should be clear that (3.55) and (3.56) hold true for smooth solution (u,p) to (3.1) and (3.2). In fact, both are true for the suitable weak solutions defined in [CKN].

Proof of Lemma 3.3. In the following, for any fixed positive numbers r and ρ, with the property that $0 < r \le \frac{1}{4}\rho$ and $\rho \le r_0$, we set $r_* = 2r \le \frac{1}{2}\rho$. We also denote by $\bar{g}_r$ the average of g on the ball B_r, i.e. $\bar{g}_r = \frac{1}{w_3 r^2}\int_{B_r} g\,dx$. We first prove the part (i) of the Lemma 3.3. This will be the consequence of the following claims. We will borrow and generalize some ideas from [CKN].

Claim 1. *For $r \le \frac{1}{4}\rho$, $\rho \le r_0$, it holds that*

$$
\begin{aligned}
E_1(r) + E_2(r) \le C \Bigg\{ & E(r_*) + r_*^2 \iint_{\mathbb{B}_{r_*}} |u|[|u|^2 - \overline{|u|^2_{r_*}}] dxdt \\
& + r_*^{-2} \iint_{\mathbb{B}_{r_*}} |u - \bar{u}_\rho| \, |p| \, dxdt + r_*^{-2} \iint_{\mathbb{B}_{r_*}} |\bar{u}_\rho| \, |p(x,t)| dxdt \Bigg\} \\
\equiv\ & C\{E(r_*) + J_1(r_*) + J_2(r_*) + J_3(r_*)\}.
\end{aligned} \tag{3.57}
$$

Proof. Let $\psi(x,t)$ be a smooth function with the property that $0 \le \psi \le 1$, $\psi \equiv 1$ on $\mathbb{B}_r$, $\psi \equiv 0$ away from $\mathbb{B}_{r_*}$ such that

$$
|\partial_x \psi| \le \frac{C}{r_*} \quad \text{and} \quad |\partial_t \psi| + |\nabla_x^2 \psi| \le \frac{C}{r_*^2}. \tag{3.58}
$$

Set $\theta = \psi^2$ in the inequality (3.54) to obtain

$$
\begin{aligned}
& \int_{B_{r_*}} (\theta |u|^2)(x,t) dx + \iint_{\mathbb{B}^t_{r_*}} \theta |\nabla u|^2 dxds \le \frac{C}{r_*^2} \iint_{\mathbb{B}^t_{r_*}} |u|^2(x,s) dxds + \\
& \quad + \left| 2 \iint_{\mathbb{B}^t_{r_*}} (\theta u) \cdot (u \cdot \nabla u) dxds \right| + 2 \left| \iint_{\mathbb{B}^t_{r_*}} \theta u \cdot \nabla p \, dxds \right| \\
& \le C r_* E(r_*) + \frac{C}{r_*} \iint_{\mathbb{B}^t_{r_*}} |u| |[|u|^2 - \overline{|u|^2_{r_*}}]| dxds \\
& \quad + \frac{C}{r_*} \iint_{\mathbb{B}^t_{r_*}} |u - \bar{u}_\rho| \, |p| dxds + \frac{C}{r_*} \iint_{\mathbb{B}^t_{r_*}} |\bar{u}_\rho| \, |p| dxds
\end{aligned}
$$

where we have used integration by parts, (3.58), and the fact that u is divergence free. Hence (3.57) follows. We now estimate each term on the right hand side of (3.57). □

Claim 2. *For $0 < \mu \le \frac{1}{2}\rho$, it holds that*

$$
\begin{aligned}
E(\mu) \le & \left(\frac{\mu}{\rho}\right) E_1^{1/2}(\rho) E^{1/2}(\rho) \\
& + C \left(\frac{\rho}{\mu}\right)^{5/2} E_1^{1/4}(\rho) E^{1/4}(\rho) E_2^{1/2}(\rho).
\end{aligned} \tag{3.59}
$$

Proof. For each fix t, $-\mu^2 < t < 0$, one has by Poincare's inequality that

$$\begin{aligned}\int_{B_\mu} |u(x,t)|^2 dx &\le \int_{B_\mu} \big|\, |u|^2 - \overline{|u|^2_\rho}\big| dx + \int_{B_\mu} \overline{|u|^2_\rho} dx \\ &\le C\rho \left(\int_{B_\rho} |u|^2(x,t)dx\right)^{1/2} \left(\int_{B_\rho} |\nabla u(x,t)|^2 dx\right)^{1/2} \\ &\quad + \left(\frac{\mu}{\rho}\right)^3 \int_{B_\rho} |u|^2(x,t)dx.\end{aligned}$$

Integrating the above inequality over $(-\mu^2, 0)$ gives

$$\begin{aligned}E(\mu) \le C\rho\mu^{-3} &\left(\int_{-\mu^2}^0 \int_{B_\rho} |u|^2 dxdt\right)^{1/2} \left(\iint_{\mathbb{B}_\rho} |\nabla u|^2 dxdt\right)^{1/2} \\ &+ \rho^{-3} \int_{-\mu^2}^0 \int_{B_\rho} |u|^2(x,t)dxdt.\end{aligned}$$

Note that

$$\begin{aligned}&\int_{-\mu^2}^0 \int_{B_\rho} |u|^2(x,t)dxdt \\ &\le \left(\sup_{-\mu^2 \le t \le 0} \int_{B_\rho} |u|^2(x,t)dx\right)^{1/2} \int_{-\mu^2}^0 \left(\int_{B_\rho} |u|^2(x,t)dx\right)^{1/2} dt \\ &\le \rho^{1/2} E_1^{1/2}(\rho)\mu \left(\iint_{\mathbb{B}_\rho} |u|^2 dxdt\right)^{1/2} = \rho^2 \mu E_1^{1/2}(\rho) E^{1/2}(\rho).\end{aligned}$$

Hence (3.59) follows. Next, we estimate the term involving the cubic non-linearity. □

Claim 3.

$$\begin{aligned}J_1(\mu) &\equiv \mu^{-2} \iint_{\mathbb{B}_\mu} |u|\, |[|u|^2 - \overline{|u|^2_\mu}]| dxdt \\ &\le C \left[\left(\frac{\rho}{\mu}\right)^2 E_1^{1/2}(\rho) E^{1/4}(\rho) E_2^{3/4}(\rho) \right. \\ &\quad \left. + \left(\frac{\rho}{\mu}\right)^{5/2} E_1^{1/2}(\rho) E^{1/2}(\rho) E_2^{1/2}(\rho)\right] \quad \textit{for } \mu \le \frac{1}{2}\rho. \end{aligned} \tag{3.60}$$

Proof. It follows from Hölder, Poincare, and Sobolev-Poincare's inequalities that

$$
\begin{aligned}
\mu^2 J_1(\mu) &\leq \int_{-\mu^2}^{0} \|u\|_{L^3(B_\mu)} \|(\,|u|^2 - \overline{|u|^2_\mu}|)\|_{L^{3/2}(B_\mu)} dt \\
&\leq C \int_{-\mu^2}^{0} \left(\|u\|^{1/2}_{L^2(B_\mu)} \|\nabla u\|^{1/2}_{L^2(B_\mu)} + \frac{1}{\mu^{1/2}} \|u\|_{L^2(B_\mu)} \right) \\
&\qquad \|u\|_{L^2(B_\mu)} \|\nabla u\|_{L^2(B_\mu)} dt \\
&= C \int_{-\mu^2}^{0} \|u\|^{3/2}_{L^2(B_\mu)} \|\nabla u\|^{3/2}_{L^2(B_\mu)} dt \\
&\quad + \frac{C}{\mu^{1/2}} \int_{-\mu^2}^{0} \|u\|^{2}_{L^2(B_\mu)} \|\nabla u\|_{L^2(B_\mu)} dt \\
&\leq C\mu^2 E_1^{1/2}(\mu) E^{1/4}(\mu) E_2^{3/4}(\mu) \\
&\quad + C\mu^2 E_1^{1/2}(\mu) E^{1/2}(\mu) E_2^{1/2}(\mu),
\end{aligned} \tag{3.61}
$$

from which (3.60) follows trivially. □

Next we turn to the terms involving the pressure. First, we have

Claim 4. *For $0 < \mu \leq \frac{1}{2}\rho$, one has*

$$
\begin{aligned}
J_2(\mu) + J_3(\mu) &\equiv \mu^{-2} \iint_{\mathbb{B}_\mu} |u - \bar{u}_\rho|\, |p| dx dt + \mu^{-2} \iint_{\mathbb{B}_\mu} |\bar{u}_\rho|\, |p| dx dt \\
&\leq C \left[\left(\frac{\mu}{\rho}\right)^{3/5} E_1^{3/10}(\rho) E^{1/5}(\rho) P_1^{1/2}(\mu) \right. \\
&\quad + \left(\frac{\rho}{\mu}\right)^{1/2} E_1^{3/10}(\rho) E^{1/10}(\rho) E_2^{1/10}(\rho) P_1^{1/2}(\rho) \\
&\quad + \left(\frac{\rho}{\mu}\right)^{1/2} E_1^{1/2}(\rho) E^{1/2}(\rho) E_2^{1/2}(\rho) \\
&\quad \left. + \left(\frac{\rho}{\mu}\right)^{7/4} E_1^{1/2}(\rho) E^{1/8}(\rho) E_2^{7/8}(\rho) \right].
\end{aligned} \tag{3.62}
$$

Proof. First, by Hölder's inequality, one has

$$
\begin{aligned}
\mu^2 J_3(\mu) &= \int_{-\mu^2}^{0} (|\bar{u}_\rho| \int_{B_\mu} |p| dx) dt \\
&\leq \frac{1}{w_3}\frac{1}{\rho^3} \left(\int_{-\mu^2}^{0} \left(\int_{B_\rho} |u| dx \right)^5 dt \right)^{1/5} \left(\int_{-\mu^2}^{0} \left(\int_{B_\mu} |p| dx \right)^{5/4} dt \right)^{4/5} \\
&\leq \frac{1}{\sqrt{w_3}} \rho^{-3/2} \mu^{13/5} \left(\int_{-\mu^2}^{0} \left(\int_{B_\rho} |u|^2 dx \right)^{5/2} dt \right)^{1/5} P_1^{1/2}(\mu) \\
&\leq \frac{1}{\sqrt{w_3}} \rho^{-3/5} \mu^{13/5} E_1^{3/10}(\rho) E^{1/5}(\rho) P_1^{1/2}(\mu).
\end{aligned}
\tag{3.63}
$$

To estimate J_2, we will use the following representation for the pressure

$$
\begin{aligned}
p(x,t) &= \int_{B_\rho} \nabla_x \Gamma(x-y) \cdot (u \cdot \nabla u)(y) dy + p_0(x,t) \\
&\equiv p_1(x,t) + p_0(x,t), \quad x \in B_\rho,
\end{aligned}
\tag{3.64}
$$

where $t \in (-\mu^2, 0)$, $\Gamma(x)$ is the normalized fundamental solution of Laplace's equation in $\mathbb{R}^3$, and p_0 is harmonic in $x \in B_\rho$ for each fixed $t \in (-\rho^2, 0)$. Since $\mu \leq \frac{1}{2}\rho$, it follows from the mean value property of a harmonic function and (3.64) that

$$
|p_0(x,t)| \leq 8(\overline{|p(\cdot,t)|}_\rho + \overline{|p_1(\cdot,t)|}_\rho) \qquad \forall x \in B_\mu. \tag{3.65}
$$

Consequently,

$$
\begin{aligned}
\mu^2 J_2(\mu) &\leq \iint_{\mathbb{B}_\mu} |u - \bar{u}_\rho|\, |p_0(x,t)| dx dt + \iint_{\mathbb{B}_\mu} |u - \bar{u}_\rho|\, |p_1(x,t)| dx dt \\
&\leq 8 \int_{-\mu^2}^{0} \left(\overline{|p|}_\rho \int_{B_\mu} |u - \bar{u}_\rho| dx \right) dt \\
&\quad + 8 \int_{-\mu^2}^{0} \left(\overline{|p_1|}_\rho \int_{B_\mu} |u - \bar{u}_\rho| dx \right) dt \\
&\quad + \iint_{\mathbb{B}_\mu} |u - \bar{u}_\rho|\, |p_1(x,t)| dx dt.
\end{aligned}
\tag{3.66}
$$

Each term on the right hand side of (3.66) can be estimated rather easily as follows

$$
\begin{aligned}
&8\int_{-\mu^2}^{0}\left(\overline{|p|}_\rho\int_{B_\mu}|u-\bar{u}_\rho|dx\right)dt\\
(3.67)\qquad &\le \frac{C}{\rho^3}\left(\int_{-\mu^2}^{0}\left(\int_{B_\mu}|u-\bar{u}_\rho|dx\right)^5 dt\right)^{1/5}\times\\
&\times\left(\int_{-\mu^2}^{0}\left(\int_{B_\rho}|p(y,t)|dy\right)^{5/4} dt\right)^{4/5}\\
&\le C\rho^{-2/5}\left(\int_{-\mu^2}^{0}\left(\int_{B_\mu}|u-\bar{u}_\rho|dx\right)^5 dt\right)^{1/5} P_1^{1/2}(\rho),
\end{aligned}
$$

but, Poincare inequality yields

$$
\begin{aligned}
&\int_{-\mu^2}^{0}\left(\int_{B_\mu}|u-\bar{u}_\rho|dx\right)^5 dt\\
&\le w_3\mu^3\int_{-\mu^2}^{0}\left(\left(\int_{B_\mu}|u-\bar{u}_\rho|dx\right)^3\int_{B_\rho}|u-\bar{u}_\rho|^2dx\right)dt\\
&\le C\rho\mu^3\int_{-\mu^2}^{0}\left(\left(\int_{B_\mu}|u-\bar{u}_\rho|dx\right)^3\int_{B_\rho}|u-\bar{u}_\rho|\,|\nabla u|dx\right)dt\\
(3.68)\qquad&\\
&\le C\rho\mu^{15/2}\left(\sup_{-\mu^2\le t<0}\int_{B_\rho}|u|^2dx\right)^{3/2}\times\\
&\times\left(\iint_{\mathbb{B}_\rho}|u|^2dxdt\right)^{1/2}\left(\iint_{\mathbb{B}_\rho}|\nabla u|^2dxdt\right)^{1/2}\\
&= C\rho^{9/2}\mu^{15/2}E_1^{3/2}(\rho)E^{1/2}(\rho)E_2^{1/2}(\rho).
\end{aligned}
$$

Hence

$$
\begin{aligned}
(3.69)\quad &8\int_{-\mu^2}^{0}\left(\overline{|p|}_\rho\int_{B_\mu}|u-\bar{u}_\rho|dx\right)dt\\
&\le C\rho^{1/2}\mu^{3/2}E_1^{3/10}(\rho)E^{1/10}(\rho)E_2^{1/10}(\rho)P_1^{1/2}(\rho).
\end{aligned}
$$

Next, using the explicit formula for (3.64), one estimates by the Young's inequality that

$$\int_{B_\rho} |p_1(x,t)|dx \le C\rho \int_{B_\rho} |u \cdot \nabla u(y)|dy \qquad (3.70)$$
$$\le C\rho \|u\|_{L^2(B_\rho)} \|\nabla u\|_{L^2(B_\rho)}.$$

Thus

$$8\int_{-\mu^2}^0 \left(\overline{|p_1|}_\rho \int_{B_\mu} |u - \bar{u}_\rho| dx \right) dt$$
$$\le C\rho^{-2} \int_{-\mu^2}^0 \left(\|u\|_{L^2(B_\rho)} \|\nabla u\|_{L^2(B_\rho)} \int_{B_\mu} |u - \bar{u}_\rho| dx \right) dt$$
$$\le C\rho^{-2}\mu^{3/2} \left(\sup_{-\mu^2 \le t < 0} \int_{B_\rho} |u|^2 dx \right)^{1/2} \times \qquad (3.71)$$
$$\times \left(\iint_{\mathbb{B}_\rho} |u|^2 dxdt \right)^{1/2} \left(\iint_{\mathbb{B}_\rho} |\nabla u|^2 dxdt \right)^{1/2}$$
$$\le C\rho^{1/2}\mu^{3/2} E_1^{1/2}(\rho) E^{1/2}(\rho) E_2^{1/2}(\rho).$$

Finally, note that (for $\mu \le \frac{1}{2}\rho$)

$$\|p_1(\cdot,t)\|_{L^{4/3}(B_\mu)} \le C\mu^{1/4} \|u \cdot \nabla u\|_{L^1(B_\rho)}$$
$$\le C\mu^{1/4} \|u(\cdot,t)\|_{L^2(B_\rho)} \|\nabla u(\cdot,t)\|_{L^2(B_\rho)} \qquad (3.72)$$

and

$$\left(\int_{B_\mu} |u - \bar{u}_\rho|^4 dx \right)^{1/4} \le C \left(\int_{B_\rho} |\nabla u|^2 dx \right)^{3/8} \left(\int_{B_\rho} |u|^2 dx \right)^{1/8} \qquad (3.73)$$

which follow from the Young's and Poincare-Sobolev's inequalities respectively. One obtain from (3.69) and (3.70) that

$$
\begin{aligned}
&\iint_{\mathbb{B}_\mu} |u-\bar{u}_\rho|\,|p_1(x,t)|dxdt \\
&\le \int_{-\mu^2}^{0}\left(\int_{B_\mu}|u-\bar{u}_\rho|^4dx\right)^{1/4}\left(\int_{B_\mu}|p_1(x,t)|^{4/3}dx\right)^{3/4}dt \\
&\le C\mu^{1/4}\int_{-\mu^2}^{0}\left(\int_{B_\rho}|\nabla u|^2dx\right)^{7/8}\left(\int_{B_\rho}|u|^2dx\right)^{5/8}dt \qquad (3.74)\\
&\le C\left(\frac{\rho}{\mu}\right)^{7/4}\mu^2E_1^{1/2}(\rho)E^{1/8}(\rho)E_2^{7/8}(\rho).
\end{aligned}
$$

(3.62) now follows from (3.63), (3.66), (3.69), (3.71), and (3.74). Finally, we estimate $P_1(r)$.

Claim 5. *For any $\eta \le \frac{1}{2}\rho$, one has*

(3.75)
$$
P_1(\eta) \le C\left\{\left(\frac{\eta}{\rho}\right)^{4/5}P_1(\rho)+\left(\frac{\rho}{\eta}\right)^{16/5}E_1^{2/5}(\rho)E^{3/5}(\rho)E_2(\rho)\right\}.
$$

Proof. We again use the representation formula (3.64) for the pressure. It then follows from (3.65) that

$$
\int_{B_\mu}|p_0(x,t)|dx \le 8\left(\frac{\mu}{\rho}\right)^3\left[\int_{B_\rho}|p_1(x,t)|dx+\int_{B_\rho}|p(x,t)|dx\right].
$$

This, together with (3.70), shows that

$$
\begin{aligned}
&\|p(\cdot,t)\|_{L^1(B_\mu)} \le \|p_1(\cdot,t)\|_{L^1(B_\mu)}+\|p_0(\cdot,t)\|_{L^1(B_\mu)} \\
&\le 8\left(\frac{\mu}{\rho}\right)^3\|p(\cdot,t)\|_{L^1(B_\rho)} \qquad (3.76)\\
&\quad +C\left(\left(\frac{\mu}{\rho}\right)^3\rho+\mu\right)\|u\|_{L^2(B_\rho)}\|\nabla u\|_{L^2(B_\rho)},
\end{aligned}
$$

where we have also used the simple estimate

$$
\int_{B_\mu}|p_1(x,t)|dx \le C\mu\int_{B_\rho}|u\cdot\nabla u(y)|dy \le C\mu\|u\|_{L^2(B_\rho)}\|\nabla u\|_{L^2(B_\rho)}
$$

which follows from the explicit representations for $\rho_1(x,t)$ by the standard estimate. As a consequence of (3.76) and Hoölder's inequality, we can derive

$$
\begin{aligned}
&\int_{-\mu^2}^{0}\left(\int_{B_\mu}|p(y,t)|dy\right)^{5/4}dt\\
&\quad\le C\left[\left(\frac{\mu}{\rho}\right)^{15/4}\int_{-\mu^2}^{0}\left(\int_{B_\rho}|p(y,t)|dy\right)^{5/4}dt\right.\\
&\quad\quad\left.+\mu^{5/4}\int_{-\mu^2}^{0}\|u\|_{L^2(B_\rho)}^{5/4}\|\nabla u\|_{L^2(B_\rho)}^{5/4}dt\right]\\
&\quad\le C\left[\left(\frac{\mu}{\rho}\right)^{15/4}\int_{-\rho^2}^{0}\left(\int_{B_\rho}|p(y,t)|dy\right)^{5/4}dt\right.\\
&\quad\quad\left.+\mu^{5/4}\left(\int_{-\mu^2}^{0}\|u(t)\|_{L^2(B_\rho)}^{10/3}dt\right)^{3/8}\left(\iint_{\mathbb{B}_\rho}|\nabla u|^2dxdt\right)^{5/8}\right]\\
&\quad\le C\mu^{13/4}\left[\left(\frac{\mu}{\rho}\right)^{1/2}(P_1(\rho))^{5/8}+\left(\frac{\rho}{\mu}\right)^{2}E_1^{1/4}(\rho)E^{3/8}(\rho)E^{5/8}(\rho)\right]
\end{aligned}
$$

which implies (3.75) immediately. □

Now, the part (i), (3.43), of the Lemma 3.3 is a direct corollary of Claims 1–5. Indeed, from (3.57), (3.59), and (3.75), one gets that for $r=\lambda\rho$, $\lambda\in\left(0,\frac{1}{4}\right]$, and $r_*=2r\le\frac{1}{2}\rho$,

$$
\begin{aligned}
\phi_1(r)\le{}&\lambda E_1^{1/2}(\rho)E^{1/2}(\rho)+C[\lambda^{-5/2}E_1^{1/4}(\rho)E^{1/4}(\rho)E_2^{1/2}(\rho)+E(r_*)\\
&+J_1(r_*)+J_2(r_*)+J_3(r_*)\\
&+\lambda^{4/5}P_1(\rho)+\lambda^{-16/5}E_1^{2/5}(\rho)E^{3/5}(\rho)E_2(\rho)].
\end{aligned}
$$

Using Claims 2–4 with $\eta=r_*=2r$, one can bound the right hand side above by

$$
\begin{aligned}
&C\{\lambda E_1^{1/2}(\rho)E^{1/2}(\rho)+\lambda^{-5/2}E_1^{1/4}(\rho)E^{1/4}(\rho)E_2^{1/2}(\rho)\\
&+\lambda^{-2}E_1^{1/2}(\rho)E^{1/4}(\rho)E_2^{3/4}(\rho)+\lambda^{-5/2}E_1^{1/2}(\rho)E^{1/2}(\rho)E_2^{1/2}(\rho)\\
&+\lambda^{4/5}P_1(\rho)+\lambda^{-16/5}E_1^{2/5}(\rho)E^{3/5}(\rho)E_2(\rho)\\
&+\lambda^{-1/2}E_1^{3/10}(\rho)E^{1/10}(\rho)E_2^{1/10}(\rho)P_1^{1/2}(\rho)
\end{aligned}
$$

$$
\begin{aligned}
&+ \lambda^{-1/2} E_1^{1/2}(\rho) E^{1/2}(\rho) E_2^{1/2}(\rho) + \lambda^{-7/4} E_1^{1/2}(\rho) E^{1/8}(\rho) E^{7/8}(\rho) + \\
&+ \lambda^{3/5} E_1^{3/10}(\rho) E^{1/5}(\rho) P_1^{1/2}(r_*)\} \\
\leq\ & C\{(\lambda E_1^{1/2}(\rho) E^{1/2}(\rho) + \lambda E_1^{3/10}(\rho) E^{1/5}(\rho) E_2^{1/2}(\rho) + \lambda^{4/5} P_1(\rho)) + \\
&+ (\lambda^{-5/2} E_1^{1/2}(\rho) E^{1/2}(\rho) E_2^{1/2}(\rho) + \lambda^{-5/2} E_1^{1/4}(\rho) E^{1/4}(\rho) E_2^{1/2}(\rho) \\
&+ \lambda^{-2} E^{1/2}(\rho) E^{1/4}(\rho) E_2^{3/4}(\rho) + \lambda^{-9/5} E_1^{3/5}(\rho) E^{1/5}(\rho) E_2^{1/5}(\rho) \\
&+ \lambda^{-16/5} E_1^{2/5}(\rho) E^{3/5}(\rho) E_2(\rho) + \lambda^{-7/4} E_1^{1/2}(\phi) E^{1/8}(\rho) E_2^{7/8}(\rho))\},
\end{aligned}
$$

where, in the last step, we have used the Claim 5 with $\mu = r_*$ and the Cauchy-Schwartz inequality. Hence (3.43) follows. So the first part, (3.43) of the Lemma 3.3 is proved.

We now turn to the proof of the part (*ii*), (3.44), of Lemma 3.3. This can be accomplished easily as follows. We first recall the following representation

(3.77)
$$
\nabla u^*(x,t) = \int_{B_\rho} \nabla_x^2 \Gamma(x-y) \wedge w(y,t)dy + |w(x,t)| + H_0(x,t)
$$

for all $(x,t) \in \mathbb{B}_\rho$, where $w(x,t) = \operatorname{curl} u(x,t)$, $H_0(x,t)$ is a harmonic function in $x \in B_\rho$ for each fixed $t \in (-\rho^2, 0)$, and the integral on the right hand side of (3.77) is in the sense of the Cauchy principal value. It then follows from the Calderon-Zygmund estimate that there is a positive constant C such that

$$
\begin{aligned}
\text{(3.78)}\quad \int_{B_r} |\nabla u(x,t)|^2 dx \leq C \int_{B_\rho} |w(x,t)|^2 dx \\
+ 3\int_{B_r} |w(x,t)|^2 dx + 3\int_{B_r} |H_0(x,t)|^2 dx.
\end{aligned}
$$

By the mean value property of harmonic functions, one has for each $t \in (-\rho^2, 0)$ that

$$
\text{(3.79)}\qquad \int_{B_r} |H_0(x,t)|^2 dx \leq \frac{\lambda^3}{(1-\lambda)^3} \int_{B_\rho} |H_0(x,t)|^2 dx.
$$

On the other hand, as for (3.78), one has from (3.77) that

(3.80)
$$
\int_{B_\rho} |H_0(x,t)|^2 dx \leq 3\int_{B_\rho} |\nabla u(x,t)|^2 dx + C\int_{B_\rho} |w(x,t)|^2 dx
$$

for each $t \in (-\rho^2, 0)$, where C is an absolute constant. It follows from (3.78)–(3.80) that for all $t \in (-\rho^2, 0)$,

$$\frac{1}{r}\int_{B_r} |\nabla u(x,t)|^2 dx \le \frac{9\lambda^2}{(1-\lambda)^3}\frac{1}{\rho}\int_{B_\rho} |\nabla u(x,t)|^2 dx + \\ + C\left(\frac{\lambda^2}{(1-\lambda)^3} + \frac{1}{\lambda}\right)\frac{1}{\rho}\int_{B_\rho} |w(x,t)|^2 dx \tag{3.81}$$

with an absolute constant C. We now integrate (3.81) over $(-r^2, 0)$ to obtain the desired estimate (3.44). This finishes the proof of the second part of Lemma 3.3. So the proof of Lemma 3.3 is complete. □

Finally, we prove the part (2) of the Theorem 3.2. As before, the key step in the proof of the inequality (3.12) will be the following iteration relation.

Lemma 3.4. *There exists an absolute constant C such that for any $\lambda \in (0, \frac{1}{4}]$, $r = \lambda\rho$, and $\rho \le r_0$, it holds that*

$$\phi_2(r) \le C\{\lambda\phi(\rho) + (E_3^{2/3}(2r) + E_3(2r) + \lambda^{-2}E_3(\rho)\}. \tag{3.82}$$

Assuming Lemma 3.4 for the moment, one can prove the part (2) of Theorem 3.2, (3.12), easily in a same way as in the previous cases. Thus it suffice to prove Lemma 3.4.

Proof of Lemma 3.4. As in the proof of Lemma 3.3, for any $r \le \frac{1}{4}\rho$, $\rho \le r_0$, we set $r_* \le 2r \le \frac{1}{2}\rho$, and divide the proof into several steps.

Step 1

$$E_1(r) + E_2(r) \le C[E(r_*) + E_3(r_*) + P_2(r_*)]. \tag{3.83}$$

Proof. In the identity (3.55), one choose $\theta = \psi^2$ with ψ satisfying (3.58) to obtain after integrations by parts that

$$E_1(r) + E_2(r) \le \frac{C}{r_*^3}\iint_{\mathbb{B}_{r_*}} |u|^2\, dxdt + \frac{C}{r_*^2}\iint_{\mathbb{B}_{r_*}} |u|^3 dxds \\ + \frac{C}{r_*^2}\iint_{\mathbb{B}_{r_*}} |u|\,|p|\, dxds \\ \le C[E(r_*) + E_3(r_*) + P_2(r_*)], \tag{3.84}$$

where in the last step one has used Hölder's inequality. □

Step 2

$$E(\mu) \leq \sqrt{w_3} E_3^{2/3}(\mu), \qquad \mu \leq \rho \leq r_0 \tag{3.85}$$

Proof. (3.85) follows from Hölder's inequality. □

Step 3 For any $0 < \mu \leq \frac{1}{2}\rho$,

$$P_2(\mu) \leq C\left\{\left(\frac{\mu}{\rho}\right) P_2(\rho) + \left(\frac{\rho}{\mu}\right)^2 E_3(\rho)\right\}. \tag{3.86}$$

Proof. We will use the following representation for the pressure which follows from the identity (3.56):

(3.87)
$$p(x,t) = \int_{B_\rho} [D_x^2\Gamma(x-y): \ (u \otimes u)(y)]dy + |u(x,t)|^2 + H_r(x,t)$$

for all $(x,t) \in \mathbb{B}_\rho$, where the integral is in the sense of the Cauchy principal value, and H_1 is harmonic on B_ρ for each fixed $t \in (-\rho^2, 0)$. Set

$$p_2(x,t) = \int_{B_\rho} [D_x^2\Gamma(x-y): \ (u \otimes u)(y)]dy. \tag{3.88}$$

Then the classical Calderon-Zygmund estimate shows that

$$\int_{B_\mu} |p_2(y,t)|^{3/2} dy \leq C \int_{B_\rho} |u(x,t)|^3 dx \tag{3.89}$$

with an absolute constant C. On the other hand, one has

$$\int_{B_\mu} |H_1(y,t)|^{3/2} dy \leq (\sqrt{2})^9 \left(\frac{\mu}{\rho}\right)^3 \int_{B_\rho} |u(x,t)|^3 dx \tag{3.90}$$

due to the mean value property of harmonic functions. It follows from (3.87)–(3.90) that

$$\begin{aligned}
\int_{B_\mu} |p(x,t)|^{3/2}dx &\leq C\left[\int_{B_\mu} |p_2(x,t)|^{3/2}dx + \int_{B_\mu} |u|^3 dx + \int_{B_\mu} |H_1|^{3/2}dx\right] \\
&\leq C\left[\int_{B_\rho} |u|^3 dx + \left(\frac{\mu}{\rho}\right)^3 \int_{B_\rho} |H_1(x,t)|^{3/2}dx\right] \\
&\leq C\left[\left(\frac{\mu}{\rho}\right)^3 \int_{B_\rho} |p(x,t)|^3 dx + \left(\frac{\mu}{\rho}\right)^3 \int_{B_\rho} |p_2|^{3/2}dx\right. \\
&\qquad \left. + \left(1+\left(\frac{\mu}{\rho}\right)^3\right)\int_{B_\rho} |u|^3 dx\right] \\
&\leq C\left[\left(\frac{\mu}{\rho}\right)^3 \int_{B_\rho} |p(x,t)|^{3/2}dx + \int_{B_\rho} |u(x,t)|^3 dx\right].
\end{aligned} \tag{3.91}$$

Integrating (3.91) over $(-\mu^2, 0)$ leads to (3.86). □

It follows from Steps 1–3 that for $r = \lambda\rho$, $\lambda \in \left(0, \frac{1}{4}\right]$, and $\rho \leq r_0$,

$$\begin{aligned}
\phi_2(r) &\equiv E_1(r) + E_2(r) + P_2(r)] \\
&\leq C[E(r_*) + E_3(r_*) + P_2(r_*)] \\
&\leq C[\lambda P_2(\rho) + \lambda^{-2}E_3(\rho) + E_3(r_*) + E_3^{2/3}(r_*)],
\end{aligned}$$

which yields the desired estimate (3.82) immediately. This completes the proof of Lemma 3.4. □

Consequently, the proof of Theorem 3.2 is completed. □

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