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Non-Haken 3-manifolds are not large with respect to mappings of non-zero degree.

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1. Introduction.

In this paper we shall be interested in the existence of maps of non-zero degree between compact 3-manifolds, and epimorphisms between their fundamental groups. We shall be particularly interested in the case when the manifolds are finite volume hyperbolic 3-manifolds. Throughout, by a hyperbolic n-manifold we shall always mean an n-manifold admitting a complete Riemannian metric of constant curvature -1. In what follows, all manifolds are assumed connected and orientable unless indicated.

There is a rich history to the study of maps of non-zero degree (see for example [14], [15] and references therein), and of particular motivation to us is the suggestion of Gromov and others that one can use maps of non-zero degree to define a partial order on the set of homeomorphism classes of compact n-manifolds. More precisely, say that a compact n-manifold M *dominates* (resp. *d-dominates*) a compact n-manifold N, and write $M \geq N$ (resp. $M \geq_d N$) if there is a map $f : M \to N$ of non-zero degree (resp. degree d). For $n \neq 3$, by H. C. Wang's famous theorem that for bounded volume there are at most finitely many closed hyperbolic n-manifolds, together with the work of Gromov (see [24]), each closed hyperbolic n-manifold dominates at most finitely many closed hyperbolic n-manifolds.

In dimension 3, it was shown by Thurston ([24]) that there exists a $V > 0$ such that there are infinitely many closed hyperbolic 3-manifolds of volume bounded above by V. The difference between dimension 3 and dimension $n \neq 3$ is further emphasized by an example in [1] of a closed non-orientable

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hyperbolic 3-manifold which was shown to admit (geometric) degree 1 maps to infinitely many closed orientable hyperbolic 3-manifolds.

It is also worth remarking that if a 3-manifold M 1-dominates N, then M can be viewed as being "larger than N", for many of the known invariants which are used to measure the complexity of 3-manifolds, say the rank of π_1 or homology groups, the number of disjoint embedded incompressible surfaces, or the Gromov norm, the numbers associated with M are not smaller than that associated with N.

Of specific interest to us are the following questions and a conjecture. Question 2 appears in Kirby's new problem set 3.100, [13] and Conjecture 3 in Kirby's (old and) new problem set 1.12 (D) [13].

Question 1: Which (hyperbolic) 3-manifolds dominate at most finitely many (hyperbolic) 3-manifolds?

Question 2: Does every compact 3-manifold 1-dominate at most finitely many 3-manifolds?

Conjecture 3: For a given knot K in S^3, there exist only finitely many knot groups G for which there is an epimorphism $\pi_1(S^3 - K) \to G$.

In an effort to resolve questions 1 and 2, it was shown in [20] that in a sequence of degree one maps $M \to M_1 \to M_2 \to M_3 \to \ldots$ among 3-manifolds satisfying Thurston's Geometrization Conjecture the number of homeomorphism types of the manifolds is finite. Also in connection with Questions 1 and 2, we refer the reader to [14], [15] (see also the references contained there) for results on mappings of non-zero degree between Seifert fibered spaces, and conditions for the existence of a degree 1 map onto a lens space.

The main results in this paper are some "substantial partial answers" to both Questions 1, 2 and Conjecture 3 above.

Before presenting the results, recall that a 3-manifold is *geometric* if it admits one of the eight geometries of Thurston [22]. It will be convenient in places to make use of the following Geometrization Conjecture of Thurston.

Conjecture 1.1. *Every closed orientable 3-manifold admits a canonical sphere-torus decomposition into pieces which have geometric structures.*

Assuming this we show (see Theorem 4.2),

Theorem 1.2. *Let M be a non-Haken 3-manifold. Then for each positive integer d, M d-dominates only finitely many 3-manifolds satisfying Conjecture* 1.1.

For non-Haken hyperbolic 3-manifolds, this can be strengthened to (see Theorem 4.4)

Theorem 1.3. *Let M be a non-Haken hyperbolic 3-manifold. Then there exists only finitely many finite volume hyperbolic 3-manifolds N for which there is an epimorphism $\pi_1(M) \to \pi_1(N)$. In particular M dominates at most finitely many hyperbolic 3-manifolds.*

In [1], there is a closed orientable Haken hyperbolic 3-manifold M admitting an epimorphism $\pi_1(M) \to \pi_1(N)$ for infinitely many closed orientable hyperbolic 3-manifolds N. However, the finiteness results may also be extended to some closed Haken 3-manifolds.

Moreover our techniques extend to give finiteness results analogous to the above for cusped hyperbolic 3-manifolds that are "non-sufficiently-large" in the sense that they do not contain a closed embedded essential surface (see Theorem 5.2 and its proof).

Theorem 1.4. *Let M be a 1-cusped hyperbolic 3-manifold of finite volume containing no closed embedded essential surface. Then there exist only finitely many finite volume hyperbolic 3-manifolds N for which there is an epimorphism $\pi_1(M) \to \pi_1(N)$ and for which the peripheral subgroup of $\pi_1(M)$ is mapped into the peripheral subgroup of $\pi_1(N)$. In particular M 1-dominates at most many finitely hyperbolic 3-manifolds.*

The proofs of our results depend on the theory of characters of representations of 3-manifold groups developed by Culler and Shalen [6]. This gives a powerful tool in detecting mappings of non-zero degree and homomorphisms on finite index subgroups, whose application in this context has previously gone unnoticed. Of particular interest is that the methods developed here seem to be the first that allow one to say anything meaningful for maps between non-Haken hyperbolic 3-manifolds.

To describe the other results of the paper we recall that a closed 3-manifold M is called *minimal* if for any compact irreducible 3-manifold N different from S^3, $M \geq_1 N$ implies that M is homeomorphic to N. For examples of minimal Seifert manifolds see [14], and for minimal knot complements see [1]. The question of which closed hyperbolic 3-manifolds are minimal was raised in [1]. In this paper we give the first such examples (for example $(1, 2)$ surgery on the figure eight knot complement).

A natural candidate for such a minimal manifold is the so-called Weeks manifold (see below), which is conjecturally the smallest volume hyperbolic 3-manifold. Here we show that it is almost minimal in the sense that (assuming the Geometrization Conjecture of Thurston), the only manifolds that the Weeks manifold can dominate by a mapping of non-zero degree are covered by S^3. Indeed if the degree is 1, the only possibilities (which exist) are the lens spaces $L(5,1)$ and $L(5,2)$. This can be viewed as some mild positive evidence that the Weeks manifold is the minimal volume hyperbolic 3-manifold.

The plan of the paper is the following. In §2 we recall some relevant facts about representation and character varieties of 3-manifold groups, and in §3 collect some facts on mappings of non-zero degree. In §4 we prove our finiteness results in the closed case, with §5 being devoted to the bounded case. Finally in §6 we perform the calculations for the examples mentioned above which provide the first known minimal hyperbolic 3-manifolds under the partial order $\geq_1$.

2. Representation and Character Varieties.

In this section we record facts about the structure of the representation and character varieties that we shall use. This section is included to make the paper more self-contained. Good references for this section are [2], [6], and [17].

2.1.

Let Γ be a finitely generated group with a generating set $\{\gamma_1, \dots, \gamma_n\}$. We denote by $\mathrm{Hom}\,(\Gamma)$ the set of all homomorphisms of Γ into $\mathrm{SL}\,(2,\mathbf{C})$. Then via the embedding

$$\mathrm{Hom}\,(\Gamma) \subset \mathrm{SL}\,(2,\mathbf{C})^n \subset \mathbf{C}^{4n},$$

$\mathrm{Hom}\,(\Gamma)$ inherits the structure of a complex algebraic variety, where the polynomials defining the variety arise from the relations in the γ_i's. $\mathrm{Hom}\,(\Gamma)$ is called the *Representation Variety* of Γ.

A related variety is the *Character Variety*. Recall that by a character of a representation $\rho \in \mathrm{Hom}\,(\Gamma)$ we mean a function $\chi_\rho : \Gamma \to \mathbf{C}$ with $\chi_\rho(\gamma) = \mathrm{tr}\,(\rho(\gamma))$. As discussed in [6], the space of characters denoted $X(\Gamma)$ also has the structure of a complex algebraic variety. We briefly recall some features of this since it will be useful in what follows.

For each $\gamma \in \Gamma$, we can define a regular function $\tau_\gamma : \mathrm{Hom}(\Gamma) \to \mathbf{C}$ by

$$\tau_\gamma(\rho) = \chi_\rho(\gamma).$$

Also following [6] we denote by I_γ the regular function on $X(\Gamma)$ defined by $I_\gamma(\chi) = \chi(\gamma)$.

Let T be the ring generated by all such functions. As shown in [6] Proposition 1.4.1, T is finitely generated. Fixing a finite set of elements $\delta_1, \dots, \delta_m$ that generate T, then $X(\Gamma)$ is described as the image of a map $t : \mathrm{Hom}(\Gamma) \to X(\Gamma)$, where

$$t(\rho) = \big(\tau_{\delta_1}(\rho), \dots, \tau_{\delta_m}(\rho)\big).$$

Hence the character χ_ρ is determined by $t(\rho)$.

The construction of a generating set for T determined by the above discussion (see [6]) contains the following well-known lemma about non-elementary subgroups of $\mathrm{SL}(2, \mathbf{C})$ going back to Fricke and Klein. Recall a subgroup G of $\mathrm{SL}(2, \mathbf{C})$ is non-elementary if G has no common fixed point for the action on $\mathbf{H}^3$.

Lemma 2.1. *Let $G = \langle g_1, \dots, g_n \rangle$ and ρ a representation into $\mathrm{SL}(2, \mathbf{C})$ whose image is non-elementary. Then $\rho(G)$ is determined up to conjugacy in $\mathrm{SL}(2, \mathbf{C})$ by the complex numbers*

$$\chi_\rho(g_i) = \mathrm{tr}\,\big(\rho(g_i)\big), \qquad \chi_\rho(g_i\, g_j) = \mathrm{tr}\,\big(\rho(g_i\, g_j)\big), \quad i < j$$

and

$$\chi_\rho(g_i\, g_j\, g_k) = \mathrm{tr}\,\big(\rho(g_i\, g_j\, g_k)\big), \quad i < j < k.$$

2.2.

Here we collect facts about $\mathrm{Hom}(\Gamma)$ and $X(\Gamma)$ when Γ is a 3-manifold group. Recall that a compact irreducible 3-manifold is Haken if it contains an embedded incompressible surface.

If M is a compact irreducible 3-manifold, we shall use the notation $X(M)$ for the character variety of $\pi_1(M)$. The following lemma is an easy consequence of the fundamental theorem of [6].

Lemma 2.2. *Let M be a non-Haken 3-manifold. Then $X(M)$ consists of a finite number of points.*

Proof. A standard fact from algebraic geometry is that a complex algebraic variety has only finitely many irreducible components. Thus suppose that some component of $X(M)$ has positive dimension. Then we can find a curve of characters in $X(M)$. By [6], this determines a non-trivial splitting of $\pi_1(M)$, and by standard 3-manifold topology an embedded incompressible surface in M contradicting the non-Haken hypothesis. □

For Haken manifolds, we observe that if the first Betti number of M is positive then $X(M)$ always has positive dimension, using representations of $\mathbf{Z}$ into $\mathrm{SL}(2, \mathbf{C})$. However, there are many Haken hyperbolic 3-manifolds with 0-dimensional character variety. The following lemma is proved as Lemma 9.1 of [2], which allows many examples to be constructed (cf. [2] Theorem 0.9, and Example 9.2).

Lemma 2.3. *Let K be a knot in S^3 whose complement is hyperbolic. Assume further that $M = S^3 \setminus \mathrm{Int}(N(K))$ contains no closed embedded essential surface. Then if $r = m/n$ is a boundary slope with $n > 1$, $M(r)$ is a Haken manifold with $X(M(r))$ consisting of a finite number of points. Moreover if $n > 2$, $M(r)$ is hyperbolic.*

We will also use the following lemma which is implicit in [6] Proposition 3.2.1. We outline the proof for completeness. Recall by an *essential* closed surface in a compact irreducible 3-manifold M we mean a closed embedded incompressible surface S in M which is not boundary parallel.

Lemma 2.4. *Let M be a compact orientable 3-manifold whose boundary consists of a disjoint union of incompressible tori. Assume $X(M)$ contains a curve of characters which take the value ± 2 on every peripheral subgroup. Then M contains a closed essential surface.*

Proof. Let $T_1, \dots T_n$ be the peripheral tori in M. Let C be a curve in $X(M)$ with the property that the characters $\chi \in C$ have value ± 2 on each peripheral subgroup $\pi_1(T_1) \dots, \pi_1(T_n)$. Using the construction of the tree of $\mathrm{SL}(2)$ over a field with a discrete valuation ring, together with the assumption that the characters χ have value ± 2 on each peripheral subgroup the techniques of [6] provide a splitting of $\pi_1(M)$, with the property that for each $i = 1, \dots n$, $\pi_1(T_i)$ is contained in a vertex group (see [6] 2.2.1). Standard 3-manifold techniques (cf. [6] Proposition 2.3.1 for instance) then imply that M must contain a closed essential surface. □

We also find it convenient to make use of $\mathrm{PSL}(2,\mathbf{C})$ versions of the above. This is discussed in [2] §2 in some detail. We will not discuss this in any detail here, other than to record the following facts, and refer the reader to [2]. Denote by $\overline{X}(M)$ the $\mathrm{PSL}(2,\mathbf{C})$-character variety. Then $X(M)$ maps to $\overline{X}(M)$ by a finite-to-one map, and in particular if $\overline{X}(M)$ consists of a finite number of points then $X(M)$ will consist of a finite number of points. It is worth pointing out that the dimension of $X(\Gamma)$ can be strictly smaller than $\overline{X}(\Gamma)$, e.g. $\Gamma = \mathbf{Z}_2 * \mathbf{Z}_n$ for any positive integer $n \geq 2$.

If M is non-Haken then Lemma 2.2 holds for $\overline{X}(M)$, to show that $\overline{X}(M)$ consists of a finite number of points.

In what follows we will always work with $\mathrm{SL}(2,\mathbf{C})$ representations unless it is absolutely necessary to pass to the $\mathrm{PSL}(2,\mathbf{C})$ versions.

3. Maps of non-zero degree.

Throughout this section, unless otherwise indicated, all manifolds are assumed to be of dimension 3 and to be irreducible.

The only facts we shall use about maps of non-zero degree between 3-manifolds are contained in the following. The first lemma is a combination of results to be found in [3], [7] and [15] for example.

Lemma 3.1. *Let $f : M \to N$ be a map of degree $d \neq 0$. If f_* denotes the homomorphism induced on fundamental groups then $[\pi_1(N) : f_*(\pi_1(M))]$ is a divisor of d. Moreover, if the degree is 1 then:*

(1) *f_* is surjective,*

(2) $\mathrm{Tor}\,(H_1(M,\mathbf{Z}))$ *is a direct summand of* $\mathrm{Tor}\,(H_1(N,\mathbf{Z}))$.

In particular if M is non-Haken, then $H_1(M,\mathbf{Z})$ is finite and $|H_1(M,\mathbf{Z})| \geq |H_1(N,\mathbf{Z})|$

Recall that a map $f : (M,\partial M) \to (N,\partial N)$ is *proper* if $f^{-1}(\partial N) = \partial M$.

Lemma 3.2. *Let $f : (M,\partial M) \to (N,\partial N)$ be a proper degree 1 map. If ∂M is a torus, then ∂N is a torus.*

Proof. Since the map is proper and ∂M is a torus, ∂N is either S^2 or a torus. Since N is irreducible, the result follows. □

The final result we record here is the following due to Gromov and Thurston, see [24], Corollary 6.2.1.

Theorem 3.3. *Let M and N be hyperbolic 3-manifolds of finite volume, and $f : M \to N$ a map of degree $d \neq 0$. Then* $\mathrm{Vol}\,(M) \geq d\,\mathrm{Vol}\,(N)$.

4. Finiteness Results: Closed 3-Manifolds.

Recall that if M is a hyperbolic 3-manifold of finite volume, then $\pi_1(M)$ admits a faithful discrete representation into $\mathrm{PSL}\,(2, \mathbf{C})$ which by Mostow Rigidity is unique up to conjugacy in $\mathrm{Isom}\,(\mathbf{H}^3)$. Furthermore this representation can be lifted to $\mathrm{SL}\,(2, \mathbf{C})$, see [6] for instance.
The results of this section can be summarized in the following theorem whose proof is given in the subsections below.

Theorem 4.1. *Let M be a closed irreducible 3-manifold such that $\overline{X}(M)$ consists of a finite number of points. Then for each positive integer d, M d-dominates only finitely many geometric 3-manifolds.*

This can be easily applied to prove:

Theorem 4.2. *Let M be a non-Haken 3-manifold and d as above. Then M d-dominates only finitely many 3-manifolds satisfying Conjecture* 1.1.

Proof. From Theorem 4.1 and Lemma 2.2 (and the discussion in §2.2), we observe that M d-dominates at most finitely many geometric 3-manifolds. In addition, if M is a non-Haken manifold, then M cannot dominate a manifold that decomposes non-trivially into geometric pieces, for otherwise we obtain a surjection of $\pi_1(M)$ onto a non-trivial free product with amalgamation over the trivial group or $\mathbf{Z} \oplus \mathbf{Z}$. Standard 3-manifold topology provides an embedded incompressible surface in M. This completes the proof. □

4.1.

We begin the proof of Theorem 4.1 by establishing the following:

Lemma 4.3. *Let M be as in the statement of Theorem 4.1 and d a fixed positive integer. Then there exists only finitely many hyperbolic 3-manifolds N for which there is an epimorphism $\pi_1(M) \to \pi_1(N)$. In particular M d-dominates at most finitely many hyperbolic 3-manifolds.*

Proof. Note first that since $\overline{X}(M)$ consists of a finite number of points any hyperbolic 3-manifold dominated by M is closed and orientable. The reason being that if a compact 3-manifold has non-empty boundary or is non-orientable then the fundamental group admits a map to $\mathbf{Z}$ (see [10] Lemma 6.7). As discussed in §2.2 this implies a positive dimensional PSL$(2,\mathbf{C})$-character variety.

Let $\Gamma = \pi_1(M)$ be generated by $\langle a_1, \ldots, a_r \rangle$. Assume $\{M_j\}$ is an infinite collection of distinct closed orientable hyperbolic 3-manifolds with surjective maps $f_{j,*} : \pi_1(M) \to \pi_1(M_j)$. Let $\Gamma_j = \pi_1(M_j)$, and let ρ_j be a faithful discrete representation of Γ_j into SL$(2,\mathbf{C})$.

By composing ρ_j with $f_{j,*}$ we get a representation ϕ_j of Γ into SL$(2,\mathbf{C})$ whose image coincides with the faithful discrete representation of Γ_j. Thus we have an infinite number of characters of representations which are described by the tuples

$$\Phi_j = \left(\chi_{\phi_j}(a_s),\, \chi_{\phi_j}(a_s a_t),\, \chi_{\phi_j}(a_s a_t a_w)\right),$$

where $s = 1 \ldots r$, $s < t < w$ – recall §2.1.

Now $\overline{X}(M)$ is assumed to consist of a finite number of points, and so by the discussion in §2.2, $X(M)$ consists of a finite number of points. We can therefore find a character χ such that infinitely many of the tuples Φ_j must coincide with χ. However Lemma 2.1 then implies that the groups $\phi_j(\pi_1(M)) = \rho_j(\Gamma_j)$ are conjugate in SL$(2,\mathbf{C})$ (since the groups $\rho_j(\Gamma_j)$ are non-elementary). Thus the groups $\rho_j(\Gamma_j)$, which are faithful, discrete representations of the groups Γ_j are conjugate subgroups of SL$(2,\mathbf{C})$ and so by Mostow Rigidity this forces the manifolds M_j to be homeomorphic.

To deal with the last part of the lemma, assume that $f_j : M \to N_j$ are maps of degree $d \neq 0$ for an infinite number of distinct hyperbolic 3-manifolds N_j. By Lemma 3.1 there are coverings M_j of N_j, of degree a divisor of d for which Γ surjects under $f_{j,*}$ onto $\pi_1(M_j)$. The argument above shows the manifolds M_j are homeomorphic, thus it suffices to show that the N_j's are homeomorphic. To see this, merely observe that a closed hyperbolic 3-manifold covers at most finitely many distinct hyperbolic 3-manifolds. Briefly, by Jørgenson and Thurston's analysis of the structure of the set of volumes of hyperbolic 3-manifolds ([24]), there are at most finitely many hyperbolic 3-manifolds of a fixed volume. In addition there is a lower bound to the volume of a hyperbolic 3-manifold. This completes the proof. □

Lemma 4.3 has the following consequence of particular interest which strengthens Theorem 4.2.

Theorem 4.4. *Let M be a non-Haken hyperbolic 3-manifold. Then there exists only finitely many closed hyperbolic 3-manifolds N for which there is an epimorphism $\pi_1(M) \to \pi_1(N)$. In particular M dominates at most finitely many hyperbolic 3-manifolds.*

Proof. The first part follows from Lemmas 2.2 and 4.3. The second claim follows from Lemma 4.3 and the observation that for a finite volume hyperbolic 3-manifold M, there exists a d_0 such that for all $d \geq d_0$, M cannot d-dominate a hyperbolic 3-manifold. This easily follows from Theorem 3.3 and the fact that there is a lower bound to the volume of a hyperbolic 3-manifold. □

We remark here that examples of closed hyperbolic 3-manifolds which map onto infinitely many distinct hyperbolic 3-manifolds with degree 2 are given in [1]. By Theorem 4.4 these examples must be Haken (as is seen from the construction in [1]).

4.2.

To continue the proof of Theorem 4.1, we recall the following facts about Seifert fibered spaces with base S^2 and 3 exceptional fibers, see for example [12].

Lemma 4.5. *Let M be a Seifert fibered space with base S^2 and 3 exceptional fibers with $H_1(M, \mathbf{Z})$ finite, and with standard form*

$$\big(l;\ a_1, b_1;\ a_2, b_2;\ a_3, b_3\big),$$

where all numbers are integers and $a_i > b_i > 0$. Then the order of $H_1(M, \mathbf{Z})$ is

$$\left| a_1 a_2 a_3 \left(\frac{b_1}{a_1} + \frac{b_2}{a_2} + \frac{b_3}{a_3} + l \right) \right| .$$

An easy corollary of Lemma 4.5 is:

Corollary 4.6. *For any given positive integers C, a_1, a_2 and a_3, there are only finitely many Seifert fibered spaces M with standard form*

$$\big(l;\ a_1, b_1;\ a_2, b_2;\ a_3, b_3\big)$$

such that the order of $H_1(M, \mathbf{Z})$ is smaller than C.

We also collect the following information about $\overline{X}(M)$ when M is a Seifert fibered space.

Lemma 4.7. *Let M be a closed orientable Seifert fibered space with infinite fundamental group. Then $\overline{X}(M)$ has positive dimension, unless the base is one of the following 2-orbifolds:*

(a) *sphere with 3 cone points or,*

(b) $\mathbf{RP}^2$ *with 2 cone points of order 2.*

Proof. If M is covered by $S^2 \times \mathbf{R}$ then there are only four possibilities for M; the two S^2-bundles over S^1, $\mathbf{RP}^2 \times S^1$, or $\mathbf{RP}^3 \# \mathbf{RP}^3$. In all but the last case we have a map from $\pi_1(M)$ onto $\mathbf{Z}$, and as pointed out in §2.2 having a such map implies the dimension of $\overline{X}(M)$ is positive. In the last case $\pi_1(M) \cong \mathbf{Z}_2 * \mathbf{Z}_2$. As for $\mathbf{Z}$, $\mathbf{Z}_2 * \mathbf{Z}_2$ admits a positive dimensional PSL$(2, \mathbf{C})$-character variety. For $\mathbf{Z}_2 * \mathbf{Z}_2$ surjects all finite dihedral groups which can be realized as subgroups of PSL$(2, \mathbf{C})$.

We now assume the base of M is a Euclidean or hyperbolic 2-orbifold. Assume first of all that the base is orientable. If the genus of the base is at least 1 then the first homology of the base is infinite and so the discussion in §2.2 shows that $\overline{X}(M)$ will have positive dimension. Also, recall that the Teichmüller Space of a hyperbolic 2-orbifold can be viewed, using Fricke co-ordinates, as a component of the real points of the corresponding character variety. Since the hyperbolic structure on a hyperbolic 2-orbifold is completely determined by the Fricke co-ordinates, and there are infinitely many distinct hyperbolic structures in the case of genus 0 and at least 4 singular fibers (and hyperbolic) we deduce that $\overline{X}(M)$ has positive dimension in this case.

There is a unique Euclidean orbifold Q with base S^2 and 4 cone points of order 2. Since the isometry group of the Euclidean plane embeds in PSL$(2, \mathbf{C})$, and since such an orbifold admits infinitely many distinct Euclidean structures we deduce, arguing as above, that M as in the hypothesis of the Lemma with base Q, has positive dimensional $\overline{X}(M)$.

Case (a) of the Lemma is well-known to contain the class of non-Haken Seifert manifolds, and so $\overline{X}(M)$ cannot have positive dimension in such cases. There are Seifert fibered spaces with base S^2 and 3 exceptional fibers which are Haken, and these manifolds automatically have positive first betti number (see [12]), so we can omit these also.

In the case of non-orientable base, the argument is almost identical to the above. If the base is hyperbolic, using the fact that Isom $(\mathbf{H}^2) = \mathrm{PGL}(2, \mathbf{R})$ embeds in $\mathrm{PSL}(2, \mathbf{C})$ we quickly reduce to the case of non-orientable 2-fold quotients of base orbifolds arising in case (a). However, the base orbifold cannot have corner reflectors (cf. [22]), and so this cannot arise as a base of a Seifert manifold. In the Euclidean case, again using the fact the base has no corner reflectors, the only possibilities are the Klein bottle or $\mathbf{RP}^2$ with 2 cone points of order 2. However, the first homology of a Seifert manifold with base a Klein bottle is infinite, and as above this will contradict the assumption on $\overline{X}(M)$. This leaves case (c). □

With these preparatory lemmas we now prove Theorem 4.1.

Proof. Suppose there exists an infinite collection of non-homeomorphic geometric 3-manifolds $\{N_j\}$ and maps $f_j : M \longrightarrow N_j$ of fixed degree $d \neq 0$.

Let N_j' be the covering space of N_j corresponding to the image of the homomorphism $f_{j,*} : \pi_1(M) \longrightarrow \pi_1(N_j')$. By Lemma 3.1 the degree of the covering is a divisor of d. Since N_j is assumed geometric, N_j' is geometric with the same geometry (see [22]).

If N_j is not hyperbolic or covered by S^3, then it and N_j' are both Seifert fibered with infinite fundamental group, or admit a Sol-structure.

Now the hyperbolic case is dealt with by Lemma 4.3. If N_j' admits a Sol structure, then there exist homomorphisms from $\pi_1(N_j')$ onto $\mathbf{Z}$ or $\mathbf{Z}_2 * \mathbf{Z}_2$ (cf. [22]). As pointed out in §2.2 having a map onto $\mathbf{Z}$ will imply the dimension of $\overline{X}(M)$ is positive. Similarly, as pointed out in the proof of Lemma 4.7, since $\mathbf{Z}_2 * \mathbf{Z}_2$ surjects all finite dihedral groups which are subgroups of $\mathrm{PSL}(2, \mathbf{C})$ we also contradict the hypothesis on $\overline{X}(M)$.

Thus it remains to deal with Seifert fibered spaces.

Claim 1. *Only finitely many of the N_j' are non-homeomorphic Seifert fibered spaces with infinite fundamental group.*

In the case where N_j' admits a geometry modelled on $S^2 \times \mathbf{R}$, Lemma 4.7 implies N_j' and hence M has a positive dimensional $\mathrm{PSL}(2, \mathbf{C})$-character variety.

Thus assume that N_j' is Seifert fibered with infinite fundamental group, and is not covered by $S^2 \times \mathbf{R}$. Since $\overline{X}(M)$ is finite, by Lemma 4.7, we see that the base of any of the Seifert manifolds N_j' is one of (a), or (b). From [22] there is a unique orientable Seifert manifold with base $\mathbf{RP}^2$ with 2 cone points of order 2, thus it suffices to show only finitely many Seifert manifolds with base as in (a) or (b) of Lemma 4.7 can arise. Consider case (a) — the argument to deal with (b) is identical.

So assume infinitely many of the N_j' have a base S^2 with 3 cone points and each base is a Euclidean or hyperbolic 2-orbifold. Composing $f_{j,*}$ with the projection map induced by the Seifert fibration, we obtain a homomorphism from $\pi_1(M)$ onto a Euclidean or Fuchsian triangle group, and hence a representation into PSL$(2, \mathbf{C})$. Since $\overline{X}(M)$ consists of a finite number of points, it follows that there at most a finite number of topological types for the base orbifolds of the Seifert fibrations of the N_j'. Furthermore, $\overline{X}(M)$ being finite implies $H_1(M, \mathbf{Z})$ is finite (recall §2.2), so Lemmas 3.1 and 4.5 together with Corollary 4.6 show that infinitely many of the N_j' are homeomorphic. It follows that some N_j' covers with degree at most d infinitely many of the manifolds N_j. By passing to a finite cover of N_j' if necessary we obtain a Seifert manifold N' that is a regular cover of bounded degree of infinitely many of the manifolds N_j. However, in [16] it is shown that any finite group action on a Seifert manifold as above is geometric, and so there are only finitely many distinct actions. Hence only finitely many homeomorphism types of the manifolds N_j. This completes the proof of Claim 1. □

Claim 2. *Only finitely many of the N_j' are non-homeomorphic manifolds covered by S^3.*

The claim will follow from the next proposition.

Proposition 4.8. *With M as above, $\pi_1(M)$ surjects the fundamental groups of at most finitely many non-homeomorphic manifolds which are covered by S^3.*

Before proving the proposition we complete the proof of Claim 2.

We are assuming that M d-dominates infinitely many manifolds covered by S^3. In the notation above, we have coverings N_j' of N_j with covering degree a divisor of d, where each N_j' is also a manifold covered by S^3 and $\pi_1(M)$ surjects $\pi_1(N_j')$.

Applying Proposition 4.8 to the epimorphisms $\pi_1(M) \to \pi_1(N_j')$, we deduce that there are only finitely many distinct N_j' up to homeomorphism.

Since $\pi_1(N_j')$ is finite and the covering degree is bounded, so $|\pi_1(N_j)|$ is bounded. Since N_j is covered by S^3, there are only finitely many homeomorphism types of the manifolds N_j with fundamental groups of bounded order [22]. This completes the proof of Claim 2. □

We break the proof of Proposition 4.8 into two cases.

(1) the groups are the fundamental groups of a manifold N with N a lens space $L(n,m)$ with $|H_1(N,\mathbf{Z})| = n$ or one of the spaces with standard form, $(l;\ 2,1;\ 3,b_2;\ 3,b_3)$ with $|H_1(N,\mathbf{Z})| = |9+6b_2+6b_3+18l|$, or $(l;\ 2,1;\ 3,b_2;\ 4,b_3)$ with $|H_1(N,\mathbf{Z})| = |12+8b_2+6b_3+24l|$, or $(l;\ 2,1;\ 3,b_2;\ 5,b_3)$ with $|H_1(N,\mathbf{Z})| = |15+10b_2+6b_3+30l|$;

(2) the groups are the fundamental groups of a manifold N with N a prism space $(l;\ 2,1;\ 2,1;\ n,b_3)$ with $|H_1(N,\mathbf{Z})| = 4|n+b|$, where $b = b_3+nl$.

To deal with case (1) we show:

Lemma 4.9. *With M as above, $\pi_1(M)$ surjects the fundamental groups of at most finitely many distinct manifolds described in* (1). *In particular M 1-dominates at most finitely many manifolds in* (1).

Proof. Since $\overline{X}(M)$ is finite, as above, it follows that $H_1(M,\mathbf{Z})$ is finite and so by 3.1 there is an upper bound on the orders of the groups $H_1(N_j,\mathbf{Z})$. Then the conclusion of the Lemma follows since there are only finitely many lens spaces whose fundamental group is of a given order, and for the other spaces we simply apply Lemma 4.5 and Corollary 4.6. □

We now handle the manifolds in case (2). The important distinction for case (2) is that there are infinitely many distinct prism spaces with the same first homology group.

For convenience we write the standard form for the manifolds in case (2) as $(2,1;\ 2,1;\ n,b)$ with first homology of order $4|b+n|$ — here b can be any integer. We first prove an analogous result to Lemma 4.9.

Lemma 4.10. *With M as above, $\pi_1(M)$ surjects the fundamental groups of at most finitely many distinct manifolds described in* (2). *In particular M 1-dominates at most finitely many distinct prism spaces.*

Proof. Assume we have infinitely many epimorphisms $f_{j,*} : \pi_1(M) \to \pi_1(N_j)$ where N_j is a prism space of type $(2,1;\ 2,1;\ n_j,b_j)$. By Lemma 3.1, $|n_j+b_j|$ is bounded by the order of $H_1(M,\mathbf{Z})$. By passing to a subsequence, we may assume that $n_j+b_j = C$ is fixed.

Consider the $\mathbf{Z}_2$ action on $(2,1;\ 2,1;\ n_j,b_j)$ which induces the identity on the base orbifold and rotation on each regular S^1 fiber by π. The quotient

$(2,1;\ 2,1;\ n_j,b_j)/\mathbf{Z}_2$ has standard form $(1,1;\ 1,1;\ n_j,2b_j)$ if n_j is odd and $(1,1;\ 1,1;\ n_j/2,b_j)$ if n_j is even.

Now $(1,1;\ 1,1;\ n_j,2b_j) = L(2(b_j - n_j),\ *)$ and $(1,1;\ 1,1;\ n_j/2,b_j) = L(b_j - n_j,\ *)$ where $* \in \mathbf{Z}$. This construction provides epimorphisms of $\pi_1(M)$ onto infinitely many finite cyclic groups (arising as the fundamental groups of the lens spaces as above).

The argument for n_j odd or even is identical, so without loss of generality we assume that infinitely many of these n_j are odd. Then $\pi_1(M)$ surjects the fundamental groups of the family of lens spaces $\{L(2(b_j - n_j),\ *)\}$. As in proof of case (1) it follows that $2|b_j - n_j|$ must be bounded. By passing to a subsequence we may therefore assume that $2(b_j - n_j) = C'$ for some constant C'.

Now substitute, $2n_j = 2b_j - C'$ into $n_j + b_j = C$ to deduce that b_j is constant. Hence it follows that n_j is a constant. Hence only finitely topological types of lens spaces $\{L(2(b_j - n_j),\ *)\}$ which is a contradiction. This contradiction finishes the proof of Lemma 4.10. □

Lemmas 4.9 and 4.10 complete the proof of Claim 2, and hence the proof of Theorem 4.1 is now complete. □

4.3.

We can extend some of the results in the previous section to other Seifert fibered spaces.

Proposition 4.11. *Every closed orientable Seifert fibered space M with $|H_1(M,\mathbf{Z}))|$ finite or $|\mathrm{Tor}\,(H_1(M,\mathbf{Z}))|$ not divisible by 4, 1-dominates only finitely many geometric 3-manifolds.*

Proof. If a 3-manifold dominated by M is geometric, then it must also be a Seifert manifold (see [25]). When $\pi_1(N)$ is infinite, it is known ([21]) that each degree one map $f : M \to N$ is a vertical pinch (see [21] for a definition). Since each vertical pinch of M either decreases the genus of the base orbifold or decreases the number of singular fibers, M admits only finitely many vertical pinches (up to homotopy). It follows that M 1-dominates only finitely many Seifert manifolds with infinite π_1.

The fact that M 1-dominates at most finitely many 3-manifolds in case (1) of the proof Theorem 4.1 follows from Lemma 3.1(2). Similarly, if $H_1(M, \mathbf{Z})$ is finite, then the proof in Theorem 4.1 applies to show that M 1-dominates at most finitely many 3-manifolds in case (2) of the proof of Theorem 4.1.

Now if $|\mathrm{Tor}\, H_1(M, \mathbf{Z})|$ is not a multiple of 4, by the fact that $|H_1(N, \mathbf{Z})|$ is a multiple of 4 for each 3-manifold in case (2) above, we deduce from Lemma 3.1(2) (i.e. that $\mathrm{Tor}\,(H_1(N, \mathbf{Z}))$ is a direct summand of $\mathrm{Tor}\,(H_1(M, \mathbf{Z}))$ for a degree 1 map), that there is no degree one map $f : M \to N$ for each N in case (2) above. □

5. Finiteness Results: Bounded 3-Manifolds.

Throughout this section, M will denote a compact 3-manifold whose boundary consists of tori, and whose interior is hyperbolic. With a slight abuse of notation, such a manifold will be referred to as *a cusped* hyperbolic 3-manifold. The boundary of M will be denoted by ∂M. If $\partial_0 M$ is a component of ∂M then $\pi_1(\partial_0 M)$ is called a peripheral subgroup of $\pi_1(M)$.

If M is a cusped hyperbolic 3-manifold, N a hyperbolic 3-manifold of finite volume and $f : \pi_1(M) \to \pi_1(N)$ an epimorphism, then f is called *peripheral preserving* if for any peripheral subgroup $P < \pi_1(M)$, $f(P)$ is conjugate into a peripheral subgroup of $\pi_1(N)$. We allow the case that $f(P) = 1$.

The only hyperbolic geometric fact we shall require in what follows is simply that with M as above, and $\rho : \pi_1(M) \to SL(2, \mathbf{C})$ a faithful discrete representation, for any component $\partial_0 M$ of ∂M, $\rho(\pi_1(\partial_0 M))$ consists of parabolic transformations so that $\mathrm{tr}\,(\rho(\pi_1(\partial_0 M)) = \pm 2$.

Following the notation of [2], we call M NSL (*Non-Sufficiently Large*) if M does not contain a closed embedded essential surface. The main result of this section is the analogue for NSL manifolds to Theorem 4.4. All maps of non-zero degree are assumed proper.

Theorem 5.1. *Let M be as above, and NSL. Then there exist only finitely many hyperbolic 3-manifolds of finite volume N for which there is a peripheral preserving epimorphism $\pi_1(M) \to \pi_1(N)$. In particular M dominates at most finitely hyperbolic 3-manifolds.*

We make some preliminary comments. The strategy is similar to that of the proof of 4.4. However in this case, by [24] (see also [6]) $X(M)$ has positive dimension. The key assumption here will be NSL.

As in the proof of Theorem 4.4, since M is a finite volume hyperbolic 3-manifold there exists a d_0 such that for all $d \geq d_0$, M cannot d-dominate a hyperbolic 3-manifold. Thus the proof of the last sentence in Theorem 5.1 will follow directly from the first part of the theorem together with the observation that a proper map induces a peripheral preserving epimorphism on a subgroup of finite index.

To avoid clutter of notation the argument is best illustrated in the proof of the following:

Theorem 5.2. *Let M have 1 cusp, and be NSL. Then there exist only finitely many hyperbolic 3-manifolds N for which there is a peripheral preserving epimorphism $\pi_1(M) \to \pi_1(N)$. In particular M 1-dominates at most finitely many cusped hyperbolic 3-manifolds.*

Proof. Fix a framing for ∂M so that $\pi_1(\partial M) = \langle a, b \rangle$. Suppose that $\pi_1(M)$ admits infinitely many epimorphisms as in the hypothesis onto groups $\{\pi_1(N_j)\}$, where N_j is hyperbolic 3-manifold of finite volume. Let $f_{j,*}$ be the surjective homomorphism induced on fundamental groups with $a_j = f_{j,*}(a)$ and $b_j = f_{j,*}(b)$. Note that, at this point we do not assume that the $N_j's$ are cusped. If N_j is closed, then peripheral preserving means that $f_{j,*}(\pi_1(\partial M)) = 1$, and the argument below will work equally well.

Let $\rho_j : \pi_1(N_j) \to \mathrm{SL}\,(2, \mathbf{C})$ be a faithful discrete representation, and $\phi_j : \pi_1(M) \to \mathrm{SL}\,(2, \mathbf{C})$ the induced representation obtained by composing ρ_j with $f_{j,*}$. Since $\langle \rho_j(a_j), \rho_j(b_j) \rangle$ is a subgroup of the peripheral subgroup of $\rho_j(\pi_1(\partial M_j))$, it consists of elements of trace ± 2. On passing to the character variety $X(M)$, we therefore have an infinite number of characters χ_{ϕ_j} which take the value ± 2 on $\langle a, b \rangle$. Now $X(M)$ has a finite number of irreducible components and so there is a component X_0 containing an infinite subsequence of the characters χ_{ϕ_j} which take the value ± 2 on $\langle a, b \rangle$. Let $W \subset X_0$ be the the set of all characters in X_0 which take the values ± 2 on $\langle a, b \rangle$. Then W is defined by a set of equations of the form $I_g(\chi)^2 - 4 = 0$ for $g \in \langle a, b \rangle$ (recall §2.1 for the definition of I_g). So W is closed algebraic subset of X_0. By construction W is known to contain an infinite number of points, and so must have positive dimension, in particular W contains a curve C say. Then Lemma 2.4 implies the existence of a closed embedded essential surface in M contrary to assumption. The proof is is now complete

Note that in the case of degree 1 maps, Lemma 3.2 forces the manifolds N_j to have one cusp. □

The proof of Theorem 5.1 follows by a similar argument. We point out the important extensions.

As above we assume the existence of infinitely many peripheral preserving epimorphisms onto groups $\pi_1(N_j)$. As before if $\langle a, b\rangle$ is a peripheral subgroup of $\pi_1(M)$, under the map induced by composing with the complete representations of $\pi_1(N_j)$ the image will still consist of elements of trace ± 2 (again we allow some of the peripheral subgroups to map to $\pm I$). Therefore as above, there is a component X_0 containing an infinite subsequence of characters which take the value ± 2 on all peripheral subgroups. Now apply Lemma 2.4 to W as constructed in the proof of Theorem 5.2 to deduce a closed embedded essential surface in M, and this contradiction completes the proof. □

Many manifolds satisfy the hypothesis of Theorem 5.1, for example by [9] all 2-bridge hyperbolic link complements in S^3. Related results to Theorem 5.1 are proved in [1]. For example, a consequence of [1] Corollary 2.7, is that the complement of a fibered knot in a homology 3-sphere with irreducible Alexander polynomial is minimal with respect to degree 1 mappings.

6. Examples: Minimal hyperbolic 3-manifolds.

In practice it would seem that for many manifolds as in Theorem 4.4, there will be no maps of non-zero degree to other closed hyperbolic 3-manifolds. As a specific example we shall deal with the so-called Weeks manifold, which has the surgery description $(-5, 1)$, $(5, 2)$ surgery on the Whitehead link (see Figure 1).

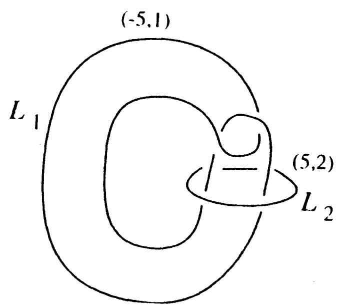

Figure 1

We denote the Weeks manifold by M_W. M_W was shown to be hyperbolic in [26], and is conjectured to have the smallest volume of a hyperbolic 3-manifold. It is known that it is smallest amongst those manifolds which are arithmetic, [4]. That the Weeks manifold is non-Haken can be deduced from the work of [5] or [8]. The point being that $(-5,1)$ surgery on one component of the Whitehead link is a 1-punctured torus bundle over the circle (the "sister" to the figure eight knot) and the methods of [5] or [8] show there is no closed embedded essential surface, and allow calculation of all boundary slopes. Let ρ_W denote the faithful discrete representation of $\pi_1(M_W)$ into $\mathrm{SL}(2,\mathbf{C})$.

We shall prove the following theorem in §6.1 below.

Theorem 6.1. *M_W does not admit any map of non-zero degree to a closed hyperbolic 3-manifold or a Seifert fibered space with infinite fundamental group. Furthermore, the only geometric manifolds 1-dominated by M_W are the lens spaces $L(5,1)$ or $L(5,2)$.*

6.1.

Before commencing on the proof we recall some salient facts. We shall call a representation of a group Γ in $\mathrm{SL}(2,\mathbf{C})$ *elliptic* if the image contains an elliptic element, that is an element x whose trace satisfies $\mathrm{tr}^2(x) < 4$. The corresponding character is also called elliptic. If M is a hyperbolic 3-manifold, and Γ denotes the faithful discrete representation in $\mathrm{SL}(2,\mathbf{C})$, then as Γ acts freely on $\mathbf{H}^3$, Γ contains no elliptic elements.

To prove the first part of Theorem 6.1 we establish:

Proposition 6.2. *$X(M_W)$ consists of a finite number of points which, apart from the characters associated to ρ_W or $\overline{\rho_W}$, consists of elliptic characters.*

Proof. As discussed above M_W is obtained by surgery on the sister to the figure eight knot complement. We find it convenient for calculations to use a presentation for $\pi_1(M_W)$ that is provided by Snap Pea ([27]), namely:

$$\pi_1(M_W) = \langle a, b \mid a^2b^2a^2b^{-1}ab^{-1} = 1,\ a^2b^2a^{-1}ba^{-1}b^2 = 1 \rangle .$$

We can conjugate a representation in $\mathrm{SL}(2,\mathbf{C})$ so that the images of a and b are the matrices:

$$\begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y & 0 \\ r & y^{-1} \end{pmatrix}$$

respectively. As we are looking for representations into $\mathrm{SL}(2,\mathbf{C})$ x and y are always non-zero.

Mathematica was used in the following calculations. Write the first relation as $w = 0$ where $w = a^2b^2a^2 - ba^{-1}b$. This gives the following equations.

- $w_{11} = r\,x^2 + r\,x^4 + r\,x\,y^2 + r\,x^2\,y^2 + r\,x^4\,y^2 - y^3 + x^5\,y^3$.
- $w_{12} = 1 + x^2 + r\,x\,y + 2\,r\,x^3\,y + r\,x^5\,y + x^3\,y^2 + r\,x\,y^3 + 2\,r\,x^3\,y^3 + r\,x^5\,y^3 + x^4\,y^4 + x^6\,y^4$.
- $w_{21} = r\,\left(x - x^2 + r\,x\,y - y^2 + x\,y^2\right)$.
- $w_{22} = 1 - x^5 + r\,x\,y + r\,x^3\,y + r\,x^4\,y + r\,x\,y^3 + r\,x^3\,y^3$.

Note from the equation for w_{21} we have either $r = 0$ or we can solve for r in terms of x and y (which as noted above are always non-zero).

Consider the case of $r = 0$. From the equation for w_{22} we note that this the forces $x^5 = 1$, so x is a 5^{th} root of unity. We claim that x is a non-trivial 5^{th} root of unity. Assume $x = 1$. This implies $w_{12} = 2 + y^2 + 2y^4$. Now use the second relation. It seems most convenient from the point of view of calculation to write the second relation as $u = 0$ with $u = a^2b^2 - b^{-2}ab^{-1}a$. Observe that $u_{11} = y^5 - 1$, and so we cannot have this holding simultaneously with w_{12}. Hence x is a non-trivial 5^{th} root of unity.

Recomputing the relations with the matrices

$$\begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$$

with x a non-trivial 5^{th} root of unity, it is easy to check that this forces $y = 1$, and hence the representation has image a cyclic group of order 5.

We now assume r is non-zero and is given from above by:

$$r = \frac{(x^2 - x + y^2 - xy^2)}{xy}.$$

Using this and re-working the above equations gives:

- $w_{11} = (-1 + x)\,x\,\left(1 + x^2 + y^2 - x\,y^2 + x^2\,y^2 + y^4 + x^2\,y^4\right)$.
- $w_{12} = \left(1 - x + x^2 - x^3 + x^4\right)\left(1 + x^2 + y^2 - xy^2 + x^2y^2 + y^4 + x^2y^4\right)$.
- $w_{21} = 0$.
- $w_{22} = (-1 + x)\,(-1 + x)\,\left(-1 - x^2 - y^2 + x\,y^2 - x^2\,y^2 - y^4 - x^2\,y^4\right)$.

Notice that the expressions for w_{11},w_{12} and w_{22} all have the common factor

$$p(x,y) = \left(1 + x^2 + y^2 - x\,y^2 + x^2\,y^2 + y^4 + x^2\,y^4\right).$$

The only way we can simultaneously satisfy all the above equations is for $p(x,y) = 0$. As before we now use the second relation together with $p(x,y)$ to describe all the possibilities. The first thing we notice is that

$$u_{12} = (x - y)\,(-1 + x\,y)\,,$$

and so we must have $x = y$ or $x = 1/y$. With $x = y$, $p(x,y)$ is simply the polynomial in x given by:

$$p(x) = 1 + 2\,x^2 - x^3 + 2\,x^4 + x^6,$$

which must solve to zero to determine a representation. Solving for $z = x + x^{-1}$ yields the polynomial in z, $z^3 - z - 1 = 0$. From the equation for r we see that $r = 2 - z$. The complex roots of the equation for z correspond to ρ_W and $\overline{\rho_W}$ (cf. [4]), and since the real root is approximately 1.32472, this will determine an elliptic representation. A similar argument applies to $y = 1/x$, which also yields the same characters as above.

These are the only possible solutions to the representation equations and the proposition is now proved. □

Note the proof actually shows the following:

Corollary 6.3. *There are* 4 *representations of* $\pi_1(M_W)$ *up to conjugacy, into* SL $(2, \mathbf{C})$, *namely* ρ_W, $\overline{\rho_W}$, *a representation with image a cyclic group of order* 5, *and the representation obtained from the real "Galois conjugate" of* ρ_W.

This last representation is actually faithful and has image in SU (2) — this is a consequence of the fact that the invariant quaternion algebra of M_W is ramified at the real place, cf. [18] for a discussion of such matters.

We now prove Theorem 6.1.

The hyperbolic case follows easily from Corollary 6.3, since if M_W dominated a hyperbolic 3-manifold N it would follow from Corollary 6.3 that the complete faithful representation would coincide with ρ_W or $\overline{\rho_W}$, and so Mostow's Rigidity Theorem implies N would be homeomorphic to M_W.

From the surgery description of M_W, $H_1(M_W, \mathbf{Z}) = \mathbf{Z}_5 \oplus \mathbf{Z}_5$. From this it follows that $\pi_1(M_W)$ cannot surject the fundamental group of a Seifert fibered space with infinite fundamental group and non-orientable base since these have double covers, nor any Seifert fibered space which has a Euclidean base since Euclidean triangle groups do not have abelianization with order divisible by 5. Thus if M_W dominates a Seifert fibered space with infinite fundamental group then we can assume the base is a co-compact Fuchsian group.

Now if $\pi_1(M_W)$ surjects a Fuchsian group (triangle group or otherwise), we obtain a homomorphism to a Fuchsian group which determines a representation of $\pi_1(M_W)$ into $\mathrm{PSL}(2, \mathbf{C})$ with image a co-compact Fuchsian group F. We claim that this representation must lift to $\mathrm{SL}(2, \mathbf{C})$. Assuming this claim, we proceed as follows. From the list of representations given by Corollary 6.3, if a representation as above lifted, it would follow that $\pi_1(M_W)$ would have to be isomorphic to a Fuchsian group, and this is impossible since $\pi_1(M_W)$ is a closed hyperbolic 3-manifold. Thus we conclude that M_W cannot dominate a Seifert fibered space with infinite fundamental group

To prove the claim, we argue as follows. From the theory of the cohomology of groups, central extensions of a group Γ are classified by elements of $H^2(\Gamma, \mathbf{Z}_2)$, (see [11] Theorem 10.3) and in particular this is zero if and only if any extension splits. Therefore given a representation of $\pi_1(M_W)$ as above with image F, there is a central extension of F by $\mathbf{Z}_2$ in $\mathrm{SL}(2, \mathbf{C})$. Now since M is a $\mathbf{Z}_2$ homology 3-sphere, we have $H^2(\pi_1(M_W), \mathbf{Z}_2) = \{0\}$, and so this holds for F as well. Thus the extension of F we see is $F \times \mathbf{Z}_2$ and we so we can then lift the representation of $\pi_1(M_W)$ into $\mathrm{SL}(2, \mathbf{C})$ with image F. This proves the claim.

To deal with the finite fundamental group case, it suffices to consider those geometric 3-manifolds with finite fundamental group which are minimal with respect to degree 1 mappings. A list of these can be found in [14]. From case (2) of the finite fundamental group considerations of the proof of Theorem 4.1, any prism manifold has first homlogy of order divisible by 4. The first homology of M_W together with Lemma 3.1 shows that M_W cannot map by degree 1 onto such a manifold. The only other possibilities provided by Lemma 3.1 are the lens spaces $L(5,1)$ and $L(5,2)$ and the Poincaré homology sphere. However, since the fundamental group of the Poincaré homology sphere is the binary icosahedral group and this embeds in $\mathrm{SL}(2, \mathbf{C})$ as the non-trivial central extension of A_5, the description of all the representations of $\pi_1(M_W)$ given above rules this out. The proof is completed by the following lemma.

Lemma 6.4. *M_W admits a degree 1 map to $L(5,1)$ and $L(5,2)$.*

Proof. We make use of the surgery description of M_W given in Figure 1. For $i = 1, 2$, let N_i denote a tubular neighborhood of L_i, and T_i denote the surgery solid tori with core curve t_i. The T_i's are to be identified with ∂N_i under the gluing maps f_i, $i = 1, 2$.

Generators of $H_1(M_W, \mathbf{Z}) = \mathrm{Tor}\,(H_1(M_W, \mathbf{Z})) = \mathbf{Z}_5 \oplus \mathbf{Z}_5$ are the homology classes represented by the two cores t_1 and t_2. To prove the Lemma, it suffices by [15] Theorem 2.2 to show that the self-linking numbers $\left|\,\mathrm{lk}\,(t_1, t_1)\right| = \frac{1}{5}$ and $\left|\,\mathrm{lk}\,(t_2, t_2)\right| = \frac{2}{5}$ (a good reference for the material on linking pairs is [23]).

We shall deal with the case of t_2, and using the fact that the two components of the Whitehead link are unknotted circles which are interchanged by a symmetry, the case of t_1 follows by a similar argument. Thus we aim to show $\left|\,\mathrm{lk}\,(t_2, t_2)\right| = \frac{2}{5}$.

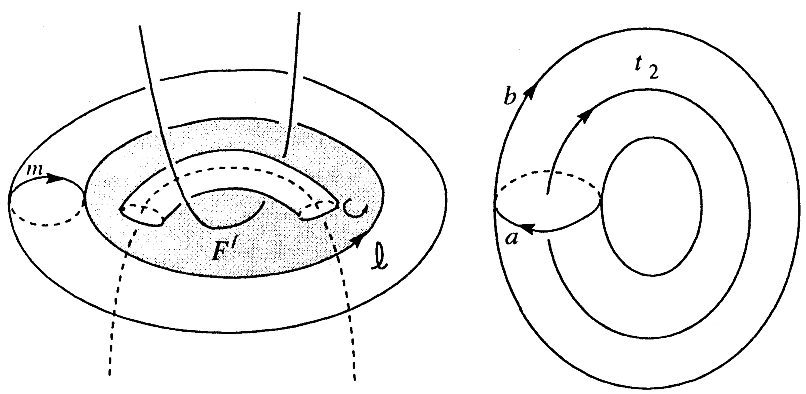

Figure 2

Let m and ℓ, be a meridian-longitude pair on ∂N_2 for the complement $S^3 \setminus \mathrm{Int}\,(N_2)$. Since the linking number $\mathrm{lk}\,(L_1, L_2) = 0$, the longitude ℓ bounds an oriented surface F' in the Whitehead link complement such that $\partial F' = \ell$ (with orientation) as shown in Figure 2.

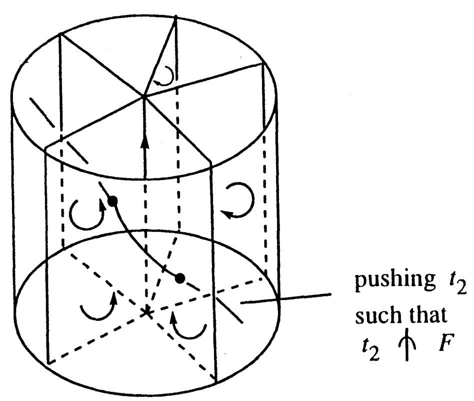

Figure 3

Let a be the meridian curve on the solid torus T_2, and choose a longitude b on T_2 such that $f_2(a) = 5m + 2\ell$ and $f_2(b) = 2m + \ell$. Then $f_2^{-1}(\ell) = -5b + 2a$. Clearly $5b - 2a$ and $5t_1$ are homologous and therefore they bound an oriented 2-chain F'' in the solid torus T_2 with $\partial F'' = 5t_2 - (5b - 2a)$. A concrete description of such an F'' can be obtained as follows. Consider the cylinder in Figure 3, where the five oriented rectangles meet along the central axis of the cylinder. When we identify the top and the bottom of the cylinder via a twist of $\frac{2\pi}{5}$, we get the solid torus T_2 and the quotiont of the five rectangles in T_2 is our required F''.

Let $F = F' \cup_l F''$ so that orientations agree. Then F is an oriented 2-chain and $\partial F = 5t_2$. Now perturb t_2 so that it becomes is transverse to F. Then from Figure 3 we see there are 2 points of intersection with the same sign. Hence $\left|\operatorname{lk}(t_2, t_2)\right| = \frac{2}{5}$. □

6.2.

We now construct a closed hyperbolic 3-manifold that is minimal with respect to the partial ordering discussed in the Introduction.

Theorem 6.5. *Let M be the result of $(1, 2)$ Dehn surgery on the figure eight knot complement. Then M is minimal in the class of geometric 3-manifolds.*

The proof follows the same idea as the proof of Proposition 6.2 in that we examine all representations into $\mathrm{SL}(2, \mathbf{C})$. In this case the calculation is easier. Again $\pi_1(M)$ is 2-generator (since the figure eight is 2-bridge) and in this case the generators are conjugate in $\pi_1(M)$. Therefore any representation ρ of $\pi_1(M) = \langle a, b\rangle$ into $\mathrm{SL}(2, \mathbf{C})$ will force $\mathrm{tr}(\rho(a)) = \mathrm{tr}(\rho(b))$. In this case a normalization for the matrices used above is:

$$\begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x & 0 \\ r & x^{-1} \end{pmatrix}$$

or

$$\begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x^{-1} & 0 \\ r & x \end{pmatrix}.$$

We remark here that it was shown in [24] that M is hyperbolic and non-Haken.

Proof. The figure eight knot complement has a presentation on two meridional generators as,

$$\left\langle a, b \;\middle|\; \left(ab^{-1}a^{-1}b\right) a \left(ab^{-1}a^{-1}b\right)^{-1} b^{-1} = 1 \right\rangle,$$

with a longitude being $\ell = a^{-1}b^{-1}aba^{-2}bab^{-1}a$. The effect of $(1,2)$-surgery is to introduce the relation $a\ell^2 = 1$. As before we employ mathematica in our calculations, and to do this it is convenient to write the relations as:

$$w = \left(ab^{-1}a^{-1}b\right) a - b\left(ab^{-1}a^{-1}b\right) = 0 \quad \text{and} \quad u = a\ell - \ell = 0.$$

Evaluating $w = 0$ using the first normalization given above we find either $r = 0$ or

$$p(x,r) = 1 - r - 3\,x^2 + 3\,r\,x^2 - r^2\,x^2 + x^4 - r\,x^4 = 0.$$

The case of $r = 0$ is quickly dealt with since in this case the representation is solvable (being a subgroup of the upper triangular matrices) and as M is a homology sphere no such non-trivial representation exists.

Thus assume $p(x,r) = 0$. Evaluating u, we find that the entries u_{11}, u_{12} and u_{22} are irreducible polynomials in x and r. u_{12} is given by:

$$r^2\,(1+x)\,\left(1+x^2\right)\,\left(1 - r - 3\,x^2 + 3\,r\,x^2 - r^2\,x^2 + x^4 - r\,x^4\right).$$

Note, this last factor is simply $p(x,r)$. We have ruled out $r = 0$ above, and $x = 1$ and $x = i$ are also easily checked not to give representations. Using resultants of $p(x,r)$ and the remaining u_{ij}'s to eliminate r, one sees that the values of x which yield representations satisfy the equation:

$$-1-2x-x^2+4x^4+8x^5+10x^6+13x^7+10x^8+8x^9+4x^{10}-x^{12}-2x^{13}-x^{14} = 0.$$

Using this we see that the trace $z = x + x^{-1}$ satisfies:

$$-1 + 10\,z^2 + 5\,z^3 - 12\,z^4 - 6\,z^5 + 2\,z^6 + z^7 = 0.$$

This polynomial has 1 pair of complex conjugate roots, and these correspond to the faithful discrete representation and its complex conjugate. The equation for z has 4 real roots which have absolute value less than 2 and one solution of absolute value greater than 2. Using the invariant quaternion algebra ([18]), once again we see that the real roots correspond to faithful real representations of $\pi_1(M)$, four into $\mathrm{SU}(2)$ and one into $\mathrm{SL}(2,\mathbf{R})$. In particular none of these can be the complete representation of the fundamental group of a closed hyperbolic 3-manifold. Thus we conclude as before that M cannot map by a map of non-zero degree to a hyperbolic 3-manifold. The case of the second normalization is handled in the same way.

Indeed, again as in the case of the Weeks manifold assuming the manifolds are geometric, the above calculation shows that M cannot map by non-zero degree to any closed orientable irreducible 3-manifold. The argument is identical to that of the Weeks manifold so we only sketch the argument. Exactly as in the case of the Weeks manifold, M cannot map by non-zero degree to a Seifert fibered space with infinite fundamental group. For the finite case, since M is a homology sphere the only possible way that M can map by non-zero degree is for the image manifold to be the Poincaré homology sphere and as noted in the proof of Proposition 6.2 its fundamental group admits a faithful representation into $\mathrm{SL}(2, \mathbf{C})$. Thus the existence of a map of non-zero degree is precluded by the discussion above. □

We have also checked the homology spheres obtained by $(1,3)$, $(1,4)$ and $(1,5)$ surgery on the figure eight knot complement. The calculations are completely analogous and show that these manifolds are also minimal with respect to the partial order $\geq_1$. It seems to natural to conjecture that all the $(1,n)$ Dehn surgeries on the figure eight knot are minimal with respect to $\geq_1$.

As a final remark we point out that we know of no explicit example of a map of degree 1 between non-Haken hyperbolic 3-manifolds. However since there are examples of non-trivial coverings between non-Haken hyperbolic 3-manifolds (see for example [19] for such a pair), there are maps of degree $d > 1$ between non-Haken hyperbolic 3-manifolds.

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