

# A rigidity theorem for periodic minimal surfaces

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We prove that the Helicoid can be characterized as the only properly embedded non rigid minimal surface in  $\mathbb{R}^3$  that is invariant by an infinite discrete group  $G$  of ambient isometries such that the quotient surface in  $\mathbb{R}^3/G$  has finite topology.

## 1. Introduction.

A classical question about submanifolds is to decide if the inclusion is, up to ambient isometries, the unique isometric immersion of such a manifold in the ambient space. When we consider this problem in minimal surface theory in  $\mathbb{R}^3$ , we find several nice theorems that give an idea of abundance of rigid surfaces, and also open questions and conjectures to be solved.

The standard notion of rigidity for minimal surfaces is the following: a properly embedded minimal surface is said to be (minimally) *rigid* if the inclusion map of the surface into  $\mathbb{R}^3$  represents the unique isometric minimal immersion of such a surface up to a rigid motion in  $\mathbb{R}^3$ .

As every minimal surface can be locally and isometrically deformed by its associate surfaces, rigidity theory has a global nature for this type of surfaces. For properly embedded minimal surfaces with more than one end, Choi, Meeks and White [2] proved that rigidity holds. However, this result fails to hold when the surface has only one end, as demonstrates the Helicoid. They also conjectured that any properly embedded nonsimply-connected minimal surface is minimally rigid. In this direction, Meeks and Rosenberg [9] obtained rigidity if the symmetry group of the surface contains two linearly independent translations, hence doubly and triply periodic properly embedded minimal surfaces are rigid. Again the Helicoid shows that this statement does not extend if we only impose that the symmetry group contains an infinite cyclic group (singly periodic minimal surfaces). In this case, Meeks [8] proved a *weak rigidity* property, provided that the induced

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quotient surface has finite topology: every intrinsic isometry is induced by a rigid motion of  $\mathbb{R}^3$ . In this paper we demonstrate that except for the Helicoid, (strong) rigidity is always satisfied in this family of surfaces. More precisely,

*If  $\tilde{M} \subset \mathbb{R}^3$  is a non flat, properly embedded minimal surface invariant by an infinite discrete group  $G$  of isometries of  $\mathbb{R}^3$  and  $\tilde{M}/G$  has finite topology, then  $\tilde{M}$  is rigid or it is the Helicoid.*

Every singly periodic properly embedded minimal surface  $\tilde{M} \subset \mathbb{R}^3$  induces a minimal surface  $M \subset \mathbb{R}^3/T$  or  $M \subset \mathbb{R}^3/S_\theta$ , where  $T$  is a non trivial translation and  $S_\theta$  is a right-hand screw motion around the  $x_3$ -axis with rotation angle  $\theta \in [0, 2\pi[$ . For such a surface, Meeks and Rosenberg [10] proved that  $M$  has finite total curvature if and only if it has finite topology, so in this case it has also finite conformal type. Moreover, they studied its behaviour at infinity: all the ends are simultaneously asymptotic to parallel planes, flat annuli or to ends of Helicoids. Following the ideas in [8] we notice that Meeks proved (strong) rigidity except when the ends are asymptotic to Helicoids. Hence we will concentrate in the remaining case. The main tool of our reasoning is the existence of a one-parameter deformation for a singly periodic minimal surface  $\tilde{M} \subset \mathbb{R}^3$  with helicoidal type ends that is not rigid. This technique has been useful for studying the index of complete minimal surfaces [12] and also for obtaining uniqueness and non existence theorems [7, 14, 15]. In fact, this deformation makes sense if we only impose that the flux of  $\tilde{M}$  along every compact cycle is a vertical vector, which gives us another characterization of the Helicoid:

*Let  $\tilde{M} \subset \mathbb{R}^3$  be a properly embedded minimal surface invariant by a screw motion  $S_\theta$ , such that the quotient surface in  $\mathbb{R}^3/S_\theta$  has finite topology and helicoidal type ends. If  $\tilde{M}$  has vertical flux, then it is the Helicoid.*

This statement generalizes a theorem in [14] where the Helicoid was characterized as the only such surface in  $\mathbb{R}^3/T$  with genus zero. Finally, we would like to point out that the general Choi-Meeks-White conjecture remains unsolved: in this line, Hoffman, Karcher and Wei [4] presented a genus-one surface in  $\mathbb{R}^3$  with only one end and infinite total curvature. It would be an interesting problem to decide if surfaces like this one also verify a rigidity condition.

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## 2. Background.

We will begin by exposing some well-known facts about minimal surfaces, besides fixing the notation in the paper. More details can be found in Osserman [13] and Hoffman and Karcher [3]. Consider a conformal minimal immersion  $\psi : M \rightarrow \mathbb{R}^3$  of a surface  $M$  into the three-dimensional Euclidean space—all surfaces in this paper are supposed to be connected and orientable—. The *flux* of  $\psi$  along a closed curve  $\Gamma \subset M$  is defined as the integral of a conormal unit field  $\eta$  along the curve, that is

$$\mathbf{flux}(\psi, \Gamma) = \int_{\Gamma} \eta \, ds,$$

where  $ds$  is measured with respect to the metric induced by  $\psi$ . This vector does not depend on the cycle  $\Gamma$  in its homology class and can be viewed as the period vector along  $\Gamma$  of the—in general, not well-defined on  $M$ —conjugate minimal surface of  $\psi$ .

From the Weierstrass representation [3, 13] we know that  $\psi$  can be determined by giving a meromorphic function  $g$  and a holomorphic one-form  $\omega$  on  $M$ , so

$$\psi = \left( \frac{1}{2} \left( \overline{\int \omega} - \int g^2 \omega \right), \operatorname{Real} \int g \omega \right) \in \mathbb{C} \times \mathbb{R} \equiv \mathbb{R}^3.$$

We recall that  $g$  is the stereographic projection from the North Pole of the Gauss map of  $\psi$ . We will also use the following notation

$$F = \frac{1}{2} \int \omega, \quad G = \frac{1}{2} \int g^2 \omega, \quad x_3 = \operatorname{Real} \int \phi_3,$$

where  $\phi_3 = g\omega$ . Hence,  $\psi = (\overline{F} - G, x_3)$ . Now we recall what has come to be called the López-Ros deformation (the reader can read the details in the work of Pérez and Ros [14]): It is defined by considering, for each positive number  $\lambda$ , the Weierstrass pair  $(g_\lambda = \lambda g, \omega_\lambda = \frac{1}{\lambda} \omega)$ . These meromorphic data define, up to an additive constant, a possibly multivalued minimal immersion  $\psi_\lambda : M \rightarrow \mathbb{R}^3$ . This map is single valued on  $M$  if and only if  $\mathbf{flux}(\psi, \Gamma)$  is vertical, for each closed curve  $\Gamma \subset M$ —in this case,  $\psi$  is said to have *vertical flux*—. Moreover, completeness and finiteness of the total curvature are preserved by this deformation. We will emphasize two geometric properties of  $\{\psi_\lambda\}_{\lambda>0}$  for later uses (their proofs can be found in [14]).

**Lemma 1.** *Let  $\psi : M \longrightarrow \mathbb{R}^3$  be a conformal minimal immersion with vertical flux. Then,*

- i) *If  $\psi$  is invariant by a screw motion  $S_\theta$  obtained by rotation around the  $x_3$ -axis by an angle  $\theta \in ]0, 2\pi[$  followed by a non trivial translation along the same axis, and this screw motion induces a holomorphic transformation of  $M$ , then all the  $\psi_\lambda$ 's can be chosen invariant by the same  $S_\theta$ .*
- ii) *If  $p \in M$  is a point where the Gauss map of  $\psi$  is vertical, then for any neighbourhood  $D$  of  $p$  there exists a positive number  $\lambda$  such that  $\psi|_D$  is not embedded.*

In the case  $\theta = 0$ , that is, when  $S_0$  is a vertical translation, Pérez and Ros proved that all the  $\psi_\lambda$ 's can be chosen invariant by translations that could depend on  $\lambda$ .

If  $\psi : M \longrightarrow \mathbb{R}^3/S_\theta$ ,  $0 \leq \theta < 2\pi$  is a proper non flat minimal embedding, the strong halfspace theorem of Hoffman and Meeks [5] insures that  $\psi$  lifts to a connected properly embedded singly periodic minimal surface  $\tilde{\psi} : \tilde{M} \longrightarrow \mathbb{R}^3$  invariant by  $S_\theta$ , that is, there exists a holomorphic transformation  $S$  of  $\tilde{M}$  such that  $\tilde{M}/S = M$  and  $S_\theta \circ \tilde{\psi} = \tilde{\psi} \circ S$ . Note that the Weierstrass representation of  $\tilde{\psi}$  can not be induced on  $M$  unless  $\theta = 0$ , but the one form  $\phi_3$  is always well-defined on the quotient surface. Suppose now that  $M$  has finite topology. From the work of Meeks and Rosenberg [10] we know that  $\psi$  has finite total curvature—in particular,  $M$  has the conformal structure of a finitely punctured compact Riemann surface—and the only allowed behaviour at infinity is one of the following:

1. All the ends of  $\psi$  are asymptotic to non vertical parallel planes (planar type ends) that lift to planar type ends in  $\mathbb{R}^3$ , as in the Riemann example. If  $\theta \neq 0$ , these planes are necessarily horizontal.
2. All the ends of  $\psi$  are asymptotic to flat vertical annuli (Scherk type ends). This case forces the angle  $\theta$  to be rational, and we can cite the Scherk's second surface as an example.
3. All ends of  $\psi$  are asymptotic to ends of Helicoids (helicoidal type ends).

We will consider the helicoidal type ends. Take an annular end  $A$  of helicoidal type of a properly embedded minimal surface in  $\mathbb{R}^3/S_\theta$  as above. Then, [10] implies that  $A$  is conformally a punctured disk  $D^*(\varepsilon) = \{z \in$

$\mathbb{C} / 0 < |z| \leq \varepsilon$  and the well-defined one form  $\phi_3 = 2\frac{\partial x_3}{\partial z} dz$  can be written as

$$\phi_3 = \left( -\frac{i\beta}{z} + f(z) \right) dz, \quad 0 < |z| \leq \varepsilon,$$

where  $\beta$  is a non zero real number called the *slope* of the end, and  $f$  is a holomorphic function in  $D(\varepsilon) = D^*(\varepsilon) \cup \{0\}$ . Moreover,

- i) If  $\theta > 0$ , the Gauss map  $g$  is multivalued on  $A$ , but it can be continuously extended to  $z = 0$ , with vertical limit normal vector. If  $g(0) = 0$ , we can write  $g(z) = z^{k+a}$ ,  $k$  being a non negative integer and  $a = \frac{\theta}{2\pi}$ . If  $g(0) = \infty$ , the expression of  $g$  is  $g(z) = z^{-(k+a)}$ , with  $k, a$  as above.
- ii) If  $\theta = 0$ ,  $g$  is singly valued on  $A$  and the remaining assertions in case *i*) hold, with  $a = 0$ . If  $g$  has a zero of order  $k$  at  $z = 0$ , then  $\frac{1}{g}\phi_3$  has a pole of order  $k + 1$  without residue at this point. Symmetrically, if  $g$  has a pole of order  $k$  at the puncture, then  $\frac{1}{g}\phi_3$  has a zero of order  $k - 1$  and the residue at  $z = 0$  of  $g\phi_3$  vanishes.

Hence the trace of the end on a vertical cylinder  $C_R$  of radius  $R$  large is very close to a helix of slope  $\beta$  that rotates an angle  $2\pi(k + a)$  when  $0 \leq \arg(z) < 2\pi$ . Thus all the slopes of a properly embedded minimal surface in  $\mathbb{R}^3/S_\theta$  with finite topology and helicoidal type ends are equal up to sign —two slopes coincide if and only if the vertical limit normal vector at the ends are the same— and the number  $k$  in the expression of  $g$  above does not depend on the end.

Finally we recall a characterization of the Helicoid as the only such surface with the simplest topology, which will be useful for later purposes. The case  $\theta = 0$  is due to Toubiana [16], while when  $\theta \neq 0$  the result was proved by Meeks and Rosenberg [10]:

**Theorem 1 [10, 16].** *The only properly embedded minimal surface in  $\mathbb{R}^3/S_\theta$ ,  $0 \leq \theta < 2\pi$ , with genus zero and two helicoidal type ends is the Helicoid.*

### 3. The theorem.

Take  $\theta \in [0, 2\pi[$ . Let  $\psi : M \rightarrow \mathbb{R}^3/S_\theta$  be a properly embedded minimal surface with finite topology and helicoidal type ends whose singly periodic lifting  $\tilde{\psi} : \tilde{M} \rightarrow \mathbb{R}^3$  has vertical flux. By lemma 2.1,  $\tilde{\psi}_\lambda$  is well-defined on  $\tilde{M}$  for each  $\lambda > 0$  and can be chosen invariant by the same screw motion  $S_\theta$

provided that  $\theta \neq 0$ . We claim that this property can be extended to the translation case. If  $\theta = 0$ , each end  $p$  of  $M$  can be conformally parametrized by the punctured disk  $D^*(\varepsilon) = \{z \in \mathbb{C} / 0 < |z| \leq \varepsilon\}$ ,  $\varepsilon > 0$  and the Gauss map  $g$  is well-defined on  $D^*(\varepsilon)$ . If  $g(0) = 0$ , we can write

$$g(z) = z^k, \quad \phi_3 = \left( -\frac{i\beta}{z} + f(z) \right) dz, \quad z \in D^*(\varepsilon),$$

where  $k$  is a positive integer,  $\beta \in \mathbb{R} - \{0\}$  and  $f$  is a holomorphic function in  $D(\varepsilon)$ . As the third coordinate function is invariant by the deformation and  $g_\lambda(z) = \lambda z^k$ , we conclude that  $(g_\lambda, \phi_3)$  are the Weierstrass data of a helicoidal type end in the same ambient space  $\mathbb{R}^3/S_0$ , with horizontal limit tangent plane and the same slope  $\beta$  as in the case  $\lambda = 1$ . This implies that all the  $\psi_\lambda$  are  $S_0$ -invariant (note that putting  $g(z) = z^{k+a}$ ,  $a = \frac{\theta}{2\pi}$  we obtain a new proof of the  $S_\theta$ -invariance for any  $\theta$ ). For any  $\theta \in [0, 2\pi[$ , we will denote the induced surfaces by  $\psi_\lambda : M \rightarrow \mathbb{R}^3/S_\theta$ .

Next we will deal with the behaviour at infinity of this curve of minimal surfaces. The normal part of the variational field for this perturbation defines a function  $u_\lambda = \langle \frac{d\tilde{\psi}_\lambda}{d\lambda}, \tilde{N}_\lambda \rangle$  (here  $\tilde{N}_\lambda$  denotes the Gauss map of  $\tilde{\psi}_\lambda$ ) satisfying the Jacobi equation, that is

$$\Delta_\lambda u_\lambda + \|\nabla_\lambda \tilde{N}_\lambda\|^2 u_\lambda = 0 \quad \text{on } \tilde{M},$$

where the subscript  $\bullet_\lambda$  means that the corresponding object is measured with respect to the metric induced by  $\tilde{\psi}_\lambda$ . As  $\tilde{\psi}_\lambda$  is  $S_\theta$ -invariant, both  $\frac{d\tilde{\psi}_\lambda}{d\lambda}$  and  $\tilde{N}_\lambda$  are invariant by a rotation of angle  $\theta$  around the  $x_3$ -axis, thus  $u_\lambda$  induces a well-defined function on  $M$ , that we will also denote by  $u_\lambda$ . We claim that this function can be continuously extended by zero through the punctures. Using the notation in section 2, we have

$$\tilde{\psi}_\lambda = \left( \frac{1}{\lambda} \bar{F} - \lambda G, x_3 \right), \quad \tilde{N}_\lambda = \left( \frac{2\lambda g}{\lambda^2 |g|^2 + 1}, \frac{\lambda^2 |g|^2 - 1}{\lambda^2 |g|^2 + 1} \right) \in \mathbb{C} \times \mathbb{R},$$

hence

$$u_\lambda = -\frac{2\lambda}{\lambda^2 |g|^2 + 1} \operatorname{Real} \left[ \left( \frac{1}{\lambda^2} F + \bar{G} \right) g \right].$$

Consider an end of  $M$  such that the extended Gauss map takes the value  $(0, 0, -1)$  at the puncture. Thus its meromorphic data can be written as

$$g(z) = z^{k+a}, \quad \phi_3 = \left( -\frac{i\beta}{z} + f_1(z) \right) dz, \quad 0 < |z| \leq \varepsilon,$$

where  $k$  is an integer greater than or equal to zero,  $\beta \in \mathbb{R} - \{0\}$ ,  $a = \frac{\theta}{2\pi}$  and  $f_1$  is a holomorphic function. This implies that the well-defined functions  $Fg$ ,  $\overline{G}g$  satisfy  $\text{Real}(Fg)(0) = 0$ ,  $(\overline{G}g)(0) = 0$ , hence  $u_\lambda(0) = 0$ , as desired. The case  $g(0) = \infty$  can be solved in a similar way, and our claim is proved.

As consequence, the ends of  $\psi_\lambda$  are asymptotic to the ones of  $\psi$ , for each  $\lambda > 0$ . This fact will play an important role in the following result:

**Lemma 2.** *In the above conditions,  $\psi_\lambda$  is a proper embedding, for each  $\lambda > 0$ .*

*Proof.* We will argue as in [14]. Denote by  $B = \{\lambda > 0 / \psi_\lambda \text{ is one-to-one}\}$ . As  $1 \in B$ , the lemma will be proved if we deduce that  $B$  is open and closed in  $]0, \infty[$ .

If  $\lambda_0 \in B$ , two distinct ends of  $\psi_{\lambda_0}$  will have the same slopes up to sign, and from the maximum principle at infinity [11] they are separated one from another by a positive vertical distance, that does not depend on  $\lambda$  by the argument before this lemma. Thus  $\psi_\lambda$  is embedded for  $\lambda$  near  $\lambda_0$  and  $B$  is open.

Now take a sequence  $\{\lambda_n\}_{n \in \mathbb{N}} \subset B$  converging to  $\lambda_0 > 0$  (recall that all the surfaces  $\psi_\lambda$  are in the same ambient space). If  $\psi_{\lambda_0}$  is not injective, by the uniform convergence on compact sets of  $\psi_{\lambda_n}$  to  $\psi_{\lambda_0}$  and the classical maximum principle we insure that the image point set of  $\psi_{\lambda_0}$  is a properly embedded minimal surface with finite total curvature in  $\mathbb{R}^3/S_\theta$  and the map  $\psi_{\lambda_0} : M \rightarrow \psi_{\lambda_0}(M)$  is a finite covering. Again the maximum principle at infinity [11] gives an embedded  $\varepsilon$ -tubular neighbourhood  $U$  of  $\psi_{\lambda_0}(M)$ . Consider the orthogonal projection onto  $\psi_{\lambda_0}(M)$ ,  $\pi : U \rightarrow \psi_{\lambda_0}(M)$  and the oriented distance to  $\psi_{\lambda_0}(M)$ ,  $l : U \rightarrow \mathbb{R}$ . For  $n$  large enough we have that  $\psi_{\lambda_n}(M) \subset U$  and  $\pi \circ \psi_{\lambda_n} : M \rightarrow \psi_{\lambda_0}(M)$  is a proper local diffeomorphism. As  $\psi_{\lambda_n}$  has finite total curvature and  $\psi_{\lambda_0}(M)$  is not flat, this covering has a finite number of sheets. This number has to be one, because the continuous function  $l \circ \psi_{\lambda_n}$  separates the points in the fibers of the covering ( $\psi_{\lambda_n}$  is embedded). By the uniform convergence of  $\pi \circ \psi_{\lambda_n}$  to  $\pi \circ \psi_{\lambda_0} = \psi_{\lambda_0}$  on compact sets of  $M$  we have that also the covering  $\psi_{\lambda_0} : M \rightarrow \psi_{\lambda_0}(M)$  has only one sheet, a contradiction. This finishes the proof of the lemma.  $\square$

Now we can state the main result:

**Theorem 2.** *Let  $\psi : M \rightarrow \mathbb{R}^3/S_\theta$  be a properly embedded minimal surface with finite topology and helicoidal type ends,  $0 \leq \theta < 2\pi$ . If the singly*

*periodic lift of  $\psi$  has vertical flux, then  $\psi$  is the Helicoid.*

*Proof.* As usual, we will denote by  $\tilde{\psi} : \tilde{M} \rightarrow \mathbb{R}^3$  the lift of  $\psi$ . As  $\tilde{\psi}$  has vertical flux, the deformation  $\{\tilde{\psi}_\lambda / \lambda > 0\}$  is well defined on  $\tilde{M}$  and each singly periodic surface induces a proper immersion  $\psi_\lambda : M \rightarrow \mathbb{R}^3/S_\theta$ . By lemma 3.2,  $\psi_\lambda$  is always embedded hence the same holds for  $\tilde{\psi}_\lambda$  for each  $\lambda > 0$ . By lemma 2.1, this implies that the (possibly multivalued) Gauss map of  $\psi$  does not take vertical values on  $M$ . As at each end of  $\psi$ , the well-defined meromorphic one-form  $\phi_3$  has a simple pole, we conclude that  $\phi_3$  has no zeroes on the compactified surface  $\overline{M}$  obtained by attaching the punctures to  $M$ . As the Euler characteristic of  $\overline{M}$  is given by  $\chi(\overline{M}) = \#\{\text{poles of } \phi_3\} - \#\{\text{zeroes of } \phi_3\}$ , we have that  $\chi(\overline{M})$  is positive hence  $\overline{M}$  is topologically a sphere and  $\psi$  has only two ends. Now the theorem follows directly from Toubiana-Meeks-Rosenberg theorem.  $\square$

We will finish by proving the following characterization of the Helicoid in terms of rigidity. As we pointed out in the introduction, there are two notions of rigidity for minimal surfaces:

**Definition 1.** A minimal surface in  $\mathbb{R}^3$  is said (minimally) rigid if the inclusion represents the unique isometric minimal immersion of such a surface up to a rigid motion in  $\mathbb{R}^3$ .

**Definition 2.** A minimal surface in  $\mathbb{R}^3$  is said weakly rigid if every intrinsic isometry extends to an isometry of  $\mathbb{R}^3$ .

Rigidity implies weak rigidity but the converse fails, as demonstrates the Helicoid. In fact, this last example lies in a large family of surfaces for which the weak rigidity is true, by the following theorem of Meeks [8] and Meeks and Rosenberg [9]:

**Theorem 3 [8, 9].** *Let  $\tilde{M} \subset \mathbb{R}^3$  be a connected, properly embedded minimal surface, invariant under an infinite discrete group  $G$  of isometries of  $\mathbb{R}^3$ . If  $\tilde{M}/G$  has finite topology, then  $\tilde{M}$  is weakly rigid.*

In fact, theorems 5.3 and 11.3 in [8], and theorem 10 in [9] prove that except when  $G$  is generated by a screw motion  $S_\theta$ ,  $0 \leq \theta < 2\pi$  and the ends are of helicoidal type, we can choose a closed curve in  $\tilde{M}$  with non zero flux. This implies that the conjugate surface of  $\tilde{M} \subset \mathbb{R}^3$  is not well-defined and by the Calabi-Lawson characterization [1, 6] this is equivalent to the (strong)

rigidity of  $\tilde{M}$ . The desirable result in the remaining case would be that the only properly embedded minimal surface in  $\mathbb{R}^3$  invariant by a screw motion, with finite topology in the quotient and ends of helicoidal type that is not rigid is the Helicoid. Now we can give a proof of this fact.

**Theorem 4.** *Let  $\tilde{\psi} : \tilde{M} \rightarrow \mathbb{R}^3$  be a non flat, properly embedded minimal surface invariant by an infinite discrete group  $G$  of isometries of  $\mathbb{R}^3$ . If  $\tilde{M}/G$  has finite topology, then  $\tilde{\psi}$  is rigid or it is the Helicoid.*

*Proof.* As we showed in the discussion above, we can restrict ourselves to the case in which  $G$  is the cyclic group generated by a screw motion  $S_\theta$ ,  $0 \leq \theta < 2\pi$ , and the ends of  $\tilde{\psi}$  are of helicoidal type. If  $\tilde{\psi}$  is not rigid, by the Calabi-Lawson characterization we have that the flux along any closed curve is zero, hence theorem 3.2 applies and we conclude that the surface is a Helicoid.  $\square$

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