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Some new examples for nonuniqueness of the evolution problem of harmonic maps

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We find some new examples to show nonuniquence for the heat flow of harmonic maps where weak solutions satisfy the same monotonicity property.

1. Introduction.

Let (M,g) be a compact Riemannian manifold (with or without boundary) and let (N,h) be another compact Riemannian manifold without boundary. Let u be a map from M to N which belongs to $H^{1,2}(M,N)$. We define the energy of u by

$$E(u,M)=\int_M |du|^2\,dM \tag{1.1}$$

where $|du|$ denotes the Hilbert-Schmidt norm of the differential $du(x)$ (see [EL]). In local coordinates (x^i) and (y^α) on M and N, we have

$$|du(x)|^2=g^{ij}(x)\,\frac{\partial u^\alpha}{\partial x^i}\,\frac{\partial u^\beta}{\partial x^j}\,h_{\alpha\beta}(u(x)).$$

The map $u:(M,g)\to(N,h)$ is called a weak harmonic map if $u\in H^{1,2}(M,N)$ and satisfies

$$\triangle_M u+A(u)(du,du)=0 \tag{1.2}$$

in the sense of distributions in M where $\triangle_M$ denotes the Laplace-Beltrami operator on M and $A(u)(du,du)$ is the second fundamental form of N. In local coordinates (x^i) and (y^α) on M and N, the harmonic map u satisfies

$$\tau^\gamma(u)=g^{ij}\left(\frac{\partial^2 u^\gamma}{\partial x^i\partial x^j}-{}^M\Gamma^k_{ij}\frac{\partial u^\gamma}{\partial x^k}+{}^N\Gamma^\gamma_{\alpha\beta}(u)\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^j}\right)=0$$

where ${}^M\Gamma$ and ${}^N\Gamma$ are the Christoffel symbols of the connections on M and N.

The heat flow for harmonic maps is defined as follows. We say $u(x,t)$: $M \times [0,\infty) \to N$ is a weak solution of the following evolution problem:

$$(1.3)\qquad \begin{aligned} &\frac{\partial u}{\partial t} = \triangle_M u + A(u)(du, du)\\ &u(x,0) = u_0(x) \quad \text{for } x \in M\\ &u(x,t) = u_0(x) \quad \text{for } (x,t) \in \partial M \times [0,\infty). \end{aligned}$$

The evolution problem for harmonic maps is introduced by Eells and Sampson in a fundamental paper [ES] to prove the existence of smooth harmonic maps in the case that the N has the nonpositive sectional curvature and u_0 is smooth. Coron and Ghidaglia in [CG] proved that if $M = N = S^k$ with $k \geq 3$ the heat flow must blow up for some smooth maps u_0. Later, Chang, Ding and Ye in [CDY] showed that the heat flow (1.3) must blow up in finite time for some smooth maps u_0 even for $M = N = S^2$. When the dimension of M is 2, Struwe in [S1] proved early the existence and uniqunence of a weak global solution to the heat flow (1.3) where the solution is smooth away from a finite singular point in $M \times [0,\infty)$.

For higher dimensional case; i.e. $\dim M \geq 3$, Chen and Struwe in [CS] (also [CL] and [S2]) proved a global existence of weak solutions of heat flow (1.3) in which the solution is partial regular.

Let $M = B^3$ and $N = S^2$ where B^3 and S^2 denote respectively the unit ball and the unit sphere in $\mathbb{R}^3$. In this case, the Problem (1.3) has the following simple form:

$$(\text{C})\qquad \begin{aligned} &\frac{\partial u}{\partial t} = \triangle u + |\nabla u|^2 u\\ &u(x,0) = u_0(x) \quad \text{for } x \in B^3\\ &u(x,t) = u_0(x) \quad \text{for } (x,t) \in \partial B^3 \times [0,\infty). \end{aligned}$$

Following Chen and Struwe [CS], $u(x,t) : B^3 \times [0,\infty) \to S^2$ is called a weak solution to problem (C) if the $u(x,t)$ satisfies the following (i)-(iv):

(i) $\dfrac{\partial u}{\partial t} \in L^2(M \times [0,\infty))$, $E((u(\cdot,t)) \leq E(u(\cdot,0)) \quad \forall t \in [0,\infty)$,

(ii) u satisfies the first equation of (C) in the weak sense on $B^3 \times (0,\infty)$,

(iii) $u(x,0) = u_0(x)$ in the trace sense,

(iv) $u(x,t) = u_0(x)$ on $\partial M \times [0,\infty)$ in the trace sense.

Coron in [Co] constructed some examples to show that the heat flow can have infinitely many weak solutions for the same maps u_0 in the case that $M = \bar{B}^3$ and $N = S^2$. Coron's idea is to show that the weak solution in [CS](also [CL]) has the following monotonicity property: For any θ in $C_0^\infty(B^3)$, any compact K of the interrior of $\{\theta = 1\}$ there exists a constant C such that for a. e. t_1, t_2 with $t_1 < t_2$ and for a in K

$$(*) \quad t_1^{-1/2} \int_{B^3} \theta^2 \left|\nabla u\right|^2 (x, t_2 - t_1) \exp \; - \frac{|x-a|^2}{4t_1}$$
$$\leq C(t_2^{1/2} - t_1^{1/2}) + t_2^{-1/2} \int_{B^3} \theta^2 \left|\nabla u_0\right|^2 \exp \; - \frac{|x-a|^2}{4t_2}.$$

Then he find some harmonic maps as weak solutions which do not satisfy the above monotonicity property $(*)$. Coron in [Co] also pointed out that his method does not allow to produce an initial data such that the heat flow has at least two weak solutions satisfiying $(*)$. Then there exists an open problem for the evolution problem of harmonic maps whether weak solutions of heat flow (1.3) with the monotonicity property $(*)$ or the monotonicity inequality for all regular points in [CS] are unique.

In this paper, we give a negative answer to the above problem.

Theorem A. *There exist some initial data u_0 such that the problem* (C) *has infinitely many weak solutions which satisfy the same monotonicity inequality* $(*)$.

Finally, we know from [F] and [CLL] that all "stationary" weak solutions to Problem (C) also satisfy the energy inequality and the monotonicity inequality defined in [CS] and [CLL], thus all weak solutions in Theorem A satisfy the energy inequality and the monotonicity inequality in regular points as in [CS].

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2. The proof of Theorem A.

Definition 1. A map u is called a weakly stationary harmonic map from B^3 into the sphere S^2 if u is a weak harmonic map and also satisfies

$$\frac{d}{ds} E\left(u_s; B^3\right)\Big|_{s=0} = 0$$

where $u_s(x) := u(x + s\,\xi(x))$ and ξ has a compact support within B^3.

Let, for k in $\{1,2,3\}$,

$$\mu^k = \left(u^i_{x_j} u^i_{x_j}\right)_{x_k} - 2\left(u^i_{x_k} u^i_{x_j}\right)_{x_j}.$$

We know (see, e. g. [Co], page 341) that u is stationary if and only if

$$\mu^k = 0 \quad \forall k \in \{1,2,3\}. \tag{2.1}$$

Lemma 1. *Assume that u_0 is a weakly stationary harmonic map from B^3 into S^2. Then the map u_0 satisfies the monotonicity inequality $(*)$.*

Proof. This results is from the Remark 3 in [Co]. Since u_0 is a weak stationary harmonic map, u_0 satisfies (2.1). Then from the proof in [Co], pages 340–341, we know that u_0 satisfies the monotonicity inequality $(*)$. □

Let us recall how a weak solution of (C) is constructed in [CS] (or [CL]). For each $\varepsilon > 0$, the Ginzburg-Landau functional E_ε is defined by

$$E_\varepsilon(u) := \frac{1}{2}\int_{B^3}\left(|\nabla u|^2 + \frac{1}{2\varepsilon^2}\left(|u|^2 - 1\right)^2\right)dx$$

for a function $u \in H^{1,2}(B^3, \mathbb{R}^3)$. The Euler-Langrange equation is

$$(\mathrm{BD}_\varepsilon) \qquad \begin{aligned} &\triangle u + \frac{1}{\varepsilon^2}\left(1 - |u|^2\right)u = 0,\\ &u(x) = u_0(x) \quad \text{for } x \in \partial B^3 \end{aligned}$$

One consider the Cauchy problem for the heat flow associated to E_ε

$$(\mathrm{C}_\varepsilon) \qquad \begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + \frac{1}{\varepsilon^2}\left(1 - |u|^2\right)u,\\ &u(x,0) = u_0(x) \quad \text{for } x \in B^3\\ &u(x,t) = u_0(x) \quad \text{for } (x,t) \in \partial B^3 \times [0,\infty) \end{aligned}$$

where u_0 is given $H^{1,2}\left(B^3, S^2\right)$. One easily sees that the evolution problem (C_ε) has a unique solution u^ε in $C^0\left([0,\infty); H^{1,2}\left(B^3, \mathbb{R}^3\right)\right) \cap C^\infty\left((0,\infty) \times B^3; \mathbb{R}^3\right)$.

Lemma 2. *The solution $u^\varepsilon(x,t)$ of (C_ε) tends to a weak solution U^ε of problem $(\mathrm{BD}_\varepsilon)$ as $t \to \infty$.*

Lemma 3. (i) *There exists a subsequence ε_i such that as $\varepsilon_i \to 0$, u^{ε_i} tends weakly in $H^{1,2}(\bar{B}^3 \times [0,\infty))$ to a map $\bar{u}(x,t)$ which $u(x,t)$ is a weak solution of* (C). *Moreover, for a sequence t_i, the solution $\bar{u}(x,t)$ converges weakly to a weak harmonic map $u_\infty(x)$ in $\bar{B}^3$ as $t_i \to \infty$.*

(ii) *The solution $u(x,t)$ of* (C) *satisfies* $(*)$.

Proof. The first part of Lemma 2 is from [CS] and [CL]. The second part of Lemma 2 is from [Co], pages 338–340. □

Let us consider a boundary value problem for harmonic maps,

$$\text{(BD)} \qquad \begin{aligned} &\triangle u + |\nabla u|^2 u = 0 \quad x \in B^3 \\ &u(x) = u_0(x) = x \text{ on } \partial B^3. \end{aligned}$$

Lemma 4. *For $x_0 \in \partial B^3$, there exists a weak harmonic maps u_0 to* (BD) *which is smooth in $\bar{B}^3 \backslash \{x_0\}$. Moreover, the u_0 is stationary.*

Proof. The existence of the weak harmonic map u_0, which is smooth $\bar{B}^3 \backslash \{x_0\}$, is due to Poon in [P]. It is straightfhtful that the u_0 is stationary since it is smooth inside B^3 (see, e.g. [GSY]). □

Proposition 5. *For $\varepsilon > 0$, let U^ε be a solution to* (BD_ε) *with $u_0|_{\partial B^3} = x$. The U^ε weakly converges to the minimizing harmonic map $\frac{x}{|x|}$ to* (BD) *in $H^{1,2}(B^3, \mathbb{R}^3)$ as $\varepsilon \to 0$.*

Proof. Using (BD_ε), we have the following Pohozaev's identity for $u = U^\varepsilon$:

(2.2)

$$\begin{aligned} \sum_{i,j=1}^{3} (u_{x_j} x_i u_{x_i})_{x_j} &= \sum_{i,j=1}^{3} u_{x_j x_j} x_i u_{x_i} + \sum_{j=1}^{3} u_{x_j}^2 + \sum_{i,j=1}^{3} u_{x_j} x_i u_{x_i x_j} \\ &= \Delta u \sum_{i=1}^{3} x_i u_{x_i} + \frac{1}{2} \sum_{i=1}^{3} \left(x_i |\nabla u|^2\right)_{x_i} - \frac{1}{2} |\nabla u|^2 \\ &= \frac{1}{\varepsilon^2} \sum_{i=1}^{3} \left[x_i \left(1 - |u|^2\right)^2\right]_{x_i} - \frac{3}{4\varepsilon^2} \left(1 - |u|^2\right)^2 \\ &\quad + \frac{1}{2} \sum_{i=1}^{3} \left(x_i |\nabla u|^2\right)_{x_i} - \frac{1}{2} |\nabla u|^2. \end{aligned}$$

Integrating both sides of the above equality gives

$$\frac{1}{2}\int_{B^3}|\nabla U^\varepsilon|^2\,dx+\frac{1}{2}\int_{\partial B^3}\left|\frac{\partial U^\varepsilon}{\partial n}\right|^2 d\sigma + \int_{B^3}\frac{3}{4\varepsilon^2}\left(1-|U^\varepsilon|^2\right)^2=\frac{1}{2}\int_{\partial B^3}|\nabla_\tau U^\varepsilon|^2\,d\tau=4\pi \tag{2.3}$$

where n is the exterior norm vector of ∂B^3, ∇_τ denote the differential of u on ∂B^3. From (2.3), there exists a function $V\in H^{1,2}(B^3,S^2)$ with $V=x$ on ∂B^3 such that U^{ε_i} weakly converges in $H^{1,2}(B^3,\mathbb{R}^3)$ to a harmonic map V for $\varepsilon_i\to 0$ and

$$\int_{B^3}|\nabla V|^2\,dx\le\liminf_{\varepsilon_i\to 0}\int_{B^3}|\nabla U^{\varepsilon_i}|^2\,dx\le 8\pi.$$

Then from the Theorem in [BCL]; Theorem 7.1, we know that for all map $u\in H^{1,2}(B^3,S^2)$ with $u|_{\partial B^3}=x$,

$$\int_{B^3}|\nabla u|^2\,dx\ge 8\pi \tag{2.4}$$

and $\frac{x}{|x|}$ is the unique minimizing harmonic map from $H^{1,2}(B^3,S^2)$. Therefore we know that V must be the map $\frac{x}{|x|}$. Since this is true for any subsequence $\varepsilon_i\to 0$, U^{ε_i} converges weakly to $\frac{x}{|x|}$ in $H^{1,2}(B^3,\mathbb{R}^3)$. □

Proof of Theorem A. Let u_0 be a Poon's harmonic map in Lemma 4. Then we choose u_0 to be a initial value to the problem (C). Then we know $u_1(x,t)=u_0(x)$ be a solution to the problem (C) which satisfying $(*)$.

Let us assume that $u^\varepsilon(x,t)$ is a solution of (C_ε). Then $u^\varepsilon(x,t)$ satisfies an energy inequality; i.e. for any $T>0$ and any $\varepsilon>0$,

$$\int_0^T\int_{B^3}|\partial_t u^\varepsilon|^2\,dx\,dt+\int_{B^3}|\nabla u^\varepsilon(\cdot,T)|^2\,dx\le\int_{B^3}|\nabla u_0|^2\,dx \tag{2.5}$$

This is easily obatined by choosing a test function ∂u^ε in (C_ε)(see [CS]). By Lemma 3, there exists a subsequence ε_k such that as $k\to\infty$, $u^{\varepsilon_k}(x,t)$ weakly converges to $u(x,t)$ in $H^{1,2}(B^3,\mathbb{R}^3)$ for $t>0$ where $u(x,t)$ is a solution of heat flow (C) by Lemma 3 and $u(x,t)=x$ on ∂B^3. From (2.5), we have for any $T>0$

$$\limsup_{k\to\infty}\int_0^T\int_{B^3}|\partial_t u^{\varepsilon_k}|^2\,dx\,dt\le\int_{B^3}|\nabla u_0|^2\,dx. \tag{2.6}$$

Then by the Fatou lemma, we know

$$\int_0^T \liminf_{k\to\infty} \int_{B^3} |\partial_t u^{\varepsilon_k}|^2\, dx\, dt \le \int_{B^3} |\nabla u_0|^2\, dx. \tag{2.7}$$

Letting $T \to \infty$ in (2.7), we have

$$\int_0^\infty \liminf_{k\to\infty} \int_{B^3} |\partial_t u^{\varepsilon_k}|^2\, dx\, dt \le \int_{B^3} |\nabla u_0|^2\, dx \tag{2.8}$$

From (2.8), there exists a subsequence t_i such that as $t_i \to \infty$,

$$\lim_{t_i\to\infty} \int_{t_i-1}^{t_i} \liminf_{k\to\infty} \int_{B^3} |\partial_t u^{\varepsilon_k}|^2\, dx\, dt = 0$$

By Hölder's inequality, energy inequality (2.5) and the above identity, we have

$$\lim_{t_i\to\infty} \int_{t_i-1}^{t_i} \liminf_{k\to\infty} \int_{B^3} |\partial_t u^{\varepsilon_k}|\, |\nabla u^{\varepsilon_k}|\, dx\, dt = 0 \tag{2.9}$$

On the other hand, using (C_ε), we have the following Pohozaev's identity for $u = u^{\varepsilon_k}(x,t)$:

$$\begin{aligned}\sum_{i,j=1}^3 (u_{x_j} x_i u_{x_i})_{x_j} &= \sum_{i,j=1}^3 u_{x_j x_j} x_i u_{x_i} + \sum_{j=1}^3 u_{x_j}^2 + \sum_{i,j=1}^3 u_{x_j} x_i u_{x_i x_j}\\ &= \Delta u \sum_{i=1}^3 x_i u_{x_i} + \frac{1}{2}\sum_{i=1}^3 \left(x_i |\nabla u|^2\right)_{x_i} - \frac{1}{2}|\nabla u|^2\\ &= \frac{1}{\varepsilon^2}\sum_{i=1}^3 \left[x_i \left(1-|u|^2\right)^2\right]_{x_i} - \frac{3}{4\varepsilon^2}\left(1-|u|^2\right)^2\\ &\quad + \frac{1}{2}\sum_{i=1}^3 \left(x_i|\nabla u|^2\right)_{x_i} - \frac{1}{2}|\nabla u|^2 + \partial_t u \sum_{i=1}^3 x_i u_{x_i}.\end{aligned}$$

Integrating both sides of the above equality (Note $u = u^{\varepsilon_k}(x,t)$) gives

(2.10)

$$\begin{aligned}&\frac{1}{2}\int_{B^3} |\nabla u^{\varepsilon_k}(\cdot,t)|^2 dx + \frac{1}{2}\int_{\partial B^3} \left|\frac{\partial u^{\varepsilon_k}(\cdot,t)}{\partial n}\right|^2 d\sigma + \int_{B^3} \frac{3}{4\varepsilon^2}\left(1-|u^{\varepsilon_k}(\cdot,t)|^2\right)^2\\ &= \frac{1}{2}\int_{\partial B^3} |\nabla_\tau u^{\varepsilon_k}(\cdot,t_k)|^2\, d\sigma + \int_{B^3} \partial_t u^{\varepsilon_k}(\cdot,t) \sum x_i u^{\varepsilon_k}_{x_k}(\cdot,t)\, dx\\ &= 4\pi + \int_{B^3} \partial_t u^{\varepsilon_k}(\cdot,t) \sum x_i u^{\varepsilon_k}_{x_k}(\cdot,t) dx\end{aligned}$$

where n is the exterior norm vector of ∂B^3, ∇_τ denote the differential of u on ∂B^3.

Next, we will prove the the heat flow (C) has at lest two different solutions by a contradicted method, i.e. assume the weak solutions of heat flow (C) are unique, i.e. $u(x,t) \equiv u_0$ (otherwise, weak solutions has at least two solutions). Letting $\varepsilon_k \to 0$ in (2.10),

$$\begin{aligned}\int_{B^3} |\nabla u_0|^2\,dx &\le \liminf_{k\to\infty} \int_{B^3} |\nabla u^{\varepsilon_k}(\cdot,t)|^2\,dx \\ &\le 8\pi + C \liminf_{k\to\infty} \int_{B^3} |\partial_t u^{\varepsilon_k}|\,|\nabla u^{\varepsilon_k}|dx\,dt\end{aligned} \tag{2.11}$$

Intergrating (2.11) on $[t_i-1,t_i]$, letting $t_i \to \infty$ and using (2.9) gives

$$\begin{aligned}\int_{B^3} |\nabla u_0|^2\,dx &\le 8\pi + C \lim_{t_i\to\infty} \int_{t_i-1}^{t_i} \liminf_{k\to\infty} \int_{B^3} |\partial_t u^{\varepsilon_k}|\,|\nabla u^{\varepsilon_k}|\,dx\,dt \\ &= 8\pi\end{aligned} \tag{2.12}$$

since u_0 does not depends on t.

From the Theorem in [BCL]; Theorem 7.1, we know that for all map $u \in H^{1,2}(B^3,S^2)$ with $u|_{\partial B^3} = x$,

$$\int_{B^3} |\nabla u|^2\,dx \ge 8\pi$$

and $\frac{x}{|x|}$ is the unique minimizing harmonic map from $H^{1,2}(B^3,S^2)$, so we know that u_0 must be the minimizing harmonic map $\frac{x}{|x|}$, this is contradicted by our initial condition. This means that the solution $u(x,t)$ is different from $u_1(x,t) = u_0(x)$ This proves that the problem (C) has two different solutions satisfying $(*)$. Infinitely many solutions can be easily proven by the same steps in [Co]. □

From the proof of Theorem A, we have

Corollary. *Aussume that $u_0 : B^3 \to S^2$ be a non-minimizing weak harmonic map with the boundary condition $u_0(x) = x$ on ∂B^3. Choosing the u_0 as a initial data to the problem* (C)*, then the problem* (C) *has infinitely many weakly solutions.*

Remark. From the result in [P], there exists a non-minimizing weak harmonic u with the boundary condition $u(x) = x$ on ∂B^3 such that $u \in C^\infty(\bar{B}^3 \backslash \{x_0\})$ for any $x_0 \neq 0 \in B^3$.

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